AN ABSTRACT OF THE THESIS OF

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(name of student) (degree)

in Mathematics presented on July 26, 1995
(major) (date)

Title: The Spaces c_0, ℓ_1, and ℓ_∞

Abstract approved: E. Ryan Warr

Given the sets c_0, all sequences of real numbers converging to zero, ℓ_1, all absolutely summable sequences of real numbers, and ℓ_∞, all bounded sequences of real numbers, the study of infinite dimensional vector spaces is developed. The use of basic analysis concepts allows for the proofs that ℓ_1 is a subset of c_0, and that c_0 is a subset of ℓ_∞. The definitions and theorems of vector spaces allow the proofs that each of these spaces are vector spaces and have norms defined on them.

Linear mappings among these spaces and from one to the set of real numbers are discussed as well as the norm of such functionals. Again using analysis, the concept of continuous functionals is developed. With this knowledge the topic of a dual space, or the space of all bounded linear functionals on a normed linear space, is investigated.

Finally, with the introduction of complete spaces it is concluded that c_0, ℓ_1, and ℓ_∞ are all Banach spaces. This result leads to the consideration of extreme points, unconditional convergence, the Dvoretsky-Rogers Theorem and the Hahn-Banach Theorem.

THE SPACES c_0, ℓ_1, AND ℓ_∞

A Thesis

Presented to

the Division of Mathematics and Computer Science

EMPORIA STATE UNIVERSITY

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Stefanie D. McKinney

July 1995

Approved for the Major Division

Approved for the Graduate Council

ACKNOWLEDGMENTS

I would like to express my gratitude to Dr. Bryan Dawson for his guidance and extraordinary patience. I would like to thank Dr. Larry Scott, Dr. Marvin Harrell and Dr. Jorge Ballester for giving of their time and energy to serve on my committee. I would also like to express my appreciation to Dr. Betsy Yanik for allowing me to borrow the computer on which I typed my thesis. Finally, I would like to thank my friends and family for their encouragement and tolerance during this stressful time.

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INTRODUCTION

This paper will develop the topic of the spaces c_0, ℓ_1, and ℓ_∞. We will study the properties of these spaces and their relationships to each other. Our inquiry will begin with the establishment of the relationships among these spaces and the consideration of them as vector spaces. It will then lead into the discovery that these spaces are Banach spaces. The topics of linear functionals, dual spaces, and extreme points will be discussed along the way. There will also be a serious treatment of the Hahn-Banach theorem.

To begin, it is important to introduce the spaces on which we will concentrate. They are:

DEFINITION 0.1: The set of all sequences of real numbers converging to zero is known as c_0.

DEFINITION 0.2: The set of all sequences of real numbers which are absolutely summable is known as ℓ_1.

DEFINITION 0.3: The set of all bounded sequences of real numbers is known as ℓ_∞.

For our purposes, the field of scalars will be the set of all real numbers and the operations defined on our spaces will be componentwise defined vector addition and scalar multiplication.

CHAPTER 1
VECTOR SPACES

The purpose of this chapter is to establish the relationship among c_0, ℓ_1, and ℓ_∞ and to determine that they are indeed vector spaces. Once this has been done we will consider some properties that are inherent to them as vector spaces.

Subsets

Let us first consider whether any of the sets are subsets of one another. If we are able to determine subset relationships, the task of verifying that they are each vector spaces will be easier. To begin, recall that the set c_0 consists of convergent sequences of real numbers. Since such sequences are bounded, it seems intuitive that c_0 is a subset of ℓ_∞. Thus, our first observation is this.

THEOREM 1.1: The set c_0 is a subset of ℓ_∞.

PROOF: Let $(a_n) \in c_0$. Let $\varepsilon > 0$. By the definition of c_0, we know that $\lim_{n\to\infty} a_n = 0$. Then, there is a natural number k such that for all $n \geq k$, $|a_n| \leq \varepsilon$. Then there are two cases to consider.

Case 1: $|a_n| \leq \varepsilon$ for all $n < k$.

In this situation, we have $|a_n| \leq \varepsilon$ for all n. Hence, (a_n) is bounded and an element of ℓ_∞.

Case 2: $|a_n| > \varepsilon$ for some $n < k$.

Since k is a positive integer, there is only a finite number of these terms. Furthermore, there is one of them with the greatest absolute value; call it a_m. Then $|a_n| \leq |a_m|$ for all $n < k$. Since $|a_n| \leq \varepsilon$ for all $n \geq k$ and $\varepsilon < |a_m|$, then $|a_n| \leq |a_m|$ for all $n \geq k$. Finally, we have $|a_n| \leq |a_m|$ for all n. Therefore, (a_n) is bounded and an element of ℓ_∞. Therefore, c_0 is a subset of ℓ_∞. Q.E.D.

Now, we have the task of determining where ℓ_1 fits. Recall the fact that the elements of ℓ_1 are absolutely summable. In other words, for each sequence (a_n) in ℓ_1 the series $\sum_{n=1}^{\infty}|a_n|$ must converge. This implies that the original sequence must converge to zero. This leads us to our next observation.

THEOREM 1.2: The set ℓ_1 is a subset of c_0.

PROOF: Let $(a_n) \in \ell_1$. Then (a_n) is absolutely convergent. Let $S_1, S_2, \ldots, S_n, \ldots$ be the partial sums of the series $\sum_{n=1}^{\infty}|a_n|$. Since (a_n) is absolutely convergent, $\lim_{n\to\infty} S_n = S$ for some real number S. Notice, for any n, $S_n = S_{n-1} + |a_n|$. So,

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(S_{n-1} + |a_n|\right) = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty}|a_n|.$$

It then follows that $\lim_{n\to\infty}|a_n| = S - \lim_{n\to\infty} S_{n-1}$. Note that as $n \to \infty$, $S_{n-1} \to S$. And so we have $\lim_{n\to\infty}|a_n| = S - S = 0$. Hence, $\lim_{n\to\infty} a_n = 0$, and $(a_n) \in c_0$. Therefore, ℓ_1 is a subset of c_0. Q.E.D.

We have thus verified that $\ell_1 \subseteq c_0 \subseteq \ell_\infty$. This relationship will continue to be of importance throughout the study of these spaces, as it will facilitate many of the proofs that follow.

Vector Spaces

Now that subset relationships are known, let us move on to the consideration of vector spaces. As ℓ_∞ is a superset of both c_0 and ℓ_1, we will first determine whether it is a vector space.

THEOREM 1.3: The set ℓ_∞ is a vector space.

PROOF: Let $(a_n), (b_n), (c_n) \in \ell_\infty$ and $r, s, t \in \mathbf{R}$. Then, by definition of ℓ_∞, there are real numbers $M, N \geq 0$ such that $|a_n| \leq M$ and $|b_n| \leq N$ for all natural numbers n. Now, since vector addition is defined componentwise, $(a_n) + (b_n) = (a_n + b_n)$. Then, considering the terms of $(a_n + b_n)$,

$$|a_n + b_n| \le |a_n| + |b_n| \text{ (by the triangle inequality)}$$
$$\le M + N \text{ for all natural numbers } n.$$

Thus, $(a_n)+(b_n)$ is bounded and an element of ℓ_∞. Therefore, ℓ_∞ is closed under vector addition.

Next, consider $r(a_n)$. Since scalar multiplication is defined componentwise, $r(a_n)=(ra_n)$. Observe that $|ra_n|=|r|\cdot|a_n|\le|r|M$ for all natural numbers n. Thus, $r(a_n)$ is bounded and an element of ℓ_∞. Therefore, ℓ_∞ is closed under scalar multiplication.

Finally, the properties of vector spaces must be verified:

i) Associativity of vector addition.

To determine if this property is satisfied in ℓ_∞, we must inspect the addition of three vectors in ℓ_∞. Note

$$\big((a_n)+(b_n)\big)+(c_n)=(a_n+b_n)+(c_n)=\big((a_n+b_n)+c_n\big)=\big(a_n+(b_n+c_n)\big)$$
$$=(a_n)+(b_n+c_n)=(a_n)+\big((b_n)+(c_n)\big).$$

Therefore, vector addition in ℓ_∞ is associative.

ii) Existence of an identity element.

Consider the sequence (0). For any $M\in\mathbf{R}^+$, $|0|=0\le M$. Therefore, (0) is bounded and $(0)\in\ell_\infty$. Let $(a_n)\in\ell_\infty$. Then, $(a_n)+(0)=(a_n+0)=(a_n)$. Likewise, $(0)+(a_n)=(a_n)$. Therefore, (0) is the identity element in ℓ_∞.

iii) Existence of an inverse element for each element of ℓ_∞.

Let (a_n) be an arbitrary element of ℓ_∞. Then $(-a_n)=(-1\cdot a_n)=-1\cdot(a_n)$ and since ℓ_∞ is closed under scalar multiplication, $(-a_n)\in\ell_\infty$. Now, $(a_n)+(-a_n)=(a_n+-a_n)=(0)$. Similarly, $(-a_n)+(a_n)=(0)$. Therefore, each element of ℓ_∞ has an inverse element also in ℓ_∞.

iv) Commutativity of vector addition.

To verify this property, we must look at the sum of two vectors in ℓ_∞. Observe $(a_n)+(b_n)=(a_n+b_n)=(b_n+a_n)=(b_n)+(a_n)$. Thus, vector addition is commutative in ℓ_∞.

v) Distributivity of scalar multiplication over vector addition.

Take note of the following:

$$r((a_n)+(b_n)) = r((a_n+b_n)) = (r(a_n+b_n)) = (ra_n+rb_n) = (ra_n)+(rb_n)$$
$$= r(a_n)+r(b_n).$$

Hence, scalar multiplication distributes over vector addition in ℓ_∞.

vi) Distributivity of scalar multiplication over scalar addition.

Notice $(r+s)(a_n) = \big((r+s)a_n\big) = (ra_n+sa_n) = (ra_n)+(sa_n) = r(a_n)+s(a_n)$.

Therefore, scalar multiplication distributes over scalar addition in ℓ_∞.

vii) Miscellaneous scalar property.

Observe $(rs)(a_n) = \big((rs)a_n\big) = \big(r(sa_n)\big) = r(sa_n)$. The scalar property connected with vector spaces holds in ℓ_∞.

viii) Existence of a scalar identity.

Consider $1 \in \mathbf{R}$; $1\cdot(a_n) = (1\cdot a_n) = (a_n)$. Thus, 1 is the scalar identity for ℓ_∞.

Therefore, ℓ_∞ is a vector space. Q.E.D.

With this fact known, the task of verifying whether c_0 is a vector space is reduced to showing that it is a subspace of ℓ_∞.

THEOREM 1.4: The set c_0 is a vector space.

PROOF: Since we have already shown that c_0 is a subset of a vector space, namely ℓ_∞, it suffices to show that c_0 is closed under vector addition and scalar multiplication. Let $(a_n),(b_n)\in c_0$ and $r\in\mathbf{R}$. Consider $\lim_{n\to\infty}(a_n+b_n)$. Since $(a_n),(b_n)\in c_0$, we know that both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are zero. Hence,

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0+0 = 0.$$

Therefore, $(a_n)+(b_n)\in c_0$. Now consider $\lim_{n\to\infty} ra_n$. Since $(a_n)\in c_0$, $\lim_{n\to\infty} a_n$ exists. Thus,

$$\lim_{n\to\infty} ra_n = r\lim_{n\to\infty} a_n = r(0) = 0.$$

Therefore, $r(a_n) \in c_0$. Hence, c_0 is a vector space. Q.E.D.

All that remains to be shown, in terms of vector spaces, is that ℓ_1 is a vector space. We will use the preceding fact to verify this.

THEOREM 1.5: The set ℓ_1 is a vector space.

PROOF: To show that ℓ_1 is a vector space it suffices to show that it is closed under vector addition and scalar multiplication. Let $(a_n), (b_n) \in \ell_1$ and $r \in \mathbf{R}$. Since $(a_n), (b_n) \in \ell_1$, they are both absolutely summable sequences. That is, $\lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$ and $\lim_{n\to\infty} \sum_{k=1}^{n} |b_k|$ both exist. Call them S and R respectively. Then,

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_k + b_k| \le \lim_{n\to\infty} \sum_{k=1}^{n} \left(|a_k| + |b_k|\right) = \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| + \lim_{n\to\infty} \sum_{k=1}^{n} |b_k| = S + R.$$

Thus, $(a_n) + (b_n)$ is absolutely convergent and an element of ℓ_1.

Now consider the sequence $r(a_n) = (ra_n)$. To show $r(a_n) \in \ell_1$, it must be an absolutely convergent sequence. To determine whether this is the case, we must evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} |ra_k|$. Observe

$$\lim_{n\to\infty} \sum_{k=1}^{n} |ra_k| = \lim_{n\to\infty} |r| \sum_{k=1}^{n} |a_k| = |r| \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| = |r| S.$$

Thus, $r(a_n)$ is absolutely convergent and an element of ℓ_1. Therefore, ℓ_1 is a vector space. Q.E.D.

Norms of Vectors

We will move on to the topic of norms on vectors. Norms are generally used in an effort to define a measurement on vectors.

DEFINITION 1.6: Let V be a vector space over the field of real numbers. Then a norm on V is a function, $\|\cdot\| : V \to \mathbf{R}$, which satisfies the following: (i) $\|v\| \ge 0$ for all v in V, (ii) $\|v\| = 0$ if and only if $v = 0$, (iii) $\|rv\| = |r|\|v\|$ for all real numbers r and vectors v in V, and (iv) $\|v + w\| \le \|v\| + \|w\|$ for all vectors v and w in V.

Since we have established that c_0, ℓ_1, and ℓ_∞ are all vector spaces, we take on the task of establishing a norm for each of these spaces. For our purposes, we shall use the absolute sum of a sequence to be its norm in ℓ_1, and the standard supremum norm will be used in ℓ_∞ and c_0. Before we go any further, we shall take the time to prove that each of these fulfill the requirements of a norm in the specified space. First, we will verify the norm for ℓ_1.

THEOREM 1.7: The function $\|\cdot\|: \ell_1 \to \mathbf{R}$ defined by $\|(a_n)\| := \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} |a_k|$ is a norm on ℓ_1.

PROOF: To show that $\|(a_n)\| = \sum_{k=1}^{\infty} |a_k|$ is a norm on ℓ_1, we need to verify that the four properties of norms do indeed hold. Let $(a_n),(b_n) \in \ell_1$ and r be a real number. First, note that $\|(a_n)\| \geq 0$ since it is a sum of absolute values which are all nonnegative. Next, consider the case where $\|(a_n)\| = 0$. That is, $\sum_{k=1}^{\infty} |a_k| = 0$, which is a sum of nonnegative terms. For that to be so, $|a_k| = 0$ for all k. Hence, (a_n) must be (0). Now, let $(a_n) = (0)$. Then, $\|(a_n)\| = \|(0)\| = \sum_{k=1}^{\infty} |0| = 0$. Thus, $\|(a_n)\| = 0$ if and only if $(a_n) = (0)$. At this time, let us look at $\|r(a_n)\|$. Observe

$$\|r(a_n)\| = \|(ra_n)\| = \sum_{k=1}^{\infty} |ra_k| = \sum_{k=1}^{\infty} |r| \cdot |a_k| = |r| \sum_{k=1}^{\infty} |a_k| = |r| \cdot \|(a_n)\|.$$

Finally, we must attend to $\|(a_n) + (b_n)\|$;

$$\|(a_n) + (b_n)\| = \|(a_n + b_n)\| = \sum_{k=1}^{\infty} |a_k + b_k| \leq \sum_{k=1}^{\infty} \left(|a_k| + |b_k|\right) = \sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} |b_k| = \|(a_n)\| + \|(b_n)\|.$$

Therefore, $\|\cdot\|$ is a norm on ℓ_1. Q.E.D.

Now, we shall establish that the supremum norm satisfies the properties of a norm for the space ℓ_∞.

THEOREM 1.8: The function $\|\cdot\|: \ell_\infty \to \mathbf{R}$ defined by $\|(a_n)\| := \sup_n |a_n|$ is a norm on ℓ_∞.

PROOF: Let $(a_n),(b_n) \in \ell_\infty$ and r be a real number. First, since $|a_n| \geq 0$ for all n, we know that $\sup_n |a_n| \geq 0$. This implies $\|(a_n)\| \geq 0$ for all $(a_n) \in \ell_\infty$. Next, let $\|(a_n)\| = 0$. That is, $\sup_n |a_n| = 0$. Then $0 \geq |a_n|$ for all n, but $|a_n| \geq 0$ for all n. Hence, $|a_n| = 0$ for all n. As a result, (a_n) must be (0). Consider if $(a_n) = (0)$, then

$$\|(a_n)\| = \|(0)\| = \sup|0| = 0.$$

Thus, $\|(a_n)\| = 0$ if and only if $(a_n) = (0)$. Now, consider $\|r(a_n)\|$;

$$\|r(a_n)\| = \|(ra_n)\| = \sup_n |ra_n| = \sup_n |r||a_n| = |r| \sup_n |a_n| = |r| \|(a_n)\|.$$

Finally, we must look at $\|(a_n) + (b_n)\|$. Observe

$$\|(a_n) + (b_n)\| = \|(a_n + b_n)\| = \sup_n |a_n + b_n| \leq \sup_n |a_n| + \sup_n |b_n| = \|(a_n)\| + \|(b_n)\|.$$

Therefore, $\|\cdot\|$ is a norm on ℓ_∞. Q.E.D.

Lastly, we need to confirm the supremum norm for c_0.

THEOREM 1.9: The function $\|\cdot\|: c_0 \to \mathbf{R}$ defined by $\|(a_n)\| := \sup_n |a_n|$ is a norm on c_0.

PROOF: Let $(a_n), (b_n) \in c_0$. Then, since c_0 is a subset of ℓ_∞, $(a_n), (b_n) \in \ell_\infty$. Since the properties of norms hold for vectors in ℓ_∞, then they also hold for (a_n) and (b_n). Therefore, $\|\cdot\|$ is a norm on c_0. Q.E.D.

CHAPTER 2
LINEAR MAPPINGS AND DUAL SPACES

In this chapter we will take our study of c_0, ℓ_1, and ℓ_∞ as vector spaces one step further. Linear transformations between two of these spaces or between one and the field of scalars will be developed in detail. We will then take on the task of developing further relationships between the spaces c_0, ℓ_1, and ℓ_∞.

Linear Mappings

DEFINITION 2.1: A linear mapping is a mapping $f: V_1 \to V_2$, where V_1 and V_2 are vector spaces over a field K, which satisfies the following two properties: (1) for any elements u and v in V_1, $f(u+v) = f(u) + f(v)$; and (2) for all c in the field K and v in the vector space V_1, $f(cv) = cf(v)$.

Let us consider a few examples of linear mappings between our three spaces.

EXAMPLE 2.2: Let $L: c_0 \to \ell_\infty$ be defined by $L((a_n)) := (a_n)$. Note that if $(a_n) \in c_0$, then $L((a_n)) = (a_n) \in \ell_\infty$ since c_0 is a subset of ℓ_∞. To verify that L is indeed a linear mapping, it is sufficient to look at $L((a_n) + r(b_n))$ where $(a_n), (b_n) \in c_0$ and r is a real number. Now,

$$L((a_n) + r(b_n)) = L((a_n + rb_n)) = (a_n + rb_n) = (a_n) + r(b_n) = L((a_n)) + rL((b_n)).$$

Since L satisfies both properties, it is a linear mapping.

EXAMPLE 2.3: Let $L: c_0 \to \ell_1$ be defined by $L((a_n)) := (a_1, 0, 0, 0, \ldots)$. Let $(a_n), (b_n) \in c_0$ and r be a real number. First, notice that $L((a_n)) \in \ell_1$ since

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left| \left(L((a_m)) \right)_k \right| = \lim_{n\to\infty} |a_1| = |a_1|.$$

Next, consider

$$L((a_n)+r(b_n))=L((a_n+rb_n))=(a_1+rb_1,0,0,\ldots)=(a_1,0,0,\ldots)+(rb_1,0,0,\ldots)$$
$$=(a_1,0,0,\ldots)+r(b_1,0,0,\ldots)=L((a_n))+rL((b_n)).$$

This proves that L is a linear mapping.

EXAMPLE 2.4: Let $L: c_0 \to \ell_1$ be defined by $L((a_n)) := (a_n)$ if (a_n) is absolutely convergent and (0) if not. Suppose that $(a_n), (b_n) \in c_0$ such that (a_n) is absolutely convergent and (b_n) is not. Then, $(a_n)+(b_n)$ is not absolutely convergent. So, $L((a_n)+(b_n)) = (0)$; but, $L((a_n)) + L((b_n)) = (a_n) + (0) = (a_n)$. Therefore, L is not a linear mapping.

Linear Functionals

DEFINITION 2.5: A linear functional is a linear mapping from a vector space into its field of scalars.

EXAMPLE 2.6: Let $L: \ell_\infty \to \mathbf{R}$ be defined by $L((a_n)) := \sum_{i=1}^{k} r_i a_i$ where $r_1, r_2, \ldots, r_k$ are all real numbers. Then, $\sum_{i=1}^{k} r_i a_i$ is a sum of real numbers and a real number itself. Let $(a_n), (b_n) \in \ell_\infty$ and s be a real number. Now consider $L((a_n)+s(b_n))$;

$$L((a_n)+s(b_n)) = L((a_n+sb_n)) = \sum_{i=1}^{k} r_i(a_i+sb_i)$$
$$= \sum_{i=1}^{k}(r_i a_i + r_i s b_i) = \sum_{i=1}^{k} r_i a_i + \sum_{i=1}^{k} r_i s b_i$$
$$= \sum_{i=1}^{k} r_i a_i + s\sum_{i=1}^{k} r_i b_i = L((a_n)) + sL((b_n)).$$

Thus, L is a linear functional.

EXAMPLE 2.7: Let k be a natural number. Let $L: c_0 \to \mathbf{R}$ defined by $L((a_n)) := a_k$. Since elements of c_0 are sequences of real numbers, $L((a_n))$ is indeed a real number for any $(a_n) \in c_0$. Now to demonstrate the property of linearity consider:

$$L((a_n)+r(b_n)) = L((a_n+rb_n)) = a_k + rb_k = L((a_n)) + rL((b_n)),$$

where r is any real number. Therefore, L is a linear functional.

Kernel and Image

Two concepts that are intrinsic to the topic of linear mappings are kernel and image. The kernel of a linear mapping is a subspace of the domain, while the image of a linear mapping is a subspace of the range. Let $F: V \to W$ be a linear map.

DEFINITION 2.8: The kernel of F is the set of all vectors v in V, the domain, such that $F(v) = 0$.

DEFINITION 2.9: The image of F is the set of all vectors w in W, the range, such that there exists an element v of V such that $F(v) = w$.

Let us relate these concepts to the examples of linear mappings we have already considered involving the spaces c_0, ℓ_1, and ℓ_∞.

EXAMPLE 2.10: Consider the linear mapping L from Example 2.3;

$$\text{Ker } L = \left\{ (a_n) \in c_0 \mid L((a_n)) = (0) \right\} = \left\{ (a_n) \in c_0 \mid a_1 = 0 \right\}$$

and

$$\text{Im } L = \left\{ (b_n) \in \ell_1 \middle| (b_n) = L((a_n)) \text{ for some } (a_n) \in c_0 \right\} = \left\{ (b_n) \in \ell_1 \middle| b_i = 0 \text{ for all } i \neq 1 \right\}.$$

EXAMPLE 2.11: Next, look at the linear functional in Example 2.6;

$$\text{Ker } L = \left\{ (a_n) \in \ell_\infty \mid L((a_n)) = 0 \right\} = \left\{ (a_n) \in \ell_\infty \mid \sum_{i=1}^{k} r_i a_i = 0 \right\}$$

and

$$\text{Im } L = \left\{ r \in \mathbf{R} \mid r = L((a_n)) \text{ for some } (a_n) \in \ell_\infty \right\} = \mathbf{R}.$$

Norms of Linear Mappings

The concept of a norm is not something that merely applies to vectors. We can also discuss the idea of a norm with respect to functions.

DEFINITION 2.12: Let $L: V \to \mathbf{R}$ be a linear functional and suppose V is a normed linear space. Let the norm of L, $\|L\|$, be given by M, where M is the smallest number such that $|L(v)| \leq M\|v\|$ for all $v \in V$, if such an M exists.

DEFINITION 2.13: A linear functional L is said to be bounded if such an M (as in Definition 2.12) exists.

We shall begin the investigation of this concept by examining some examples.

EXAMPLE 2.14: Let $L: \ell_1 \to \mathbf{R}$ be defined by $L((a_n)) := a_5$. To find $\|L\|$ we must find the smallest real number M such that $|L((a_n))| \leq M\|(a_n)\|$ holds. First, note that $|L((a_n))| = |a_5|$. Next, recognize that $\|(a_n)\| = \sum_{i=1}^{\infty} |a_i|$, as $(a_n) \in \ell_1$. Since $|a_5|$ is a summand in that sum, we know that the inequality $|a_5| \leq 1 \cdot \sum_{i=1}^{\infty} |a_i|$ is true. If there are terms of (a_n), other than a_5, that are nonzero, M may be less than one. However, the same M must work for all elements of ℓ_1. Consider elements of ℓ_1 in which all the terms of (a_n) are zero other than a_5. Then $|a_5| = \sum_{i=1}^{\infty} |a_i|$. In which case, M must be one for $|L((a_n))| \leq M\|(a_n)\|$ to hold. Therefore, $\|L\| = 1$.

EXAMPLE 2.15: Let $L: \ell_1 \to \mathbf{R}$ be defined by $L((a_n)) := a_1 + 4a_3$. Notice that for any $(a_n) \in \ell_1$, $|a_1 + 4a_3| \leq |a_1| + 4|a_3| \leq \sum_{i=1}^{\infty} 4|a_i| = 4\|(a_n)\|$. That is, $|L((a_n))| \leq 4\|(a_n)\|$ for all $(a_n) \in \ell_1$. Hence, $\|L\| \leq 4$ for this linear functional. Now, let $(a_n) \in \ell_1$ such that a_3 is the only nonzero term. Then $|L((a_n))| = |4a_3| = 4|a_3| = 4\|(a_n)\|$. Note that $M = 4$ is the smallest number for which the inequality $|L((a_n))| \leq M\|(a_n)\|$ holds for this particular element of ℓ_1. Therefore, $\|L\| = 4$.

EXAMPLE 2.16: Let $L: c_0 \to \mathbf{R}$ be defined by $L((a_n)) := a_5$. Then for any $(a_n) \in c_0$, $|L((a_n))| = |a_5| \leq \sup_n |a_n| = \|(a_n)\|$. Thus, $\|L\| \leq 1$. Now, consider the sequence $(a_n) = (0,0,0,0,1,0,0,\ldots) \in c_0$. Then $|L((a_n))| = |a_5| = |1| = 1 = \|a_n\| = 1 \cdot \|a_n\|$. That is, for (a_n) $M = 1$ is the smallest number for which $|L((a_n))| \leq M\|(a_n)\|$ is true. Hence, $\|L\| = 1$.

Now we shall look at a characteristic of norms that will facilitate any future use of norms.

LEMMA 2.17: Let $L:V \to \mathbf{R}$ be a linear functional and V a normed linear space. Then $\|L\| = \sup_{\|v\|\le 1} |L(v)|$.

PROOF: Let V be a normed linear space and $L:V \to \mathbf{R}$ be a linear functional. Let $v \in V$ such that $\|v\| \le 1$. By the definition of norm of L, we know that $|L(v)| \le \|L\|\|v\|$. Then, since $\|v\| \le 1$ implies that $x\|v\| \le x$ for all real numbers x, we have $|L(v)| \le \|L\|$. Now, since $|L(v)| \le \|L\|$ is true for all $v \in V$ such that $\|v\| \le 1$, we can conclude that $\sup_{\|v\|\le 1} |L(v)| \le \|L\|$. Next, we need to show that $\|L\| \le \sup_{\|v\|\le 1} |L(v)|$. Let $\sup_{\|v\|\le 1} |L(v)| = \alpha$. Then,

$$|L(v)| = \left|L\left(\tfrac{\|v\|v}{\|v\|}\right)\right| = \|v\|\left|L\left(\tfrac{v}{\|v\|}\right)\right| \text{ (since L is linear)}$$

$$\le \|v\|\alpha \text{ (since } \left\|\left(\tfrac{v}{\|v\|}\right)\right\| = 1).$$

Therefore, $|L(v)| \le \alpha\|v\|$ for all $v \in V$. Since $\|L\|$ is the least M such that $|L(v)| \le M\|v\|$ is true for all $v \in V$, it may be deduced that $\|L\| \le \alpha$. That is, $\|L\| \le \sup_{\|v\|\le 1} |L(v)|$. Therefore, $\|L\| = \sup_{\|v\|\le 1} |L(v)|$. Q.E.D.

<u>Continuous Linear Functionals</u>

As linear functionals have already been defined, and we have already looked at examples as well as defined norms on linear functionals, we are now ready to develop the concept of linear functionals one step further. The next thing we will do is determine what it means for a linear functional to be continuous.

DEFINITION 2.18: Let X be a normed linear space. Let $x^*:X \to \mathbf{R}$ be a linear functional. Then x^* is continuous if for each $x \in X$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|x - y\| < \delta$ then $|x^*(x) - x^*(y)| < \varepsilon$. Furthermore, x^* is said to be continuous at some point $x_0 \in X$ if given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X$ such that $\|x - x_0\| < \delta$ then $\|x^*(x) - x^*(x_0)\| < \varepsilon$.

We will begin this discussion by developing some properties of continuous functionals.

THEOREM 2.19: Let X be a normed linear space. Let $x^*: X \to \mathbf{R}$ be a linear functional. Then x^* is continuous if and only if x^* is continuous at some point $x_0 \in X$.

PROOF: Let X be a normed linear space. Let $x^*: X \to \mathbf{R}$ be a linear functional. Suppose that x^* is continuous. Then x^* is continuous at each $x \in X$. Clearly then x^* is continuous at some $x_0 \in X$.

Suppose that x^* is continuous at some point $x_0 \in X$. Then given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X$ and $\|x - x_0\| < \delta$ then $\|x^*(x) - x^*(x_0)\| < \varepsilon$. Now, let $x \in X$ such that $\|x - 0\| < \delta$. Then $\|x - 0\| = \|(x + x_0) - x_0\| < \delta$. Since x^* is continuous at x_0, we have $\|x^*(x + x_0) - x^*(x_0)\| < \varepsilon$. Since x^* is additive, we arrive at $\|x^*(x) + x^*(x_0) - x^*(x_0)\| < \varepsilon$ which yields $\|x^*(x)\| < \varepsilon$. Since $x^*(0) = 0$, $\|x^*(x)\| = \|x^*(x) - x^*(0)\| < \varepsilon$. Therefore, x^* is continuous at $0 \in X$.

Now, consider any $y \in X$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in X$ such that $\|x\| < \delta$ we have $\|x^*(x)\| < \varepsilon$, since x^* is continuous at $0 \in X$. Suppose $\|x - y\| < \delta$. Then $\|(x - y) - 0\| < \delta$, and $\|x^*(x - y) - x^*(0)\| < \varepsilon$. Then, since x^* is additive, $\|x^*(x) - x^*(y) - x^*(0)\| < \varepsilon$. This implies then that $\|x^*(x) - x^*(y)\| < \varepsilon$. Therefore, x^* is continuous at each $y \in X$. Q.E.D.

Before making our next observation of continuous linear functionals, we need to consider the following characteristic of bounded functionals.

LEMMA 2.20: Let X be a normed linear space and $x^*: X \to \mathbf{R}$ be a linear functional. Then x^* is bounded if and only if $\{x^*(x) \mid \|x\| \le 1\}$ is bounded.

PROOF: Let X be a normed linear space and $x^*: X \to \mathbf{R}$ be a linear functional. Suppose that x^* is bounded. Then, by Definition 2.13, we know that there is a real number M such that

$$|x^*(x)| \le M\|x\| \text{ for all } x \in X.$$

From this fact, we can infer that for those x such that $\|x\| \le 1$

$$|x^*(x)| \le M.$$

Hence, $\{x^*(x) \mid \|x\| \le 1\}$ is bounded.

Now, suppose that $\{x^*(x) \mid \|x\| \le 1\}$ is bounded. That is, there exists a real number $M \ge 0$ such that $|x^*(x)| \le M$ for all $x \in X$ such that $\|x\| \le 1$. Let x' be an arbitrary element of X such that $\|x'\| \ne 0$. Observe that $\left\| \frac{x'}{\|x'\|} \right\| = 1$ and thus it is an element of $\{x^*(x) \mid \|x\| \le 1\}$. Hence,

$$\left| x^*\left(\frac{x'}{\|x'\|} \right) \right| \le M.$$

Since x^* is linear,

$$\frac{1}{\|x'\|} \cdot |x^*(x')| \le M.$$

This implies

$$|x^*(x')| \le M\|x'\|.$$

Therefore, by Definition 2.13, x^* is bounded. Q.E.D.

Now that we have the above fact at our disposal, the proof of the following observation will be made easier.

THEOREM 2.21: Let X be a normed linear space and $x^*: X \to \mathbf{R}$ be a linear functional. Then x^* is continuous if and only if x^* is bounded.

PROOF: Let X be a normed linear space and $x^*: X \to \mathbf{R}$ be a linear functional. First, suppose that x^* is continuous. Then specifically, x^* is continuous at 0. Let $\varepsilon = 1$. Then there exists a $\delta > 0$ such that for all $x \in X$ such that $\|x\| \le \delta$ then $|x^*(x)| < 1$. Let $x \in X$ such that $\|x\| \le 1$. Observe

$$|x^*(x)| = \left| x^*\left(\tfrac{1}{\delta} \cdot \delta x \right) \right|.$$

Since x^* is linear,

$$\left| x^*\left(\tfrac{1}{\delta} \cdot \delta x \right) \right| = \left| \tfrac{1}{\delta} \cdot x^*(\delta x) \right|$$

By the properties of absolute value,

$$\left| \tfrac{1}{\delta} \cdot x^*(\delta x) \right| = \tfrac{1}{\delta} |x^*(\delta x)|.$$

Now, since $\|x\| \le 1$ we know that $\|\delta x\| \le \delta$. Consequently, $|x^*(\delta x)| < 1$ which implies

$$\tfrac{1}{\delta} |x^*(\delta x)| < \tfrac{1}{\delta}.$$

Thus, substitution yields

$$|x^*(x)| < \tfrac{1}{\delta} \text{ for all such } x \in X.$$

Hence, $\{x^*(x) \mid \|x\| \le 1\}$ is bounded. Therefore, x^* is bounded.

Now, suppose that x^* is bounded. Then we know that $\{x^*(x) \mid \|x\| \le 1\}$ is bounded. That is, there exists a real number $M > 0$ such that $|x^*(x)| \le M$ for all $x \in X$ such that $\|x\| \le 1$. Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{M}$. Now, for all $x \in X$ such that $\|x\| < \frac{\varepsilon}{M}$ we have

$$|x^*(x)| = |x^*(\tfrac{\varepsilon}{M} \cdot \tfrac{M}{\varepsilon} x)| = \tfrac{\varepsilon}{M} |x^*(\tfrac{M}{\varepsilon} x)|.$$

Since $\|x\| < \frac{\varepsilon}{M}$, we have $\|\frac{M}{\varepsilon} x\| < 1$, and so $|x^*(\frac{M}{\varepsilon} x)| < M$. Thus

$$\tfrac{\varepsilon}{M} |x^*(\tfrac{M}{\varepsilon} x)| < \varepsilon.$$

Therefore, for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in X$ such that $\|x\| < \delta$ then $|x^*(x)| < \varepsilon$. That is, x^* is continuous at 0. Therefore, x^* is continuous by Theorem 2.19 since X is a normed linear space. Q.E.D.

Dual Spaces

The next step in our reflection on the spaces c_0, ℓ_1, and ℓ_∞ will be the consideration of dual spaces.

DEFINITION 2.22: Let X be a vector space. Then the dual of X, denoted X^*, is the space of all bounded linear functionals on X.

Since we have recently completed a discussion on the norm of a function and are now considering a space of functionals, let us begin by recognizing a connection between the two.

THEOREM 2.23: The function $\|\cdot\| : X^* \to \mathbf{R}$ defined in Definition 2.12 is a norm on X^*.

PROOF: Let $L \in X^*$. Then by definition of dual space, L is a bounded linear functional. That is, $\|L\|$ exists. From Lemma 2.17, we have $\|L\| = \sup_{\|x\| \le 1} |L(x)|$. Since $|L(x)| \ge 0$ for all $x \in X$, $\sup_{\|x\| \le 1} |L(x)| \ge 0$ as well. Hence,

$\|L\| \geq 0$ for all $L \in X^*$.

Now, let $L \in X^*$ such that $\|L\| = 0$. By definition of norm of L, $|L(x)| \leq \|L\|\|x\|$ for all $x \in X$. Substitution yields $|L(x)| \leq 0$ for all $x \in X$. Since $|L(x)|$ is a nonnegative value, $L(x)$ must be zero for all $x \in X$. Hence, L is the zero functional. Next, let $0 = L \in X^*$. To determine $\|L\|$ we consider the previously proven fact $\|L\| = \sup_{\|x\| \leq 1} |L(x)|$. Since $L(x) = 0$ for all $x \in X$, we can conclude $\|L\| = 0$. Thus, $\|L\| = 0$ if and only if $L = 0$.

Let r be a real number. Then, by Lemma 2.17, we have

$$\|rL\| = \sup_{\|x\| \leq 1} |(rL)(x)|.$$

But, by definition of rL, $(rL)(x) = r(L(x))$ and so

$$\sup_{\|x\| \leq 1} |(rL)(x)| = \sup_{\|x\| \leq 1} |r(L(x))|.$$

Now, by the properties of absolute values,

$$\sup_{\|x\| \leq 1} |r(L(x))| = \sup_{\|x\| \leq 1} |r||L(x)|.$$

Since r is a constant,

$$\sup_{\|x\| \leq 1} |r||L(x)| = |r| \sup_{\|x\| \leq 1} |L(x)|.$$

Hence, we have

$$|r| \sup_{\|x\| \leq 1} |L(x)| = |r|\|L\|.$$

Therefore, $\|rL\| = |r|\|L\|$.

Finally, let $L_1, L_2 \in X^*$. Then $\|L_1\|$ and $\|L_2\|$ both exist. Consider $\|L_1 + L_2\|$. By Lemma 2.17,

$$\|L_1 + L_2\| = \sup_{\|x\| \leq 1} |(L_1 + L_2)(x)|.$$

Then, by linearity of L_1 and L_2,

$$\sup_{\|x\| \leq 1} |(L_1 + L_2)(x)| = \sup_{\|x\| \leq 1} |L_1(x) + L_2(x)|.$$

Now, by the triangle inequality,

$$\sup_{\|x\| \leq 1} |L_1(x) + L_2(x)| \leq \sup_{\|x\| \leq 1} \left(|L_1(x)| + |L_2(x)|\right).$$

Next, by the triangle inequality,

$$\sup_{\|x\| \leq 1} \left(|L_1(x)| + |L_2(x)|\right) \leq \sup_{\|x\| \leq 1} |L_1(x)| + \sup_{\|x\| \leq 1} |L_2(x)|.$$

Finally, by Lemma 2.17,

$$\sup_{\|x\|\le 1} |L_1(x)| + \sup_{\|x\|\le 1} |L_2(x)| = \|L_1\| + \|L_2\|.$$

That is, $\|L_1 + L_2\| \le \|L_1\| + \|L_2\|$. Therefore, $\|\cdot\|: X^* \to \mathbf{R}$ satisfies the requirements for a norm. Q.E.D.

Before considering the first relationship among our spaces and their duals, we will want to know the following fact. Although we are focusing on the space ℓ_1 in the next proof, it will be useful to consider the corresponding fact for our other spaces as well.

THEOREM 2.24: Let $(a_k) \in \ell_1$. Then $(a_k) = \sum_{n=1}^{\infty} a_n \cdot e_n$, where e_n is the unit sequence in which the nth term is the only nonzero term, and it is 1.

PROOF: To show $(a_k) = \sum_{n=1}^{\infty} a_n \cdot e_n$ is equivalent to showing $\lim_{n\to\infty} \sum_{k=1}^{n} a_k \cdot e_k = (a_k)$. Thus, it suffices to show that $\lim_{n\to\infty} \left\| (a_k) - \sum_{k=1}^{n} a_k \cdot e_k \right\| = 0$.

Let $(a_k) \in \ell_1$. Then $\sum_{k=1}^{\infty} |a_k| = S$ for some real number S. Now consider

$$\begin{aligned}
\lim_{n\to\infty} \left\| (a_k) - \sum_{k=1}^{n} a_k \cdot e_k \right\| &= \lim_{n\to\infty} \left\| (a_1, a_2, a_3, \ldots) - (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \right\| \\
&= \lim_{n\to\infty} \left\| (0, 0, \ldots, a_{n+1}, a_{n+2}, \ldots) \right\| \\
&= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |a_k| \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} |a_k| - \sum_{k=1}^{n} |a_k| \right) \\
&= S - \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| = S - S = 0.
\end{aligned}$$

Therefore, $(a_k) = \sum_{k=1}^{\infty} a_k \cdot e_k$. Q.E.D.

We now begin the task of determining any relationships among our three spaces and their duals. The next theorem states $\ell_\infty = \ell_1^*$. In other words, for each bounded linear functional $L: \ell_1 \to \mathbf{R}$ (i.e., each element of ℓ_1^*) there must be some element in ℓ_∞ that could be considered the "same" as L and each element of ℓ_∞ must represent some bounded linear functional on ℓ_1.

Perhaps, in the proof that follows, we will develop a better grasp of this idea.

THEOREM 2.25: The dual space of ℓ_1 is ℓ_∞.

PROOF: To prove that two spaces are equal we must show that each is a subspace of the other. We shall begin by letting $L \in \ell_1^*$. By definition, $L: \ell_1 \to \mathbf{R}$ is a bounded linear functional. Let $k_i = L(e_i)$ for $i = 1,2,3,\ldots$. Then $k_i \in \mathbf{R}$ for all i. Consider the sequence (k_n) with entries defined as above. Let $(a_n) \in \ell_1$. Then, from Theorem 2.24, $(a_n) = \sum_{i=1}^{\infty} a_i \cdot e_i$. So, by substitution we have

$$L((a_n)) = L\left(\sum_{i=1}^{\infty} a_i e_i\right).$$

Since L is bounded Theorem 2.21 applies and we may infer that L is continuous. Next, we can take advantage of a theorem found in the Bartle and Sherbert text. In summary, the theorem states: For any subset A of the real numbers, $f: A \to \mathbf{R}$ and c element of A the statement that f is continuous at c is equivalent to the statement that if (x_n) is a sequence of elements of A that converges to c, then $(f(x_n))$ converges to $f(c)$ (Bartle and Sherbert 141-142). Although this theorem is stated for subsets of the reals, it can be extended to apply to sequences of real numbers as well. Consider the sequence (s_n) defined by $s_n = \sum_{i=1}^{n} a_i \cdot e_i$ for all n. Note that $(s_n) \to (a_n)$. Then, making use of the above mentioned theorem, we have

$$L((a_n)) = \lim_{n\to\infty} L((s_n)) = \lim_{n\to\infty} L\left(\sum_{i=1}^{n} a_i \cdot e_i\right).$$

Since L is linear, we arrive at

$$L((a_n)) = \lim_{n\to\infty} \sum_{i=1}^{n} L(a_i \cdot e_i).$$

That is,

$$L\left(\sum_{i=1}^{\infty} a_i \cdot e_i\right) = \sum_{i=1}^{\infty} L(a_i \cdot e_i).$$

Now, since L is linear,

$$\sum_{i=1}^{\infty} L(a_i \cdot e_i) = \sum_{i=1}^{\infty} a_i \cdot L(e_i).$$

By definition of k_i,

$$\sum_{i=1}^{\infty} a_i \cdot L(e_i) = \sum_{i=1}^{\infty} a_i k_i .$$

Therefore, (k_n) is the sequence associated with the linear functional L. Since L is bounded, $|L((a_n))| \leq M\|(a_n)\|$ for all $(a_n) \in \ell_1$, where M is a real number. In particular, $|L(e_n)| \leq M$ for all n. Substitution yields $|k_n| \leq M$ for all n. Therefore, (k_n) is bounded and thus an element of ℓ_∞. Hence, every $L \in \ell_1^*$ can be associated with a sequence $(k_n) \in \ell_\infty$. Therefore, $\ell_1^* \subseteq \ell_\infty$.

Now, let $(k_n) \in \ell_\infty$. Define $L: \ell_1 \to \mathrm{R}$ by $L((a_n)) = \sum_{n=1}^{\infty} k_n a_n$. Note that L is linear. Consider $|L((a_n))|$. By the definition of L,

$$|L((a_n))| = \left|\sum_{n=1}^{\infty} a_n k_n\right|.$$

By the triangle inequality and properties of absolute values,

$$\left|\sum_{n=1}^{\infty} a_n k_n\right| \leq \sum_{n=1}^{\infty} |a_n| \cdot |k_n|.$$

Since $(k_n) \in \ell_\infty$, there exists a real number M such that $|k_n| \leq M$ for each n. Therefore,

$$\sum_{n=1}^{\infty} |a_n| \cdot |k_n| \leq M \sum_{n=1}^{\infty} |a_n|.$$

Now, $\sum_{n=1}^{\infty} |a_n| = \|(a_n)\|$ since $(a_n) \in \ell_1$. Thus, we arrive at

$$|L((a_n))| \leq M\|(a_n)\|.$$

Hence, L is bounded, by Definition 2.13, and thus an element of ℓ_1^*. Therefore, every element of ℓ_∞ can be associated with a bounded linear functional of ℓ_1^*. That is, $\ell_\infty \subseteq \ell_1^*$. Q.E.D.

We will now proceed to explore the relationship between c_0 and its dual space.

THEOREM 2.26: The dual space of c_0 is ℓ_1.

PROOF: Let $(k_n) \in \ell_1$. Define $f: c_0 \to \mathbf{R}$ by $f((a_n)) = \sum_{n=1}^{\infty} k_n a_n$. Note that f is linear. Consider $|f((a_n))|$ for all $(a_n) \in c_0$. By definition of f, we can infer

$$|f((a_n))| = \left|\sum_{n=1}^{\infty} k_n a_n\right|.$$

By the triangle inequality,

$$\left|\sum_{n=1}^{\infty} k_n a_n\right| \le \sum_{n=1}^{\infty} |k_n a_n|.$$

Since $(a_n) \in c_0$, there exists a real number $M > 0$ such that $\|(a_n)\| = \sup_n |a_n| = M$. Therefore,

$$\sum_{n=1}^{\infty} |k_n a_n| \le \sum_{n=1}^{\infty} M|k_n|.$$

Observe that, by the property of distribution,

$$\sum_{n=1}^{\infty} M|k_n| = M\sum_{n=1}^{\infty} |k_n|.$$

Now, note that since $(k_n) \in \ell_1$, $\sum_{n=1}^{\infty} |k_n|$ exists. Call it N. Then we have

$$|f((a_n))| \le N\|(a_n)\|.$$

Therefore, by Definition 2.13, f is bounded. Hence, ℓ_1 is a subset of c_0^*.

Now, let $L \in c_0^*$. Let $k_n = L(e_n)$. Then, as demonstrated in the proof of Theorem 2.25, it suffices to show that $(k_n) \in \ell_1$. Thus, we must consider $\sum_{n=1}^{\infty} |k_n|$. Fix j. Let (b_n) be the sequence defined as follows: for $n \le j$, $b_n = 1$ if $k_n \ge 0$ and $b_n = -1$ if $k_n < 0$; for $n > j$, $b_n = 0$. Then for any natural number j,

$$L((b_n)) = L\left(\sum_{n=1}^{j} b_n e_n\right).$$

By the linearity of L,

$$L\left(\sum_{n=1}^{j} b_n e_n\right) = \sum_{n=1}^{j} L(b_n e_n) = \sum_{n=1}^{j} b_n L(e_n).$$

From the definition of k_n's, we have

$$\sum_{n=1}^{j} b_n L(e_n) = \sum_{n=1}^{j} b_n k_n;$$

and from the definition of (b_n),

$$\sum_{n=1}^{j} b_n k_n = \sum_{n=1}^{j} |k_n|.$$

That is, $L((b_n)) = \sum_{n=1}^{j} |k_n|$ for all j. Since L is bounded Definition 2.13 tells us that there exists a real number M such that $|L((b_n))| \leq M\|(b_n)\|$ for all j. Note that $\|(b_n)\| = 1$ for all j and so we have $|L((b_n))| \leq M$ for all j. Finally, by a property of absolute values, $L((b_n)) \leq |L((b_n))|$ and transitivity yields $\sum_{n=1}^{j} |k_n| \leq M$ for all j. Thus, $\sum_{n=1}^{\infty} |k_n|$ exists and (k_n) is an element of ℓ_1. Therefore, c_0^* is a subset of ℓ_1. Q.E.D.

Now that we have discovered some relationships among our spaces and their dual spaces, we will discuss linear functionals on dual spaces. Note that such a linear functional would take a bounded linear functional (from the dual space) to a real valued constant.

EXAMPLE 2.27: Let X be a normed linear space and $x \in X$. Define $\eta(x): X^* \to \mathbf{R}$ by $\eta(x)(x^*) = x^*(x)$ for all $x^* \in X^*$. To check the linearity of $\eta(x)$, we will consider $\eta(x)(x^* + ry^*)$ where $x^*, y^* \in X^*$ and $r \in \mathbf{R}$. From the definition of $\eta(x)$ we have

$$\eta(x)(x^* + ry^*) = (x^* + ry^*)(x).$$

Notice that $x^* + ry^*$ is a sum of two functionals. Thus,

$$(x^* + ry^*)(x) = x^*(x) + ry^*(x).$$

Since ry^* is a scalar multiple of a functional,

$$x^*(x) + ry^*(x) = x^*(x) + r(y^*(x)).$$

Again using the definition of $\eta(x)$,

$$x^*(x) + r(y^*(x)) = \eta(x)(x^*) + r\eta(x)(y^*).$$

Therefore, $\eta(x)$ is linear.

To determine $\|\eta(x)\|$, we use the previously stated definitions and facts. If we begin by considering Lemma 2.17, then we have

$$\|\eta(x)\| = \sup_{\|x^*\| \leq 1} |\eta(x)(x^*)|.$$

From the definition of $\eta(x)$,

$$\sup_{\|x^*\| \leq 1} |\eta(x)(x^*)| = \sup_{\|x^*\| \leq 1} |x^*(x)|.$$

Since $x^* \in X^*$, it is a bounded linear functional. Then by Definition 2.13, $|x^*(x)| \le \|x^*\| \cdot \|x\|$ for all $x^* \in X^*$. Therefore,

$$\sup_{\|x^*\|\le 1} |x^*(x)| \le \sup_{\|x^*\|\le 1} \|x^*\| \cdot \|x\| \le \|x\|$$

and we have $\|\eta(x)\| \le \|x\|$. The reverse inequality can also be shown, as in previous examples, so that $\|\eta(x)\| = \|x\|$.

Then, since $\eta(x)$ is a bounded linear functional on X^*, $\eta(x) \in X^{**}$. We can then consider $\eta: X \to X^{**}$ and whether it is a linear mapping. Let $w, u \in X$, $c \in \mathbf{R}$ and $x^* \in X^*$. Then to determine if η is linear we must consider $\eta(w+cu)(x^*)$. By the definition of $\eta(x)$,

$$\eta(w+cu)(x^*) = x^*(w+cu).$$

Now, since x^* is linear,

$$x^*(w+cu) = x^*(w) + cx^*(u).$$

Again using the definition of $\eta(x)$, we have

$$x^*(w) + cx^*(u) = \eta(w)(x^*) + c\eta(u)(x^*).$$

Finally, since $\eta(x)$ and $c\eta(x)$ are both functionals on the same space,

$$\eta(w)(x^*) + c\eta(u)(x^*) = \left(\eta(w) + c\eta(u)\right)(x^*).$$

Therefore, η is a linear mapping.

CHAPTER 3
BANACH SPACES

The purpose of this chapter is to introduce the idea of Banach spaces and to propose and prove that each of the spaces c_0, ℓ_1, and ℓ_∞ are indeed Banach spaces. To begin, we must first define a Banach space.

DEFINITION 3.1: A Banach space is a complete normed linear space.

A normed linear space is simply a vector space on which a norm has been defined. The concept of a vector space was covered extensively in Chapter 1 where it was proven that c_0, ℓ_1, and ℓ_∞ are all vector spaces. In Chapter 1, we also discussed norms and defined a norm on each of the spaces c_0, ℓ_1, and ℓ_∞. Therefore, the only thing left to discuss before we further develop the idea of Banach spaces is the concept of complete spaces.

Complete Spaces

We begin our discussion with a definition.

DEFINITION 3.2: A space X is complete if every Cauchy sequence converges to a limit in X.

To develop the concept of a complete space it will then be crucial to know the definition of a Cauchy sequence.

DEFINITION 3.3: A sequence (x_n) in a normed linear space is Cauchy if for each $\varepsilon > 0$, there is some $N \in \mathbf{N}$ such that for all $n, m \geq N$, $\| x_n - x_m \| < \varepsilon$.

Let us examine some Cauchy sequences in our three spaces.

EXAMPLE 3.4: Let (x_n) be the sequence in the space c_0 such that

$$x_1 = (1, 0, 0, 0, \ldots)$$
$$x_2 = \left(1, \tfrac{1}{2}, 0, 0, \ldots\right)$$
$$x_3 = \left(1, \tfrac{1}{2}, \tfrac{1}{4}, 0, 0, \ldots\right)$$
$$\vdots$$
$$x_n = \left(1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \tfrac{1}{2^{n-1}}, 0, 0, \ldots\right)$$
$$\vdots$$

Let $\varepsilon > 0$. We wish to consider $\|x_n - x_m\|$. There exists a natural number N such that $\frac{1}{2^N} < \varepsilon$. Without loss of generality suppose that $n > m$. Then for all natural numbers $n, m \geq N$ we have

$$\|x_n - x_m\| = \left\|\left(0,0,\ldots,\tfrac{1}{2^m},\ldots,\tfrac{1}{2^{n-1}},0,0,\ldots\right)\right\|.$$

Since $x_n, x_m \in c_0$, we know that $x_n - x_m \in c_0$ also, and thus

$$\|x_n - x_m\| = \tfrac{1}{2^m}.$$

Now, because $m \geq N$ we have $2^m \geq 2^N$ which implies $\frac{1}{2^m} \leq \frac{1}{2^N} < \varepsilon$. Therefore, (x_n) is a Cauchy sequence.

EXAMPLE 3.5: Since c_0 is a subset of ℓ_∞ with the same norm, the sequence (x_n) from Example 3.4 is also a Cauchy sequence in ℓ_∞.

Keeping the preceeding example in mind, we can make an observation regarding the completeness of a certain subspace of c_0.

PROPOSITION 3.6: Let S be the subspace of c_0 consisting of sequences that are finitely nonzero. Then S is not complete.

PROOF: To show that S is not complete, it suffices to find a Cauchy sequence in S that does not converge to a limit in S. Consider the sequence (x_n) from Example 3.4. As shown in Example 3.4, this sequence is Cauchy. However, (x_n) converges in c_0 to the sequence $\left(1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-1}}, \ldots\right)$ which is not in S. Therefore, S is not complete. Q.E.D.

It may be difficult to show that every Cauchy sequence in a space converges to a limit in that same space. Therefore, we shall consider the following theorem which will facilitate proving completeness. The following theorem as well as the majority of the proof can be found in Royden's text (124-125).

THEOREM 3.7: A normed linear space X is complete if and only if every absolutely summable series is summable.

PROOF: Let X be a complete normed linear space. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely summable series of elements of X. By definition of absolutely summable, $\sum_{n=1}^{\infty} \|x_n\| = M < \infty$ where $M \in \mathbf{R}$. Now, since $\lim_{m\to\infty} \sum_{n=1}^{m} \|x_n\| = M$, we have for any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbf{N}$ such that for all $N-1 \geq N_\varepsilon$, $\left| M - \sum_{n=1}^{N-1} \|x_n\| \right| < \varepsilon$. Hence, $\sum_{n=N}^{\infty} \|x_n\| < \varepsilon$.

Let $s_n = \sum_{i=1}^{n} x_i$ be the nth partial sum of the series $\sum_{n=1}^{\infty} x_n$. Then, for $n \geq m \geq N$,

$$\| s_n - s_m \| = \left\| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right\| = \left\| \sum_{i=m+1}^{n} x_i \right\|.$$

By the triangle inequality, we have

$$\left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\|.$$

Now, by addition of infinitely many nonnegative terms,

$$\sum_{i=m+1}^{n} \|x_i\| \leq \sum_{i=m+1}^{\infty} \|x_i\|$$

(where equality is achieved only if $\sum_{i=n+1}^{\infty} \|x_i\| = 0$). Next, by addition of $m+1-N$ nonnegative terms we arrive at

$$\sum_{i=m+1}^{\infty} \|x_i\| \leq \sum_{i=N}^{\infty} \|x_i\|$$

(in which case equality will occur only if $\sum_{N}^{m} \|x_i\| = 0$). Finally, from above, we have

$$\sum_{i=N}^{\infty} \|x_i\| < \varepsilon.$$

Therefore, $\|s_n - s_m\| < \varepsilon$. Thus, the sequence (s_n) of partial sums is a Cauchy sequence by definition. Since X is complete, (s_n) converges to an element s of X. That is, $\sum_{n=1}^{\infty} x_n$ is summable.

Now, let X be a normed linear space such that every absolutely summable series of elements of X is summable. Let (x_n) be a Cauchy sequence in X. For all $\varepsilon > 0$, in particular for $\varepsilon = 2^{-k}$ where $k \in \mathbf{Z}$, there is some integer N_k such that for all $n,m \geq N_k$, $\|x_n - x_m\| < \varepsilon = 2^{-k}$. Now, we may choose n_k's such that $n_{k+1} > n_k$, $n_k \geq N_k$, and $n_{k+1} \geq N_{k+1}$. In doing so, we can then consider the sequence $\left(x_{n_k}\right)$ which is a subsequence of (x_n). Now, let us define the sequence (y_n) such that $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$ for $k > 1$. Since $n_k > n_{k-1} \geq N_{k-1}$, we have $\| y_k \| = \| x_{n_k} - x_{n_{k-1}} \| \leq 2^{-k+1}$ for $k > 1$. Thus,

$$\sum_{k=1}^{\infty} \| y_k \| = \| y_1 \| + \sum_{k=2}^{\infty} \| y_k \| \leq \| y_1 \| + \sum_{k=2}^{\infty} 2^{-k+1} = \| y_1 \| + 1$$

(as $\sum_{k=2}^{\infty} 2^{-k+1} = 1$). So then, by definition, (y_n) is absolutely summable. Now, we have that (y_n) is summable; that is, the sequence of partial sums of the series $\sum_{n=1}^{\infty} y_n$ (the sequence $\left(x_{n_k}\right)$) converges to an element x of X.

Now, all that is left to show is that (x_n) converges to an element of X. Since (x_n) is a Cauchy sequence, given $\varepsilon > 0$ there exists a natural number N such that for all $n,m \geq N$, $\|x_n - x_m\| < \frac{\varepsilon}{2}$. Since $\left(x_{n_k}\right)$ coverges to x, there is a natural number K such that for all $k \geq K$, $\left\| x_{n_k} - x \right\| < \frac{\varepsilon}{2}$. Now, let k be such that $k > K$ and $k \geq N$. Then

$$\| x_k - x \| \leq \| x_k - x_{n_k} \| + \| x_{n_k} - x \| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Hence, (x_n) converges to $x \in X$. Therefore, X is complete. Q.E.D.

Banach Spaces

We have already defined a Banach space as a complete normed linear space and discussed the characteristics that make a set a Banach space. Now we are ready to further develop the study of the spaces c_0, ℓ_1, and ℓ_∞ by considering them as Banach spaces.

Before proving c_0, ℓ_1, and ℓ_∞ are Banach spaces, we consider some of the ideas we will encounter in those proofs. To begin, let us contemplate what might be involved in showing that any of our three spaces is complete. We have already revealed two ways to prove that a space is complete. The first would be to prove that every Cauchy sequence in the space converges to a limit in that space. The second way would be to show that every absolutely summable series in the space is summable. Let us concentrate on this second method. A series in any of our three spaces would necessarily be a series of sequences. Hence, absolute summability would then rely on norms instead of entirely on absolute values. Also, to show that such a series was summable, you would need to show that it converges to a sequence in the space. In any of our three spaces an absolutely summable series would be a series such as:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(x_k^n\right)_{k=1}^{\infty} = \; & \left(x_1^1, x_2^1, x_3^1, \ldots, x_n^1, \ldots\right) \\ & +\left(x_1^2, x_2^2, x_3^2, \ldots, x_n^2, \ldots\right) \\ & +\left(x_1^3, x_2^3, x_3^3, \ldots, x_n^3, \ldots\right) \\ & \quad \vdots \\ & +\left(x_1^n, x_2^n, x_3^n, \ldots, x_n^n, \ldots\right) \\ & \quad \vdots\end{aligned}$$

where $\sum_{n=1}^{\infty}\|x^n\| = M$ for some real number M. As mentioned above, the dilemma of showing such a series is summable lies in showing that it converges to an element of the space. For example, in ℓ_∞ the task would be to demonstrate that

$$\left(\sum_{n=1}^{\infty} x_1^n, \sum_{n=1}^{\infty} x_2^n, \sum_{n=1}^{\infty} x_3^n, \ldots, \sum_{n=1}^{\infty} x_k^n, \ldots\right) \in \ell_\infty .$$

That would entail showing that $\left(\sum_{n=1}^{\infty} x_1^n, \sum_{n=1}^{\infty} x_2^n, \sum_{n=1}^{\infty} x_3^n, \ldots, \sum_{n=1}^{\infty} x_k^n, \ldots\right)$ is first of all a sequence of real numbers and secondly that it is a bounded sequence. A similar argument would have to be followed when considering each of the other spaces. Keeping this in mind, we will proceed in the task of showing ℓ_∞ is a Banach space.

THEOREM 3.8: The space ℓ_∞ is a Banach space.

PROOF: It has already been shown that ℓ_∞ is a normed linear space. Thus, to show that ℓ_∞ is a Banach space, it suffices to show that ℓ_∞ is complete. To do this, we take advantage of the previous theorem and show that every absolutely summable series in ℓ_∞ is also summable.

Let $\sum_{n=1}^{\infty} x^n$, where $x^n = \left(x_k^n\right)_{k=1}^{\infty}$, be an absolutely summable series in ℓ_∞. Then, by definition of absolutely summable,

$$\sum_{n=1}^{\infty} \|x^n\| = M < \infty \text{ for some } M \in \mathbf{R}.$$

Now, examine the series $\sum_{n=1}^{\infty} x^n$ as it is a sequence with a kth term of $\sum_{n=1}^{\infty} x_k^n$. For this sequence to be an element of ℓ_∞ it must be a sequence of real numbers and be bounded. We must first show that $\left|\sum_{n=1}^{\infty} x_k^n\right| < \infty$ for all natural numbers k. Note that by the triangle inequality the following relationship holds for all k:

$$\left|\sum_{n=1}^{\infty} x_k^n\right| \leq \sum_{n=1}^{\infty} \left|x_k^n\right|.$$

Observe that $\left|x_k^n\right| \leq \left|\sup_i x_i^n\right|$ for each k. And so we have

$$\sum_{n=1}^{\infty} \left|x_k^n\right| \leq \sum_{n=1}^{\infty} \sup_i \left|x_i^n\right| \text{ for all } k.$$

Since each $x^n \in \ell_\infty$, we know that $\|x^n\| = \sup_i \left|x_i^n\right|$. Hence,

$$\sum_{n=1}^{\infty} \sup_i \left|x_i^n\right| = \sum_{n=1}^{\infty} \|x^n\|.$$

Finally, as previously stated, we have

$$\sum_{n=1}^{\infty} \|x^n\| = M.$$

Therefore, $\left|\sum_{n=1}^{\infty} x_k^n\right| < M$ for all k. Thus, $\sum_{n=1}^{\infty} x^n$ is a sequence of real numbers. Also, since the absolute value of each term of that sequence is bounded by M, we may conclude that $\sum_{n=1}^{\infty} x^n$ is bounded and hence an element of ℓ_∞. Therefore, $\sum_{n=1}^{\infty} x^n$ is summable. Consequently, ℓ_∞ is a Banach space. Q.E.D.

Next we shall consider the space ℓ_1. Recall from previous discussion that our task will include showing that a sequence such as

$$\left(\sum_{n=1}^{\infty} x_1^n, \sum_{n=1}^{\infty} x_2^n, \sum_{n=1}^{\infty} x_3^n, \ldots, \sum_{n=1}^{\infty} x_k^n, \ldots\right)$$

is a sequence of real numbers that is absolutely summable.

THEOREM 3.9: The space ℓ_1 is a Banach space.

PROOF: As previously demonstrated, ℓ_1 is a normed linear space. Thus, to show that ℓ_1 is a Banach space it is sufficient to show that ℓ_1 is complete. Let $\sum_{n=1}^{\infty} x^n$ be an absolutely summable series in ℓ_1, where $x^n = \left(x_k^n\right)_{k=1}^{\infty}$. Now, since $\sum_{n=1}^{\infty} x^n$ is a series in ℓ_1, it is a series of sequences and thus a sequence itself. We shall call this sequence x. Note that the kth term of x is $\sum_{n=1}^{\infty} x_k^n$. For x to be an element of ℓ_1, x must be an absolutely summable sequence of real numbers. First, to show that each term of x is a real number we must have $\left|\sum_{n=1}^{\infty} x_k^n\right| < M_k$ for each k, where each M_k is a real number. By the triangle inequality,

$$\left|\sum_{n=1}^{\infty} x_k^n\right| \leq \sum_{n=1}^{\infty} \left|x_k^n\right| \text{ for all } k.$$

Next, recall that the series $\sum_{n=1}^{\infty} x^n$ is absolutely summable. That is,

$$\sum_{n=1}^{\infty} \left\|x^n\right\| = M < \infty \text{ for some real number } M.$$

Since $x^n \in \ell_1$, for all natural numbers n we have

$$\sum_{n=1}^{\infty} \left\|x^n\right\| = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \left|x_k^n\right|\right).$$

Observe the following inequality does indeed hold

$$\sum_{n=1}^{\infty} \left|x_k^n\right| \leq \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \left|x_k^n\right|\right).$$

Hence, we arrive at the conclusion that

$$\left|\sum_{n=1}^{\infty} x_k^n\right| \leq M \text{ for all } k.$$

Therefore, x is a sequence of real numbers. Next, for x to be absolutely summable, we must have $\sum_{k=1}^{\infty}\left|\sum_{n=1}^{\infty} x_k^n\right|$ be a real number. By the triangle inequality we have

$$\sum_{k=1}^{\infty}\left|\sum_{n=1}^{\infty} x_k^n\right| \leq \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\left|x_k^n\right|\right).$$

Using previous inequalities, we arrive at

$$\sum_{k=1}^{\infty}\left|\sum_{n=1}^{\infty} x_k^n\right| \leq M.$$

Thus, x is an element of ℓ_1 and $\sum_{n=1}^{\infty} x^n$ is summable. Therefore, ℓ_1 is a Banach space. Q.E.D.

We may now turn our attention to the space c_0. Our strategy this time will be different. We must still show that any absolutely summable series such as $\sum_{n=1}^{\infty}\left(x_k^n\right)_{k=1}^{\infty}$ in c_0 is also summable in c_0, but to do so we will take advantage of the fact that c_0 is a subset of a known Banach space, namely ℓ_∞, and shares the same norm with that Banach space.

THEOREM 3.10: The space c_0 is a Banach space.

PROOF: As it is already known that c_0 is a normed linear space, the task of showing it is a Banach space is reduced to showing that it is complete. Let $\varepsilon > 0$. Let $\sum_{n=1}^{\infty} x^n$ be an absolutely summable sequence in c_0. Then, since c_0 is a subset of ℓ_∞ and they have the same norm, $\sum_{n=1}^{\infty} x^n$ is also an absolutely summable series in ℓ_∞. Since ℓ_∞ is a Banach space, $\sum_{n=1}^{\infty} x^n$ is summable in ℓ_∞. That is, $\left\|\sum_{n=1}^{\infty} x^n\right\| = M$ for some real number M. Then there exists a natural number N_1 such that for all $k \geq N_1$, $\left|\left\|\sum_{n=1}^{k} x^n\right\| - M\right| < \frac{\varepsilon}{2}$. Fix k. Since $x^n \in c_0$ for all n, $\sum_{n=1}^{k} x^n \in c_0$ for $k \in \mathbf{N}$. That is, there exists a natural number N_2 such that for all $i \geq N_2$, $\left|\sum_{n=1}^{k} x_i^n\right| < \frac{\varepsilon}{2}$. Now let $N = \sup\{N_1, N_2\}$. Then, for all

$i \geq N$, consider $\left|\sum_{n=1}^{\infty} x_i^n\right|$. Note

$$\left|\sum_{n=1}^{\infty} x_i^n\right| = \left|\sum_{n=1}^{k} x_i^n + \sum_{n=k+1}^{\infty} x_i^n\right|.$$

By the triangle inequality,

$$\left|\sum_{n=1}^{k} x_i^n + \sum_{n=k+1}^{\infty} x_i^n\right| \leq \left|\sum_{n=1}^{k} x_i^n\right| + \left|\sum_{n=k+1}^{\infty} x_i^n\right|.$$

Substitution yields

$$\left|\sum_{n=1}^{k} x_i^n\right| + \left|\sum_{n=k+1}^{\infty} x_i^n\right| \leq \left|\sum_{n=1}^{k} x_i^n\right| + \left\|\sum_{n=k+1}^{\infty} x^n\right\| \leq \left|\sum_{n=1}^{k} x_i^n\right| + \left|M - \left\|\sum_{n=1}^{k} x^n\right\|\right|.$$

Using previously stated relationships,

$$\left|\sum_{n=1}^{k} x_i^n\right| + \left|M - \left\|\sum_{n=1}^{k} x^n\right\|\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In summary, we have $\left|\sum_{n=1}^{\infty} x_i^n\right| < \varepsilon$ for all $i > N$. Therefore, $\lim_{i \to \infty} \sum_{n=1}^{\infty} x_i^n = 0$ and $\sum_{n=1}^{\infty} x^n \in c_0$. We can thus conclude that every absolutely summable series in c_0 is also summable in c_0. Hence, c_0 is a Banach space. Q.E.D.

<u>Additional Properties of Banach Spaces</u>

At this time, we have verified that each of the spaces c_0, ℓ_1, and ℓ_∞ are Banach spaces. As a result, we are ready to extend our consideration of Banach spaces. Our next topic will be extreme points.

DEFINITION 3.11: Let K be a subset of a Banach space. Then x is an extreme point of K if whenever $y, z \in K$ and $\lambda \in (0,1)$ with $x = \lambda y + (1-\lambda)z$ (i.e., x is between y and z), $x = y = z$.

Extreme points of a subset of a Banach space can be seen as those points which are not between any other two points of that subset. Before we begin investigating this issue in our three spaces, we will comment on extreme points of spaces that may be more familiar. For instance, in $\mathbf{R}^2$ let K be the closed unit circle. Then every point on the circle itself is an extreme point of K. Next, in $\mathbf{R}^3$ let K be the closed unit sphere. Then every point on the surface of the sphere is an extreme point. Note that each of these subsets have contained elements with norms less than or equal to one. Following this pattern, we will now examine subsets of c_0 and ℓ_∞ for extreme points.

EXAMPLE 3.12: Consider the set $K = \{x \in c_0 | \|x\| \leq 1\}$. Let us assume that K has extreme points. Let $(a_n) \in c_0$ be an extreme point of the set K. Now consider $(b_n), (c_n) \in K$ such that each differs from (a_n) in only one term, specifically the first term such that $|a_n| < 1$. If a_i is that term, then b_i and c_i should be chosen such that $b_i + c_i = 2a_i$ and $|b_i|, |c_i| < 1$. Note that $\frac{1}{2}(b_n) + \frac{1}{2}(c_n) = (a_n)$. Therefore, by definition, (a_n) is not an extreme point of the set. As a result, $K = \{x \in c_0 | \|x\| \leq 1\}$ has no extreme points.

EXAMPLE 3.13: Consider the set $K = \{x \in \ell_\infty | \|x\| \leq 1\}$. Let us consider extreme points for K. Note that the problem experienced above can be avoided in a subset of ℓ_∞. Let $(a_n) \in K$ such that $|a_i| = 1$ for all i. Then for any $\lambda \in (0,1)$ and $(b_n), (c_n) \in K$ the relationship $\lambda(b_n) + (1-\lambda)(c_n) = (a_n)$ holds only when $(a_n) = (b_n) = (c_n)$. Therefore, each element of the set $\{(a_n) \in K \,\big|\, |a_i| = 1 \text{ for all } i\}$ is an extreme point of K and $K = \{x \in \ell_\infty | \|x\| \leq 1\}$ has infinitely many extreme points.

Another issue that can be undertaken when discussing Banach spaces is unconditionally convergent series.

DEFINITION 3.14: A series $\sum_{n=1}^{\infty} x_n$ in a Banach space is called unconditionally convergent if any rearrangement of the order of the sum results in a convergent series.

To understand what is meant by unconditionally convergent, we must also define a rearrangement.

DEFINITION 3.15: A rearrangement of $\sum_{n=1}^{\infty} x_n$ is $\sum_{n=1}^{\infty} x_{\varphi(n)}$, where $\varphi : \mathrm{N} \to \mathrm{N}$ is one-to-one and onto.

The following is a famous theorem connecting Banach spaces and unconditionally convergent sequences. It was proven by Dvoretsky and Rogers in 1950.

THEOREM 3.16: (The Dvoretsky-Rogers Theorem) Every infinite dimensional Banach space has a series which is unconditionally convergent but not absolutely convergent (Dvoretsky and Rogers 192-196).

Since each of our spaces are infinite dimensional, the Dvoretsky-Rogers Theorem applies to each of them. We will now entertain such a series in c_0.

EXAMPLE 3.17: Let $\sum_{n=1}^{\infty} x_n$ be the series in c_0 defined by

$$x_1 = (1,0,0,0,0,\ldots),$$
$$x_2 = \left(0,\tfrac{1}{2},0,0,0,\ldots\right),$$
$$x_3 = \left(0,0,\tfrac{1}{3},0,0,\ldots\right),$$
$$x_4 = \left(0,0,0,\tfrac{1}{4},0,0,\ldots\right),$$
$$\vdots$$

Observe that

$$\sum_{n=1}^{\infty} x_{\varphi(n)} = \left(1,\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{4},\ldots\right) \in c_0$$

for any bijection $\varphi\colon \mathbf{N} \to \mathbf{N}$. However,

$$\sum_{i=1}^{\infty} \|x_i\| = \sum_{i=1}^{\infty} \tfrac{1}{i} = \infty$$

does not converge.

CHAPTER 4
THE HAHN-BANACH THEOREM

The intent of this chapter will be to cover the remaining concepts needed to discuss the Hahn-Banach theorem. Then, we shall state and prove that theorem.

Extensions of Linear Functionals

A concept that is essential to the Hahn-Banach theorem is the extension of a linear functional.

DEFINITION 4.1: Let f be a function with a domain of S and $X \supseteq S$. Then any function $\hat{f}$ with a domain of X is an extension of f to the set X if $\hat{f}(x) = f(x)$ for all $x \in S$.

Another way to state $\hat{f}(x) = f(x)$ for all $x \in S$ is to say that $\hat{f}$ restricted to the set S is the same as f. This is denoted by $\hat{f}|_S = f$.

For our purposes, we will be concerned with extensions of linear functionals that retain the same norm. That is, given a linear functional f on S with a norm of N we want to find a linear extension of f to X, where $X \supseteq S$, such that the norm of the extension is N also.

Consider the linear functional $f: \mathbf{R}^5 \rightarrow \mathbf{R}$ given by

$$f((a_1, a_2, a_3, a_4, a_5)) = \sum_{i=1}^{5} a_i .$$

We will discuss finding such extensions of f first to all of c_0 and then to all of ℓ_1. The topics from the section on dual spaces will be helpful in this discussion.

EXAMPLE 4.2: Before we can begin finding an extension of f to all of c_0, we must first find a way to regard $\mathbf{R}^5$ as a subset of c_0. To do this we identify each element of $\mathbf{R}^5$ with the sequence of c_0 in which the first five terms are the same and the remaining terms are zero. That is, we identify $(a_1,a_2,a_3,a_4,a_5)\in \mathbf{R}^5$ with $(a_1,a_2,a_3,a_4,a_5,0,0,0,\ldots)\in c_0$. Therefore, $\mathbf{R}^5$ is a subset of c_0. Another thing that must be done is finding $\|f\|$ so that we know what norm to preserve. To find $\|f\|$ in c_0 we will make use of Definition 2.12. In which case, we have

$$\left|\sum_{n=1}^{5} a_n\right| \leq M\left(\sup\{|a_1|,|a_2|,|a_3|,|a_4|,|a_5|\}\right).$$

However, $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$, and so

$$\sum_{n=1}^{5} |a_n| \leq 5\left(\sup\{|a_1|,|a_2|,|a_3|,|a_4|,|a_5|\}\right).$$

Thus, $\|f\| \leq 5$. Consider the sequence $(x_n)=(1,1,1,1,1,0,0,0,0,\ldots)$. Notice that $\sum_{n=1}^{5} |x_n| = 5$ while $\sup\{|x_1|,|x_2|,|x_3|,|x_4|,|x_5|\}=1$. Hence, $\|f\|$ must be 5 for the above inequality to hold. Recall that $\hat{f}:c_0 \to \mathbf{R}$ must be linear and $\|\hat{f}\|=5$. That is, $\hat{f}\in c_0^* = \ell_1$. Keeping this in mind, let $\hat{f}:c_0 \to \mathbf{R}$ be defined by $\hat{f}((x_n))=\sum_{n=1}^{5} x_n$ for all $(x_n)\in c_0$. That is, $\hat{f}=(1,1,1,1,1,0,0,0,\ldots)$.

EXAMPLE 4.3: To find an extension of f to all of ℓ_1 with the same norm we must again begin by considering $\mathbf{R}^5$ as a subset of ℓ_1 and determining $\|f\|$ this time in ℓ_1. We shall consider elements of $\mathbf{R}^5$ as we did in Example 4.2. Now, we can turn our attention to $\|f\|$. By applying Definition 2.12, we arrive at

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq M\sum_{n=1}^{\infty} |a_n|.$$

It is clear that equality can be achieved when all the a_n's are positive. Hence, $\|f\|=1$. Now, we can find an extension of f to all of ℓ_1. This time, note that $\hat{f}\in \ell_1^* = \ell_\infty$. Let $\hat{f}:\ell_1 \to \mathbf{R}$ be $\hat{f}=(1,1,1,1,1,0,0,0\ldots)$.

Now, to gain a better understanding of these extensions consider the following questions. Is there only one such extension into all of c_0? How about into all of ℓ_1? Both of these answers depend on the original linear functional and its norm. In the above example, there is no other extension of f into all of c_0; but, there are infinitely many more extensions of f into all of

ℓ_1. In fact, any sequence (x_n) such that $x_n = 1$ for $1 \le n \le 5$ and $|x_n| \le 1$ for $n > 5$ represents an extension of f into all of ℓ_1. Observe that $\|(x_n)\| = 1$ and $(x_n) \in \ell_\infty = \ell_1^*$.

Let us continue by examining another more example.

EXAMPLE 4.4: Let S be the set of all finitely nonzero sequences. Let S have the same norm as ℓ_1. Define $f: S \to \mathbf{R}$ by $f((a_n)) = \sum_{n=1}^{\infty} a_n$, and note that this is actually a finite sum. Find an extension of f to all of ℓ_1 with the same norm.

Let us begin by determining $\|f\|$. By Definition 2.12, $\|f\|$ will be the smallest M such that $|f(s)| \le M\|s\|$ for all $s \in S$. Let (a_n) be an arbitrary element of S. Without loss of generality, let $a_1', a_2', \ldots, a_k'$ be the nonzero terms of (a_n). Then $f((a_n))$ becomes $f((a_n)) = \sum_{i=1}^{k} a_i'$. By the triangle inequality, $|f((a_n))| \le \sum_{i=1}^{k} |a_i'|$. However, $\sum_{i=1}^{k} |a_i'| = \|a_n\|$. So, we have the inequality $|f((a_n))| \le 1 \cdot \|(a_n)\|$. Then the question becomes one of determining if there is a smaller $M > 0$ such that $|f((a_n))| \le M\|(a_n)\|$ for all $(a_n) \in S$. Suppose (a_n) is a positive valued sequence. Then $\left|\sum_{i=1}^{k} a_i'\right| = \sum_{i=1}^{k} |a_i'|$ in which case M must be 1. Thus, we may conclude that $\|f\| = 1$.

Now, to find an extension of f to all of ℓ_1, we should recognize that $\hat{f} \in \ell_1^* = \ell_\infty$. Let $\hat{f}: \ell_1 \to \mathbf{R}$ be given by $\hat{f}((a_n)) = \sum_{n=1}^{\infty} a_n$. Then $\hat{f} = (1,1,1,\ldots)$.

In this case, there is no other extension into all of ℓ_1. The nonzero terms in the elements of S do not have to be in specific places. Since $\hat{f}|_S = f$, $\hat{f}$ must allow for the nonzero terms to occur anywhere in the sequence hence the reason for a sequence of all ones.

The Hahn-Banach Theorem

The last item that should be mentioned before delving into the Hahn-Banach Theorem is an axiom equivalent to the Axiom of Choice.

ℓ_1. In fact, any sequence (x_n) such that $x_n = 1$ for $1 \le n \le 5$ and $|x_n| \le 1$ for $n > 5$ represents an extension of f into all of ℓ_1. Observe that $\|(x_n)\| = 1$ and $(x_n) \in \ell_\infty = \ell_1^*$.

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Let us begin by determining $\|f\|$. By Definition 2.12, $\|f\|$ will be the smallest M such that $|f(s)| \le M\|s\|$ for all $s \in S$. Let (a_n) be an arbitrary element of S. Without loss of generality, let $a_1', a_2', \ldots, a_k'$ be the nonzero terms of (a_n). Then $f((a_n))$ becomes $f((a_n)) = \sum_{i=1}^{k} a_i'$. By the triangle inequality, $|f((a_n))| \le \sum_{i=1}^{k} |a_i'|$. However, $\sum_{i=1}^{k} |a_i'| = \|a_n\|$. So, we have the inequality $|f((a_n))| \le 1 \cdot \|(a_n)\|$. Then the question becomes one of determining if there is a smaller $M > 0$ such that $|f((a_n))| \le M\|(a_n)\|$ for all $(a_n) \in S$. Suppose (a_n) is a positive valued sequence. Then $\left|\sum_{i=1}^{k} a_i'\right| = \sum_{i=1}^{k} |a_i'|$ in which case M must be 1. Thus, we may conclude that $\|f\| = 1$.

Now, to find an extension of f to all of ℓ_1, we should recognize that $\hat{f} \in \ell_1^* = \ell_\infty$. Let $\hat{f}{:}\,\ell_1 \to \mathbf{R}$ be given by $\hat{f}((a_n)) = \sum_{n=1}^{\infty} a_n$. Then $\hat{f} = (1,1,1,\ldots)$.

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The Hahn-Banach Theorem

The last item that should be mentioned before delving into the Hahn-Banach Theorem is an axiom equivalent to the Axiom of Choice.

LEMMA 4.5: (Zorn's Lemma) If each chain in a partially ordered set $(X,\leq)$ has an upper bound in X, then X has a maximal element (Willard 10).

Now that some of the basic ideas behind the Hahn-Banach theorem have been covered, we state and prove that theorem.

THEOREM 4.6: (The Hahn-Banach Theorem) Suppose X is a normed linear space, S is a subspace of X, and $f:S\to\mathbf{R}$ is a linear functional with norm 1. Then there is a linear functional $\hat{f}:X\to\mathbf{R}$ such that $\hat{f}\big|_S=f$ and $\|\hat{f}\|=1$.

PROOF: Let $x_0\in X\setminus S$ and $u\in S$. Then, if we need $\|\hat{f}\|=\|f\|$, by Definition 2.12, we need the following:

$$\left|\hat{f}(x_0+u)\right|\leq\|x_0+u\| \text{ and } \left|\hat{f}(x_0-u)\right|\leq\|x_0-u\|.$$

If we need $\hat{f}\big|_S=f$, then

$$\hat{f}(x_0+u)=\hat{f}(x_0)+f(u) \text{ and } \hat{f}(x_0-u)=\hat{f}(x_0)-f(u).$$

By substitution, we arrive at the following inequalities:

$$\left|\hat{f}(x_0)+f(u)\right|\leq\|x_0+u\| \text{ and } \left|\hat{f}(x_0)-f(u)\right|\leq\|x_0-u\|.$$

Algebraic manipulation results in

$$\hat{f}(x_0)\leq\|x_0+u\|-f(u) \text{ and } \hat{f}(x_0)\geq f(u)-\|x_0-u\|.$$

Consider the sets $\{f(u)-\|x_0-u\| \mid u\in S\}$ and $\{\|x_0+u\|-f(u) \mid u\in S\}$. Since $\|f\|=1$ we can apply Definition 2.12 to attain $|f(u)|\leq\|u\|$ for all $u\in S$. From properties of absolute value inequalities, we have $-\|u\|\leq f(u)\leq\|u\|$. Concentrating on $f(u)\leq\|u\|$ we arrive at $f(u)-\|u\|\leq 0$ and $0\leq\|u\|-f(u)$. Therefore, $f(u)-\|u\|\leq\|u'\|-f(u')$ for all $u,u'\in S$. We may then conclude that $f(u)-\|x_0-u\|\leq\|x_0+u\|-f(u)$ for all $u\in S$. Therefore, $\{f(u)-\|x_0-u\| \mid u\in S\}$ is bounded above and by the Supremum Property of $\mathbf{R}$ has a supremum. Likewise, $\{\|x_0+u\|-f(u) \mid u\in S\}$ is bounded below and by the Infimum Property of $\mathbf{R}$ has an infimum (Bartle and Sherbert 46). Since $\|x_0+u\|-f(u)$ is an upper bound of $\{f(u)-\|x_0-u\| \mid u\in S\}$ for all $u\in S$, then we know that

$$\sup\{f(u)-\|x_0-u\| \mid u\in S\}\leq \|x_0+u\|-f(u) \text{ for all } u\in S.$$

That is, $\sup\{f(u)-\|x_0-u\| \mid u\in S\}$ is a lower bound for $\{\|x_0+u\|-f(u) \mid u\in S\}$.

Therefore,

$$\sup\{f(u)-\|x_0-u\| \mid u\in S\}\leq \inf\{\|x_0+u\|-f(u) \mid u\in S\}.$$

Let $\alpha\in\mathbf{R}$ such that $\sup\{f(u)-\|x_0-u\| \mid u\in S\}\leq\alpha\leq \inf\{\|x_0+u\|-f(u) \mid u\in S\}$. Define $f':\langle S,x_0\rangle\to\mathbf{R}$ by $f'(s)=f(s)$ for all $s\in S$ and $f'(x_0)=\alpha$. Then f' is an extension of f to $\langle S,x_0\rangle$ such that $\|f'\|=\|f\|=1$. Using this process, we can continue to extend f one dimension at a time. However, at this rate we can only extend f in a finite manner. We must do something else in order to extend f to all of X.

Consider the set $\mathcal{A}=\left\{(S_n,f_n)\;\middle|\;\begin{array}{l} S_n \text{ is a subspace of } X \text{ such that } S\subseteq S_n \\ f_n:S_n\to\mathbf{R} \text{ such that } f_n|_S=f \text{ and } \|f_n\|=1\end{array}\right\}$.

Define the following order on $\mathcal{A}$. For $(S_n,f_n),(S_m,f_m)\in\mathcal{A}$, let $(S_n,f_n)<(S_m,f_m)$ only if $S_n\subseteq S_m$ and $f_m|_{S_n}=f_n$. Let $(S_1,f_1)<(S_2,f_2)<\ldots<(S_n,f_n)<\ldots$ be an arbitrary chain in $\mathcal{A}$. Let $S'=\cup_{n=1}^{\infty}S_n$. Notice $S_n\subseteq S'$ for all n. Define $f':S'\to\mathbf{R}$ by $f'(x)=f_n(x)$ for any n such that $x\in S_n$. Note that f' is well-defined and that $f'|_{S_n}=f_n$ for all n. Therefore, $(S',f')>(S_n,f_n)$ for all n and (S',f') is an upper bound for the chain. Hence, by Zorn's lemma, $\mathcal{A}$ has a maximal element. Assume that $(M,\hat{f})$ is the maximal element, where $M\neq X$. Then, as proven above, we may extend $\hat{f}$ one dimension to the space $\langle M,x'\rangle$. This is a contradiction, as $(M,\hat{f})$ is maximal. Therefore $(X,\hat{f})$ is the maximal element of $\mathcal{A}$. Q.E.D.

CONCLUSION

Throughout this paper we examined the sets c_0, ℓ_1, and ℓ_∞. We began by considering their definitions and building our understanding of them from there. The development of each set progressed from vector space to dual space to Banach space.

The set of all absolutely summable sequences is ℓ_1. Each absolutely summable sequence must converge to zero and so ℓ_1 is a subset of the set of all sequences that converge to zero, otherwise known as c_0. Likewise, since each sequence that converges to zero is bounded, c_0 is a subset of the set of all bounded sequence, or ℓ_∞. Working in the opposite order, we were able to prove that each set is a vector space. Using concepts and definitions corresponding to those found in any study of finite vector spaces, we were able to show that the analogous results hold in these spaces.

Since a Banach space is a complete normed linear space and normed linear spaces were covered under the concepts of vector spaces, the introduction of complete spaces allowed us to consider c_0, ℓ_1, and ℓ_∞ as Banach spaces. Upon showing that each space was complete, we concluded that c_0, ℓ_1, and ℓ_∞ are each Banach spaces. This knowledge opened up the field of concepts to consider in relation to these three spaces. With the basic understanding of the information presented in this paper, one can now move on to topics such as Banach limits.

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The Spaces c_0, ℓ_1, and ℓ_∞

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August 10, 1995

Date Received