AN ABSTRACT OF THE THESIS OF

Janet Marie Sharp Laird for the Master of Science

in Mathematics presented on May 1, 1990

TITLE: MATHEMATICS HISTORY AS A MOTIVATIONAL TOOL IN THE MATHEMATICS CLASSROOM

Abstract Approved : George L. Downing

The purpose of this thesis is to encourage the use of mathematics history as a motivational tool in the mathematics classroom. An overview of some educational literature regarding motivation theory is presented. A brief discussion regarding some mathematics history which is related to chosen mathematical topics follows the literature review. Finally, a sketch of some historical topics of the mathematics branch of Number Theory is given. The conclusion includes the writer's views towards utilizing mathematics history in the mathematics classroom.

MATHEMATICS HISTORY

AS A

MOTIVATIONAL TOOL

IN THE

MATHEMATICS CLASSROOM

A Thesis

Presented to

The Division of Mathematics and Computer Science

Emporia State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

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May, 1990

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ACKNOWLEDGMENTS

I would like to thank Dr. George Downing for being a fabulous teacher. He used some history of Non-Euclidean Geometry in class and this served to enhance the motivation of my own study. Always ready with an encouraging word, he was of valuable assistance in the completion of the requirements for the degree I have earned. He also read this thesis an infinite (or n + 1) number of times, for this he will always have a special place in my heart.

I wish to also thank my parents for their eternal belief and faith in God. This attitude made it easy to believe that everything **would** work out. Without them, I would have been lost many times. My parents have lovingly offered continual support and encouragement throughout my life and I wish to acknowledge their unending love and support. I only hope that I can make them proud with this work.

Finally, without the tear-soaked shoulder of my husband, this thesis would have been only a dream. He sat through countless, no doubt repetitious, discussions about the mystical Pythagoras and the mysterious Euclid, and always managed to have that grin on his face.

My friends, loved ones, and teachers have been flexible and pleasantly helpful throughout this semester. Thank you.

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CHAPTER I

INTRODUCTION

Introduction

Every teacher in every classroom in America has probably attempted at least one lesson that was designed to enhance student motivation. Motivational techniques have been changed and recycled within the context of educational theory ever since people have been awarded teaching certificates. The teaching of students about mathematics has certainly not been without its student motivational shortcomings. Among the list of motivational strategies useful in creating a positive mathematics atmosphere in the regular classroom is the combination of teaching mathematics concepts in conjunction with related historical content.

Statement of the Problem

Although all students are motivated to channel their energies in one direction or another, it is unfortunate for the mathematics teacher that a considerable number of pupils are not motivated to learn the art of mathematics. Consequently, it behooves the teacher to employ motivational strategies in order to guide students toward choosing to receive and learn mathematics well. Many avenues of methodology exist for use by the teachers in the schools, and the teacher must choose carefully the methods he or she will utilize. This choice must cause each individual student's interest in mathematics and eagerness to do mathematics to increase. Thus, the purpose of this thesis is to explore the realm of student motivation within the mathematics classroom through the use of mathematics history lessons for a selected set of mathematical concepts and content.

Importance of the study

Success in learning mathematics relies greatly on the skills of the teacher to explain mathematics and to motivate students. That is, students should leave their mathematics classroom feeling confident and secure with their abilities to do mathematics as well as experiencing enjoyment of the field itself.

In today's society it is imperative that people be able to understand and utilize mathematics. Mathematics is necessary to perform day to day tasks which could range from budgeting money to programming computers. The need for mathematics is, therefore, unmistakably clear. This study will establish a possible methodology that teachers of mathematics may use to enhance the learning of mathematics by including some of its history in presentations to students.

Motivating students to learn and enjoy mathematics may lead students to consider careers which they may have previously ignored. Students who have learned mathematics and enjoyed doing so may simply lose any mathematics anxiety or fear and, consequently, think about day to day living with more logic and continuity than they would have otherwise. It is the opinion of the writer that the suggestions included in this study for enhancing student motivation could be of considerable value in the classroom environment.

Limitations of the study

It is not feasible in this study to consider all historical occurrences related to the field of mathematics. Neither is it feasible to discuss all motivational techniques used by educators. Certainly, not all strategies work on all mathematics students' learning styles. Consequently, five mathematics classroom topics along with the historical motivational strategy most appropriate for that lesson have been chosen as small representations of the vast number of possibilities. The use of mathematics began centuries ago and continues to expand to date. Many exciting discoveries have been made; yet, only a few have been selected for this study. The writer is aware of the existence of many other motivational ideas besides the ones discussed in this study and she acknowledges their usefulness as well.

Background of the Problem

As stated previously, there are several methods endorsed by educators for motivating students to learn. However, some of those methods do not seem to lend themselves well to the mathematics classroom. In 1969, The National Council of Teachers of Mathematics published its yearbook, Historical Topics for the Mathematics Classroom. (37 : 1-524) An excellent work in and of itself, yet, it lacked the practical aspect of being immediately useful in the classroom. Brilliant explanations and dramatic diagrams were concisely recorded on nearly every page. However, aside from an introduction explaining that the mathematics history compilation in the book would be very useful in a classroom, there were no actual classroom examples of those uses given in the yearbook. Once a teacher had become accustomed to using history in his or her mathematics classroom, the writer believes that the aforementioned text would be a wonderful source of information.

While teaching a college algebra course, the writer found that many sections of the text were preceded by a brief historical sketch of the mathematical concepts that were to follow. The brief sketch centered upon either the mathematician to whom the credit has been granted for that particular problem, or else an interesting anecdote relating to the material. Those synopses brought mathematics to life for the writer. Suddenly, mathematics had roots, it had been developed, not inherited. In an effort to transfer those new positive feelings about mathematics to the students, the writer often included mathematics history in her mathematics presentations. Encouragement from many of the students to continue the history insertions resulted in the premise of this thesis.

Development of the Study

The general method used in the development of the study grew from the literature search. Much historical research proceeded in the following manner: (1) a mathematical concept was considered, and (2) some aspects of the history surrounding the development of the concept were described. Consequently, this thesis follows a similar format.

Sources of Information

The principal sources of information used in developing this study were books written about mathematics history, secondary mathematics texts, and secondary education methodology texts. These various sources will be referenced as they are discussed in subsequent chapters. Also, the writer will draw from some of her teaching experiences as an additional source.

Organization of the Thesis

Chapter I has been devoted to presenting a clear description and development of the problem addressed in this thesis. Chapter II will present a discussion of current literature regarding educational trends and suggestions concerning the motivation of students. In Chapter III, the first historical topics of the thesis will be presented. Specific concepts from the intermediate grades are stated, followed by the historical development. Chapter IV will be a continuation of Chapter III in that, the mathematical concepts stated will be drawn from the high school level. This will be followed by a brief account of some history involved with that particular concept. Chapter V will contain a historical perspective of Number Theory. The mathematics involved in this historical account should be clear without the aid of a mathematics textbook. Chapter VI will be a summary of the study together with recommendations by the writer.

CHAPTER II

SURVEY OF RELATED LITERATURE

Introduction

A lack of motivation on the part of students to learn mathematics is a problem encountered by many mathematics teachers. However, there are strategies which mathematics teachers may employ to overcome these deficiencies of student motivation in their classrooms. According to Lawrence R. Lyman, Alfred P. Wilson, C. Kent Garhart, Max O. Heim, and Wyonna O. Winn, "Teachers cannot make students learn. Teachers can however, use motivation theory to make a student more likely to want to learn." (33 : 55) The theory to which they alluded consists of seven factors as follows:

1. Student Success
2. Teacher Feedback
3. Classroom Climate
4. Student Concern
5. Learning Payoff
6. Student Interest and Involvement
7. Teacher Enthusiasm

Although this list comes from the work of the previously mentioned authority, it is echoed by many other educational experts in various contexts.

In this chapter, the writer will present a brief review of some pertinent literature with respect to the noted motivational factors. A survey of some literature regarding mathematics history as it pertains to motivation will conclude the discussion.

Student Success as a Motivational Factor

Student success is one of the biggest motivators in any classroom. When a particular activity is successfully completed by a student, and he or she experiences a good feeling of accomplishment, it is called student success. It is the yardstick by which even many teachers measure their own effectiveness. Quite obviously then, (especially if teachers tout it as important) students would be quickly propelled either positively or negatively by the presence or absence of success.

"All students need to experience success in school on a regular basis." (33 : 56) This can be their reason for coming to class. However, according to M. C. Wittrock, there is a need for students to "attribute success or failure to their own effort, rather than to factors over which they have no control..." (52 : 31) It seems quite obvious that being successful at a particular activity would invite more work by the student in that area. An individual's willingness to pursue tasks at which he or she has experienced or expects to experience success, certainly exemplifies motivation by student success. A major element of this type of motivation is self esteem. Donald R. Grossnickle and William B. Thiel stated that "Some educators believe that a healthy self-esteem is the single most important quality of successful people." (26 : 24) They go on to say, "Students who have a poor self concept are likely to also have motivational and learning problems." (26 : 24) The importance of self-esteem cannot be denied; it is certainly one of the most important aspects of motivation. Grossnickle and Thiel continue, "...positive self esteem [promotes] higher rates of success in students academic performance..." (26 : 24) In one sense, student success can create student motivation and even elicit good teaching, while in another sense, motivated students can achieve student success. However, the stimulus for the motivated student, ideally, is not ultimately the responsibility of an outside force. Whenever possible, motivation should culminate in student success, so that a circular relation can occur which would lead again to increased student motivation. This certainly affects the learning process in a positive manner.

It is one of the teacher's responsibilities to help the student be successful. When a student finds success, the student also finds personal satisfaction. Personal satisfaction is essential to a positive self-concept. Again, a cyclic series of events occurs. Much research tends to support that the road to this goal lay paved with teacher expectations. Some educational researchers seem to agree with this position. Thomas R. McDaniel states "the principle of inviting success depends on high expectations..." (34 : 47) McDaniel continues with praise for teachers who understand "the importance of self-fulfilling prophecies and of teacher behaviors that communicate high expectations." (34 : 47) Teachers need to have high expectations of students in order for the students to feel worthy and to raise their own motivational levels. Ernest R. Boyer even quoted one student, who said "we don't want more busy work, but we would like to have harder work..." (6 : 148) Consequently, there is a challenge here, and a good teacher should recognize and meet it. Belief in each individual student on the part of the teacher must be obvious to the student. Students must be treated with dignity, respect, and understanding. In other words, every student should feel that the teacher considers each student in the mathematics classroom important. The good teacher will see that this happens.

Lest teachers be hasty in putting stock in the use of easy activities, Wittrock states, "Praise for success of an easy task often implies that the teacher has low expectations for that student." (52 : 31) Tasks below students' levels are not the answer for achieving that precious success. The tasks must grant success to low-ability students while concurrently challenging high-ability students. Activities must be "both challenging and realistically attainable." (26 : 26) Although this sounds impossible, it is not. It is difficult, but it is also one of the things which makes teaching worthwhile. Teacher expectation can actually reduce motivation if it is used in a fashion which indicates to students that the expectations are low or not "from the heart." For example, repeated use of the mundane praise "good" or "right" may cause these words to lose their potency if students believe that the teacher did not individualize the praise for their particular achievement.

Teacher Feedback as a Motivational Factor

Teacher feedback was alluded to as an important factor of student success. However, it is in its own right a legitimately autonomous factor of student motivation. An indication to the student as to his or her current grade in the course, a remark to the student such as "that was a dumb question," or giving a big smile to a student are all examples of teacher feedback. Much research in this area discusses "appropriate" feedback. Many teachers give feedback; however, it is often inappropriate feedback. For example, unforgiving criticism and cruel sarcasm, can be a part of teachers' feedback. The cliche, "sarcasm-the whipping board of the eighties!" probably relates to what teachers say more often than they like to admit. Feedback is effective when it "maintains the momentum of learning, provides encouragement to students, and lets the student know when he or she needs additional help." (33 : 57)

Teacher feedback is a strong motivator because it indicates to the students that the teacher is concerned and is not going to give up on them. This type of motivation precedes quality learning. Students' motivation to do homework can be affected by teacher comments written on their paper, and according to one authority, "a variety of means to...communicate student progress must be used." (26 : 27) Thus, appropriate teacher feedback should be a requirement of highly motivated learning.

Teachers should use effective feedback in all possible instances. A primary suggestion revolves around grading student work, whether it be homework, projects or tests. Often, students receive papers with only a grade placed on one page. Students continually need comments, corrections, and indications as to how to correct and improve their work. Research also suggests that a teacher must possess effective abilities to express the view that the student is considered an important person through non-evaluative comments. According to Grossnickle and Thiel, (26 : 26) the teacher must:

> help students feel that they belong...help students feel worthy, understood, and appreciated for who they are; tactfully correct student errors...

For example, appropriate feedback should be delivered soon after the behavior occurs, should focus on the individual's behavior, and should also involve communication with parents.

The feedback procedure is hindered when "methods of correcting student misbehavior and errors [are] embarrassing, humiliating, or stressful [to] students." (26 : 27) For instance, comparing the work of various students to a particular student is less effective than suggesting that each student look at his or her work and consider whether or not it was the result of worthwhile expended effort. If a particular student with a good paper is to be praised for good work, it can be done in pen on the paper. Also, motivational feedback might be inappropriate if the teacher emits feelings which imply that the student displeased the teacher. Two different messages may evoke two different interpretations, even though the teacher may be trying to convey the same thought in both instances. For example, the teacher might say, "I didn't like the way you did this" as opposed to saying, "your homework needs to be better organized." Obviously, the last message is clear and precise; hence, it is the better feedback. It does not imply that the student has shortcomings--but that the behavior of the student should be improved.

Classroom Climate as a Motivational Factor

Classroom climate might be termed the environment in which a room full of students must live and learn for at least one hour per day. What feelings does a student have upon entering the classroom? Is there chaos and disruption, or is there a sense of dedication to learning and a feeling of acceptance? The answers to these questions might define classroom climate in any given classroom.

A "positive classroom climate makes students more comfortable and encourages them to work to achieve." (33 : 60) If students enter a room and feel eager toward the learning in which they will participate that day, positive classroom climate motivation has been achieved. Bobby Moore even says that "it will be the environment that determines the levels of motivation as well as the extent to which desired levels of motivation are maintained." (36 : 257) Margaret Cohen explains that "how teachers structure the classroom environment has also been shown to affect the attributions students make." (12 : 24) According to the aforementioned research, certainly, the classroom climate is a significant motivational factor.

Teachers who promote comfortable, safe, and secure feelings among students in their classrooms have a better opportunity for achieving a positive classroom climate. Then, the teacher should follow up by promoting an environment which is stimulating and challenging. Thus, students will exhibit a "zest for learning" which will most likely "occur if teachers differentiate their instructional process and materials." (36 : 257) Students who work in a classroom in which the teacher uses various styles of instruction are not easily bored, and they will likely view their experiences in a positive manner. Teachers fostering this type of classroom climate will listen to, make requests of, and gently encourage students. The existence of a sense of continuity and predictability allows students to expect a well-organized learning activity which culminates with a thought-provoking conclusion.

A poor classroom climate may be evident to students if they feel dominated by the teacher or the subject. Goodlad (24 : 242) states his concerns :

> I wonder about the impact of the flat, neutral emotional ambience of most of the classes we studied. Boredom is a disease of epidemic proportions. Many of their escapes from boredom leave people unsatisfied, unfulfilled, and fretful.

Students who have grown accustomed to an unproductive classroom climate may not respond well to most initial motivational attempts made by the classroom teacher in the area of classroom environment.

Student Concern as a Motivational Factor

Student concern is "the amount of concern that students feel about a learning experience." (33 : 62) Teachers must recognize that this job might entail both raising and lowering student concern, depending on the circumstances. A teacher who chooses to raise the concern of the student is indicating that he or she believes that the current level of student attention is insufficient for learning to take place. In brief, it can "increase student attention and promote careful work habits." (33 : 62) Although it is designed to increase student stress to some extent, the new level of concern should not be detrimental to the learning process. A teacher who chooses to lower student concern may be attempting to ease the student's tension so he or she might continue or begin their learning activity. Lyman, et al, believe that lowering student concern is a strategy utilized by teachers "to reduce frustration and worry" or "to keep the students working when learning is temporarily difficult." (33 : 62)

Teachers promote this motivational tool by making decisions which indicate that they either condone or reject certain student behavior. The teacher's decision should better serve the students' need to learn. If an inappropriate decision is made by the teacher, a student's low concerns may be dismissed in favor of even lower ones, and destructively high concerns may be ascended to uncontrollable heights. Teachers can enhance students' concern for their own learning by making learning lively and interesting. Teachers need to "provide ample opportunities for students to get involved and not just sit passively while the teacher teaches by showing and telling." (36 : 261)

Learning Payoffs as a Motivational Factor

Learning payoffs involves "rewarding students for learning." (33 : 63) Learning payoffs can be either intrinsic or extrinsic. Intrinsic rewards come from within the student. Feelings of accomplishment and pride are intrinsic examples of learning payoffs. Extrinsic rewards are supplied by the teacher or other outside influences. A smiling sticker, a reduced homework assignment, or a "Zowie! Excellent Answer!!!" are all examples of extrinsic learning payoffs. They are designed to make the students acknowledge their accomplishments.

As a motivational factor, a learning payoff is a strategy which makes the student feel successful and creates in him or her a desire to do more activities of a similar nature. Consequently, when students are rewarded with prizes or praise, positive extrinsic motivation can occur. Thomas R. McDaniel states that intrinsic motivation occurs when "students somehow become more interested in and committed to their own educational improvement." (34 : 47) Research seemed to indicate that learning for the sake of acquiring the knowledge necessary to achieve personal goals is a characteristic of strongly motivated students.

For a teacher to utilize learning payoffs as a method of motivating students, the teacher must decide whether to strive for intrinsic results, extrinsic results, or both. Extrinsic rewards may be easiest for teachers to employ. A box full of stickers and a brain full of verbal reinforcers could suffice. However, intrinsic motivation, although sometimes more difficult for the teacher to inspire in students, may be the choice for the teacher. Continuous effort on the part of the teacher and a constant striving for that special learning payoff may be exhausting; however, it can be a worthwhile quest. Adele Eskeles Gottfried states that "children who reported higher academic intrinsic motivation [in his study] had significantly higher school achievement..." (25 : 642) Teachers who require students to solve challenging problems along a route to a solution and allow failure to be encountered and overcome by the student, promote intrinsic motivation. This is not to say the teacher should not guide the discovery; however, it is to say the teacher could focus on the progress and process and emphasize its importance.

There are problems associated with exclusive use of extrinsic motivational tools. The teacher should not reward "students with tangible items ... [because] when the reward is removed, the desire to learn decreases." (33 : 63) Also, Carol S. Dweck notes that "errorless learning has been found to produce bizarre emotional responses following non-reinforcement." (15 : 1045) Errorless learning occurs if the classroom teacher so strongly guides the students that they make no mistakes in their work towards a solution. If student activities are inadequately designed by the teacher, and if "confusion does accompany the initial attempt to learn new material, mastery of the material is seriously impaired." (15 : 1044)

Student Interest as a Motivational Factor

Much research showed that student interest was highly discussed and a desired factor in student motivation. This element of motivation theory can be utilized in two different ways. The first pertains to teaching which relates material to past, present, and future student experiences. The second regards teaching plans which will increase involvement in the assigned learning activity. From either viewpoint, the goal of the teacher is to interest students in that which they are learning through creative means.

Students who feel that a particular concept has relevance to their lives will attempt to understand it. Hence, motivation has occurred. According to Margaret M. Gullette (27 : 18) teachers should be aware that:

> students learn best when they are intent on learning; relating new material to [their] accumulated knowledge or current goals helps them recognize the importance of the topic and increases their interest.

With mathematics, mastery of some concepts which may not have past or present relevance to their lives is desirable. Consequently, the teacher may either persuade them of the futuristic uses of the material or entertain them with an interesting activity designed to engage their participation. According to McDaniel (34 : 48):

> motivation depends on a teacher's skill in getting students to attend to the objectives, skills, knowledge, and values that constitute any given lesson. The idea is to prepare students for learning by grabbing their attention with an activity that is arresting.

Consequently, a classroom which is full of interested, involved students would exhibit a strong motivational level. Boyer (6 : 151) cites an example:

> In a large math class...the teacher demanded creativity and individual participation. The students were excitedly engaged in their own personally constructed strategies and arguments with the teacher doing all he could to keep up with the various approaches.

In Mortimer J. Adler's Paideia Proposal, he states that "the well taught class that awakens lively interest in learning and gives students a sense of accomplishment will help to promote decorum." (1 : 55) Diane Ravitch (40 : 53) makes note of the difficulty with which this objective is achieved.

> To embrace the interests of children... textbooks [have] to be supplemented or replaced by newspapers, magazines, excursions, projects, audio-visual aids, and activities.

Grossnickle and Thiel (26 : 18) state the following with regard to teaching in general:

> It is the job of the teacher to "appeal to both the right and left brain-dominant learner, such as a demonstration, a riddle, role playing, or problem solving activities that confront the question, 'Why do we have to learn this'?"

A few of Grossnickle and Thiel's suggestions and examples include: (26 : 42)

(1) begin each class with an interest-focusing activity

(2) use a variety of methods for going over homework

(3) develop interesting and appealing (to all abilities) worksheets,...puzzles, and humor

(4) plan interpretations of abstract ideas into understandable, concrete ideas

Although use of these suggestions may evoke a difficult task for teachers, McDaniel encourages that unusual "questions can establish a state of disequilibrium, a state of tension that motivates students to resolve a problem or dilemma." (34 : 48) Goodlad (24 : 231) also presents an appeal:

> ...part of the brain, known as Magoun's Brain, is stimulated by novelty. It appears to me that students spending twelve years in the schools we studied would not be likely to experience much novelty. Does part of the brain just sleep then?

This, in addition to his research which indicated that mathematics textbooks dominated the mathematics curriculum and that "basic skills previously acquired [were not used in upper years' activities] but reappeared as ends in themselves" certainly could compel teachers to create interesting activities. (24 : 209)

There are factors which could contribute to the teachers inability to enhance student interest. Just as student interest can be aroused by teacher creativity in preparation of activities, it can be crushed by teaching plans which express a limited creative appeal. Termination of a good mathematical idea for a learning activity can occur through a teaching plan which stagnates possibilities. Hindrance can also occur if the exercise promotes disruptive behavior. A drastic change in the class routine by an unexpected activity can open the door to chaos. The new creation should be easy to use in a classroom setting and should fit in normally. It should not take an inordinate amount of allotted class time.

Teacher Enthusiasm as a Motivational Factor

Teacher enthusiasm certainly should not be overlooked as a motivational factor. This realm of effective motivation is evident when the teacher enters the room with a smile and projects an attitude of love for both the subject and for teaching. Enthusiastic teaching can turn into enthusiastic learning. Evidence that a teacher is enthralled with mathematics may cause a student to become more interested and to more actively participate.

Dan C. Lortie (32 : 152) briefly explains the teacher's job:

> The teacher must 'motivate' students...and enjoy his efforts. He cannot count on voluntary enthusiasm; the teacher must generate much of the positive feeling that animates purposeful effort.

Consequently, establishment of excitement in a classroom rests upon the shoulders of the teachers.

Suggestions from Lyman, et al, (33 : 64) include:

1. A Variation of the Tone of Voice and Gestures
2. Effective use of Humor
3. Activities Interesting to teachers and students
4. Teacher Mastery of Subject Matter
5. Knowledge of Current Literature in the Subject
6. Appropriate Personal Interjections

Much literature remarked on the necessity of teacher enthusiasm for effective motivation, Boyer (6 : 149) averred:

> ...there remain some old fashioned yet enduring qualities in human relationships that still work...enthusiasm for the material to be taught, contagious enthusiasm for the work to be done,...

Grossnickle and Thiel associate a motivation-related problem with "dull and unenthusiastic teacher performance." (26 : 32) The pitfalls involved in total reliance upon teacher enthusiasm for effective student motivation include three areas.

The first area involves the teaching of a concept with which a teacher does not feel comfortable. The teacher who is more concerned with the mathematics material may misplace his or her enthusiasm. The lesson delivery might usurp the teacher's usual expression of fascination leaving the students with a feeling of abandonment. Consequently, student motivation could possibly dissolve.

The second area revolves about the inevitable "off day" that a teacher experiences. As teachers themselves are human, they will experience personal problems and gloomy days. If these attitudes are allowed to influence the teacher's lesson, the result may be a quite dispassionate lecture.

Finally, if there is a substitute teacher, that teacher may not be inclined to project enthusiasm for the subject matter. Students who are taught by a substitute teacher might feel upset about the change in routine from the beginning of the day. Then, if the students are normally expectant of zealous teaching and do not receive that from their substitute, they may not feel motivated.

Mathematics History as a Motivational Tool

Mathematics history topics in the regular mathematics classroom could be presented in a manner that would challenge the abilities of individual students. In addition, multi-solution activities, similar to problems solved by famous mathematicians could be used. As evidenced by Abraham Arcavi, "the answer will be approached differently by different students." (2 : 15) To present a problem which famous mathematicians faced years ago could lure the high ability students into finding interesting solutions or to expand the original work of the mathematician, while the lower ability students could concentrate on finding the elementary solution before they move toward more involved work. Thus, historical topics of interest could be the starting point for enhancing student motivation of this important factor -- ultimate student success.

Mathematics history activities may liven classroom climate in many ways. First, a lively history discussion might relieve some degree of boredom and enhance the teacher's personal attitude. Also, a special history lesson could convey a message to the students that the instructor was taking a new interest in the subject and may help students to realize that learning can be exciting.

Sharon Kunoff and Sylvia Pines relate an excellent example of teaching a mathematical concept while making use of the particular mathematics history involved with that topic. They taught some elementary probability concepts by utilizing the history surrounding those particular concepts from probability. They used the historical perspective by relaying a story to their students regarding a gambling problem that a famous Duke had asked Galileo to solve. They state that "we have found that students are very impressed by the fact that they, like the great Galileo, have solved the Duke's problem." (30 : 211) Intrinsic motivation may have been achieved through this motivating activity. Kunoff and Pines (30 : 216) continue:

> Many students are anxious to try Fra Paccioli's problem when they learned that it was not solved by either Cardan or Tartaglia, both being important mathematicians of the sixteenth century who worked on it extensively. Mathematical history lends itself quite well to enhancing this motivational factor.

According to Frank J. Swetz, "historical notes and justifications can easily be injected into lectures." (46 : 696) Consequently, the eccentricities of mathematicians and humorous anecdotes can be easily found in nearly every great discovery and are adaptable to nearly all grade levels. Duplicating the circumstances under which the great mathematical minds worked "enliven and enrich classroom presentations." (45 : 33) If a mathematics problem is presented by the teacher, then Kunoff and Pines claim that its "historical perspective make it interesting to consider." (30 : 211) A brief history lesson related to the mathematics topic can offer a change of pace and can place the problem at hand into a historical perspective that can enliven student interest.

<u>Summary</u>

The reader may wish to consider these seven aforementioned motivational factors in light of daily classroom activities. While it may prove impossible for a teacher to employ each motivational strategy everyday, at least one technique might be attempted each day. Variation in praise, or varying tone of voice may become effortless over a short period of time. Concentration can then fall upon the somewhat difficult and time consuming elements of motivation. Lortie consolidates the complex requirements of motivation-oriented teaching by stating that "control must be maintained, work must be ordered and the students' interest must be aroused and sustained." (32 : 152)

It is the premise of this study that many of the motivational factors previously discussed may be applied in the mathematics classroom through the use of mathematics history. Many great ideas in mathematics were discovered throughout history due to a particular mathematician's or even a layman's curiosity. Good teachers might use the rich history of mathematics to good advantage as they go about the business of motivating students to learn.

CHAPTER III

NUMBER THEORY, GEOMETRY, AND ALGEBRA

Introduction

In this chapter, concepts from Number Theory, Geometry, and Algebra will be described briefly as they might be presented to students of the intermediate grades. Some mathematics history pertaining to these fields will be discussed. These discussions could offer classroom teachers of mathematics a historical sketch which might be included in any lesson pertaining to the given concept. In terms of student motivation, these synopses could be used to enhance student interest, classroom climate, and teacher enthusiasm.

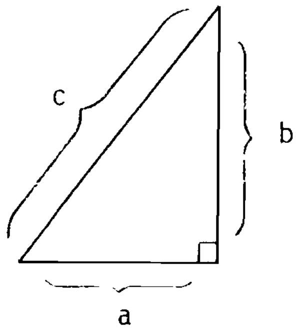
Number Theory

Elementary Number Theory involves a wide range of mathematical topics. Certainly, one concept from this field about which a lesson may be presented is the Pythagorean Theorem. This theorem involves work with right triangles. Although it was proven using geometry, Pythagoras was considered by most mathematics historians to be a number theorist. Hence, the writer chose to classify the section as Number Theory.

Figure 3 shows a pictorial representation of the Pythagorean Theorem along with the statement of the theorem.

The Pythagorean Theorem:

> A triangle is a right triangle, if and only if the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



$$a^2 + b^2 = c^2$$

$$a, b, c \in \mathbf{R}^+$$

Figure 1

The Pythagorean Theorem

A lesson relating to the Pythagorean Theorem could be enhanced by including mathematics history into the focusing remarks of the lesson. One historical topic which may be presented in conjunction with the Pythagorean Theorem is the study of the personal life of Pythagoras, the man to whom the credit for the proof of the theorem has been granted.

Pythagoras. According to Eric T. Bell, Pythagoras was born about 569 B.C. (5 : 20) According to another author, Tobias Dantzig, he studied under Thales, the only mathematician among the seven wise men of antiquity. (13 : 21) Thales encouraged the young Pythagoras to travel to other countries in order to gain experience and knowledge. One of the places he visited was Egypt, where he learned much of the layman's mathematics--simple computations. He also learned much from the priests. His previous education had been devoid of superstition. But, when he studied from these men of God, he discovered many things. The religious leaders of ancient Egypt had kept their knowledge within the church in order to "miraculously" predict supernatural events and to do the computations as only a person "appointed by God" should.

When he returned to his homeland of Samos, an island, he found it under the control of Persia. Consequently, he relocated in Crotona, Italy, a Greek seaport. Pythagoras believed that this new community needed moral reform. He was living during the early era of the Golden Age of Greece and, as if driven by some unseen force, he wanted to positively affect his environment. With moral reform as an objective, a secret society was formed with Pythagoras as the hero; it was even named The Pythagorean Brotherhood. This society took it upon itself to positively affect the community. Within the confines of their realm, many scientific discussions took place alongside the moral issues which were discussed daily. A higher level of mathematical thinking had been born to non-priests.

The main ideas of the brotherhood revolved around numbers themselves, specifically integers. To them and even more specifically to Pythagoras, each number had a significant meaning, and all things could be reduced to the number with which they were associated. For example:

1 - reason
2 - masculine
3 - feminine
4 - justice
5 - marriage

Pythagoras assigned these meanings, as he thought that numbers were the basis of everything. He possessed a "bizarre number mysticism in which he clothed his cosmic speculations." (5 : 20) He believed that numbers were the rhyme and reason; all nature could be explained by using numbers. Lucas N. H. Bunt, Phillip S. Jones, and Jack D. Bedient note that: (7 : 73)

> In the course of history, it has happened several times that people were so much impressed and surprised by an original point that they lost sight of its special nature and believed they could apply the new concept everywhere. This may also be said of the Pythagoreans.

For example, a Pythagorean, Philolaus in 450 B.C. (cited in 7 : 82) wrote that:

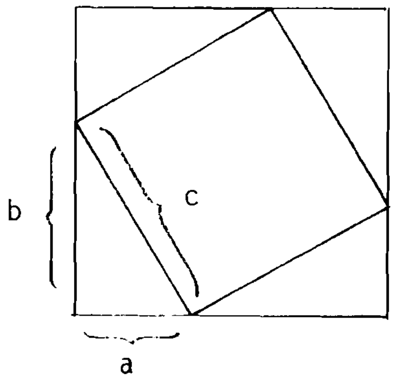
> ...everything that is known has a number. For it is impossible that without it anything can be known or understood by reason. The One (1) is the foundation of everything.

The reader should note, ironically, that the writing of this statement occurred after Pythagoras was dead. The originator of the brotherhood required students to commit all mathematical knowledge to memory. All teaching was oral. He took no chances of a non-member happening upon their work. Ironically, he became as secretive about mathematical knowledge as the priests had been in his early scholarly days.

The brotherhood members seemed to be producing the most logical thoughts and learning of the decade. Consequently, their meeting grounds became known as more than a moral reform house. It actually was considered a "school." The most famous finding of the school was the proof of the aforementioned theorem--The Pythagorean Theorem. They did not originate the statement since the relationship between the two legs and the hypotenuse of a right triangle had been known for several centuries. It was the proof of the theorem for which the society was famous. It was a requirement of the school that all discoveries were subject to Pythagoras's scrutiny. Consequently, when the proof became known, Pythagoras himself was credited with the finding. Most writers believe that Pythagoras, or the member of the brotherhood who first found the proof, used the following method, or one similar to it. Either way, it is the belief of many scholars, that the concept of area was used in the proof of the Pythagorean Theorem. Figure 2 shows a proof of the theorem. The hypotenuse of the right triangle (c) is associated with a square of area c^2 square units. Next, the sum of the two legs of the right triangle (a + b) is associated with a square of area

$(a + b)^2$ square units. Note: it is written here making use of variables and using simple algebra, although during time of Pythagoras such symbols had not been discovered.

> Construct four identical right triangles and arrange them as shown. Then on the basis of areas, the following algebraic statement is true.



$$(a + b)^2 = 4(1/2)ab + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

Figure 2

Geometric Representation of Proof

This great master, had originated a society which had produced a magnificent proof. He had claimed that everything in the universe came from natural numbers and this magnificent proof indicated that this idea was taking shape, and was surely true.

Next Pythagoras took his favorite number, the number of reason, and constructed a right triangle which had a length of one for its legs and set about finding the length of the hypotenuse. Surely, it would be a truly great number. When he worked on his theorem, the broad smile probably grew thin as he happened upon his solution, for he found a non-integer length for the hypotenuse. He obtained an irrational number, specifically $\sqrt{2}$, as a representation of its length. An unspeakable situation had occurred, and Pythagoras was devastated. There were lengths which could not be expressed as integers, yet could be mechanically drawn with a straightedge. A simple length could be drawn and not be an integer, or at least a ratio of integers -this certainly upset Pythagoras. Since he could hardly stand this new result, he forbade the members of his society to ever discuss these terrible values which constituted a "logical scandal." According to Howard Eves, the Pythagorean, Hippasus, allegedly disclosed this secret to a non-Pythagorean. Soon after his betrayal, he perished at sea. (17 : 63)

The Crotona elite and pious protested his bringing of enlightenment to non-priests or "unworthy" people. About 500 B.C., their hatred climaxed and his school buildings were burned to the ground. There are at least three stories regarding the death of Pythagoras. Some such as Bell claim that he perished amidst the flames. (5 : 20) Dantzig relates yet another story claiming that, although Pythagoras escaped the fiery blazes, he ran into a field of beans. Pythagoras believed that beans had souls, so rather than trample the beans to their deaths, he let the mob catch him and send him to his death. (13 : 23-24) Still others such as Eves say that he escaped the fiery walls and migrated to Metapotum where he was ultimately murdered at the advanced age of 80. (17 : 55)

Nevertheless, the Pythagorean influence won in the long run. David Eugene Smith relates information that certainly indicated that Pythagoras' mysticism and hero worship continues today.

The influence which Pythagoras had over his followers may have been excessive. However, that charismatic, mystical mathematician may elicit followers even today. David Eugene Smith relates a story which may help emphasize this mystical possibility. (43 : 10)

> [In what is left of San Mathesis, there is a] gothic chapel known as the Capelia Pittagora. In this chapel, in the floor in front of the chancel, is a block of marble worn by centuries of time and by centuries and centuries of pilgrims, of the inscription only a few letters can be deciphered '-HI-CET-OS-----PYT--GO--,--AM-S', evidently showing that, in generations past, tradition asserted that 'Here lie the bones of Pythagoras of Samos'. The sole occupant of the chapel is a long-robed priest...

Geometry

In the intermediate grades, the study of geometry may include many aspects of plane figures, including circles. Inherent in the study of circles is a discussion relating to the number pi. The irrational number pi is defined as follows:

> Given any circle, the ratio of the circumference to the diameter is denoted by the Greek letter pi(π).

Although a rough approximation of pi may be 3 millenia old, much about pi is attributed to the mathematician who first computed pi to as many decimal places as he desired. Archimedes was quite a colorful mathematician. Partly legend, partly fact, he nonetheless possesses a royal chair in mathematics history. Certainly, a discussion of this character could enhance student motivation.

Archimedes. According to Bell, Archimedes was born in 287 B.C. to an astronomer named Pheidias. (5 : 19) He grew up in Syracuse, a Greek city on the island of Sicily. He spent time in Egypt and probably studied at the great school in Alexandria. (5 : 30) Eves hints that it is even possible that he learned from Euclid, the famous geometer. (17 : 142)

Archimedes has been dubbed "the father of mathematics", "the great thinker", "the old man", and "the undisputed chieftain", to name a few descriptive titles. (5 : 28) All are intended to show complete respect and admiration for his style and skill. Author Dirk Struik claims Archimedes to be the most original thinker and the master of computational technique. Many debts by many great mathematicians are owed to Archimedes. According to Eves the discovery of pi was attributed to Archimedes. (17 : 91)

Pi is the ratio of the circumference of any given circle to its diameter, Consequently, Archimedes found that this ratio had a constant value for all circles. It is an irrational number. He concluded this by using two regular polygons, each with the same number of sides. One he inscribed in the circle--the other he circumscribed about the circle. Using this method he was the first to come up with a highly accurate estimation of pi. According to 'the old man' it was between 3 10/71 and 3 10/70, when he used polygons with 96 sides. If he would have needed a more accurate value (more decimal places), he certainly could have done it, by choosing polygons with increasingly more sides.

Archimedes did not have scratch paper or a chalkboard, as we know them today on which to do his configurations. But he improvised in many ways. Most often, he went to the beaches armed with his drawing tools and drew in the sand. It was commonplace for his friends to enter his house and be forbidden to walk through certain piles of ashes or dust. It would also have been impossible for them to **ever** attempt to clean his home. They might unwittingly destroy some of his work. For instance he might be sitting by the fireplace and rake out some ashes in which he might draw his figures. (5 : 30) Many of these collections of dust contained constructions relating to the current problem in which he was engaged. A story is related by Bell of genuine Archimedean eccentricity. Upon removing himself from a bath one evening, he anointed himself with olive oil. (This was the custom of the day.) He accidently drew a line with his fingernail through the oil on his skin. Immediately, he was lost in thought. He began retracing a problem on which he had been working. Soon his pondering had taken up a few hours of his time and he was still unclothed, sitting down, and drawing on his oily skin. (5 : 30)

The fact that he could get completely lost in his work was his trademark. He could sit and gaze at his work for hours, even days upon end. He would forget to eat or sleep when he was truly involved with his circles. Many times, friends had to go to his house and force him to eat. (17 : 143-144)

Bell relates that Archimedes possibly was related to King Hieron II, king of Syracuse during the third century B.C.. One day, King Hieron received a gift from a less than reputable "friend." It was a golden crown with many jewels. Needless to say, the king questioned whether or not the crown was of pure gold or just gold plated. He went to 'the chieftain' for advice. He gave Archimedes the crown and told him to find the true composition of the crown without damaging it. This quest was eagerly undertaken by our hero and it consumed his time. One day, when Archimedes was particularly frustrated with this problem he innocently took a bath. Little did he suspect that upon entering the tub of water, his eyes would light up and he would bound from his home. "Eureka!! Eureka!!", he shouted as he ran through the streets of town. (This means "I have found it".) He had discovered the first law of hydrostatics. Unfortunately, as he later discovered, he had forgotten to clothe his body before he had run about Syracuse. But, the king had his answer, the crown did not displace enough water, when placed in a tub, for it to be made of nothing but gold.

Another event which took place during the life of Archimedes was the Roman effort to conquer much of the world. The Roman army was ready to conquer Sicily which was in their way. Consequently, Syracuse came under fire. But Syracuse had something that Rome did not. They had Archimedes. His infinite wisdom served King Hieron well. Roman ships were met by strong Archimedean levers and tossed upside down into the sea. Archimedes had once boasted of his levers, "Give me a place to stand and I could move the earth." Individual Roman soldiers were pummelled with rocks thrown from mighty Archimedean catapults. Archimedes's wisdom held Roman thugs at bay for three years. (17 : 142). Ultimately though, in 212 B.C., the Roman determination held, and the Romans took the city. They attacked the city from the rear and captured it with very little effort.

After the city was conquered, one resident remained unaware of the Roman victory. Archimedes was, as usual, alone on a beach contemplating a problem. He had figures drawn all about him, in the sand. A Roman soldier cast a shadow on his work and Archimedes berated the man, "Don't spoil my circles." Enraged, the fighter drew his shiny sword and slew one of the greatest thinkers of all time. (8 : 75)

According to Vera Sanford, many years later, the tomb of Archimedes was found. (41 : 13) It had been erected by the Romans as compensation for his murder. Figures of a cylinder and a sphere were etched into the epitaph. These were symbolic of Archimedes' countless investigations.

Algebra

Basic Algebra contains concepts which are centered upon the rectangular (Cartesian) coordinate system. Credit is often granted to Rene DesCartes for his instrumental influence in the development of this link between Geometry and Algebra.

Rene DesCartes. According to Bell, Rene DesCartes was born on March 31, 1596 in LaHaye, a town near Tours, France. He was born in the days of the early Renaissance. Consequently, it was also a time of religious and political reconstruction. His mother died a few days after his birth. Rene, himself, was never very healthy. Author Eric Bell speaks of his "frail health" in many instances in his extensive biographical sketch. In an effort to compensate for his weak physical state, young Rene switched his energy to thought and spent many hours simply thinking. His father kept a watchful eye on this "young philosopher." That is what his father called him, because he always wanted to know "why." Why did that happen; or this happen, why are some things this way and other things are not? His questions seemed endless. (5 : 36)

At the age of eight he began to get some of these questions answered. It was then that his father sent him to a Jesuit School at LaFleche. The head father, Father Charlet took him under his wing as did another father, Father Mersenne. They allowed Rene to sleep until very late in the day so that his mind would be ready to work hard during at least some of the day. Rene later claimed that these times were the times when he happened upon his most brilliant philosophical and mathematical thoughts. (5 : 36-37)

For instance, one morning, he was lying in bed and noticed a fly on the ceiling of his room. He gazed at the fly for a long time. He pondered how he could explain to Father Mersenne the location of the insect. After much thought he decided that the best chance of explaining it was to begin in one corner of the room and to count units "over" (along the edge of the wall) until he was directly beneath the fly, and then to count "up" in a straight line toward the fly, until he "met" the fly. Thus, was born the Cartesian Coordinate System. He left school in 1612 and embarked on his many journeys. (5 : 37)

His first quest was to understand how people know anything. DesCartes was every bit the philosopher as the mathematician. It was through these philosophical thoughts that he decided that the reason he existed was because he could think; **Cogito ergo sum** (I think, therefore I am.) This was his first philosophical addition to the existing world of deep thought. (5 : 38)

In 1613 he moved to Paris and wholeheartedly undertook the sport of gambling. (5 : 38) He was actually pretty good at it, but it did not satisfy his quest for a greater understanding of life. After only a few months he moved away and studied mathematics incessantly for two years. Soon though, his gambling friends found him. To get away from them, he joined the army. During his tour with the army, three dreams passed through his head on the eve of November 10, 1619. (5 : 39-40)

> Dream 1 - He was blown by evil winds from church and college to a third party which the winds could not budge.
>
> Dream 2 - He observed a terrific storm without being superstitious (unusual for his day) and saw that it could not harm him.
>
> Dream 3 - He was reciting a poem by Ausonius which begins "What way of life should I follow?"

He claimed that these dreams changed his life. He never explained how they were connected to his forthcoming fantastic mathematical and philosophical ideas, but it suffices to say that they were related in a manner which allowed DesCartes to ponder and enlighten the world.

Bell relates that in 1621, he quit soldiering and headed for Northern Europe. Through some mishaps, he stayed in Austria and just visited Holland from time to time. Later he visited Rome where he had the honor of meeting the great Galileo. For some reason, he always held contempt for the man, even though he agreed with him on many mathematical grounds, even the ones which Galileo was forced by the church to withdraw. Needless to mention, DesCartes never confided his agreement with Galileo to the church.

Even so, DesCartes, being fairly impressed with his own skills, publicly implied unfelt humility when it came to his mathematics. He spoke modest words; however, they were words which could have been interpreted as boastful. B.L. van der Waerden (48 : 74) claims:

> According to Pappos, Apollonius says in the third book of his treatise on 'the locus of three or four lines', that Euclid had not solved this problem, and that he himself too had not been able to solve it completely, nor had anyone else. 'This', says DesCartes, 'led me to try to find out whether by my own method, I could go as far as they had gone'.

Not reminiscent of most philosophers, he dressed in the latest fashions. He could not be accused of looking like the absent-minded professor. His outfit was complete with both a sword and a broad-rimmed, ostrich-plumed hat. Once while out with a beautiful lady, they were confronted by a drunken lout. DesCartes went rashly after him, plucked his sword from his hand, but opted to spare the poor fool's life only because he was entirely too filthy to be butchered in front of a refined lady. "A rational mind is sometimes the queerest mixture of rationality and irrationality on earth." (5 : 42)

Bell recounts the rest of DesCartes life stating that DesCartes moved to Holland in 1628 and wandered around for twenty years. He never stayed too long in any one place, but kept in contact with the scientific world through continual correspondence with Father Mersenne. It was during this peaceful time that he gave the world his Analytic Geometry, which included his fabulous graphing system, in the form of a book called **Discourse on Method.** Although this book is by far his most useful and well-respected mathematical work, it could be considered his deathwish. Queen Christina of Sweden found the book. She requested that DesCartes come to her court and teach her all that he knew. DesCartes declined as long as he could. A strong longing for and respect for royalty had been taught to him. It is sometimes referred to as DesCartes snobbery. Eventually, he left the solitude of his wanderings and went to Sweden. The iron-willed queen habitually studied at 5am and expected no less from her new employee. DesCartes, the man who had normally slept until noon, was faced with a dilemma which went totally against all the things in his life which had kept him well. But, the Swedish winter that year was one as seldom seen before. The icy cold rooms in which Christina preferred to study and learn from the great Rene DesCartes, were the very rooms which killed him. (5 : 49) He died in 1650 at the age of 54. (5 : 51)

Summary.

In this chapter, the writer considered elementary concepts in Number Theory, Geometry, and Algebra. Upon consideration of these mathematical branches, biographical sketches of Pythagoras, Archimedes, and DesCartes, respectively, were presented. These short historical sketches of the men's lives might be useful for teachers attempting to positively influence the motivation of the students studying the previously mentioned mathematical areas.

CHAPTER IV

GEOMETRY, ALGEBRA, TRIGONOMETRY, AND CALCULUS

Introduction

In this chapter, mathematical objectives will be chosen from Geometry, Algebra, Trigonometry, and Calculus, as they might be presented to high school students. The chosen lesson will be explained briefly, and some history surrounding that concept will be presented. These discussions could offer classroom teachers of mathematics a historical sketch which might be included in lessons pertaining to the given concept. In terms of student motivation, these sketches might enhance student interest, classroom climate, and teacher enthusiasm.

Geometry

Often, geometry is taught to sophomores of American high schools. The course usually offered is a concise study of Euclidean Geometry. This geometry is the study of figures in a plane and their relationships to one another, in particular those properties that remain invariant under rigid motions. Perhaps a historical survey relating to Euclid, the man for whom the geometry is named, could be presented to whet the enthusiasm of the students.

<u>Euclid.</u> Although the information in much research is brief, Dirk Struik said Euclid "lived probably during the time of the first Ptolemy (306 - 283 B.C.)." (44 : 58) Euclid is said to have been one of the few ancient mathematicians who received a salary for pursuing knowledge. Struik goes on to say that "among the first scholars associated with Alexandria was Euclid, one of the most influential mathematicians of all time." (44 : 58) Alexandria was a city named for "Alexander the Great," and was located in Egypt. It was in this fair new city that a great center for learning was founded. The new capital became "the intellectual and economic center of the Hellinistic world." (44 : 57)

In 306 B.C., Ptolemy began his reign and chose Alexandria as his capital. "In order to attract learned men to his city, he immediately began the erection of the famed University of Alexandria." (17 : 113) Many years later, the great school began to fade, and it was reported that "hectic days followed the fight of Christianity against paganism, and finally, in 641 A.D., Alexandria was taken by the Arabs." (17 : 142) Many literary works were destroyed. Fortunately, the **Elements**, the work for which Euclid is famous, was preserved. Struik speculates that **Elements** was partly original work and partly commentary on already existing work. In either case, these volumes represent "the first full mathematical texts that have been

preserved from Greek Antiquity." (44 : 58)

"Before Pythagoras, it had not been clearly realized that proof must proceed from assumptions." (5 : 20) Euclid followed this Pythagorean requirement and called these assumptions postulates and axioms. He listed five of each. He then listed twenty-three definitions and proceeded to prove his forty-eight propositions (theorems) of Book I accordingly.

His rigorous method of proof was by and large, errorless, and it has received much acclaim throughout history. However, the fifth of his five postulates has been a point of question on his wonderful **Elements** ever since people of mathematics have had the ability to reason deductively. Consequently, it is this fifth postulate which distinguishes Euclidean Geometry from other geometries. Consider the following five postulates:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than the two right angles.

Clearly, whereas the first four are succinct and seem to be relatively straight-forward, the fifth has over four times as many words as the longest of the other four postulates. This fact aside, the fifth postulate became a target for many Geometers as early as 1500. They wished to prove the fifth postulate, as this would better serve geometry and make it more perfect. Postulates are statements assumed without reproach, and the fifth postulate of Euclid was deemed to be of different character than the other by some mathematicians. In particular, many of the mathematicians of the early renaissance questioned its validity. Consequently, many students of geometry believed that the fifth postulate should be a proposition rather than a postulate and set out to find a proof of this statement. In the process, other unexpected discoveries were made.

The use of the fifth postulate, often referred to as the "parallel postulate," in proving the propositions does not occur in **Elements** until the twenty-ninth proposition. Consequently, the first twenty-eight propositions, along with any others that do not use the fifth postulate, are also part of other geometries. Those geometries use the first four postulates as Euclid decreed, but alter the fifth postulate. Thus, new geometries were born and are referred to as Non-Euclidean geometries.

Carl F. Gauss, a mathematician of the late

Renaissance was one of the mathematicians to question this fifth postulate. In letters to friends dated around 1816, it was clear that Gauss was in possession of new ideas regarding Geometry. "The appearance of his diaries and of some of his letters has shown that he kept some of his most penetrating thoughts to himself." (44 : 208) The letters indicated much work with an altered postulate to Euclid's fifth postulate. Sanford cites the following quote made by Gauss in 1817, "I am becoming more and more convinced that the necessity of our Geometry cannot be proved." (41 : 69)

However, this fact need not rob Euclid of deserved credit. His work is, "next to the Bible, probably the most reproduced and studied in the history of the Western world." (44 : 58-59) Many scholars believe that even the great Euclid struggled with the fifth postulate, and after seeing no other alternative, he chose to keep it a postulate rather than to make it a proposition. Certainly, his monumental work is worthy of continued study. Euclid's style and grace of writing and documentation of theorems has earned him entry onto the list of great mathematicians of the world.

Even many non-mathematicians may find Euclid worthy of study. The elegance and simplicity inherent in his **Elements** might prove to be enjoyable to students of Geometry, as well as to students of other structured disciplines. The techniques he utilized to build his geometry are very structured. Smith relates a quote made by Abraham Lincoln. (43 : 22)

> In the course of my law reading I constantly came upon the word **demonstrate.** I thought at first that I understood its meaning, but soon became satisfied that I did not. ... I consulted all the dictionaries and works of reference I could find. ...At last I said, 'Lincoln you can never make a lawyer, if you do not understand what **demonstrate** means', and I left my situation in Springfield, went home to my father's house and stayed there till [sic] I could give any proposition in the six books of Euclid at sight. I then found out what **demonstrate** means and went back to my law studies.

Ian D. Macdonald wrote a play which may have described Euclid's early years. Although he offers neither references nor documentation, the play includes some historical occurrences of Athens in 300 B.C.. This subjective description of Euclid could bring the man, about whom so little is known, to life for students of Geometry. It could easily be produced for or by a high school geometry class. It offers a view, possibly fictitious, of the man who had so much influence in the construction of their geometry textbook.

The writer alluded earlier to the fact that very little is known about Euclid. Much of what is known may be legend alone. Most writers agree that Ptolemy appointed Euclid to the head of the mathematics department of his great University of Alexandria. Euclid was known for his "modesty and consideration of others." (17 : 113) There are a few legendary anecdotes regarding Euclid, but the truth is unknown to us. On one hand, Ptolemy requested a short cut to geometric knowledge and Euclid told this king that there was no "royal road" to geometry. On the other hand, a student studying under Euclid questioned what he could get from learning geometry, and Euclid ordered a slave to give the student a penny so that the student would gain from this geometric knowledge. (17 : 113-114)

Much of what is believed to be true regarding Euclid's life, is merely speculation made by historians familiar with that particular era. As elusive as some proofs may seem to the present day students of geometry, so is the man, Euclid, to mathematics history researchers. Therefore, geometry students may appreciate the elusiveness of Euclid in the form of the following poem, "EUCLID" by Vachel Lindsay, as quoted by Clifton Fadiman. (19 : 274)

EUCLID

Old Euclid drew a circle
On a sand-beach long ago.
He bounded and enclosed it
With angles thus and so.
His set of solemn graybeards
Nodded and argued much
Of arc and of circumference,
Diameters and such.
A silent child stood by them
From morning until noon
Because they drew such charming
Round pictures of the moon.

Another work which adds to the mystery surrounding Euclid's life is "PLANE GEOMETRY" by Emma Rounds also cited by Clifton Fadiman. (19 :279)

PLANE GEOMETRY

'Twas Euclid, and the theorem pi
  Did plane and solid in the text,
All parallel were the radii,
  and the ang-gulls convex'd

"Beware the Wentworth-Smith, my son,
  And the Loci that vacillate;
Beware the Axiom, and shun
  The faithless Postulate."

He took his Waterman in hand;
  Long time the proper proof he sought;
Then rested he by the XYZ
  And sat awhile in thought.

And as in inverse thought he sat
  A brilliant proof, in lines of flame,
All neat and trim, it came to him.
  Tangenting as it came.

"AB, CD," reflected he-
  The Waterman went snicker-snack-
He Q.E.D.-ed, and, proud indeed,
  He trapezoided back.

"And hast thou proved the 29th?
  Come to my arms, my radius boy!
O good for you! O one point two!"
  He rhombused in his joy.

'Twas Euclid, and the theorem pi
  Did plane and solid in the text;
All parallel were the radii,
  And the ang-gulls convex'd.

Perhaps the students, themselves, could be encouraged to be creative and poetic in their geometric work. These poems are from the twentieth century. However, to aid in studying Euclid, the students might choose to study the poetry and prose of the time in which he lived and worked. This too, could enhance student motivation.

This brief historical sketch offers glimpses as to what Euclid may have been like. Certainly, students of geometry may find it motivating indeed to learn these things on their road to geometric knowledge.

Algebra

Many High school juniors may study Algebra. Basic Algebra involves the study of finding solutions to simple equations by using algorithms developed for that purpose. These algorithms are primarily arithmetic in nature. The history of Algebra is very interesting. In order to begin a study of the history of Algebra one must study the Hindu contributions to this field. Much of early Algebra is attributed to the Hindus. As early as 500 A.D., the elder Aryabhata stated the following Algebra problem. (17 : 186)

> Beautiful maiden with beaming eyes, tell me as thou understandst the right method of inversion which is the number which multiplied by three, then increased by 3/4 of the product, then divided by 7, diminished by 1/3 of the quotient, multiplied by itself, diminished by 52, by the extraction of the square root, addition of 8 and division by 10 gives the number 2?

This of course is a lengthy version of the following problem:

$$\left\{\sqrt{[(2/3)(7/4)(3x)/(7)]^2 - 52} + 8\right\}/10 = 2$$

This problem "illustrates the Hindu Method of clothing arithmetical problems in poetic garb." (17 : 186) Eves goes on to claim that this was done due to the fact that Algebra problems were frequently used for social amusement and fun. This alone is a fact which high school juniors may find interesting. They may not claim that Algebra is fun. However, they may wish to solve this 1500 year old problem.

It may also offer young students consolation to know that the Hindu forefathers of Algebra used complicated symbolism. (17 : 187) The following are equivalent:

$8xy + \sqrt{10} - 7$ and $y\bar{a}\ k\bar{a}\ 8\ bha\ ka\ 10\ r\bar{u}\ \dot{7}$

In this case, addition was indicated by juxtaposition

subtraction by a dot over the minuend

multiplication by bha after the factors

division by writing the dividend over the divisor

square root by ka (means irrational) before the quantity

Unknowns by yā

Known integers by rū

Additional unknowns were indicated by initial symbols of words for different colors
ka (kālaka-"black")

The word algebra developed from an Arab word "**al-jabr** (transpose of a negative quantity) also **al-muqabalah** (transpose of a negative quantity and the combining of terms.)" (41 : 144)

Mohammed ibn Musa al-Khowarizmi lived in 825 A.D. and wrote **Al-Jabr w'al Muqabalah.** This treatise classified equations and contained geometric demonstrations of his algebraic rules for solving the quadratic and linear equations. (41 : 167-168) The word **al-jabr** became known as algebra and exists today as the word describing various ways to manipulate terms of and find solutions of equations of various degrees. (41 : 144)

Eves also claims that the Hindus knew that quadratic equations with real solutions possessed two solutions. However, if one of the solutions was complex, it was dismissed as impossible. This fact may be of interest to high school students studying mathematics. The famous mathematicians instrumental in the development of algebra also struggled with non-real solutions. (17 : 187)

Although Europe was deep into the middle ages at the time of al-Khowarizmi, the term **al-jabr** made its way to that continent. There were, in fact, many people known as **algebristas** at work during that non-mathematical period. Much to the mathematician's chagrin, the term was used in conjunction with the occupation of barber. Barbers of that time also performed bonesetting and bloodletting. (18 : 161). Students of Algebra may make humorously ironic comparisons between the job performed by their classroom teacher and the medieval algebristas. They might also care to notice the red and white barber poles found outside most barber shops.

Trigonometry

Aside from, or in addition to algebra, many juniors in high school will study Trigonometry. They will need this discipline in order to pursue Calculus or any other higher level mathematics course. The Houghton-Mifflin Dictionary defines Trigonometry to be the study of properties and applications of functions involved with right triangles.

Hipparchus (140 B.C.) "made a catalogue of 850 stars, placing each by its own latitude and longitude. He seems to have been the first man to make a systematic use of Trigonometry. He knew that "$\sin^2 (x) + \cos^2 (x) = 1$" for all real numbers (written here in modern terms.) (41 : 292) Consequently, Hipparchus was probably the first person to fully use and understand Trigonometric ideas in his work.

Around 150 A.D., Ptolemy, the astronomer, wrote a book named **Syntaxis**, which later became known as **Almagest.** It contained an extensive study of heavenly bodies. "The most interesting part of the **Almagest** is in the work of construction of tables." (41 : 294) He "computed a table of chords for arcs from 0 to 180 degrees in steps of half a degree." (41 : 295)

"The important role played by Arabs in geometry was not one of discovery, but one of preservation." (17 : 195) Considering all that was lost at the University of Alexandria's library, this would not be an easy task. A very interesting event occurred due to this preservation quest: many scholars studied Hipparchus's work.

Aryabhata of Hindu renown, studied these works and called sine "ardha-jya" which means half chord. Eventually, he shortened it to jya ("chord".) From jya, the Arabs phonetically derived jiba. The Arab practice of omitting vowels meant jiba would be written as jb. Jiba is a meaningless word in Arabic, it is merely the correct way to pronounce Aryabhata's jya. Later, writers decided to use jaib since, after all, it did contain the correct letters (from jiba) and it did have a meaning in Arabic. Eves claims it meant "bosom" (17 : 198), and Sanford indicates that it meant "fold" or "bay". (40 : 295)

Sicily was a major point of contact between east and west. At this place, Western merchants became acquainted with Islamic civilization. It was not long before "scholars flocked to Toledo to learn the science of the Arabs." (44 : 102) Many translators disguised themselves as Mohammedan students in order "to obtain the jealously guarded knowledge." (17 : 210) Gherado of Cremora, one of the most industrious translators, translated many works from Arabic to Latin. This list included "Euclid's **Elements**, Ptolemy's **Almagest**, and al-Khowarizmi's algebra."

(17 : 210). In fact, he took one of the literal meanings of jaib and in 1150 A.D., made his translation into Latin. The word became **sinus.** Eventually, the word became **sine,** and any extraneous definitions were abolished. Sine has simply come to mean the function of an angle of a right triangle which is equal to the ratio of the length of the opposite leg to the length of the hypotenuse. Thus, one of the many interesting developments for the world of mathematics was the word sine.

Many students of Trigonometry may find this development highly interesting, if not a bit odd, and be henceforth, motivated to learn more about this subject.

Calculus

Through the course of the study of mathematics, students will inevitably encounter Calculus. Most mathematical study in the public schools centers upon Number Theory and Algebra and Geometry. However, the next step usually involves Calculus. Calculus is "concerned with the behavior and applications of functions." (20 : 1) It "is distinguished from algebra, geometry, and other precalculus topics by the concept of the limit." (20 : 47) When the study of Calculus begins, it may prove motivating to both students and teacher to study its history.

The history of early Calculus is a very powerful and moving story. Its very beginnings are rooted in controversy. There was for over a century an argument about who should receive the credit for its discovery. The two men involved were Isaac Newton and Gottfried Wilhelm Leibniz. From the beginning, it should be noted that much research indicates there is no argument in terms of who invented it first. That credit goes to Newton. However, Leibniz published first. The question is then: To whom should the credit be granted? Had Newton simply published what he knew when he knew it, the entire undignified dispute may have been avoided.

There were many differences in the Calculus used by Newton and that used by Leibniz. Cajori (9 : 197) states the following

> Newton, holding to the conception of velocity or fluxion used the infinitely small increment as a means of determining it, while with Leibniz the relation of the infinitely small increments is itself the object of determination.

In essence, Newton needed Calculus to do his physics while Leibniz was more impressed with the beauty of the mathematical manipulations.

Isaac Newton. According to Eves, Newton was born in 1642. (17 : 335) Perhaps as an omen to the controversy which would surround his life, it was also the year in which Galileo would breathe his last breath. Newton's father was a farmer who died before his son was born. It was expected that Newton would follow in his father's footsteps and run the family farm. However, when Newton was a small child, he invented a toy gristmill powered by a mouse which turned wheat into flour. Also, he invented a clock powered by water. (17 : 336) Obviously, farming was not in his future.

In elementary school, Newton was slow and inattentive. Sanford tells us (41 : 191) that one day,

> The little Isaac received a severe kick upon his stomach from a boy who was above him, so he labored hard until he ranked higher in school than his offender. From that time he continued to rise until he was the head boy.

While attending this grammar school, Newton lived with Mr. Clarke, the village apothecary. Mr. Clarke had a stepdaughter, Miss Storey. Newton fell in love with Miss Storey, and subsequently they became engaged. However, when Newton was unable to push his studies to the background, Miss Storey married another. Newton remained in love with studies most of his life, but for Miss Storey he "cherished a warm affection." (5 : 92)

When he was 15, his mother took him from his school to tend the farm. Newton did not do any work. Instead, his mother always found him studying. She then sent him back to school. He attended Trinity College in Cambridge. At 18 years of age, Newton attended a county fair where he acquired a book on Astrology. (41 : 55) While in school, he then studied Euclid's **Elements** to better understand the astrology. Finding the masterpiece by Euclid too obvious, he continued in other studies.

By the age of 23, he had created his **method of fluxions** (differential calculus). But then his school closed for two years as the bubonic plague was ravaging the people. Newton then found himself back on his mother's farm, and still studying. It was during those two years away from the closed school that he perfected his **method.** By the time he was 27, he received a professorship at Cambridge. He lectured there for eighteen years.

In 1672, at age 30, Newton became a member of the Royal Society of London. Then, the next year he declined a membership, claiming travel to London was too far. The secretary of the Royal Society believed the real reason to be monetary and persuaded the Society to waive Newton's fees. (41 : 56) Newton remained a member of the Society until his death.

During his stint with Cambridge, he continued to work on his Calculus. "For a long time, Newton's **method** remained unknown except to his friends and their correspondents." (9 : 193) It would remain unpublished in English until 1736, nearly 65 years after it was written.

Beginning in 1689, Newton represented Cambridge in Parliament until 1692. It was in that year that he began a two-year battle with a "form of mental derangement" (17 : 337) Eves speaks of Newton's obsessive attempts to avoid controversy at any cost. (17 : 336) He makes the following statement:

> [Newton's] tremendous dislike of controversy, which seems to have bordered on the pathological, and had an important bearing on the history of mathematics, for the results of his writings remained unpublished until many years after their discovery.

Newton's followers and friends were extremely loyal. He was a mathematician who spent 18 hours a day writing. Bell (5 : 109) commented on Newton's lifestyle.

> Never careful of his bodily health, Newton seems to have forgotten that he had a body which required food and sleep when he gave himself up to the composition of his masterpiece. Meals were ignored or forgotten, and on arising from a snatch of sleep he would sit on the edge of the bed half-clothed for hours, threading the mazes of his mathematics.

His work ranged from optics to dynamics to theology. About Newton, Pope wrote a short poem. (cited in 17 : 341)

"Nature and nature's laws lay hid in night;

God said, 'Let Newton Be', and all was light."

But, forever humble, and timid about controversy, Newton sums up his work as follows: (cited in 17 : 342)

> I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

In 1687, Newton published **Principia**, a description of all of his astronomical and dynamic discoveries. To briefly discuss any element of this great work would not do the work justice. However, it should suffice to say that Eric T. Bell described it as "the system of the world." (5 : 109).

Smith found it noteworthy to mention that Thomas Jefferson studied Newton's **Principia.** "There have been not many, if any, other presidents of the United States who have cared to add this treatise to their libraries." (43 : 64)

In 1696 Newton left the college classroom to become warden of the mint. (17 : 337) The minted coins, in circulation at that time were of imperfect weight and size due to the imperfect machinery producing them. People clipped coins in circulation to use the valuable shavings; the metal in the coin was worth more than the coin. Newton set about fixing the machines, and within three years, he had withdrawn all the imperfect coins from society and the machinery was producing uniformly perfect coins. (41 : 57) By 1699, he was master of the mint.

1703 marked the year for his election to president of the Royal Society, a post which he held until he died. Queen Anne bestowed a magnificent honor upon Newton in 1705. It was in that year that he was knighted. In 1727 a tombstone was erected for the Sir Isaac Newton in Westminster Abbey.

Gottfried Wilhelm Leibniz. According to Ball, Leibniz was born in 1646, four years after Newton. He lived then in Leipzig, Germany. His father died before Leibniz was six. By the age of 12, the young Gottfried had taught himself Latin. Ball implies that the elementary schools of Germany at that time were not sound. Consequently, excellence in student work was quite rare. "...the teaching at the school to which he was then sent was inefficient, but his industry triumphed over all difficulties..." (41 : 354)

The University of Leipzig was where, at the age of 15, he began formal study with better teaching. His primary interest was law, but he studied every branch of knowledge diligently. Higher mathematics was not taught, and as a case in point - "a certain John Kuhn lectured on Euclid's **Elements** but that his lectures were so obscure that none except Leibniz understood them." (9 : 205) Despite all of these obstacles, Leibniz kept studying, and by the time he was 20 he had mastered ordinary texts on mathematics, philosophy, theology, and law.

He grew up in an historically magnificent time. Constantly, he was "in contact with the best of the culture then existing." (9 : 205)

In 1666 Leibniz published a treatise, **De Arte Combinatoria.** Although the treatise contained little new mathematical material, it laid a momentous groundwork in terms of new symbols for mathematics. According to Cajori, P.E.B. Jourdain, (cited in 9 : 211) makes the following statement regarding Leibniz:

> Leibniz made important contributions in the notation of mathematics. Not only is our notation of the differential and integral calculus due to him, but he used sign of equality in writing proportions, thus a:b = c:d. In Leibnizian manuscripts occurs $\sim$ for 'similar' and $\simeq$ for 'equal and similar' or 'congruent'.

At the time of the publication of Leibniz's **De Arte Combinatoria**, he was refused a doctorate by those jealous of his youth, and he relocated in Nuremberg. It was there that he wrote an essay on the study of law which prompted the Elector of Mainz to employ him to rewrite some statutes for him. Through the course of time, he eventually was employed in diplomatic service. This led him to Paris. While in France, around 1672, C. Huygens gave Leibniz a copy of his work on pendulum oscillations. Consequently, Leibniz again studied geometry, "which he described as opening a new world to him." (4 : 354)

1673 saw two extremely important works of Leibniz. He presented plans to the Royal Society for a computing machine. Although it was similar to Pascal's, it was also more efficient and perfect. One of his other accomplishment is that he derived the series:

$$\arctan(x) = x - 1/3\, x^3 + 1/5\, x^5 \ldots$$

where $-1 \leq x \leq +1$

This series is the one "from which most practical methods of computing pi have been obtained." (9 : 206) In the same year, the Elector of Mainz died and Leibniz was quickly employed by the influential Brunswick family for 3 years. Finally in 1676, he moved to Hanover where he spent the rest of his days. He "occupied the well-paid post of librarian in the ducal library." (4 : 355) Ball goes on to explain that Leibniz was considered invaluable with regard to his writing on political, historical, and theological questions of the times. The Hanoverian court worked his pen very hard. His vast number of works produced during this time were recognized, and he received many awards of various kinds.

The acquisition of this leisurely job afforded Leibniz time to pursue more favorite hobbies, mathematics being just one of them. Leibniz's works in mathematical fields may be exceeded by his work in philosophy and literature. His philosophical beliefs were something different for his time. According to Ball, Leibniz viewed "ultimate elements of the universe as individual percipient beings whom he called monads. The monads are centres of force and substance is force, while space, matter, and motion are merely phenomenal; finally, the existence of God is inferred from the existing harmony among the monads." (4 : 355)

While pursuing his mathematical quests, he wrote to

Isaac Newton and received some of the following information in a letter:

(1) June 1676 - This letter contained

* some calculations and information regarding the binomial theorem
* nothing directly related to Newton's **method of fluxions**

After receiving a praise-filled letter from Leibniz requesting more information, Newton wrote another letter to Leibniz:

(2) October 1676 - This letter contained

* his method to determine the binomial theorem
* an awkward anagram explaining the **method**

In 1684, Leibniz shared differential calculus with the world via **Acta Eruditorum.** The publication brought Calculus to public knowledge and this work was notationally concise. He presented the present day integration sign, $\int$, alonng with the "d" for derivatives. Jourdain claimed that "Leibniz himself attributed all of his mathematical discoveries to his improvements in notation." (cited in 9 : 211)

Leibniz never stated that he was the first to do Calculus. However, he did state that he discovered independently of Newton. His publication was written simply because he wanted to share with other mathematicians what he had discovered. But, in 1699, a Swiss mathematician, Fatio de Duillier, sent a letter to the Royal Society which implied that Leibniz had plagiarized. Then, in 1704 an anonymous review of Newton's work implied that Newton had stolen from Leibniz. It was that review which drew the line between the two men. Prior to 1704 Leibniz had been given credit for the calculus, but when it was suggested that Newton's honor was questioned in that he had stolen from Leibniz, the entire question was taken under review by the Royal Society.

Any protests which Leibniz may have had took the form of only private letters to friends. He declared that he would not answer arguments so weak. He offered his defense and left the arguments to the others.

In the end, the opinion of most eighteenth century mathematicians was decidedly against Leibniz from Germany. The NCTM 31st yearbook (38 : 418) states that

> When the Duke of Hanover went to London to become George I, the first German King Of England, Leibniz was left behind in neglect, and two years later he died; it is said that only his faithful secretary attended the funeral.

<u>Summary of the Controversy</u>. The answers to the questions relating to who discovered the Calculus, will never be known for certain. Many mathematics historians have pondered this controversy and wondered what did happen. Today, the primary belief is that both Newton and Leibniz discovered the Calculus independently of one

another. However, in 1712, the Royal Society of London ruled thus: (9 : 215)

(1) Newton was the first to discover calculus

(2) There is no mention of Leibniz having any other Differential Method before the June letter...

(3) This June letter arrived a year after a letter from Newton dated the 10th of December, 1672, had been sent to Paris to be communicated to Leibniz...

(4) Leibniz's introduction of Calculus occurred about four years after a letter in which "the method of fluxions was sufficiently describ'd [sic] to any intelligent person" was shared with the mathematics community.

At first, Cajori says, (9 : 213) Newton agreed:

> That most excellent geometer, Leibniz... had fallen upon a method of the same kind, and communicated his method, which had hardly differed from mine, except in his forms of words and symbols.

According to this statement, it may certainly be the case that Newton did not feel as though Leibniz had stolen from him.

Some authors claim that it was Leibniz's style to write down everything he learned about a particular topic. This was also true regarding his study of mathematics. However, the letters mentioned previously, which Leibniz had received from Newton, were not among the other mathematical papers which had been transcribed by Leibniz. This may indicate that Leibniz saw nothing new in those two letters. Cajori (9 : 2115) cites some recent findings regarding the controversy:

> About 1850 it was shown that what H. Oldenburg sent to Leibniz was not Newton's letter of Dec. 10, 1672, but only excerpts from it which omitted Newton's method of drawing tangents and could not possibly convey an idea of fluxions.

However, according to another source, (4 : 360) Leibniz may have performed some questionable activities:

> [He did not acknowledge] possession of a copy of part of one of Newton's manuscripts. [Leibniz] deliberately altered or added to important documents before publishing and...a material date in one of his manuscripts had been falsified (1675 being altered to 1673), [this] makes his own testimony on the subject of little value.

The calculus controversy brought virtually all sharing of mathematical ideas to a halt. Letters sent from one mathematician to another no longer contained new ideas or unpublished methods. The English adhered to Newton and his knightly honor. Refusing to use non-English originated methods, they adamantly remained ignorant of discoveries made in the rest of Europe until at least 1820.

If a small bit of progress was made due to this controversy, it was "through the challenge problems by which each side attempted to annoy its adversaries." (9 : 217)

Certainly, this study of Calculus history might prove of interest to students of the mighty Calculus.

Summary

In this chapter, the writer included a discussion of Geometry, Algebra, Trigonometry, and Calculus. As a reinforcement for the study of Geometry or Calculus, concise biographical sketches of Euclid, Newton, and Leibniz were presented. Regarding Algebra and Trigonometry, the writer traced the development of some symbolism and mathematical terms. It was the writer's intention that these brief historical overviews might enhance the motivation of the students studying the previously mentioned mathematical areas.

CHAPTER V

NUMBER THEORY

Introduction.

Inherent in nearly every mathematics course taught in the public schools is Number Theory. Whether students are learning to add single digit numbers or to integrate polynomials, conceptual understanding of the theory of numbers is a required part of the process. Set theory, operations with integers, simple algebraic equations, counting, logic, and congruences include a few of the topics which public school mathematics students may study in Number Theory.

Students might be motivated to learn more about Number Theory if they were granted the opportunity to study its elegant history. Mathematics includes many diverse areas of study. Although the birth of mathematics revolved about Number Theory, this branch has not died. It is a healthy and continually growing field. Students may, therefore, find the mathematicians who helped develop Number Theory worthy of study.

Some form of number theorizing has been traced as far back as 3400 B.C., when an elementary positional number system is believed to have been used. The Sumerian system "which was sexagesimal but whose numeral representation was a simple grouping system" was probably the first positional system. (39 : 16) Although this was very rudimentary, it was a start, and the study of number theory continues today.

Carl F. Gauss may have summed up the magnitude of number theory better than any other person. (cited in 31 : 1)

> Mathematics is the queen of the sciences, but Number Theory is the queen of the Mathematics.

Henri Poincare made the following remark. (cited in 31 : 1)

> A scientist worthy of the name, above all a a mathematician, experiences in his work the same sensation as an artist; his pleasure is as great and of the same nature.

Long, too, added a statement. (31 : 1)

> Yet surprising results, economically stated and subtly proved have been a source of pleasure and satisfaction to the minds of men throughout the ages.

J.V. Uspensky and M.A. Heaslet speak of Number Theory as having the ability to lure people into study. (48 : 20)

> The theory of numbers, unlike some other branches of mathematics, is purely a theoretical science without practical applications. [There are mathematicians who] boast about its lack of application. Only very few among the great mathematicians did not work at one time or another in the theory of numbers. The answer is that the whole beauty of this science becomes apparent only to those who penetrate deep into it.

Ore speaks about number theory as having all the qualities that delight both mathematicians and non-mathematicians. It is ruled by simple and easily understood ideas and yet many unsolved problems still exist. Number Theory includes both concrete calculation and abstract thought.

> The innumerable individual contributions, calculations, speculations, and conjectures bears witness to the continued interest in this field of mathematics throughout the centuries.

Counting.

Counting is perhaps the most rudimentary aspect of Number Theory. It seems to have been a part of almost every culture. Counting usually takes the form of "matching the objects to be counted with some familiar set of objects like fingers, toes, pebbles, sticks, notches or the number words." (39 : 1) For example, it may well have been the case that shepherds kept a pouch of stones equal in number to their flock in order to keep track of their sheep. When one sheep came in, one pebble was cast into a pile. When all the pebbles had been cast into the pile, all the sheep should be home.

Concerning smaller numbers, "almost all people seem to have used their fingers as the most convenient and natural counters." (39 : 1) This is evidenced by the representing and grouping of number into base 10 used by most of the world today. Obviously, base 5 (one hand) and base 20 (both hands and both feet) could have been the prevalent base. Certainly, there are examples of non-base ten systems in society today. A dozen and a gross are both from base twelve. Minutes and seconds would be reminiscent of base 60. Lincoln's famous speech "four score and seven..." implies common knowledge that a score is twenty.

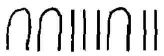
Several methods of counting have been found in researching ancient societies. According to one authority, "the method of using knots tied on strings has been used quite widely..." (39 : 8)

Finally, society's bookkeeping needs forced cultures into finding a method for counting which could be saved and consulted again at a later date. As early as 3400 B.C., the Egyptian cultures used the following symbols and method. (39 : 10)

1	10	100	1000	10000

Figure 3

Egyptian Numbers

Using this method, the number thirty-five might be written as  . It should be noted that there is no concern about the positional importance of the symbols. Methods such as these were in use in many societies, each different culture possessing a unique set of symbols.

According to Ore, although positional numbers (those dependent upon place-value) were used in some early societies, none had a zero. It was not until late in the first millenia A.D. that the zero is known to have existed. "Often the true value of a number can be decided upon only through the context, although at times the spacing of the symbols may be of assistance." (39 : 18) For example, without a zero, the numeral 24 might mean 204 or 24 or even 2004, depending on the context surrounding the numeral. Finally, though, "the use of a positional system with a zero seems to have made its appearance in India in the period A.D. 600- 800." (39 : 19)

Al-Khowarizmi contributed much to the spreading usage of this new system which he used to do his calculations in his algebraic work, **Al-Jabr w'al-Muqabalah.** By 1000 A.D., Gobar Numerals were being used in Spain. (39 : 20)

1 2 3 4 5 6 7 8 9 0

Figure 4

Gobar Numerals

Admiration of common peoples' abilities to use the positional number system, which were far superior to the use of counters, seems to have crept into non-mathematical literature. People who could easily use the new system in its early days, were respected. Cajori (8 : 122) relates a specific example:

> In the **Winter's Tale** (IV.3), Shakespeare lets the clown be embarrassed by a problem which he could not do without counters. Iago (in **Othello**,i,I) expresses his contempt for Michael Cassio, 'forsooth a great mathematician' by calling him a 'counter-caster'.

Then in 1202, Leonardo of Pisa, published **Liber Abaci** which was a "compendium of arithmetic algebra and number theory." (39 : 20) This book helped more than any other scholarly work to secure the widespread usage of the positional numeral system among different societies. However, the needs of merchants and traders for a universal number system contributed as much as anything to the usage and acceptance of the Hindu - Arabic system at use in today's society. It may well be the case that Fibonacci was the only mathematician of extraordinary talent of the middle ages.

Leonardo of Pisa. Leonardo of Pisa was also called Fibonacci. He was born in Pisa around 1175. His father was connected to the mercantile business as a customs manager. As his father was knowledgeable of numerical calculations, he made the young Fibonacci learn the use of the abacus. The education was continued in Bougie on the North Coast of Africa, where he lived for a short time.

Fibonacci also traveled much in Egypt, Syria, Greece, and Sicily. It was in these areas that Fibonacci came into contact with leading Arabic mathematical works. During the twelfth century, Europe was still embracing the medieval period, which was not conducive to original mathematical thought. Consequently, all that the Greeks had done which had been preserved was alive primarily in the eastern and southern Mediterranean countries only. Fibonacci's studies had a decidedly Hindu influence, and he learned the Hindu method of calculation--which incorporated place value.

Fibonacci wanted to share this new positional number system method with society. He did not study to solve a great physics problem, nor for prestige among members of society. He "treats the subject in a free and independent way", "he was not merely a compiler, nor like other writers of the middle ages, a slavish imitator of the form in which the subject had not been previously presented." (9 : 120) For example, he developed a sequence of numbers, fittingly known as the Fibonacci Sequence. This sequence begins with 1, 1 and continues in a manner such that each number is the sum of the preceding two numbers:

$$\{ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \}$$

"Probably in 1224, Leonardo was summoned to take part in a mathematical tournament which was to be held in the presence of the Emperor". Frederick II was the Holy Roman

Emperor in Italy beginning in 1212. He "was a sincere patron of learning and actively promoted the diffusion of Arabic knowledge in Europe". The challenge problems offered Fibonacci the chance to exercise the place value system in front of a lofty audience. "Leonardo easily carried off the laurels by solving all problems in the most admirable manner." (39 : 187-188)

Following the tournament, Fibonacci again published an original mathematical work entitled **Liber quadratorum.** He continued to work throughout his life on basic number theory. He was a bright spot in mathematic's dark period. It might be interesting to note that many people "considered him a blockhead (for his interest in the new Hindu - Arabic numerals), and it pleased him to show these critics what a blockhead could accomplish." (18 : 168) He took to signing some of his works with the name "Bigollo," which means traveler - certainly applicable to Fibonacci. However, it also means blockhead.

Around 1250, the Fibonacci light was suffocated, and mathematics again fell into darkness for over two centuries. Long wars were partly to blame for occupying peoples' minds on non-mathematical thought. Some blame may fall on the intellectual leaders of the time who misdirected mathematical thinkers to ponder the number of angels which would fit on a needle point. (9 : 125) Although there were some writers of consequence, they scarcely used the ideas of Euclid or Fibonacci.

Special Numbers.

There are many types of numbers, and many numbers which can be of more than one type. As it had been with Pythagoras, in 500 B.C., mathematical study of the fifteenth and sixteenth centuries again turned towards properties of numbers. Although Pythagoras worshipped numbers and found mysticism in numbers, that would not happen again. During the renaissance, numbers were studied formally. One type of number, which was studied by the Pythagoreans was triangular numbers. Figure 5 shows that these numbers can be represented by equilateral triangles.

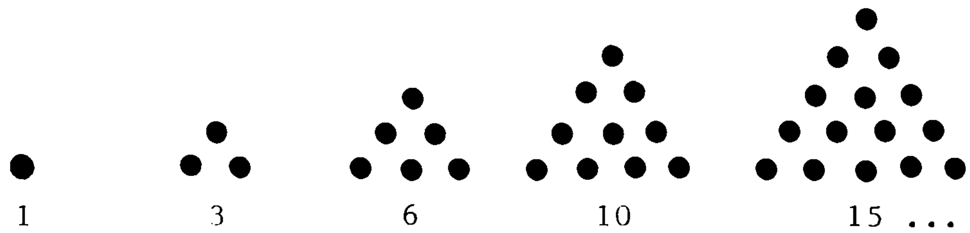


Figure 5

Triangular Numbers

With regard to triangular numbers, Sir Thomas Heath (28 : 76) relates that:

> It was probably Pythagoras who discovered that the sum of any number of successive terms of the series of natural numbers 1,2,3,... beginning from 1 makes a triangular number.

Other types of numbers include square and pentagonal numbers, whose pictorial representations reflect squares and pentagons, respectively. These too, were known to Pythagoras.

Among the Number Theory concepts first studied with regard to natural numbers was the inspection of a number's divisors. Several definitions regarding the factorization of whole numbers occurred due to this study:

1. Prime number - A number greater than one which has no divisors other than 1 and itself

2. Composite number - A number greater than one which has divisors other than one and itself

3. Aliquot divisors - Divisors of a number which are less than the number itself (sometimes called aliquot parts)

4. Perfect number - A number which is equal to the sum of its aliquot parts

5. Deficient number - A number which is greater than the sum of its aliquot parts

6. Abundant number - A number which is less than the sum of its aliquot parts

7. Friendly numbers - Two numbers which are equal to the sum of each other's aliquot parts

8. Greatest Common Divisor - The largest number which divides two given numbers

9. Relatively Prime - Two numbers whose greatest common divisor is one

10. Least Common Multiple - The smallest number for which two given numbers are factors

These definitions are less mystical than the ones cited by Pythagoras. However, many sources indicated that he probably knew of these relationships. The bulk of Pythagorean number theory study seemed to be with addition and its relationship to multiplication. The numerical notation used in 500 B.C. did not lend itself well to studying factorization techniques and consequences. Euclid was aware of many facts regarding Number Theory and included these in **Elements.** For instance, the idea for the Fundamental Theorem of Arithmetic and its proof can be found in Books VII and IX. (39 : 52)

> Fundamental Theorem of Arithmetic:
>
> Every integer greater than one can be written as a unique product of prime numbers, if the order of the factors is not taken into consideration.

But, when Fibonacci introduced an elegant method of calculating by using the Hindu - Arabic numeration system all the ideas founded by Pythagoras and Euclid began to find their way into rigorous mathematics.

Pierre de Fermat. Pierre de Fermat was an innovative force on the history of Number Theory and on the introduction of rigor into 17th century mathematics. "Fermat must be awarded the honor of being the founding father of Number Theory as a systematic science." (39 :54) Among other things, he took hold of the previously mentioned factorization ideas and developed a technique to easily factor large odd numbers. "It becomes one of the most effective factorization methods available." (39 : 58) Using that method, he first found a way to write the odd numbers as the difference of two squares and then that, of course, factors quite easily.

He also issued the following conjecture and stated that he had a proof of his statement. (14 : 59)

Conjecture: If p is a prime number and x is a positive integer such that x and p are relatively prime,

Then $x^{(p-1)}$ is divisible by p.

Fermat did not provide his proof of the statement, Leibniz later gave a proof of the above conjecture.

Another example of Fermat's work was with respect to prime numbers. It is important to note that the next claim was stated without any type of claim regarding the possession of any type of proof.

It was his belief that:

All numbers of the form $2^{2^n} + 1$ were prime numbers,

when n is a positive integer.

To this day, all numbers of this form are bestowed with the honor of being called 'Fermat Numbers'. Since the time of the issuance of Fermat's belief, mathematicians have discovered that some of these numbers are prime and that some are not.

An interesting application of the Fermat Numbers, which involves geometric constructions, may be of interest to young students of Number Theory. Bell (5 : 67) states that an eighteenth century teenager, Carl F. Gauss, found the following statement to be true:

> A straight-edge and compass construction of a regular polygon having an odd number of sides is possible when and only when that number is either a prime Fermat number or a product of prime Fermat numbers.

Pierre de Fermat was born near Toulouse France around 1601 to a leather merchant. He received his early education at home with his father. When he was thirty years of age, he married his mother's cousin. Also, in that same year, he obtained the post of councillor for the local parliament. During the next seventeen years he settled into a quiet life; he served his country honorably as a fine lawyer. Fermat's reputation was one of modesty and proper protocol; he was known for his honest, hard work. He then was promoted "to a king's councillorship in the parliament of Toulouse, a position which he filled with dignity, integrity, and great ability for seventeen years." (5 : 58)

He spent these last years as a humble lawyer who spent his leisure hours on the study of mathematics. The type of work in which he was engaged by the state lent itself well to the pursuit of mathematics. He was encouraged to "abstain from unnecessary social activities and hold himself aloof from fellow townsmen lest he be corrupted by bribery." (5 : 58) Although his mathematical work was not evidenced by formal publications, he was in contact with many other mathematicians of the time, including Rene DesCartes. Consequently, Fermat may have influenced the mathematicians who were publishing mathematical discoveries.

Eves claims that "in this field (Number Theory), [Fermat] possessed extraordinary intuition and ability." (18 : 289) Certainly, many claims he made were later proved. "Without doubt, he is one of the foremost amateurs in the history of science, if not the very first." (5 : 57)

Number Theory deals with integers. It is the pure mathematical branch, most often practiced by amateurs and laymen. To be sure, Number Theory contains many difficult concepts. However, the simplicity of the field alone may encourage many people to study mathematics for the joy and art inherent in these numerical manipulations and relationships. Bell claims that Fermat did much of his work "for the sheer love of it" and that his work made it all seem so simple. (5 : 58)

After Fermat's death in 1665, his son Clement-Samuel saw to it that his father's margin notes in Diophantus's **Arithmetica** were published in 1670.

Within the pages of this work was the following Diophantine equation:

> $x^n + y^n = z^n$ where x, y, z, n are natural numbers has no solution when $n > 2$

Beside it, Fermat wrote in the margin the following remark: (cited in 17 : 293)

> I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.

Once again, Fermat suppressed his proof. Throughout Fermat's mathematically productive life, he many times claimed that he had a proof of his statements. But, he would choose not to share that with his contemporaries. Sometimes, he would offer a conjecture but make absolutely no claim regarding a proof. "Whenever Fermat asserted that he had proved anything, the statement with one exception noted has been subsequently proved." (5 : 71) The above Diophantine equation represents the one needle in the Fermat haystack which is unproven to date (1990). It has come to be known as Fermat's Last Theorem.

The validity of Fermat's claim of possession of a proof may never by verified. Many mathematicians declare that Fermat deceived himself, while others proclaim that his honest character would not have allowed him to do so. Students who ponder the questions surrounding Fermat's Last Theorem and consider Fermat's previous record may feel that this indicates that he indeed may have unraveled the proof.

However, the fact that the proof has eluded mathematicians for over 400 years after the conjecture, may indicate that Fermat's alleged proof may have indeed contained some type of error.

Mathematicians of the Renaissance Era shared a lot of ideas. During this time when periodicals did not exist, many mathematicians communicated their thoughts by writing letters to one another. Many dates of proofs and conjectures are known solely due to the collected mounds of personal correspondence between the mathematicians. Also, discussion circles played an important role in the dissemination of information. Some of these mathematical get-togethers occurred by chance, some were strictly by invitation only. To be sure, many of the famous mathematicians discussed in this thesis knew one another.

Father Marin Mersenne of the seventeenth century, probably knew almost all of the mathematicians of his day. He was a center of exchange of ideas for many of these people who were pursuing mathematics. As mentioned in Chapter III, he was a favorite of Rene DesCartes. Struik says that "to inform Mersenne of a discovery meant to publish it throughout the whole of Europe." (44 : 140) He was a necessary and useful component of the progress of mathematical development and the dispersion of mathematical knowledge during the Renaissance Era.

Two mathematicians who wrote letters to one another were Christian Goldbach and Leonhard Euler. Their correspondence dated back to at least 1742. In one letter, Goldbach wrote to Euler and stated the following conjectures:

1. Every even number greater than five can be written as the sum of two odd primes.

2. Every odd number greater than eight can be written as the sum of three odd primes.

"Euler, whose mathematical intuition was acute, states in reply that he also is convinced of the truth of these propositions, but he is unable to find any proof." (39 : 81) These statements have come to be called simply "Goldbach's Conjecture." To date (1990), a proof has not yet been found for Goldbach's Conjecture.

Also in 1742, possibly in correspondence with Euler, Goldbach used the symbol π (pi) to represent the irrational number 3.141592... which represents the ratio of the circumference of a circle to its diameter. In his **Analysis**, Euler made the usage of this symbol official. (4 : 395)

Christian Goldbach. Christian Goldbach was a mathematician of the eighteenth century. Not much is known of his life. A German mathematician, he lived from 1690 to 1764. Ore states that Goldbach was "a one-time teacher of Peter II" and a secretary for the academy in St.

Petersburg in Russia. (39 : 60) As Euler was also employed by the academy of St. Petersburg, it is the possible reason that Goldbach was in correspondence with Euler so often. Eventually, Goldbach left the realm of scientific work and research to finish out his life with a career in the Russian Civil Service.

Until 1976, a formula which generates only primes and all primes had not yet been discovered. James P. Jones, Daihachiro Sato, Kideo Wada, and Douglas Weins discovered a twenty-six variable polynomial of degree twenty-five which generates all primes and only primes when positive integers are substituted for the variables. Many statements had been made regarding the existence of such a function. Many attempts had been made through the centuries to discover such a function. In an effort to locate a prime number generator, often students would ponder a table of primes. While doing this, a conjecture was delivered in 1792 from a fifteen year old boy:

Conjecture: Let A_n be the number of primes less than n where n is a positive integer.

then $\frac{A_n(\log_e(n))}{n}$ approaches 1 as n increases without bound

A_n/n is referred to as the density of primes among the first n integers. It is approximated by $\frac{1}{\log_e(n)}$ which becomes more exact as the value of n increases.

This conjecture was proved in 1896 independently by two different mathematicians: J. Hadamard and C. J. delavallee Poussin. (18 : 116) The fifteen year old boy was named Carl Friederich Gauss.

Carl F. Gauss. He was born in Brunswick, Germany on April 23, 1777. His father was a bricklayer and his family, in general, was poor. His parents expected him to be a common laborer. But thanks to the alertness of his grammar school teacher, he was encouraged to study in the academic world. Charles William, Duke of Brunswick took upon the responsibility of Gauss's education. It was the Duke who sent Gauss to the Collegium Carolinum in 1792. (4 : 447)

By 1795, Gauss's professors said that he had learned all that it was possible for them to teach. At that time, Gauss went to Gottingen and studied under Kastner, about whom Gauss proclaimed as being "the first mathematician among the poets and the first poet among the mathematicians." (9 : 435) It was during this educational stint that Gauss made many of his discoveries in the theory of numbers. Finally, in 1798, he returned to Brunswick where he made his living by the private tutoring of other students.

Disquistiones Arithmeticae, published in 1801, contained much of Gauss's early work on Number Theory.

This had previously been submitted to the French Academy of Science, but it was "rejected in a most regrettable manner; Gauss was deeply hurt, and his reluctance to publish his investigations may be partly attributable to this unfortunate incident." (4 : 448)

During his life, Gauss wrote much. However, several of his notes and letters were not published until after his death. The treatises he did publish included work in Algebra, Astronomy, Geodesy, Electricity & Magnetism, Electrodynamics, Optics, and Number Theory. It should be noted that Gauss is considered by many to be the last mathematician whose interests and expertise were so broad. "The ground covered by Gauss's research was extraordinarily wide, and it may be added that in many cases his investigations served to initiate new lines of work." (5 : 451) It should be noted that Gauss is considered by many to be the last mathematician whose interests and expertise were so broad. Anymore, mathematicians are forced to concentrate on one area of specialization within the branches of mathematics, due to the vast amount of literature available in each field. Struik (44 : 203) emphasizes this point:

> Gauss's comparative isolation, his grasp of 'applied' as well as 'pure' mathematics, his preoccupation with astronomy and his frequent use of Latin have the touch of the eighteenth century, but his work breathes the spirit of a new period.

The emperor of Russia offered Gauss the mathematics chair in the academy of St. Petersburg in 1807. But Gauss refused Euler's old post to take instead the position of director of Gottingen Observatory. He was also a professor of Astronomy at this school at Gottingen. Ball states that Gauss never slept away from the observatory save once when he attended a seminar in Berlin. (4 : 449) This is the job he held until his death on February 23, 1855.

Congruences.

Much of more recent number theory has developed around the concept of congruences. In the public schools these concepts are taught as clock arithmetic. The definitions are as follows:

1. Congruent - If m is a positive integer and m divides the difference of two numbers a and b, $m \mid (a-b)$, then a is congruent to b modulo m, (mod m).

   A. Notation - $a \equiv b \pmod{m}$

2. Residue Class - A set of congruent integers (mod m)

3. Complete Residue System - The set containing precisely one element or representative from each residue class modulo m

Leonhard Euler. Leonhard Euler also contributed much to this aspect of Number Theory. He developed a function which when performed on a number gave the number of positive integers which were relatively prime to m and less than m. Called the Euler - Phi (ϕ) function, it is useful in work with mathematical congruences. For example, $\phi(8) = 4$ because {1, 3, 5, 7} are the only four numbers less than 8 and relatively prime to 8. The set {1, 3, 5, 7} is said to form a reduced residue system modulo 8.

Euler generalized Fermat's theorem, by stating and proving that if a and n are relatively prime positive integers, then $a^{\phi(n)} \equiv 1 \pmod{n}$. (30 : 62)

Leonhard Euler was born in 1707 in Basel, Switzerland. His father was a Calvinist minister. Euler was sent to the university of Basel after his father could no longer offer new instruction to the young man. His father had expected the young Euler to go into the ministry upon his return from school. While at the University, he became a favorite pupil of the famous mathematician, Johann Bernoulli. Bernoulli managed to convince the elder Euler that Leonhard possessed a gift in the area of mathematics and that his destiny was to be a mathematician. (9 : 232)

At the age of nineteen, Euler wrote a dissertation on masting ships. This effort was awarded a second prize by the French Academy of Science. Some historians claim that Euler was perhaps a bit idealistic in his work. He was, after all, from Switzerland; he had never seen an ocean. (5 : 144) No matter, for Euler, in subsequent years, won the French Academy of Science award twelve times. The reader might recall that, ironically, Gauss too was rejected by the French Academy.

In 1725, Johann Bernoulli's sons went to Russia. Euler too, was granted a job, due to the Bernoulli influence. Eight years later, Euler became the chairperson of the mathematics department at St. Petersburg academy. (4 : 393) Euler lost sight in one eye at the age of twenty eight. He did not let that slow his pace. At the age of sixty-two, he was completely blind. He would dictate to a secretary and do the calculations in his mind. During his life, he wrote 886 works. (44 : 168) Much of this volume of written work was due to the historical time in which he lived. Rather than worry about bloody battles or government spies, he stayed quitely in his home and wrote. His working conditions at home were ones which many mathematicians might find difficult. He "would often compose his memoirs with a baby in his lap while the older children played all about him." (5 : 146)

Euler wrote the mathematics books used in the Russian schools. He "supervised the government department of Geography, helped reform weights and measures, and devised practical means for testing scales." (5 : 147)

Although Euler did much work in many fields of knowledge, most of his study was mathematically based. Contributions to mathematics in general include denoting the irrational number 2.71828... (it is called the natural number even though it is not an integer) as "**e.**" Although he was not the first mathematician to use a single symbol to represent that value, he was the first to use **e.** Also to his credit go the symbolism of i for $\sqrt{-1}$ (40 : 187) and the current abbreviations for the trigonometric functions. (4 : 393) Ball also grants Euler the credit for the relationship between sine, cosine, and e: $e^{i\theta} = \cos(\theta) + i[\sin(\theta)]$. He "revised almost all branches of pure mathematics, adding proofs, and arranging the whole in a consistent form." (4 : 393)

On September 18, 1783, Euler contemplated Hershey's planet (Uranus). It was a new discovery, and Euler outlined the calculation of its orbit using his mental powers and help from his children. Bell (5 : 152) relates the story of Euler's death in a dramatic fashion:

> A little later he asked that his grandson be brought in. While playing with the child and drinking tea he suffered a stroke. The pipe dropped from his hand, and with the words 'I die' Euler ceased to live and calculate.

<u>Summary.</u>

When Gauss was seven and in grammar school, the teacher asked the class to add the numbers from 1 to 100. Apparently, the young boy Gauss, had discovered the formula for adding consecutive natural numbers. He had the problem done within moments of the issuance of the assignment. (36 : 158)

The formula for adding the first n natural numbers is:

$$\frac{n(n + 1)}{2}$$

where n is a positive integer

The Pythagoreans had knowledge of this summation. It was the same formula as the one which generated their precious triangular numbers. Also, this had been the password which Pythagoras required of his brethren to gain entrance into his world.

Perhaps it is fitting that the history of Number Theory would seem somewhat cyclic. From Pythagoras to Gauss and back to Pythagoras, a circle was made. The circle is, after all, that special regular geometric figure which has no corners, no beginning, and no ending.

In this chapter, the writer presented a condensed historical account of elementary Number Theory. The history was traced through brief developments of the concepts related to counting, special numbers, and congruences. In particular, mathematical contributions by Fibonacci, Fermat, Goldbach, Gauss, and Euler were cited. Views of the lives of these mathematicians were included in the historical sketch of Number Theory.

As Number Theory is inherent in nearly every branch of mathematics, teachers of mathematics will inevitably include its concepts in many lesson plans. Perhaps inclusion of some historical notes of the type found in this chapter might enhance the motivation of these students of mathematics.

CHAPTER VI

SUMMARY AND RECOMMENDATIONS

Summary of the Thesis

It was the purpose of this thesis to review motivation theory and to suggest possible mathematics history useful in meeting the needs of certain motivational strategies. This was accomplished in the preceding chapters.

In particular, Chapter I was an introduction to the study. In Chapter II, the writer explored pertinent literature relating to motivation in the classroom. Chapter III gave biographical sketches filled with oddities related to Pythagoras, Archimedes, and DesCartes. Chapter IV contained discussion of some mathematics history in light of specific mathematical concepts, Algebra, Geometry, Trigonometry, and Number Theory. The mathematicians studied in that chapter included Euclid, Newton, and Leibniz. Chapter V was dedicated to Number Theory and a brief view of that mathematical branch was presented. A sketch of the lives of the mathematicians Fibonacci, Fermat, Goldbach, Gauss, and Euler were presented in that chapter. This chapter, Chapter VI presents a brief summary of the study in addition to providing the reader with recommendations relating to the study.

Recommendations.

When the writer first learned of the historical underpinnings of mathematics, she was instantly intrigued. In high school and early in college, she believed that mathematical manipulations, definitions, and proofs were dropped from some perennial entity sitting among the unknown reaches of the universe. Toward the end of her college years, she studied Geometry and was introduced to the mysterious Euclid. At this time the term "Euclidean" Geometry finally made sense.

While teaching college algebra, several mathematicians were highlighted in that particular algebra text. By utilizing these relevant historical notes within the lesson plans, the writer was able to present a more enthusiastic lecture. Students' responses indicated a vigorous desire to learn more about the mathematicians discussed in class. Some students prepared research reports regarding the mathematicians studied in the text. Pursuant to the requests of the students, the writer's lesson plans grew to include much mathematics history. It is the opinion of the writer that their increased interest was due to biographies of famous mathematicians which were included in the lectures. Thus, their motivation was enhanced.

In order to correct the motivational difficulties sometimes encountered by students when studying irrational numbers, the writer used mathematics history to enhance student learning.

In order to discuss the irrational number pi (π), she assumed the personality of Archimedes for an entire class period. Without disturbing the atmosphere created by this act, representative of Archimedes's life, the writer presented the lesson as Archimedes may have wished. She developed the concept of pi by inscribing and circumscribing a circle in four and six sided regular polygons.

Pi was then defined to be the ratio of the circumference of the circle to the diameter of the circle.

One month later, the lesson turned to the study of arc length and circumference The students reacted very quickly to the use of pi and the definition of circumference. They did not use the 3.14 approximation, and they seemed more comfortable doing calculations with the symbol than previous students which were not offered Archimedes's perspective.

Suggestions for Further Study

This study was very brief when compared to the volumes which exist on the subjects of motivation and mathematics history. Other mathematical topics of interest to secondary students could be chosen, and the history of the lives of other mathematicians relating to such topics could be researched. Specifically, the thirty-first NCTM yearbook, Historical Topics for the Mathematics Classroom, should be consulted for further classroom discussion possibilities.

The topics presented were very elementary. The study of higher algebra and its history might prove interesting. The geometry topics presented in this study are very different from the geometry studied by Gauss. The elementary Number Theory topics presented in the preceding chapters, although treated rather extensively, certainly did not reach the depth which the field entails. Alternatives for study might include topology, graph theory, fractals, or analysis. As these fields are themselves rather extraordinary, there may be interesting ways in which they may be studied by relating them to the lives of appropriate mathematicians.

Finally, this study was limited primarily to research involving historical works written in the twentieth century. To study original manuscripts directly translated into the researcher's native language, may be a valuable method through which to gain a more comprehensive understanding of mathematics history and its relevance to the motivation of students of mathematics.

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