AN ABSTRACT OF THE THESIS OF

Hiroshi Shinoyama for the Master of Science

in Mathematics presented on March 1993

Title: Geometric Groups

Abstract approved: Essam Abotteen

Our objective in this paper is to discuss and illustrate basic properties of geometric groups and some of their applications in chemistry. In Chapter 1, we provide the readers with basic concepts from linear algebra and abstract algebra that are needed in later chapters. In Chapter 2, we study two types of length (or distance) preserving transformations of a finite-dimensional Euclidean space, namely, orthogonal transformations and Euclidean transformations. In Chapter 3, we state and prove Cartan's Theorem and apply it to the classification of orthogonal and Euclidean transformations on *2*- and *3*-dimensional Euclidean spaces. In Chapter 4, we define the symmetry group of a set in a Euclidean space and classify the finite symmetry groups of bounded sets in the *2*- and *3*-dimensional Euclidean spaces, $\mathbb{R}^2$ and $\mathbb{R}^3$. In Chapter 5, we present two applications of geometric groups namely, the study of the trigonometric functions of a *2*-dimensional Euclidean space and isomer enumeration in organic chemistry via Pólya's Theorem.

GEOMETRIC GROUPS

A Thesis

Presented to

The Division of

Mathematics and Computer Sciences

EMPORIA STATE UNIVERSITY

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Hiroshi Shinoyama

March 1993

Approved for the Major Division

Approved for the Graduate Council

ACKNOWLEDGMENTS

My deepest thanks to Dr. Essam Abotteen. His continuous support in the writing of this thesis will always greatly appreciated. I would also like to express sincere gratitude to Dr. Bryan Dawson, Dr. Linda Fosnaugh, and Dr. Charles Greenlief for their help and encouragement.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Introduction

In this paper, we present an elementary introduction to geometric groups. These are groups having their origin in some branch of geometry. Geometric groups are useful in many applications of group theory in science. From a mathematical point of view, they provide a better understanding of the interaction between different branches of mathematics, in particular between group theory, linear algebra, and geometry. Our goal in this paper is to discuss and illustrate the basic properties of geometric groups and some of their applications. The nature of this study is more that of a compilation of existent ideas than that of a development of original ideas. The theorems, proofs, and examples constitute a modeling of materials from multiple sources so that, even if possible, in most instances, crediting a single source would not be appropriate, though occasionally a specific source is cited. We provide a list of references that we consulted in this study at the end of the paper.

The material in this paper is intended for readers familiar with the contents of standard courses in linear algebra and abstract algebra. In particular, we assume that the reader is familiar with the following notations: finite-dimensional vector spaces, subspaces, linear transformations and matrices, determinants, eigenvalues, groups, subgroups, group homomorphisms, group isomorphisms, kernels, normal subgroups, direct products, permutations, cycle decompositions of permutations. Accounts of these topics may be found in most linear algebra and abstract algebra books; for example, [2], [16], and [21]. Partly in order to establish notation, we devote the rest of this chapter to review without proofs, basic notations, and terminology of Euclidean spaces and some related topics.

THROUGHOUT THIS STUDY, *E* DENOTES AN *n-DIMENSIONAL EUCLIDIAN SPACE OVER THE FIELD OF REAL NUMBERS* UNLESS STATED OTHERWISE.

Definitions:

1.1 A *Euclidean space* is a finite-dimensional real vector space E together with a function $< \, , \, >: E\times E\to \mathbb{R}$, called the *inner product* on E, that satisfies the following conditions:

(1) $<A,B>=<B,A>$ for all vectors $A,B\in E$;

(2) $<A+B,C>=<A,C>+<B,C>$ for all vectors $A,B,C\in E$;

(3) $<cA,B>=<A,cB>=c<A,B>$ for all vectors $A,B\in E$ and all scalars $c\in\mathbb{R}$;

(4) $<A,A>\geq 0$ for all vectors $A\in E$, and $<A,A>=0$ if and only if $A=0$, the zero vector of E.

1.2 Let E be a Euclidean space. The *length* (or *Euclidean norm*) of the vector A is the number $\|A\|=\sqrt{\langle A,A\rangle}$.

1.3 Let E be a Euclidean space. The *distance function* on E is a function $d:E\times E\to\mathbb{R}$ defined by setting $d(A,B) =\|A-B\|$ for all $A,B\in E$.

1.4 Let E be a Euclidean space. Two vectors $A,B\in E$ are said to be *orthogonal* if $\langle A,B\rangle=0$, in which case we write $A\perp B$.

1.5 Let E be a Euclidean space, and let U be a subspace of E. The *orthogonal complement* of U is the set $U^{\perp}=\{X\in E\,|\,X\perp A \text{ for every vector } A\in U\}$.

1.6 Let E be a Euclidean space. An *orthonormal basis* for E is a basis $\{A_1,A_2,\ldots,A_n\}$ of E such that $\langle A_i,A_j\rangle=\delta_{ij}$, where δ_{ij} is the Kronecker symbol (or delta); δ_{ij} equals 1 when $i=j$ and 0 otherwise.

1.7 A *linear isomorphism* of a vector space V over a field F is a one-to-one linear transformation mapping V onto V.

1.8 Let V be a finite-dimensional vector space over F, and let W be a subspace of V. Let T be a linear transformation of V into V. Then the subspace W is called *invariant* with respect to T if $T(W)\subseteq W$.

1.9 Let V be a vector space, and let T be a linear transformation of V into V. A polynomial $\varphi_T(t)$ is called a *minimal polynomial* for T if $\varphi_T(T)=0$ and $\varphi_T(t)$ is the lowest degree of such polynomials of t.

Theorems:

1.1 Let E be a Euclidean space, and let $A,B\in E$, then

$$\langle A,B\rangle=\frac{1}{2}[d(A,0)^2+d(B,0)^2-d(A,B)^2].$$

1.2 Let E be a Euclidean space, and let U be a subspace of E. Then $U^{\perp}$ is a subspace. Moreover $E=U\oplus U^{\perp}$; that is every vector in E may be written uniquely in the form $A+B$, where $A\in U$ and $B\in U^{\perp}$.

1.3 The set $GL(V)$ of all linear isomorphisms on a vector space V is a group under function composition called the *general linear group* of V.

1.4 $\mathbb{R}^n=\{(x_1,x_2,\ldots,x_n)/x_i\in\mathbb{R}\}$ is an n-dimensional Euclidean space with the inner product defined by setting

$$\langle(a_1,\ldots,a_n),(b_1,\ldots,b_n)\rangle=a_1b_1+\ldots+a_nb_n$$

for all vectors $(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in\mathbb{R}^n$. If $A=(a_1,\ldots,a_n)$ and $B=(b_1,\ldots,b_n)$, then

$$|A|=\sqrt{a_1^2+\ldots+a_n^2},$$

$$d(A,B)=\sqrt{(a_1-b_1)^2+\ldots+(a_n-b_n)^2},$$

and

$$\langle A,B\rangle=|A||B|\cos\theta,$$

where θ is the angle between A and B.

In the sequel, $\mathbb{R}^n$ with this inner product is called the ordinary Euclidean n-space.

1.5 Let V be a vector space, and let T be a linear transformation of V. Let $\varphi_T(t)$ be a minimal polynomial of T. Then

1. If $f(t)$ is any polynomial for which $f(T)=0$, then $\varphi_T(t)$ divides $f(t)$. In particular, $\varphi_T(t)$ divides the characteristic polynomial of T.
2. There is only one minimal polynomial for T; i.e., $\varphi_T(t)$ is unique.

1.6 Let V be a vector space, and let T be a linear transformation of V. Let $\varphi_T(t)$ be a minimal polynomial of T. A scalar λ is an eigenvalue of T if and only if $\varphi_T(\lambda)=0$. Hence the characteristic polynomial and the minimal polynomial for T have the same roots.

1.7 Let V be a vector space, and let T be a linear transformation of V. Let $f(t)$ be a characteristic polynomial and $\varphi_T(t)$ be a minimal polynomial of T. Suppose that $f(t)$ factors as

$$f(t)=(\lambda_1-t)^{n_1}(\lambda_2-t)^{n_2}\cdots(\lambda_k-t)^{n_k},$$

where $\lambda_1,\lambda_2,\ldots,\lambda_k$ are the distinct eigenvalues of T. Then there exist integers $m_1,m_2,\ldots,m_k$ such that $1\le m_i\le n_i$ for all i and

$$\varphi_T(t)=(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}.$$

1.8 Let V be a vector space over F, and let T be a linear transformation of V. Assume that the minimal polynomial $\varphi_T(t)$ of T is a product of two polynomials $\varphi_1(t)$ and $\varphi_2(t)$ which are relatively prime over F. Let $V_1=\{\alpha\in V/\varphi_1(T)(\alpha)=0\}$ and $V_2=\{\alpha\in V/\varphi_2(T)(\alpha)=0\}$. Then

1. $V=V_1\oplus V_2$ and $T(V_1)\subseteq V_1$, $T(V_2)\subseteq V_2$
2. $V_1=\{\varphi_2(T)(\alpha)/\alpha\in V\}$, $V_2=\{\varphi_1(T)(\alpha)/\alpha\in V\}$.
3. The restriction T_1 [T_2] of T to V_1 [V_2] has $\varphi_1(t)$ [$\varphi_2(t)$] as its minimal polynomial.

1.9 Let V be a finite vector space over F, and let T be a linear transformation of V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of $T/_W$ divides the characteristic polynomial of T.

1.10 Let V be a finite vector space over F, and let T be a linear transformation of V, and suppose that $V=W_1\oplus\ldots\oplus W_k$, where W_i is a T-invariant subspace of V for each i $(1\leq i\leq k)$. If $f(t)$ denotes the characteristic polynomial of T and $f_i(t)$ denotes the characteristic polynomial of $T|_{W_i}$ $(1\leq i\leq k)$, then

$$f(t)=f_1(t)f_2(t)\ldots f_k(t).$$

CHAPTER 2

The Orthogonal Group

Throughout this chapter, E denotes a finite dimensional real Euclidean space. Thus E is a vector space together with an inner product $\langle\ ,\ \rangle : E \times E \to \mathbb{R}$. If $A \in E$, then the number $|A| = \sqrt{\langle A, A \rangle}$ is called the length (or Euclidean norm) of A. Our objective in this chapter is the study of length-preserving transformations of E. There are two types of length preserving transformations that are of primary concern to us in this chapter: orthogonal transformations and Euclidean transformations.

In Section 2.1, we define and study the basic properties of orthogonal transformations. In Section 2.2, an algebraic structure on the set of orthogonal transformations $O(E)$ is introduced and it is shown that with respect to this structure $O(E)$ is a group, called the orthogonal group of the space. Section 2.3 is devoted to the study of Euclidean transformations (or rigid motions). Euclidean transformations, like orthogonal transformations, form a group, called the Euclidean group of the space, and the study of its structure is given in Section 2.3.

2.1 Orthogonal Transformations

In this section, we are going to discuss special kinds of linear transformations of a Euclidean space E, namely orthogonal transformations. We first characterize these transformations geometrically and note that they extend the concept of rotations and reflections of ordinary plane Euclidean geometry to higher dimensional Euclidean spaces. Among the most important results to be proved in this section is the spectral decomposition theorem for orthogonal transformations.

Definition 2.1:

Let E be a finite dimensional real Euclidean space. An *isometry* (*Euclidean transformation* or *rigid motion*) of E is a mapping $\sigma: E \to E$ such that $d(A,B)=d(\sigma(A),\sigma(B))$ for all $A,B \in E$, where d is the distance function on E.

Thus, an isometry of E is a function from E into E that preserves the distance between all points in E.

Theorem 2.1:

Let $\sigma: E \to E$ be an isometry of E. Then

1. σ is one-to-one.
2. $\sigma^{-1}: E \to E$ is also an isometry.
3. If $\tau: E \to E$ is another isometry of E then $\sigma \circ \tau$ is an isometry of E.

Proof:

1. Assume $\sigma(A)=\sigma(B)$ for some $A,B \in E$. Then $d(\sigma(A),\sigma(B))=0$; therefore, $d(A,B)=\|A-B\|=0 \Rightarrow A=B$. Hence σ is one-to-one.

2. Since σ is one-to-one and we will prove later σ is also onto, then σ^{-1} exists. And we know that $d(A,B)=d(\sigma(A),\sigma(B))$ for all $A,B\in E$.
Then $d(\sigma^{-1}(A),\sigma^{-1}(B))=d(\sigma(\sigma^{-1}(A)),\sigma(\sigma^{-1}(B)))=d(A,B)$. Therefore, σ^{-1} is an isometry.

3. We are given that σ and τ are isometries on E. We wish to show $\sigma\circ\tau$ is an isometry. Let $A,B\in E$. Since σ and τ are isometries, $d(A,B)=d(\sigma(A),\sigma(B))$ and $d(A,B)=d(\tau(A),\tau(B))$. Therefore, $d(\sigma\circ\tau(A),\sigma\circ\tau(B))=d(\tau(A),\tau(B))=d(A,B)$; hence, $\sigma\circ\tau$ is an isometry.

Corollary:

Let $Iso(E)$ be set of all isometries of E. Then $Iso(E)$ is a group under composition of functions.

Proof:

1. $Iso(E)\neq\emptyset$. The identity map $1_E\in Iso(E)$.
2. $(Iso(E),\circ)$ is a mathematical system since composition of isometries is an isometry by Theorem 2.1.
3. The associative property holds since a composition of functions in general is associative.
4. The identity map $1_E:E\to E, \forall A\in E, 1_E(A)=A$, is the identity element.
5. $\forall\sigma\in Iso(E)$, σ^{-1} is an isometry by Theorem 2.1.

Therefore, $(Iso(E),\circ)$ is a group.

Definition 2.2:

Let E be a Euclidean space. Suppose $A \in E$. The map $T_A : E \to E$ given by $T_A(x) := x + A$ is called the *translation* of E by A.

Theorem 2.2:

The translation of a space E by A is an isometry of E.

Proof:

Let $T_A(x) = x + A$. We wish to show $T_A \in Iso(E)$. Let $B, C \in E$, then $T_A(B) = B + A$ and $T_A(C) = C + A$.

So, $d(T_A(B), T_A(C)) = d(B+A, C+A)$

$$= \|(B+A) - (C+A)\|$$

$$= \|B - C\|$$

$$= d(B, C).$$

Definition 2.3:

An *orthogonal transformation* of E is a linear isometry of E; that is, σ is an orthogonal transformation of E if σ is a linear transformation from E to E and σ is an isometry.

Lemma 2.3:

An isometry $\sigma : E \to E$ is a linear isometry if and only if $\sigma(0) = 0$.

Proof:

($\to$) Assume σ is a linear transformation. Then $\forall A, B \in E$, $\sigma(A) + \sigma(B) = \sigma(A+B)$. Therefore, $\sigma(0) = \sigma(0+0) = \sigma(0) + \sigma(0)$, and thus $\sigma(0) = 0$.

($\Leftarrow$) Assume σ is an isometry such that $\sigma(0)=0$. First, we need to show σ preserves the inner product of E. Let $A,B\in E$, then

$$<\sigma(A),\sigma(B)>=1/2[d(\sigma(A),0)^2+d(\sigma(B),0)^2-d(\sigma(A),\sigma(B))^2]$$
$$=1/2[d(\sigma(A),\sigma(0))^2+d(\sigma(B),\sigma(0))^2-d(\sigma(A),\sigma(B))^2]$$
$$=1/2[d(A,0)^2+d(B,0)^2-d(A,B)^2]$$
$$=<A,B>.$$

Next, to show σ is a linear transformation, let $A,B,C\in E$ and $c\in\mathbb{R}$. Since σ is an onto map, then $\exists C'\in E$ such that $C=\sigma(C')$. So

$$<\sigma(A+B)-\sigma(A)-\sigma(B),C>=<\sigma(A+B)-\sigma(A)-\sigma(B),\sigma(C')>$$
$$=<\sigma(A+B),\sigma(C')>-<\sigma(A),\sigma(C')>$$
$$-<\sigma(B),\sigma(C')>$$
$$=<A+B,C'>-<A,C'>-<B,C'>$$
$$=<A+B-A-B,C'>$$
$$=<0,C'>$$
$$=0.$$

Therefore, $\forall C\in E, \sigma(A+B)-\sigma(A)-\sigma(B)\perp C$; thus $\sigma(A+B)-\sigma(A)-\sigma(B)$ must be the zero vector. Hence $\sigma(A+B)=\sigma(A)+\sigma(B)$. Thus σ preserves vector addition. Similarly,

$$<\sigma(cA)-c\sigma(A),C>=<\sigma(cA)-c\sigma(A),\sigma(C')>$$
$$=<\sigma(cA),\sigma(C')>-<c\sigma(A),\sigma(C')>$$
$$=<cA,C'>-<cA,C'>$$
$$=0.$$

Therefore, $\forall c\in\mathbb{R}$, $\sigma(cA)=c\sigma(A)$.

The following theorem shows that in studying isometries of a space E, it will often suffice to study only linear isometries.

Theorem 2.4:

Every isometry $\sigma: E \to E$ can be expressed as a composition of a linear isometry of E and a translation of E.

Proof:

Let $\sigma: E \to E$ be an isometry of E, and let $A=\sigma(0)$. By Theorem 2.2, the translation T_{-A} is an isometry of E. Thus, the map $\tau = T_{-A} \circ \sigma$ is an isometry of E. Since $\tau(0)=(T_{-A} \circ \sigma)(0)$ $=T_{-A}(\sigma(0))=T_{-A}(A)=0$, Lemma 2.3 implies that τ is a linear isometry. Moreover, $\sigma = T_{-A}^{-1} \circ \tau = T_A \circ \tau$. This completes the proof.

Corollary 1:

The factors τ and T_A are uniquely determined by σ.

Corollary 2:

Every isometry σ of E is onto.

Proof:

Since $\sigma: E \to E$ is an isometry, then by Theorem 2.4, we have $\sigma = T_A \circ \tau$, where τ is a linear isometry and T_A is a translation of E. By Theorem 2.1 (1), τ is one-to-one, and since τ is a linear transformation and E is finite dimensional, τ is onto. Now we are going to show that T_A is onto. Let $X \in E$. Then

$X=(T_A \circ T_{-A})(X)=T_A(T_{-A}(X))$ and $T_{-A}(X)\in E$. Hence, T_A is onto. Thus, $\sigma=T_A \circ \tau$ is onto.

Next, we are going to give a variety of characterizations of orthogonal transformations. The following theorem gives a geometric characterization of an orthogonal transformation.

Theorem 2.5:

Let σ be a linear transformation on a finite dimensional Euclidean space E. Then the following statements are equivalent:

1. σ is an orthogonal transformation.
2. $\|\sigma(A)\|=\|A\| \ \forall A\in E$; that is, σ preserves the length of all vectors in E.
3. If $A\in E$ is a unit vector (*i.e.*, if $\|A\|=1$), then $\|\sigma(A)\|=1$.
4. For any vectors $A,B\in E$, $\langle\sigma(A),\sigma(B)\rangle=\langle A,B\rangle$; that is, σ preserves the inner product on E.

Proof:

(1→2) Assume σ is an orthogonal transformation. We wish to show $\forall A\in E$, $\|\sigma(A)\|=\|A\|$. Let $A\in E$. Then

$$\|\sigma(A)\|^2=d(0,\sigma(A))^2$$
$$=d(\sigma(0),\sigma(A))^2$$
$$=d(0,A)^2$$
$$=\|A\|^2.$$ Therefore, $\|\sigma(A)\|=\|A\|$.

(2→1) Assume σ is a linear transformation such that $\forall A\in E$, $\|\sigma(A)\|=\|A\|$. We wish to show σ is an orthogonal transformation. Let $A,B\in E$. Then

$$d(\sigma(A),\sigma(B))^2=\|\sigma(A)-\sigma(B)\|^2$$
$$=<\sigma(A)-\sigma(B),\sigma(A)-\sigma(B)>$$
$$=<\sigma(A-B),\sigma(A-B)>$$
$$=\|\sigma(A-B)\|^2$$
$$=\|A-B\|^2$$
$$=d(A,B)^2.$$

Therefore, σ is an isometry, and hence σ is an orthogonal transformation.

(2→3) Assume $\forall A\in E$, $\|\sigma(A)\|=\|A\|$. If $\|A\|=1$, then $\|\sigma(A)\|=1$.

(3→2) Assume σ is a linear transformation such that if $A\in E$ is a unit vector then $\|\sigma(A)\|=1$. We wish to show $\forall B\in E$, $\|\sigma(B)\|=\|B\|$. If $B\in E$, and $B=0$, then $\|\sigma(B)\|=\|\sigma(0)\|=\|0\|=\|B\|$. Let $B\in E$, and $B\neq 0$ and let $A=B/\|B\|$. Then A is a unit vector; therefore,

$$\|\sigma(B)\|=\|\sigma(\|B\|A)\|$$
$$=\|B\|\|\sigma(A)\|$$
$$=\|B\|.$$

(1→4) Assume σ is an orthogonal transformation. We wish to show $\forall A,B\in E$, $<\sigma(A),\sigma(B)>=<A,B>$. Let $A,B\in E$. Then

$$<\sigma(A),\sigma(B)>=1/2[d(\sigma(A),0)^2+d(\sigma(B),0)^2-d(\sigma(A),\sigma(B))^2]$$
$$=1/2[d(\sigma(A),\sigma(0))^2+d(\sigma(B),\sigma(0))^2-d(\sigma(A),\sigma(B))^2]$$
$$=1/2[d(A,0)^2+d(B,0)^2-d(A,B)^2]$$
$$=<A,B>.$$

(4→2) Assume σ is a linear transformation such that $\forall A,B\in E$, $<\sigma(A),\sigma(B)>=<A,B>$. We wish to show $\forall C\in E$, $\|\sigma(C)\|=\|C\|$. Let $C\in E$. Then

$$\|\sigma(C)\|^2=<\sigma(C),\sigma(C)>$$
$$=<C,C>$$
$$=\|C\|^2.$$

Corollary:

Let $\sigma:E\to E$ be a linear transformation. Then σ is an orthogonal transformation if and only if for some orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ of E, the set of vectors $\{\sigma(e_1),\sigma(e_2),\ldots,\sigma(e_n)\}$ forms an orthonormal set.

Proof:

Assume that σ is an orthogonal transformation. Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal basis of E. Then $<e_i,e_j>=0$ if $i\neq j$ and $<e_i,e_i>=1$. By Theorem 2.5 (4), we have $<\sigma(e_i),\sigma(e_j)>=0$ if $i\neq j$ and $<\sigma(e_i),\sigma(e_i)>=1$, or $<\sigma(e_i),\sigma(e_j)>=\delta_{ij}=\begin{cases}1 & if\ i=j\\ 0 & if\ i\neq j\end{cases}$

(δ_{ij} is called the Kronecker delta). Thus $\{\sigma(e_1),\sigma(e_2),\ldots,\sigma(e_n)\}$ is an orthonormal set.

Conversely, suppose that for some orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ of E, the set $\{\sigma(e_1),\sigma(e_2),\ldots,\sigma(e_n)\}$ is an orthonormal set. Let $A=a_1e_1+\dots+a_ne_n=\sum_{i=0}^{n} a_ie_i$ be an arbitrary vector in E. Then

$$\begin{aligned}\|A\|^2 &= \langle A, A\rangle \\ &= \langle \sum_{i=1}^{n} a_ie_i, \sum_{j=1}^{n} a_je_j\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\langle e_j, e_j\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\delta_{ij} \\ &= \sum_{i=1}^{n} a_i^2 .\end{aligned}$$

and

$$\begin{aligned}\|\sigma(A)\|^2 &= \langle\sigma(A), \sigma(A)\rangle \\ &= \langle \sum_{i=1}^{n} a_i\sigma(e_i), \sum_{j=1}^{n} a_j\sigma(e_j)\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\langle\sigma(e_j), \sigma(e_j)\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\delta_{ij} \\ &= \sum_{i=1}^{n} a_i^2 .\end{aligned}$$

Hence $\|A\|=\|\sigma(A)\|$. Thus, by theorem 2.5 (2), σ is an orthogonal transformation.

Remarks:

In the literature, an orthogonal transformation is also called a unitary map or unitary operator. The reason why some authors use the terminology unitary is that they are characterized by the fact that they map unit vectors into unit vectors as we have shown in Theorem 2.5 (3). The reason most authors use the term, orthogonal transformations, is that these transformations preserve orthogonality of vectors.

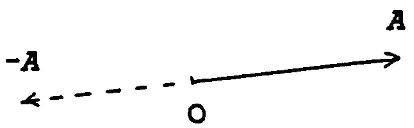
Throughout the paper we are going to follow this standard terminology. Unfortunately, this choice is not the best choice for the following reason: Let $U:E\to E$ be a unitary map; that is, U satisfies the condition: If $X\in E$ such that $\|X\|=1$, then $\|U(X)\|=1$. Then it follows that U preserves orthogonality of vectors in E. On the other hand, it does not follow that a map which preserves orthogonality of vectors is necessarily unitary. For example, the map $\sigma:\mathbb{R}^2\to\mathbb{R}^2$ given by $\sigma(X)=2X$, preserves orthogonality but is not unitary.

Before we continue with other characterizations of orthogonal transformations, let us give some examples.

Example 1:

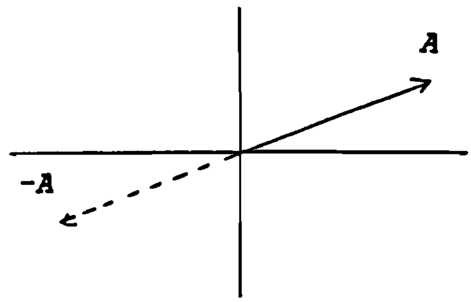
Let E be any n-dimensional Euclidean space. The identity transformation on E is the map $1_E:E\to E$ defined by $1_E(A)=A$ for any vector $A\in E$, and the inversion transformation on E is the map $-1_E:E\to E$ defined by $-1_E(A)=-A$ for any vector $A\in E$. These are both orthogonal transformations of E since both $\pm 1_E$ are linear transformation of E and $\langle\pm 1_E(A),\pm 1_E(B)\rangle=\langle\pm A,\pm B\rangle=\langle A,B\rangle$ for all vectors $A,B\in E$.

Geometrically, the inversion transformation maps every vector to its inverse. The figures below illustrate the inversion transformation on the *1*-dimensional Euclidean space $\mathbb{R}^1$, and on the *2*-dimensional Euclidean plane $\mathbb{R}^2$.



$-1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$

Figure 2.1



$-1_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$

Figure 2.2

Example 2:

Let E be an n-dimensional real Euclidean space. Let H be a hyperplane in E; that is, H is an $(n-1)$-dimensional subspace of E. Then its orthogonal complement $H^\perp$ is a *1*-dimensional subspace of E. Let $L=H^\perp$. Then $E=H\oplus L$ and hence every vector $X\in E$ can be written uniquely in the form $X=X_H+X_L$, where $X_H\in H$ and $X_L\in L$. The map $R_H:E\to E$ defined by $R_H(X)=X_H-X_L$ for every $X\in E$ is an orthogonal transformation of E. The map R_H is called a hyperplane reflection of E through (or in) the hyperplane H. If $E=\mathbb{R}^2$ or $\mathbb{R}^3$, then $R_H(X)$ is the mirror image of X obtained by regarding H as a two sided mirror. The figure below illustrates a typical hyperplane reflection of the *3*-dimensional Euclidean space $\mathbb{R}^3$.

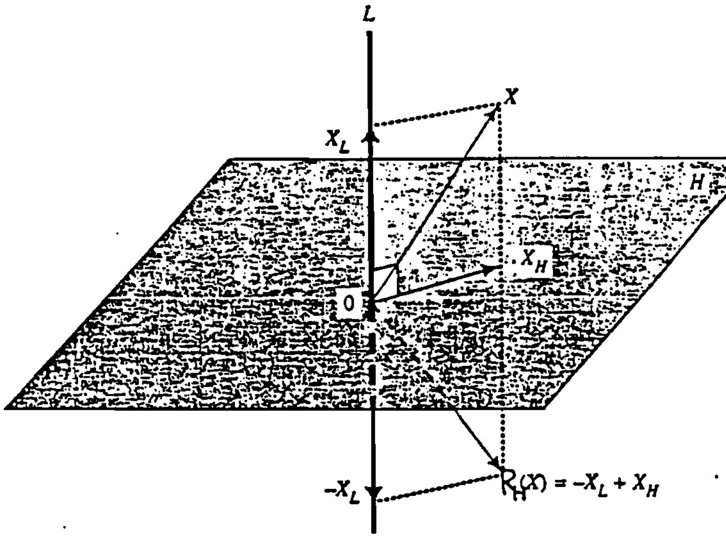


Figure 2.3

Example 3:

Let E be an n-dimensional Euclidean space, and let A be a nonzero vector in E. The map $S_A:E\to E$ defined by $S_A(X) = X-2\frac{\langle X,A\rangle}{\langle A,A\rangle}A$ for every $A\in E$ is an orthogonal transformation of E. For example, let $E=\mathbb{R}^2$ and $A=(1,2)$, then $S_{(1,2)}:\mathbb{R}^2\to\mathbb{R}^2$ is given by

$$\begin{aligned} S_{(1,2)}((x,y)) &= (x,y) - 2\frac{(1,2)\cdot(x,y)}{(1,2)\cdot(1,2)}(1,2) \\ &= (x,y) - \frac{2}{5}(x+2y)(1,2) \\ &= (\frac{3}{5}x - \frac{4}{5}y, -\frac{4}{5}x - \frac{3}{5}y) \ . \end{aligned}$$

Thus $S_{(1,2)}$ is a hyperplane reflection of the space (or plane) $\mathbb{R}^2$ through the line $H=A^{\perp}=\langle(1,2)\rangle^{\perp}=\langle(2,-1)\rangle$. See the figure below.

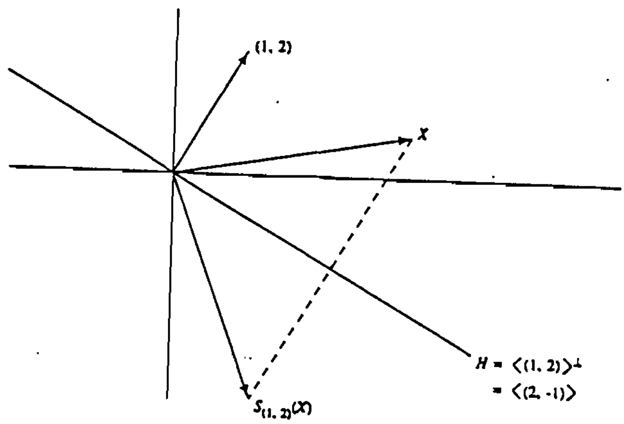


Figure 2.4

The preceding example illustrates that the orthogonal transformation S_A on $\mathbb{R}^2$ is just the reflection of $\mathbb{R}^2$ through a line $H=A^\perp$ passing through the origin; that is, in the notation of example 2, $S_A=R_{A^\perp}$. This type of orthogonal transformation is called a symmetry of E and it will be discussed later.

The following theorem gives a characterization of orthogonal transformations in terms of matrices.

Theorem 2.6:

Let σ be a linear transformation of E. Then σ is an orthogonal transformation if and only if the matrix A of σ with respect to some orthonormal basis of E satisfies the condition $A^tA=I$, where A^t is the transpose of A.

Proof:

Suppose that σ is an orthogonal transformation. Let $\{e_1,...,e_n\}$ be an orthonormal basis of E. Let $\sigma(e_i)=\sum_{k=1}^{n}\alpha_{ki}e_k$.

By the corollary to Theorem 2.5, the set $\{\sigma(e_1),\sigma(e_2),...,\sigma(e_n)\}$ is an orthonormal set; thus, we have $<\sigma(e_i),\sigma(e_j)>=\delta_{ij}$. Also we have $\langle\sigma(e_i),\sigma(e_j)\rangle=\langle\sum_{k=1}^{n}\alpha_{ki}e_k,\sum_{k=1}^{n}\alpha_{kj}e_k\rangle=\sum_{k=1}^{n}\alpha_{ki}\alpha_{kj}$. These equations imply that $A^tA=I$ since the (i,k)th entry of A^t is α_{ki}. Conversely, assume that $A^tA=I$. Thus the equations above are satisfied and hence the set $\{\sigma(e_1),\sigma(e_2),...,\sigma(e_n)\}$ is an orthonormal set. This completes the proof.

Corollary:

$AA^t=I$.

Definition 2.4:

An nxn matrix A with real entries is called an *orthogonal matrix* if $A^tA=I$.

Example 4:

Let us use the matrix approach to show that every rotation of the Euclidean plane $\mathbb{R}^2$ about the origin is an orthogonal transformation of the plane. Recall that the rotation of the plane through an angle of θ radians about the origin is a mapping $\rho_\theta:\mathbb{R}^2\to\mathbb{R}^2$, where $\rho_\theta(x,y)=(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$ for all vectors $(x,y)\in\mathbb{R}^2$. It follows easily from this formula that ρ_θ is a linear transformation of $\mathbb{R}^2$ whose matrix representation relative to the standard orthonormal basis $\{e_1,e_2\}$ is

$$A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

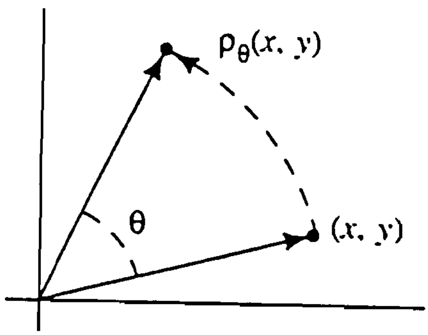


Figure 2.5

Since the inverse of a rotation is the rotation in the opposite direction, $\rho_\theta^{-1}=\rho_{-\theta}$; therefore,

$$A^{-1}=\begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=A^t.$$

It follows that A is an orthogonal matrix and hence that the rotation ρ_θ is an orthogonal transformation of the Euclidean plane $\mathbb{R}^2$.

Now we are going to show that the determinant of an orthogonal transformation is always ± 1, and then use this fact to extend the notion of rotations and reflections of ordinary plane geometry to higher dimensional Euclidean spaces.

Theorem 2.7:

Let σ be an orthogonal transformation of a Euclidean space E. Then $det(\sigma)=\pm 1$.

Proof:

Let $M(\sigma)$ be a matrix representation of the orthogonal transformation σ. Since σ is an orthogonal transformation of E, we have $M(\sigma)^t M(\sigma)=I$. Therefore, $det(M(\sigma)^t M(\sigma))=det(I)$, so $det(M(\sigma)^t)det(M(\sigma))=1$. Hence $det(M(\sigma))^2=1$. It follows that $det(M(\sigma))=\pm 1$. Thus $det(\sigma)=\pm 1$.

Lemma 2.8

Let E be a 2-dimensional real Euclidean space. Let σ be an orthogonal transformation of E. Then there exists an orthonomal basis B of E such that the matrix of σ with respect to the basis B is either one of the following two matrices:

(1) $[\sigma]_B=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, where $0\le\theta\le 2\pi$.

(2) $[\sigma]_B=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

Furthermore, type (1) occurs if and only if $det(\sigma)=1$, and type (2) occurs if and only if $det(\sigma)=-1$.

Proof:

By Theorem 2.6, we can choose an orthonormal basis $\{\beta_1,\beta_2\}$ of E such that the matrix of σ with respect to this basis $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is an orthogonal matrix. Since A is orthogonal, we have $A^tA=AA^t=I$. Thus

$$\begin{array}{ll} a^2+b^2=1 & a^2+c^2=1\\ ac+bd=0 & ab+cd=0\\ c^2+d^2=1 & b^2+d^2=1 \end{array}$$

The equation $a^2+b^2=1$ implies that there is a unique real number θ, with $0\le\theta\le 2\pi$, such that $a=\cos\theta$ and $b=\sin\theta$. If $det(\sigma)=1$, then $ad-bc=1$. Using this equation together with the equations $ac+bd=0$ and $c^2+d^2=1$, we have $c=-b=-\sin\theta$ and $d=a=\cos\theta$. Hence the matrix of σ relative to the base $B=\{\beta_1,\beta_2\}$ is of type (1). Conversely, if the matrix of σ relative to an orthonormal basis B is of type (1) then $det(\sigma)=\cos^2\theta+\sin^2\theta=1$. On the other hand, if $det(\sigma)=-1$ then $ad-bc=-1$. Using equation together with the equations $ac+bd=0$ and $c^2+d^2=1$, we obtain $c=b=\sin\theta$ and $d=-a=-\cos\theta$. Hence the matrix of σ relative to the basis $\{\beta_1,\beta_2\}$ has the form $A=\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}$. Consider the system of linear equations:

$$\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}x \\ y\end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix}\cos\theta-1 & \sin\theta \\ \sin\theta & -(1+\cos\theta)\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

Since $\det\begin{bmatrix}\cos\theta-1 & \sin\theta \\ \sin\theta & -(1+\cos\theta)\end{bmatrix}=-(\cos\theta-1)(1+\cos\theta)-\sin^2\theta$

$$=-(\cos^2\theta-1)-\sin^2\theta=0,$$

the system has a nonzero solution, say $\beta=\begin{bmatrix}x_1 \\ y_1\end{bmatrix}$. By normalizing this solution, we obtain a unit vector, call it $\alpha_1=\frac{\beta}{\|\beta\|}$.

Clearly $\sigma(\alpha_1)=\alpha_1$. Since E is 2-dimensional, we can choose another unit vector, say α_2, such that $B=\{\alpha_1,\alpha_2\}$ is an orthonormal basis of E. Thus the transformation σ can be represented by an orthogonal matrix with respect to the basis

$B=\{a_1,a_2\}$ of the form $\begin{bmatrix}1 & p\\ 0 & q\end{bmatrix}$. $\det\begin{bmatrix}1 & p\\ 0 & q\end{bmatrix}=q$, but $\begin{bmatrix}1 & p\\ 0 & q\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

for some θ. $\det\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}=-1$. Thus $q=-1$. Hence $p=0$.

Therefore, in this case, $[\sigma]_B$ is of type 2. Conversely, if the matrix of σ is type (2) then $det(\sigma)=-1$.

Now we are going to generalize the notions of rotation and reflection in ordinary plane geometry to higher dimensional Euclidean spaces. First, let us examine the 2-dimensional Euclidean plane $\mathbb{R}^2$.

Let $E=\mathbb{R}^2$ be the 2-dimensional ordinary Euclidean plane. Let σ be an orthogonal transformation of E. Then by Lemma 2.8, the matrix of σ with respect to an orthonormal basis $B=\{a_1,a_2\}$ is either of the following form $[\sigma]_B=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ or

$[\sigma]_B=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Observe that in the first case, $det(\sigma)=cos^2\theta+sin^2\theta$

$=1$, and in the second case, $det(\sigma)=-1$. In the first case, if A is a nonzero vector in $\mathbb{R}^2$, we calculate the angle between A

and $\sigma(A)$. If $[A]_B=\begin{bmatrix}x\\y\end{bmatrix}$ then $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\cos\theta-y\sin\theta\\ x\sin\theta+y\cos\theta\end{bmatrix}$. The

angle ϕ between the vectors A and $\sigma(A)$ is given by

$$\cos\phi=\frac{\langle A,\sigma(A)\rangle}{|A||\sigma(A)|}=\frac{\langle A,\sigma(A)\rangle}{|A|^2}.$$

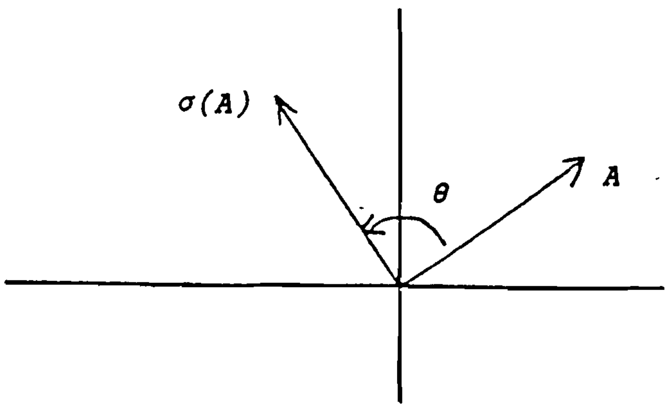


Figure 2.6

$$\langle A,\sigma(A)\rangle=[x\ y]\begin{bmatrix}x\cos\theta-y\sin\theta\\ x\sin\theta+y\cos\theta\end{bmatrix}$$

$$=(x^2\cos\theta-xy\sin\theta)+xy\sin\theta+y^2\cos^2\theta$$

$$=(x^2+y^2)\cos\theta$$

$$=|A|^2\cos\theta.$$

Thus $\cos\phi=\frac{|A|^2\cos\theta}{|A|^2}=\cos\theta$. Hence $\phi=\theta$ if $0\le\theta\le\pi$ and $\theta=2\pi-\phi$ if $\pi\le\theta\le 2\pi$. Thus σ in this case is a counterclockwise rotation of the plane $\mathbb{R}^2$ about the origin through an angle of radian measure θ.

In the second case, σ satisfies the following properties:

1. $\sigma^2=1_{\mathbb{R}^2}$ (the identity transformation of $\mathbb{R}^2$).

2. The line $L=\{r\alpha_1 : r\in\mathbb{R}\}$ is pointwise fixed by σ; that is, $\sigma(x)=x$ for every $x\in L$.

3. Let $A\in\mathbb{R}^2$. Then $A=x\alpha_1+y\alpha_2$, and since $[\sigma]_B\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-y\end{bmatrix}$, then $\sigma(A)=x\alpha_1-y\alpha_2$. Thus σ maps A into its mirror image with respect to the line L; that is, σ is a reflection of $\mathbb{R}^2$ thorough the line L. Note that $\sigma(A)$ can be written as $\sigma(A)=A-2\langle A,\alpha_2\rangle\alpha_2$ for all $A\in\mathbb{R}^2$.

Since rotations of the plane have determinant $+1$ and reflections have determinant -1, we formally define rotations and reflections of n-dimensional real Euclidean space as follows:

Definition 2.4:

Let E be an n-dimensional real Euclidean space. An orthogonal transformation σ of E is called a rotation if $det(\sigma)=+1$ and a reflection if $det(\sigma)=-1$.

Remark:

We caution the reader about the terms rotation and reflection as just defined for arbitrary real Euclidean spaces. In the context of an arbitrary real Euclidean space, they simply express a formal property of an orthogonal transformation and should not mislead the reader into thinking, for example, that a rotation "rotates" the space about an "axis" or that a reflection "reflects" the space through some hyperplane. Let us illustrate this remark with examples.

Example:

Let E^4 be a 4-dimensional real Euclidean space, and let $\sigma:E^4\to E^4$ be the inversion transformation given by $\sigma=-1_{E^4}$. Since $det(\sigma)=(-1)^4=1$, σ is a rotation of E^4. But σ does not "rotate" E^4 about a fixed axis since σ fixes no nonzero vector in E^4.

Example:

Let E^3 be a 3-dimensional Euclidean space and $\sigma=-1_{E^3}$. Since $det(\sigma)=(-1)^3=-1$, σ is a reflection of E^3. But σ does not reflect the space through a hyperplane since σ fixes only the zero vector and hence it does not fix the vectors of the hyperplane. Thus $\sigma=1_{E^3}$ is a reflection in the general sense of the word, but σ is not a hyperplane reflection.

Theorem 2.8:

Let E be an n-dimensional real Euclidean space. Then there exist a rotation and a reflection of E.

Proof:

$\sigma=1_E$, the identity transformation of E, is obviously a rotation. Suppose $B=\{e_1,\ldots,e_n\}$ is an orthonormal basis for E. Let $\sigma:E\to E$ be the linear transformation defined on B by $\sigma(e_1)=-e_1$ and $\sigma(e_i)=e_i$ for $i=2,\ldots,n$. Then $<\sigma(e_i),\sigma(e_i)>=1$ for $i=2,\ldots,n$, and $<\sigma(e_i),\sigma(e_j)>=0$ for $i\neq j$. Hence σ is an isometry and the matrix of σ with respect to B is given by

$$[\sigma]_B=\begin{bmatrix}-1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{bmatrix}$$

Thus $det(\sigma)=-1$ and hence σ is a reflection.

Corollary:

Let E be an n-dimensional Euclidean space. then E has at least n reflections.

Proof:

Let $\sigma_i:E\to E$ be a linear transformation defined on B by $\sigma_i(e_i)=-e_i$ for a fixed $i\in\{1,\ldots,n\}$, and $\sigma_i(e_j)=e_j$ for all $j\neq i$. Then $det(\sigma_i)=-1$ and hence σ_i is a reflection. Since there exist n such linear transformation σ_i's, E has at least n reflections.

Theorem 2.9:

Let E be an n-dimensional Euclidean space. Then

1. If n is even then $\sigma=-1_E$ is a rotation, and if n is odd then σ is a reflection.
2. The product of two rotations is a rotation.
3. The product of two reflections is a rotation.
4. The product of a rotation and a reflection is a reflection.

Proof:

1. Assume n is even. Then $det(\sigma)=det(-1_E)=(-1)^n=1$; therefore, σ is a rotation. If n is odd, then $det(\sigma)=det(-1_E)=(-1)^n=-1$; therefore, σ is a reflection.
2. Let σ and τ be rotations. Then $det(\sigma)=1$ and $det(\tau)=1$; therefore, $det(\sigma\tau)=det(\sigma)det(\tau)=1$, and hence $\sigma\tau$ is a rotation.
3. Let σ and τ be reflections. Then $det(\sigma)=-1$ and $det(\tau)=-1$; therefore, $det(\sigma\tau)=det(\sigma)det(\tau)=1$, and hence $\sigma\tau$ is a rotation.
4. Let σ be a rotation and τ be a reflection. Then $det(\sigma)=1$ and $det(\tau)=-1$; therefore, $det(\sigma\tau)=det(\tau\sigma)=det(\sigma)det(\tau)=-1$, and hence $\sigma\tau$ and $\tau\sigma$ are reflections.

Our next objective is the proof of an important result called the spectral decomposition theorem for orthogonal transformations. This theorem is concerned with the matrix representation of an orthogonal transformation by choosing a suitable orthonormal basis of E.

First, we need to prove some preliminary results which we shall need.

Lemma 2.10:

Let σ be an orthogonal transformation on n-dimensional Euclidean space E.

1. If σ has a real eigenvalue λ, then $\lambda=\pm 1$.
2. Every eigenvalue of σ has absolute value 1.
3. If U is a subspace of E invariant under σ, then $U^\perp$ is invariant under σ.

Proof:

1. Let $\sigma(x)=\lambda x$, where $x\neq 0$. Then $\langle x,x\rangle=\langle\sigma(x),\sigma(x)\rangle=\langle\lambda x,\lambda x\rangle =\lambda^2\langle x,x\rangle$. Thus $(\lambda^2-1)\langle x,x\rangle=0$. Since $x\neq 0$, $\langle x,x\rangle\neq 0$, and hence $\lambda^2-1=0$ or $\lambda=\pm 1$.
2. Let $\sigma(x)=\lambda x$, where $x\neq 0$. Then $\|x\|=\|\sigma(x)\|=\|\lambda x\|=|\lambda|\|x\|$. Thus $|\lambda|=1$.
3. Since U is invariant under σ, then $\sigma(U)=U$ and $\sigma^{-1}(U)=U$. Let $y\in U^\perp$. Then for any $x\in U$, we have $\langle\sigma(y),x\rangle=\langle y,\sigma^{-1}(x)\rangle =0$. Thus $\sigma(y)\in U^\perp$ for every $y\in U^\perp$, and hence $\sigma(U^\perp)=U^\perp$.

Lemma 2.11:

Let σ be an isometry on E. If $\sigma^p(x)=0$ for some $x \in E$ and for some integer $p \geq 1$, then $\sigma(x)=0$.

Proof:

Let q be the smallest integer such that $\sigma^q(x)=0$. Assume that $q>1$ and let $y=\sigma^{q-1}(x)$. Then $y \neq 0$. $\sigma(y)=\sigma(\sigma^{q-1}(x))=\sigma^q(x)=0$. Thus $\|\sigma(y)\|=0$. Since σ is an isometry on E, then $\|\sigma(y)\|=\|y\|=0$ a contradiction since $y \neq 0$. Thus $\sigma(x)=0$.

Lemma 2.12:

Let σ be an isometry on E. If ϕ_1 is an irreducible factor of the minimal polynomial m of σ then

$$m=\phi_1 \cdot \psi,$$

where ϕ_1 and ψ are relatively prime. That is, the factors of m are distinct.

Proof:

Assume that $m=\phi_1^p \cdot \psi$, where ϕ_1 and ψ are relatively prime. We shall show that $p=1$. Since m is the minimal polynomial of σ, then for $x \in E$, we have $m(\sigma)(x)=0$, thus $m(\sigma)(x) =\phi_1^p(\sigma)(\psi(\sigma)(x))=0$. Now we apply Lemma 2.11 to the isometry $\phi_1(\sigma)$ and the vector $\psi(\sigma)(x)$, we obtain $\phi_1(\sigma)(\psi(\sigma)(x))=0$. Since this is the case for any $x \in E$, then $\phi_1(\sigma)\psi(\sigma)=0$. Thus $p=1$.

We are now in a position to prove the spectral decomposition theorem for orthogonal transformations.

Theorem 2.13 (Spectral Theorem):

Let σ be an orthogonal transformation of an n-dimensional real Euclidean space E. Then there exists an orthonormal basis $B=\{\alpha_1,\ldots,\alpha_n\}$ of E with respect to which σ can be represented by a matrix of the form

$$\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & & & & & & \vdots \\
\vdots & & -1 & & & & & & \vdots \\
\vdots & & & \ddots & & & & & \vdots \\
\vdots & & & & \cos\theta_1 & -\sin\theta_1 & & & \vdots \\
\vdots & & & & \sin\theta_1 & \cos\theta_1 & & & \vdots \\
\vdots & & & & & & \ddots & & 0 \\
\vdots & & & & & & & \cos\theta_k & -\sin\theta_k \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \sin\theta_k & \cos\theta_k
\end{bmatrix}$$

where 1 and -1 appear the same number of times as their multiplicities as eigenvalues and $\theta_1,\ldots,\theta_k$, $0\le\theta_j<2\pi$, are such that $\cos\theta_j\pm i\sin\theta_j$, $1\le j<k$, are the distinct eigenvalues of A other than ± 1, each block appearing the same number of times as the multiplicity of $\cos\theta_j+i\sin\theta_j$ as a characteristic root.

Proof:

The proof is by induction on the dimension of E. If $dim(E)=1$, then the only orthogonal transformations of E are $\sigma=\pm 1_E$ and the matrix representation of σ has the form $[\sigma]_B=[\pm 1]$. Assume that the theorem is valid for any orthogonal transformation on a Euclidean space E' with $dim(E')=k$, where $k\geq 1$. Let E be a Euclidean space and $dim(E)=k+1$. Let σ be an orthogonal transformation on E. Since E is a real Euclidean space, the minimal polynomial m of σ has irreducible factors that are either linear or quadratic. That is, m is either the product of factors of the form $x-c$ or $(x-a)^2+b^2$, where $b\neq 0$ and a,b,c are real numbers. Lemma 2.12 implies the factors of m are distinct. We consider the following two cases.

1. m has a linear factor, say $(x-c)$, with c as a real eigenvalue. Lemma 2.10 (1) implies $c=\pm 1$. Let α be an eigenvector corresponding to c. Then $U=\langle\alpha\rangle$ is a 1-dimensional subspace of E and U is invariant under σ. Thus by Lemma 2.10, $U^{\perp}$ is invariant under σ. We have $E=U\oplus U^{\perp}$. Since $dim(U^{\perp})=k$, we may apply the inductive assumption to the restriction of σ to $U^{\perp}$ to obtain an orthonormal basis, say $B'=\{\alpha_1,\ldots,\alpha_k\}$, with respect to which the matrix of $\sigma|_{U^{\perp}}$ is of the form given in the theorem. Let $B=\{\alpha',\alpha_1,\ldots,\alpha_n\}$, where $\alpha'=\alpha/\|\alpha\|$. Then B is an orthonormal basis of E. Then the matrix of σ with respect to the basis B has a block diagonal form

$$[\sigma]_B=\begin{bmatrix}\pm 1 & 0\\ 0 & [\sigma|_{U^{\perp}}]_{B'}\end{bmatrix}.$$

Thus the theorem is valid for $dim(E)=k+1$.

The minimal polynomial m has only quadratic factors of the form $(x-a)^2+b^2$, where $b\neq 0$. Then by Theorem 1.8, there exists an invariant subspace under σ, say U, on which the minimal polynomial of the restriction of σ to U, say σ', is $m'=(x-a)^2+b^2$. Thus σ', the restriction of σ to U, can be expressed in the form $\sigma'=aI+bJ$, where I is the identity transformation and $J=(\sigma'-aI)/b$. Since $(\sigma'-aI)^2+b^2I=0$, we have $J^2=(\sigma'-aI)^2/b^2=(-b^2I)/b^2=-I$. Let β_1 be an arbitrary nonzero vector in U, and set $\beta_2=J(\beta_1)$. Let $W=<\{\beta_1,\beta_2\}>$.

Claim 1: W is a subspace of U invariant under σ'. Clearly W is a subspace of U. To show that W is invariant under σ', let $w\in W$. Then $w=a_1\beta_1+a_2\beta_2$, where $a_1, a_2 \in \mathbb{R}$.
$\sigma'(w)=a_1\sigma'(\beta_1)+a_2\sigma'(\beta_2)=a_1(aI+bJ)(\beta_1)+a_2(aI+bJ)(J(\beta_1))$
$=a_1a(\beta_1)+a_1bJ(\beta_1)+a_2aJ(\beta_1)+a_2bJ^2(\beta_1)=a_1a\beta_1+a_1b\beta_2-a_2a\beta_1-a_2b\beta_1$
$=(a_1a-a_2a-a_2b)\beta_1+(a_1b)\beta_2\in W$. Therefore, W is invariant under σ'.

Claim 2: $\{\beta_1,\beta_2\}$ is linearly independent. Assume the contrary, namely β_1 and β_2 are linearly dependent. Then $\beta_1=a_2\beta_2$ for some nonzero scalar a_2. Thus $J(\beta_1)=J(a_2\beta_2)$ $=a_2J(\beta_2)=a_2J(J(\beta_1))=-a_2\beta_1$. This is equivalent to $\beta_2=-a_2\beta_1$ $\Rightarrow \beta_1/a_2=-a_2\beta_1 \Rightarrow \beta_1+a_2^2\beta_1=0 \Rightarrow (1+a_2^2)\beta_1=0$, a contradiction since $1+a_2^2\neq 0$ and $\beta_1\neq 0$. Hence $\{\beta_1,\beta_2\}$ is linearly independent and $dim(W)=2$.

By Lemma 2.8, there exists an orthonormal basis $\{\alpha_1, \alpha_2\}$ of W with respect to which the matrix of σ' has the form

$$[\sigma']_{B'} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad 0 \le \theta \le 2\pi.$$

(Note that the other matrix form of σ' does not occur since σ' does not have ± 1 as an eigenvalue.) The characteristic polynomial of $[\sigma']_{B'}$ is given by

$$C(x) = \begin{vmatrix} x-\cos\theta & \sin\theta \\ -\sin\theta & x-\cos\theta \end{vmatrix} = (x-\cos\theta)^2 + \sin^2\theta.$$

Claim 3: $\sin\theta \neq 0$. Assume the contrary, namely $\sin\theta = 0$. Then $C(x) = (x-\cos\theta)^2$ and hence σ' has real eigenvalues which contradicts the fact that σ' has no real eigenvalues. Thus $\sin\theta \neq 0$. Therefore, $C(x) = (x-\cos\theta)^2 + \sin^2\theta$ is the minimal polynomial of σ'. Thus $\cos\theta = a$ and $\sin\theta = b$. Since $\dim(W) = 2$ and $E = W^{\perp} \oplus W$, we apply the inductive hypothesis to the restriction of σ to $W^{\perp}$. This implies there exists an orthonormal basis B'' of $W^{\perp}$ with respect to which the matrix of $\sigma|_{W^{\perp}}$ has the form given in the theorem. Then the matrix of σ with respect to the basis $B = B'' \cup B'$ has a block diagonal form

$$[\sigma]_B = \begin{bmatrix} [\sigma|_{W^\perp}]_{B''} & \begin{matrix} & 0 & \end{matrix} \\ 0 & \begin{matrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{matrix} \end{bmatrix}.$$

This completes the proof.

2.2 The Orthogonal Group

Let E be an n dimensional real Euclidean space. The set of nonsingular linear transformations of E forms a group under compositions of maps. This group is one of the "classical groups", and it is called the general linear group, and it is denoted by $GL(E)$. In this section our objective is to study two groups that are subgroups of $GL(E)$.

Theorem 2.14:

Let E be an n-dimensional Euclidean space, and let $O(E)$ be the set of all orthogonal transformations of E. Then $O(E)$ is a group under function composition.

Proof:

1. $O(E) \neq \emptyset$ since $1_E \in O(E)$, and clearly $(O(E), \circ)$ is a mathematical system since the composition of two orthogonal transformations is an orthogonal transformation.
2. $(O(E), \circ)$ is associative since composition of functions is associative.
3. $1_E \in O(E)$ is the identity element.
4. If $\sigma \in O(E)$ then $det(\sigma)=\pm 1$; therefore, σ^{-1} exists, and $det(\sigma^{-1})=\pm 1$; hence, $\sigma^{-1} \in O(E)$.

Definition 2.5:

The group $O(E)$ of orthogonal transformations of E is called the *orthogonal group* of E.

Example 1:

Let us find the orthogonal group of a 1-dimensional Euclidean space E^1. The general linear group $GL(E^1)$ consists of all linear transformations $\sigma_c: E^1 \to E^1$, where c is a nonzero scaler and $\sigma_c(x)=cx$. To find the elements of $GL(E^1)$ that are orthogonal transformations, note that $<\sigma_c(x),\sigma_c(y)>=c^2<x,y>$, thus σ_c is orthogonal if and only if $c^2=1$ or $c=\pm 1$. Hence $O(E^1)=\{-1_{E^1}, 1_{E^1}\}$.

Example 2:

Let E^2 be a 2-dimensional Euclidean plane. We have shown earlier in Lemma 2.8 that if σ is an orthogonal transformation of E^2 then there exists an orthonormal basis B of E such that the matrix of σ relative to B is either $A_1=[\sigma]_B=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

or $A_2=[\sigma]_B=\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}$. $det(A_1)=+1$ and $det(A_2)=-1$; hence, if σ is

a rotation, then its matrix is A_1 and $\sigma=\rho_\theta$, the rotation of the plane through an angle of θ radians about the origin. If, on the other hand, σ is a reflection, then its matrix must be A_2. Thus, $O(E^2)$ consists of the usual rotations about the origin of E^2 and the reflections of E^2 through a line passing through the origin.

Let E be an n-dimensional Euclidean space. We denote the subgroup of rotations of E by $O^+(E)$ and the set of reflections of E by $O^-(E)$.

Theorem 2.15:

Let E be an n-dimensional Euclidean space. Then $O^+(E)$ is a normal subgroup of $O(E)$. Moreover, $O(E)/O^+(E) \cong Z_2$, and therefore the index of $O^+(E)$ in $O(E)$ is 2; that is, $[O(E):O^+(E)]=2$.

Proof:

Consider the map $det:O(E) \to \{+1,-1\}$. If $\sigma, \tau \in O(E)$ then $det(\sigma\tau)=det(\sigma)det(\tau)$; therefore, det is a group homomorphism of $O(E)$ onto $\{+1,-1\}$. By definition, the kernel of det is the set of rotations $O^+(E)$ of E. Therefore, $O^+(E) \triangleleft O(E)$ and $O(E)/O^+(E) \cong Z_2$.

Definition 2.6:

The subgroup $O^+(E)$ of the group $O(E)$ is called the rotation group of E.

Remark:

The set $O^-(E)$ of reflection of E is not a subgroup of $O(E)$. However, since $[O(E):O^+(E)]=2$, the rotation group $O^+(E)$ partition $O(E)$ into two cosets, namely $O^+(E)$ and $O^-(E)$.

Example:

The rotation group of the 2-dimensional Euclidean plane $\mathbb{R}^2$ consists of the rotations ρ_θ through an angle of θ radians about the origin. That is $O^+(\mathbb{R}^2)=\{\rho_\theta \in O(\mathbb{R}^2) | \theta \in \mathbb{R}\}$. Moreover, $O^+(\mathbb{R}^2)$ is isomorphic to the circle group $\mathbb{R}/2\pi Z$. If ρ_θ and $\rho_{\theta'}$ are rotation of $\mathbb{R}^2$, then $\rho_\theta\rho_{\theta'}=\rho_{\theta+\theta'}=\rho_{\theta'}\rho_\theta$; therefore, $O^+(\mathbb{R}^2)$ is abelian. Define a function $f:\mathbb{R}\to O^+(\mathbb{R}^2)$ by $\forall\theta\in\mathbb{R}$, $f(\theta)=\rho_\theta$. Then f is a group homomorphism from the additive group of real numbers into the rotation subgroup $O^+(\mathbb{R}^2)$, and f is onto since every rotation of $\mathbb{R}^2$ has the form ρ_θ for some $\theta\in\mathbb{R}$. Observe that $\rho_\theta=1_{\mathbb{R}^2}$ if and only if $\theta=2\pi k$ for some integer k; therefore, $Ker(f)=2\pi Z$ and hence, by the fundamental theorem of group homomorphism, $O^+(\mathbb{R}^2)\cong\mathbb{R}/2\pi Z$.

The study of the structure and more properties of the orthogonal group and the rotation group of an n-dimensional Euclidean space E will be given in other chapters of this paper.

2.3 Euclidean Motions and the Euclidean Group

In Sections 2.1 and 2.2, we discussed orthogonal transformations and the orthogonal group of a Euclidean space. Orthogonal transformations are the distance-preserving linear transformations of the space. In this section, we are going to discuss the distance-preserving mapping on a Euclidean space. These mappings are called Euclidean transformations or (Euclidean) rigid motions of the space.

First, let us recall definition 2.1: let E be a finite dimensional real Euclidean space. A Euclidean transformation (or a Euclidean motion) of E is a mapping $m:E\to E$ such that $d(A,B)=d(m(A),m(B))$ for all $A,B\in E$, where d is the distance function on E.

Clearly, every orthogonal transformation of E is a Euclidean motion of E. However, there are Euclidean motions that are not linear transformations and hence are not orthogonal transformations of E. For example, the translation of the space E by a nonzero vector A, $T_A:E\to E$, given by $T_A(x)=x+A$ is a Euclidean motion, but T_A is not an orthogonal transformation. However, it follows from Lemma 2.3 that if $m:E\to E$ is a Euclidean motion and $m(0)=0$ then m is an orthogonal transformation.

Let $M(E)$ be the set of Euclidean motions of E. We have shown in Section 2.1, that $M(E)$ is a group under function composition. Moreover, the orthogonal group $O(E)$ is a subgroup of $M(E)$. The group $M(E)$ is called the Euclidean group of E.

Theorem 2.16:

The set of translations of E, $T(E)$ is a normal subgroup of the Euclidean group $M(E)$. The subgroup $T(E)$ is called the translation group of E.

Proof:

We have already shown that $T(E)$ is a subgroup of $M(E)$. We need to show $T(E) \triangleleft M(E)$. Let $T_A \in T(E)$ and $f \in M(E)$. Then $f=T_B\sigma$ for some $\sigma \in O(E)$; therefore, $fT_Af^{-1}=T_B\sigma T_A\sigma^{-1}T_{-B}$. Now, $\sigma T_A\sigma^{-1}=T_{\sigma(A)}$. If $X\in E$, then $(\sigma T_A\sigma^{-1})(X)=\sigma(\sigma^{-1}(X)+A)=X+\sigma(A)=T_{\sigma(A)}(X)$, and hence $\sigma T_A\sigma^{-1}=T_{\sigma(A)}$. It follows that $fT_Af^{-1}=T_B(\sigma T_A\sigma^{-1})T_{-B}=T_BT_{\sigma(A)}T_{-B}=T_{\sigma(A)}$. Hence $fT_Af^{-1}\in T(E)$. Therefore, $T(E)\triangleleft M(E)$.

Corollary 1:

$M(E)=T(E)O(E)$.

Proof:

We know that by Theorem 2.4, every $f\in M(E)$ is of the form $T_A\sigma$ for some $T_A\in T(E)$ and $\sigma\in O(E)$.

Corollary 2:

$M(E)/T(E)\cong O(E)$.

Proof:

Define the function $\alpha: M(E)\to O(E)$ by setting $\alpha(T_A\sigma)=\sigma$ for every $T_A\in T(E)$ and every $\sigma\in O(E)$. Let $T_B\sigma_1$ and $T_C\sigma_2$ be elements of $M(E)$. Then $\alpha(T_B\sigma_1T_C\sigma_2)=\alpha(T_{B+\sigma_1(C)}\sigma_3)=\sigma_3$ for some $\sigma_3=\sigma_1\sigma_2$. Also $\alpha(T_B\sigma_1)\alpha(T_C\sigma_2)=\sigma_1\sigma_2=\sigma_3$; therefore, α is a group-homomorphism mapping $M(E)$ onto $O(E)$ and $Ker(\alpha)=T(E)$. Hence $M(E)/T(E)\cong O(E)$.

The preceding results give us a complete description of the Euclidean motions of a Euclidean space E in terms of the orthogonal transformations of E. They are precisely the orthogonal transformations of the space E followed by any translation of the space.

Let $A\in E$, and let $M(E)_A$ be the set of Euclidean motions that fix the vector A. That is, $M(E)_A=\{m\in M(A) | m(A)=A\}$.

Corollary 3:

1. Let $m\in M(E)_A$. Then there exists $\sigma\in O(E)$ such that $m=T_A\sigma T_A^{-1}$.
2. $M(E)_A=T_AO(E)T_A^{-1}$.
3. $M(E)_A\cong M(E)$.

Proof:

1. Let $\sigma=T_A^{-1}mT_A$. Then σ is a Euclidean motion of E since it is the product of three elements in $M(E)$ and fixes the zero vector since $\sigma(0)=T_A^{-1}mT_A(0)=T_A^{-1}(A)=0$. Therefore, σ is an orthogonal transformation of E. Hence $m=T_A\sigma T_A^{-1}$.
2. Note that $M(E)_A\subseteq T_AO(E)T_A^{-1}$. Since every motion in $T_AO(E)T_A^{-1}$ fixes the vector A, $T_AO(E)T_A^{-1}\subseteq M(E)_A$. Therefore, $M(E)_A=T_AO(E)T_A^{-1}$.

3. Define $\alpha : M(E)_A \to M(E)$ by $\alpha(T_A \sigma T_A^{-1}) = T_A \sigma$. Then α is group homomorphism from $M(E)_A$ onto $M(E)$.

Example:

Let E^1 be a 1-dimensional Euclidean space. We found in Section 2.1, the orthogonal group of E^1, $O(E^1)=\{1_E, -1_E\}$. The translation group of E^1 is given by $T(E^1)=\{T_a(x)=x+a \mid a\in\mathbb{R}\}$, $M(E^1)=O(E^1)T(E^1)=\{\sigma\circ T_a \mid \sigma\in O(E^1), T_a\in T(E^1)\}=\{1_E\circ T_a, -1_E\circ T_a\}$. But $(1_E\circ T_a)(x)=1_E(T_a(x))=1_E(x+a)=x+a=T_a(x)$ and $(-1_E\circ T_a)(x)$ $=-1_E(T_a(x))=-1_E(x+a)=-(x+a)=-T_a(x)$. Thus $M(E^1)=\{\pm T_a \mid a\in\mathbb{R}\}$.

Chapter 3

Symmetries and Cartan's Theorem

The objective of this chapter is to prove Cartan's Theorem and to apply it to the classification of orthogonal (and Euclidean) transformations on *2*- and *3*-dimensional Euclidean spaces.

3.1 Symmetries

In this section, we are going to study a very important class of orthogonal transformations of E, namely the symmetries of E. The symmetries of a Euclidean space E are especially important because, as we will show later in this section, they are the basic building blocks from which all orthogonal transformations are constructed. This result, known as Cartan's Theorem, sates that the symmetries of E generate all the orthogonal transformations of E; that is, every orthogonal transformation of E is a product of a finite number of symmetries.

First, we need to prove the following lemma.

Lemma 3.1:

Let U and W be orthogonal subspaces of E. Let $F=U\oplus W$. Suppose that $\sigma:U\to U$ and $\tau:W\to W$ are orthogonal transformations. Then the map $\rho:F\to F$ defined by $\rho(A+B)=\sigma(A)+\tau(B)$ for $A\in U$ and

$B\in W$, is an orthogonal transformation of F extending both σ and τ. We shall denote ρ by $\rho=\sigma\oplus\tau$.

Proof:

Since σ and τ are both linear transformations, ρ is a linear transformation. We wish to show ρ preserves the inner product of F. Let $X,Y\in F$. Then $X=A_1+B_1$ and $Y=A_2+B_2$ for some $A_1,A_2\in U$ and $B_1,B_2\in W$.

$$\begin{aligned}<\rho(X),\rho(Y)> &= <\rho(A_1+B_1),\rho(A_2+B_2)> \\ &= <\sigma(A_1)+\tau(B_1),\sigma(A_2)+\tau(B_2)> \\ &= <\sigma(A_1),\sigma(A_2)>+<\sigma(A_1),\tau(B_2)> \\ &\quad +<\tau(B_1),\sigma(A_2)>+<\tau(B_1),\tau(B_2)> \\ &= <\sigma(A_1),\sigma(A_2)>+<\tau(B_1),\tau(B_2)> \\ &= <A_1,A_2>+<B_1,B_2> \\ &= <A_1,A_2>+<A_1,B_2>+<B_1,A_2>+<B_1,B_2> \\ &= <A_1+B_1,A_2+B_2> \\ &= <X,Y>.\end{aligned}$$

Therefore, ρ is an orthogonal transformation.

Corollary:

Let U be a subspace of E, and let $\sigma:U\to U$ and $\tau:U^\perp\to U^\perp$ be orthogonal transformations. Then

1. $\sigma\oplus\tau$ is an orthogonal transformation.
2. Let M_σ and M_τ be matrix representations of σ and τ with respect to orthogonal bases B_1 and B_2 of U and $U^\perp$

respectively. Then $M=\begin{bmatrix} M_\sigma & 0 \\ 0 & M_\tau \end{bmatrix}$ is the matrix representation of $\sigma\oplus\tau$ with respect to the basis $B_1 \cup B_2$.

3. $det(\sigma\oplus\tau)=(det(\sigma))(det(\tau))$.

4. $\sigma\oplus\tau$ is a rotation of E if and only if either σ and τ are rotations or σ and τ are reflections.

5. $\sigma\oplus\tau$ is a reflection of E if and only if either σ is a reflection and τ is a rotation or σ is a rotation and τ is a reflection.

Proof:

1. Since σ and τ are orthogonal transformations, $\sigma\oplus\tau$ is a linear transformation. Let $A,B\in E$; then $A=A_U+A_{U^\perp}$ and $B=B_U+B_{U^\perp}$ for some $A_U,B_U\in U$, $A_{U^\perp},B_{U^\perp}\in U^\perp$.

$$\begin{aligned} \langle\sigma\oplus\tau(A),\sigma\oplus\tau(B)\rangle &= \langle\sigma\oplus\tau(A_U+A_{U^\perp}),\sigma\oplus\tau(B_U+B_{U^\perp})\rangle \\ &= \langle\sigma\oplus\tau(A_U)+\sigma\oplus\tau(A_{U^\perp}),\sigma\oplus\tau(B_U)+\sigma\oplus\tau(B_{U^\perp})\rangle \\ &= \langle\sigma(A_U)+\tau(A_{U^\perp}),\sigma(B_U)+\tau(B_{U^\perp})\rangle \\ &= \langle A_U+A_{U^\perp},B_U+B_{U^\perp}\rangle \\ &= \langle A,B\rangle \end{aligned}$$

Therefore, $\sigma\oplus\tau$ is an orthogonal transformation.

2. Since B_1 and B_2 are bases of U and $U^\perp$ respectively, $B=B_1\cup B_2$ is an orthogonal basis of $E=U\oplus U^\perp$. We take B by listing the elements of B_1 and B_2 in succession. In this case the matrix of $\sigma\oplus\tau$ relative to B is $M=\begin{bmatrix} M_\sigma & 0 \\ 0 & M_\tau \end{bmatrix}$.

3. $det(\sigma\oplus\tau)=det(M)$

$$=det(M_\sigma)det(M_\tau)$$

$$=det(\sigma)det(\tau).$$

4. By (3), $det(\sigma\oplus\tau)=1$ if and only if either $det(\sigma)=1$ and $det(\tau)=1$ or $det(\sigma)=-1$ and $det(\tau)=-1$.

5. By (3), $det(\sigma\oplus\tau)=-1$ if and only if either $det(\sigma)=-1$ and $det(\tau)=1$ or $det(\sigma)=1$ and $det(\tau)=-1$.

Let E be an n-dimensional Euclidean space, and let H be a hyperplane in E; that is, H is an $(n-1)$-dimensional subspace of E. Then $H^\perp$ is a 1-dimensional subspace of E and $E=H\oplus H^\perp$.

Definition 3.1:

The orthogonal transformation $S:E\to E$ given by $S=\mathbf{1}_H\oplus-\mathbf{1}_{H^\perp}$ is called the *symmetry* of E with respect to H (or the *hyperplane reflection* of E with respect to H).

Theorem 3.2:

Let S be a symmetry of E with respect to H. Then

1. S is an involution; that is, $S^2=1_E$.
2. S is a reflection of E.
3. S leaves every vector in H fixed and reverses each vector in $H^\perp$; that is, $S(X)=X$ for every $X\in H$ and $S(Y)=-Y$ for every $Y\in H^\perp$.

Proof:

1. Let $X\in E=H\oplus H^{\perp}$. Then $X=Y+Z$ for some $Y\in H$ and $Z\in H^{\perp}$.

$S^2(X)=S^2(Y+Z)$

$=S(1_H(Y)+(-1_{H^{\perp}})(Z))$

$=S(Y+(-Z))$

$=1_H(Y)+1_{H^{\perp}}(-Z)$

$=Y+Z=X.$

Therefore, $S=1_E$.

2. By the above Corollary (3), $det(S)=det(1_H)det(-1_{H^{\perp}})$ $=(1)(-1)=-1$. Therefore, S is a reflection of E.

3. By definition of symmetry $S=1_H\oplus -1_{H^{\perp}}$, it is clear that $S(X)=X$ for every $X\in H$ and $S(Y)=-Y$ for every $Y\in H^{\perp}$.

Theorem 3.3:

Let $\sigma:E\to E$ be an orthogonal transformation leaving a hyperplane H pointwise fixed. Then either $\sigma=1_E$ or σ is a symmetry of E with respect to H.

Proof:

$E=H\oplus H^{\perp}$. Since σ leaves H pointwise fixed, $\sigma(H^{\perp})=H^{\perp}$, and thus $\sigma=1_H\oplus\sigma'$ where σ' is an orthogonal transformation of $H^{\perp}$. But $dim(H^{\perp})=1$. Thus $\sigma=\pm 1_{H^{\perp}}$. Therefore, either $\sigma=1_H\oplus 1_{H^{\perp}}=1_E$ or $\sigma=1_H\oplus -1_{H^{\perp}}$, in which case σ is a symmetry of E with respect to H.

Corollary:

Let σ_1 and σ_2 be both rotations or both reflections of E. Suppose that there is a hyperplane H in E such that $\sigma_1|_H=\sigma_2|_H$. Then $\sigma_1=\sigma_2$.

Proof:

Since $det(\sigma_2^{-1}\sigma_1)=(det\sigma_2^{-1})(det\sigma_1)=(\pm 1)(\pm 1)=1$, then $\sigma_2^{-1}\sigma_1$ is a rotation of E. Since $\sigma_1|_H=\sigma_2|_H$, then for every $A\in H$, $\sigma_1(A)=\sigma_2(A)$ and thus $\sigma_2^{-1}\sigma_1(A)=A$. Hence $\sigma_2^{-1}\sigma_1$ leave the hyperplane H pointwise fixed. By Theorem 3.3, either $\sigma_2^{-1}\sigma_1=1_E$ or $\sigma_2^{-1}\sigma_1$ is a symmetry, but it can not be a symmetry since $\sigma_2^{-1}\sigma_1$ is a rotation. Hence $\sigma_1=\sigma_2$.

Theorem 3.4 (Existence of Symmetries):

Let A be a non-zero vector in E. Then the map $S_A(X)=X-2\frac{\langle X,A\rangle}{\langle A,A\rangle}A$ for every $X\in E$ is a symmetry of E with respect to $H=\langle A\rangle^{\perp}$; that is, $S_A=1_{\langle A\rangle^{\perp}}\oplus(-1_{\langle A\rangle})$.

Proof:

First, we need to show S_A is a linear transformation of E. Let $Y,Z\in E$. Then

$$\begin{aligned}S_A(Y+Z) &= (Y+Z)-2\frac{\langle Y+Z,A\rangle}{\langle A,A\rangle}A\\ &= Y+Z-2\frac{\langle Y,A\rangle+\langle Z,A\rangle}{\langle A,A\rangle}A\\ &= Y-2\frac{\langle Y,A\rangle}{\langle A,A\rangle}A+Z-2\frac{\langle Z,A\rangle}{\langle A,A\rangle}A\\ &= S_A(Y)+S_A(Z).\end{aligned}$$

Let $Y\in E$ and $c\in\mathbb{R}$. Then

$$\begin{aligned} S_A(cY) &= (cY) - 2\frac{\langle cY,A\rangle}{\langle A,A\rangle}A \\ &= c\left(Y - 2\frac{\langle Y,A\rangle}{\langle A,A\rangle}A\right) \\ &= cS_A(Y). \end{aligned}$$

Therefore, S_A is a linear transformation.

Next, we need to show S_A is a symmetry of E. Let $X \in E$. Then $X=Y+Z$ for some $Y \in H=\langle A\rangle^{\perp}$ and $Z \in \langle A\rangle$ since $E=\langle A\rangle^{\perp} \oplus \langle A\rangle$. Note that $A, Z \in \langle A\rangle$; therefore, $Z=cA$ for some $c \in \mathbb{R}$.

$$\begin{aligned} S_A(X) &= S_A(Y+Z) \\ &= S_A(Y) + S_A(Z) \\ &= Y - 2\frac{\langle Y,A\rangle}{A,A}A + Z - 2\frac{\langle Z,A\rangle}{\langle A,A\rangle}A \\ &= Y - 0 + Z - 2\frac{\langle cA,A\rangle}{\langle A,A\rangle}A \\ &= Y + Z - 2\frac{\langle A,A\rangle}{\langle A,A\rangle}cA \\ &= Y + Z - 2Z \\ &= Y - Z. \end{aligned}$$

Hence, $S_A(X)=S_A(Y+Z)=Y-Z$; that is, $S_A = 1_{\langle A\rangle^{\perp}} \oplus (-1_{\langle A\rangle})$.

Remark:

In the previous theorem, we have shown that every non-zero vector A in E gives rise to a symmetry of E, namely S_A. Conversely, every symmetry $\sigma = 1_H \oplus -1_{H^{\perp}}$ of E with respect to the hyperplane H has the form S_B for some non-zero vector B in E.

Given two non-zero vectors A and B in E, a natural question one may ask is under what conditions $S_A=S_B$? To give necessary and sufficient conditions for S_A to equal S_B, we need the following lemma:

Lemma 3.5:

let A be a non-zero vector in E. Then

1. A vector $X \in E$ is left fixed by S_A if and only if $X \in \langle A \rangle^\perp$.
2. A vector $X \in E$ is reversed by S_A; that is, $S_A(X) = -X$ if and only if $X \in \langle A \rangle$.

Proof:

1.$\Leftarrow$ Assume $X \in \langle A \rangle^\perp$, then

$$S_A(X) = X - 2\frac{\langle X,A \rangle}{\langle A,A \rangle}A$$
$$= X \text{ since } \langle X,A \rangle = 0.$$

Therefore, X is left fixed by S_A.

$\Rightarrow$ Assume a non-zero vector $X \in E$ is left fixed by S_A, then $S_A(X) = X$. By definition of S_A, $S_A(X) = X - 2\frac{\langle X,A \rangle}{\langle A,A \rangle}A = X$. Thus

$\frac{\langle X,A \rangle}{\langle A,A \rangle}A = 0$. Since $A \neq 0$, we have $\langle A,A \rangle \neq 0$ and hence

$\langle X,A \rangle = 0$; that is, $X \in \langle A \rangle^\perp$.

2.$\Leftarrow$ Assume $X \in \langle A \rangle$. Then $X = cA$ for some $c \in \mathbb{R}$. Then

$$S_A(X) = X - 2\frac{\langle X,A \rangle}{\langle A,A \rangle}A$$
$$= X - 2\frac{\langle cA,A \rangle}{\langle A,A \rangle}A$$
$$= X - 2c\frac{\langle A,A \rangle}{\langle A,A \rangle}A$$
$$= X - 2cA$$
$$= X - 2X$$
$$= -X$$

Therefore, X is reversed by S_A.

$\Rightarrow$ Assume $X \in E$ is reversed by S_A. Then $S_A(X) = -X$. By definition of S_A, $S_A(X) = X - 2\frac{\langle X,A \rangle}{\langle A,A \rangle}A$.

Therefore, $-2\frac{\langle X,A\rangle}{\langle A,A\rangle}A=-2X$.

That is, $\frac{\langle X,A\rangle}{\langle A,A\rangle}A=X$. Hence $X\in\langle A\rangle$.

Theorem 3.6:

Let A and B be non-zero vector in E. Then $S_A=S_B$ if and only if A and B are linearly dependent.

Proof:

⇒ Assume $A,B\in E$ and $S_A=S_B$; we need to show A and B are linearly dependent. By Lemma 3.5, $S_A(A)=-A$ and since $S_A(X)=S_B(X)$ then $S_B(A)=-A$, and hence $A\in\langle B\rangle$. Therefore, $A=bB$ for some scalar b and hence A and B are linearly dependent.

⇐ Assume that A and B are linearly dependent. Then $\langle A\rangle=\langle B\rangle$ and hence $\langle A\rangle^{\perp}=\langle B\rangle^{\perp}$; therefore, $1_{\langle A\rangle^{\perp}}\oplus(-1_{\langle A\rangle})=1_{\langle B\rangle^{\perp}}\oplus(-1_{\langle B\rangle})$. That is, $S_A=S_B$.

Theorem 3.7:

Let E be a Euclidean space, and let U be a subspace of E. Let A be a non-zero vector in U, and let S_A^{*} and S_A stand for the symmetries of U and E respectively. Then

1. U is a Euclidean space under the restriction of the inner product on E to U.
2. $S_A/_U=S_A^{*}$, where $S_A/_U$ stands for the restriction of S_A to U.

Proof:

1. By definition of subspace, U is a Euclidean space under the restriction of the inner product on E to U.
2. By definition of $S_A/_U$, $S_A/_U = S_A^*$.

Finally we are in a position to proof Cartan's Theorem.

Theorem 3.8 (Cartan's Theorem):

Let E be an n-dimensional Euclidean space. Then every orthogonal transformation of E is the product of at most n symmetries of E.

Proof:

The proof is by induction on the dimension n of E. Let σ be an orthogonal transformation of E. Suppose $n=1$. Then $\sigma=\pm 1_E$. But -1_E is the only symmetry of E. Since $1_E=(-1_E)^0$ and $-1_E=(-1_E)^1$, every orthogonal transformation of E is a product of at most one symmetry.

Now suppose $n > 1$ and assume that every orthogonal transformation of a Euclidean space E of dimension $k < n$ can be written as a product of at most k symmetries of the space. Let σ be an orthogonal transformation of E. There are two cases to consider.

1. Suppose that σ fixes some non-zero vector $A \in E$. Let $H=\langle A \rangle^\perp$ be the hyperplane orthogonal to A. Then H is a Euclidean space of dimension $n-1$ and the restriction $\sigma/_H$ of σ to H is an orthogonal transformation of H. Hence,

by assumption, there are symmetries $S^*_{A_1},\dots,S^*_{A_m}$ of H such that $\sigma|_H=S^*_{A_1}\dots S^*_{A_m}$, where $m\leq n-1$ and $A_1,\dots,A_m\in H$. Let $S_{A_1},\dots,S_{A_m}$ be symmetries of E corresponding to the vectors $A_1,\dots,A_m$. Now we claim: $\sigma=S_{A_1}\dots S_{A_m}$. Let $X\in E$. Since $E=A\oplus H$ then $X=X_A+X_H$, where $X_A\in\langle A\rangle$ and $X_H\in H$. Since $X_A\in\langle A\rangle$, $X_A=cA$ for some $c\in\mathbb{R}$, and hence $\sigma(X_A)=\sigma(cA)=A$. Also $\sigma(X_H)=(\sigma|_H)(X_H)=S^*_{A_1}\dots S^*_{A_m}(X_H)$. But $S_{A_i}|_H=S^*_{A_i}$ and $S_{A_i}(X_A)=X_A$ since $X_A\perp A_i$. Therefore,

$$\begin{aligned}\sigma(X)&=\sigma(X_A)+\sigma(X_H)\\&=X_A+S^*_{A_1}\dots S^*_{A_m}(X_H)\\&=X_A+S_{A_1}\dots S_{A_m}(X_H)\\&=S_{A_1}\dots S_{A_m}(X_A+X_H)\\&=S_{A_1}\dots S_{A_m}(X)\ .\end{aligned}$$

Thus $\sigma=S_{A_1}\dots S_{A_m}$; that is, σ is a product of at most $n-1$ symmetries of E.

2. Suppose that σ fixes no vector in E other than the zero vector. Let A be any non-zero vector in E, and let $B=\sigma(A)-A$. Then $B\neq 0$.

$$\begin{aligned}S_B(\sigma(A))&=\sigma(A)-2\frac{\langle\sigma(A),B\rangle}{\langle B,B\rangle}B\\&=\sigma(A)-2\frac{\langle\sigma(A),\sigma(A)-A\rangle}{\langle\sigma(A)-A,\sigma(A)-A\rangle}(\sigma(A)-A)\\&=\sigma(A)-2\frac{\langle\sigma(A),\sigma(A)\rangle-\langle\sigma(A),A\rangle}{\langle\sigma(A),\sigma(A)\rangle-\langle\sigma(A),A\rangle-\langle A,\sigma(A)\rangle+\langle A,A\rangle}(\sigma(A)-A)\\&=\sigma(A)-\frac{2(\langle A,A\rangle-\langle\sigma(A),A\rangle)}{2(\langle A,A\rangle-\langle\sigma(A),A\rangle)}(\sigma(A)-A)\\&=A.\end{aligned}$$

Therefore, $S_B\sigma$ is an orthogonal transformation of E that fixes the non-zero vector A and hence, by part (1), there exist symmetries $S_{A_1}, \ldots, S_{A_m}$ such that $S_B\sigma = S_{A_1} \ldots S_{A_m}$, where $m \leq n-1$. It follows that $\sigma = S_B^{-1} S_{A_1} \ldots S_{A_m}$. That is, σ is the product of at most n symmetries.

Corollary:

The orthogonal group $O(E)$ of E is finitely generated by the symmetries of E.

Proof:

Since every $\sigma \in O(E)$ is the product of at most n symmetries, $O(E)$ is finitely generated by the symmetries of E.

3.2 The Classification of Orthogonal Transformation in Dimensions 2 and 3:

In this section, we apply the theorems we have established to classify the orthogonal transformations of a *2*-dimensional Euclidean plane as well as a *3*-dimensional Euclidean space. Above all, we want to identify those orthogonal transformations which are rotations, and those which are reflections (in a point, in a line, or in a plane). We are also interested in the question of whether these exhaust the set of orthogonal transformations. We shall see that in a *2*-dimensional Euclidean plane, an orthogonal transformation is either a rotation or a reflection in a line; however in a *3*-dimensional Euclidean space there are orthogonal transformations which are neither rotations nor reflections in a hyperplane.

I. *2*-Dimensional Euclidean Plane:

Let E be a *2*-dimensional Euclidean space (or plane). By Cartan's Theorem, every orthogonal transformation of E is either 1_E, a symmetry, or the products of two symmetries. The next theorem gives a complete description of all the orthogonal transformations of E.

Theorem 3.9:

Let E be a 2-dimensional Euclidean space. Let σ be an orthogonal transformation of E. Then

1. σ is a reflection if and only if σ is a symmetry.
2. σ is a reflection (hence symmetry) if and only if $\sigma \neq 1_E$ and σ has a non-zero fixed vector in E.
3. A reflection is completely determined by the image of one non-zero vector.
4. σ is a rotation of E if and only if σ is the product of two symmetries of E in which case the first of these can be chosen arbitrarily.
5. σ is a rotation if and only if $\sigma = 1_E$ or σ has no non-zero fixed vector in E.
6. A rotation is completely determined by the image of one non-zero vector in E.

Proof:

Note that a hyperplane of E is a line through the origin since E is a 2-dimensional Euclidean space.

1. If σ is a symmetry of E then by Theorem 3.2, σ is a reflection. Assume that σ is a reflection. By Cartan's Theorem, σ is a product of 1 or 2 symmetries. If σ is a product of two symmetries, say $\sigma=\tau_1\tau_2$ then $det(\sigma) = det(\tau_1\tau_2) = det(\tau_1)det(\tau_2) = (-1)(-1) = 1$, and this is not possible. Hence σ must be a symmetry.
2. If σ is a reflection (hence by (1) a symmetry) then $\sigma \neq 1_E$, and by Lemma 3.5 (1), σ leaves a line in E pointwise

fixed and hence has a nonzero fixed vector in E. Conversely, if $\sigma \neq 1_E$ and σ has a nonzero fixed vector $A \in E$, then σ leaves the line $\langle A \rangle$ pointwise fixed. Thus by Theorem 3.3, either $\sigma = 1_E$ or σ is a symmetry. But $\sigma \neq 1_E$ by assumption, σ must be a symmetry of E.

3. Let σ_1 and σ_2 be reflections of E such that $\sigma_1(A) = \sigma_2(A)$ for some nonzero vector $A \in E$. Then $\sigma_1|_{\langle A \rangle^\perp} = \sigma_2|_{\langle A \rangle^\perp}$, and hence by the corollary to Theorem 3.3, $\sigma_1 = \sigma_2$.

4. If $\sigma = \tau_1\tau_2$ where τ_1 and τ_2 are symmetries, then $det(\sigma) = det(\tau_1\tau_2) = (det\tau_1)(det\tau_2) = (-1)(-1) = 1$, and hence σ is a rotation. Conversely, assume σ is a rotation. If $\sigma = 1_E$ then $\sigma = \tau\tau$ for every symmetry τ. If $\sigma \neq 1_E$ then by Cartan's Theorem, σ is the product of one or two symmetries. But σ cannot be the product of one symmetry. Thus σ must be the product of two symmetries. It remains to show one of these two symmetries can be chosen arbitrarily. Let σ be a rotation and let τ be any symmetry. Then $det(\sigma\tau) = (det\sigma)(det\tau) = (1)(-1) = -1$. Thus $\sigma\tau$ is a reflection. By part (1), $\sigma\tau$ is a symmetry. Moreover, $\sigma = (\sigma\tau)\tau$ for any symmetry τ.

5. Assume σ is a rotation and $\sigma(A) = A$ for some nonzero vector $A \in E$. By part (4), σ is a product of two symmetries. If $\sigma = \tau\tau$ then $\sigma = 1_E$, and if $\sigma = \tau\tau'$, where $\tau \neq \tau'$, then $\sigma(A) = A$ implies $\tau\tau'(A) = A$, and thus $\tau(\tau\tau'(A)) = \tau(A)$. Hence, $\tau'(A) = \tau(A)$ and by part (3) above, this implies $\tau = \tau'$, a contradiction. Conversely, if $\sigma = 1_E$ then σ is a rotation.

Assume $\sigma \neq 1_E$ has no nonzero fixed vector in E. By Cartan's Theorem, σ is the product of one or two symmetries. But every symmetry has a nonzero fixed vector. Thus σ is the product of two symmetries, and hence σ is a rotation.

6. The proof of this part is similar to the proof of part (3) above.

Theorem 3.10:

Let σ be a rotation, and let τ be a reflection of a Euclidean plane E. Then $\tau\sigma\tau^{-1}=\sigma^{-1}$.

Proof:

$\tau\sigma$ is a reflection and hence by Theorem 3.9 (1), is a symmetry. Thus by Theorem 3.2 (1), $\tau\sigma$ is an involution, and hence $(\tau\sigma)^{-1}=\tau\sigma$. On the other hand, $(\tau\sigma)^{-1}=\sigma^{-1}\tau^{-1}=\sigma^{-1}\tau$. So $\sigma^{-1}\tau=\tau\sigma$ or $\sigma^{-1}=\tau\sigma\tau^{-1}$.

Corollary:

The plane rotation group $O^+(E)$ of a 2-dimensional Euclidean space is commutative.

Proof:

Let τ be a reflection of E. Since $O^+(E)$ is a normal subgroup of $O(E)$, the mapping $\varphi: O^+(E) \to O^+(E)$ given by $\varphi(\sigma)=\tau\sigma\tau^{-1}$ is an automorphism. By Theorem 3.10, $\varphi(\sigma)=\tau\sigma\tau^{-1}=\sigma^{-1}$. Now let $\sigma_1, \sigma_2 \in O^+(E)$. Then $\varphi(\sigma_1\sigma_2)=(\sigma_1\sigma_2)^{-1}=\sigma_2^{-1}\sigma_1^{-1}$, and on the other hand, $\varphi(\sigma_1)\varphi(\sigma_2)=\sigma_1^{-1}\sigma_2^{-1}$. Thus $\sigma_2^{-1}\sigma_1^{-1}=\sigma_1^{-1}\sigma_2^{-1}$, and hence $\sigma_1\sigma_2=\sigma_2\sigma_1$.

II. *3*-Dimensional Euclidean Spaces:

By Cartan's Theorem, every orthogonal transformation of a *3*-dimensional Euclidean space E is either a symmetry of E, or a product of two or three symmetries of E. Since a product of two symmetries is a rotation, a rotation of E is either 1_E or the product of exactly two symmetries. Hence a reflection of E is either a symmetry or the product of three symmetries and not less than three symmetries. Thus there are two types of reflections in *3*-dimension Euclidean spaces: those that are symmetries and those that are product of three symmetries. For example, $\sigma=-1_E$ is a reflection of the latter type since it fixes no vector except the zero vector, and hence σ cannot be a symmetry of E. Thus we are led to study another class of orthogonal transformations, namely the involutions of E.

Definition 3.2:

Let E be an n-dimensional Euclidean space. An orthogonal transformation σ of E is called an *involution* if $\sigma^2=1_E$.

We have already worked with examples of involutions, namely the symmetries of E, 1_E and -1_E. A question one may ask: are these all the involutions? As we are going to show this is not the case if $n\geq 3$.

Theorem 3.11:

Let U and W be subspaces of E and $E=U\oplus W$. Let $\sigma=-1_U\oplus 1_W$. Then σ is an involution.

Proof:

Let $a\in E$, then $a=b+c$, $b\in U$ and $c\in W$. $\sigma^2(a)=\sigma(\sigma(a)) =\sigma(\sigma(b+c))=\sigma(-b+c)=b+c=a$; therefore, $\sigma^2=1_E$. Hence σ is an involution.

Thus it follows from this theorem that every orthogonal splitting of E gives rise to an involution. Now we want to show that different orthogonal splittings give rise to different involutions.

Theorem 3.12:

Let E be an n-dimensional Euclidean space, and let $\sigma=-1_U\oplus 1_W$ and $\sigma=-1_{U'}\oplus 1_{W'}$. Then $U=U'$ and $W=W'$.

Proof:

Since $\sigma=-1_U\oplus 1_W$, then $U=\{A\in E|\sigma(A)=-A\}$ and $W=\{A\in E|\sigma(A)=A\}$. Also since $\sigma=-1_{U'}\oplus 1_{W'}$, $U'=\{A\in E|\sigma(A)=-A\}$ and $W'=\{A\in E|\sigma(A)=A\}$. Thus $U=U'$ and $W=W'$.

The question now, does every involution of E come from an orthogonal splitting of E? The answer to this question is yes and is given in the following theorem.

Theorem 3.13:

Every involution of E comes from an orthogonal splitting of E.

Proof:

Let σ be an involution of E. Then $(\sigma-1_E):E\to E$ and $(\sigma+1_E):E\to E$ are linear transformations.

Let $U=Im(\sigma-1_E)=\{A\in E|A=(\sigma-1_E)(B) \text{ for some } B\in E\}$, and let $W=Im(\sigma+1_E)=\{A\in E|A=(\sigma+1_E)(B) \text{ for some } B\in E\}$. Then U and W are subspaces of E.

Claim: σ is the involution coming from U and W.

To prove this claim we need to show:

a. U and W are orthogonal; that is, $\forall A_1\in U, A_2\in W, \langle A_1,A_2\rangle=0$.

b. $U\cap W=\{0\}$.

c. $E=U+W$.

d. $\sigma=-1_{U^\perp}\oplus 1_W$.

a. Let $A_1\in U$ and $A_2\in W$. Then $A_1=(\sigma-1_E)(B_1)$ for some $B_1\in E$ and $A_2=(\sigma+1_E)(B_2)$ for some $B_2\in E$.

$$\begin{aligned}\langle A_1,A_2\rangle&=\langle(\sigma-1_E)(B_1),(\sigma+1_E)(B_2)\rangle\\&=\langle\sigma(B_1)-B_1,\sigma(B_2)+B_2\rangle\\&=\langle\sigma(B_1),\sigma(B_2)\rangle+\langle\sigma(B_1),B_2\rangle+\langle -B_1,\sigma(B_2)\rangle+\langle -B_1,B_2\rangle\\&=\langle B_1,B_2\rangle+\langle\sigma(B_1),\sigma(\sigma(B_2))\rangle-\langle B_1,\sigma(B_2)\rangle-\langle B_1,B_2\rangle\\&=\langle B_1,B_2\rangle+\langle B_1,\sigma(B_2)\rangle-\langle B_1,\sigma(B_2)\rangle-\langle B_1,B_2\rangle\\&=0.\end{aligned}$$

Therefore, U and W are orthogonal.

b. To show $U\cap W=\{0\}$, let $x\in U\cap W$. Then $x\in U$ and $x\in W$. Hence there exists $y\in E$ such that $x=(\sigma-1_E)(y)$. Then $x=\sigma(y)-y$.

Also there exists $z \in E$ such that $x=(\sigma+1_E)(z)$. Then $x=\sigma(z)+z$. Hence $\sigma(y)-y=\sigma(z)+z$; then $\sigma^2(y)-\sigma(y) =\sigma^2(z)+\sigma(z)$. Thus $y-\sigma(y)=z+\sigma(z)$. Then we have $\sigma(y)-y =y-\sigma(y)$. Hence $\sigma(y)=y$ or $x=\sigma(y)-y=0$.

c. To show $E=U+W$, let $x \in E$. Then $(\sigma-1_E)(-x/2)=-\sigma(x)/2+x/2 \in U$, and $(\sigma+1_E)(x/2)=\sigma(x)/2+x/2 \in W$; therefore, $x=(\sigma-1_E)(-x/2)+(\sigma+1_E)(x/2)$; hence $E=U+W$.

d. To show $\sigma=-1_{U^\perp}\oplus 1_W$, let $X \in E$. Then $X=u+w$ for some $u \in U$ and $w \in W$. Also $u=\sigma(A)-A$ and $w=\sigma(B)+B$, where $A,B \in E$. Thus $X=(\sigma(A)-A)+(\sigma(B)+B)$. Then

$$\begin{aligned}
\sigma(X)&=\sigma(\sigma(A)-A+\sigma(B)+B)\\
&=\sigma^2(A)-\sigma(A)+\sigma^2(B)+\sigma(B)\\
&=A-\sigma(A)+B+\sigma(B)\\
&=-(\sigma(A)-A)+(\sigma(B)+B)\\
&=-1_{U^\perp}(\sigma(A)-A)+1_W(\sigma(B)+B)\\
&=(-1_{U^\perp}\oplus 1_W)((\sigma(A)-A)+(\sigma(B)+B))\\
&=(-1_{U^\perp}\oplus 1_W)(X).
\end{aligned}$$

Hence $\sigma=-1_{U^\perp}\oplus 1_W$.

Definition 3.3:

Let σ be an involution of E. Then $\sigma=-1_U\oplus 1_W$, where $E=U\oplus W$. We define the *type of* σ to be the dimension of U. That is, the *type* of σ is the dimension of the space of vectors inverted by σ, namely $U=\{A \in E \mid \sigma(A)=-A\}$. An involution of *type 2* is also called a *180°* rotation.

Theorem 3.14:

If σ is an involution of E of *type p*, then if p is even σ is a rotation and if p is odd σ is a reflection.

Proof:

$$det(\sigma)=det(-1_U)det(1_W)$$

$$=(-1)^p=\begin{cases}1 \text{ if } p \text{ is even}\\ -1 \text{ if } p \text{ is odd.}\end{cases}$$

Theorem 3.15:

There is only one involution of *type 0*, namely 1_E, and there is only one involution of *type n*, namely -1_E. The involutions of *type 1* are the symmetries of E.

Proof:

Let σ be an involution of *type 0*. Then $\sigma=-1_U\oplus 1_W$ where $E=U\oplus W$ and $dim(U)=0$. Hence $E=W$; therefore, $\sigma=1_W=1_E$. If σ' is an involution of *type n*, then $\sigma=-1_U\oplus 1_W$, where $E=U\oplus W$ and $dim(U)=n$. Hence $dim(W)=0$ and $E=U$; therefore, $\sigma=-1_U=-1_E$.

Now we begin the investigation of the orthogonal transformations of the *3*-dimensional Euclidean space E.

A. Rotations in *3*-Dimensional Euclidean Spaces

In the three-dimensional Euclidean space $\mathbb{R}^3$, we are accustomed to the fact that rotations different from $1_{\mathbb{R}^3}$ rotate the space around an axis of rotation. In fact, it turns out this is true for rotations of *3*-dimensional Euclidean spaces. This result is due to Euler, and before we prove it, we need the following lemma.

Lemma 3.16:

Let E be an n-dimensional Euclidean space, and let $\sigma:E\to E$ be an orthogonal transformation. Let $F=\{x\in E|\sigma(x)=x\}$ be the fixed space of σ. Then σ cannot be factored as a product of less than s symmetries, where $s=dim(F^{\perp})=dim(E)-dim(F)$.

Proof:

Let $\sigma=\tau_1\tau_2\ldots\tau_r$ where τ_i is the symmetry of E with respect to the hyperplane H_i. We have to show that $r\geq s$. Since τ_i is a symmetry then τ_i leaves H_i fixed pointwise for $i=1,2,\ldots,n$. Thus $\sigma=\tau_1\tau_2\ldots\tau_r$ leaves $(H_1\cap H_2\cap\ldots\cap H_r)$ pointwise fixed. Thus $(H_1\cap H_2\cap\ldots\cap H_r)\subseteq F$. Hence $dim(H_1\cap H_2\cap\ldots\cap H_r)\leq dim(F)=n-s$. Since $dim(H_1+H_2+\ldots+H_r)=dim(H_1)+dim(H_2)+\ldots+dim(H_r)-dim(H_1\cap H_2\cap\ldots\cap H_r)$, and $dim(H_1+H_2+\ldots+H_r)\leq n$ and $dim(H_i)=n-1$ for $i=1,2,\ldots,r$ then $n\geq(n-1)+\ldots+(n-1)-dim(H_1\cap H_2\cap\ldots\cap H_r)$ or $dim(H_1\cap H_2\cap\ldots\cap H_r)$ $\geq r(n-1)-n\geq n-r$. Therefore, $n-s\geq dim(H_1\cap H_2\cap\ldots\cap H_r)\geq n-r$ and hence $n-s\geq n-r$ or $s\leq r$.

Corollary 1:

If σ leaves no nonzero vector fixed, then σ is the product of exactly n symmetries.

Proof:

Since σ leaves no nonzero vector fixed, the fixed space of σ is $F=\{0\}$. Thus $F^{\perp}=E$ and hence $dim(E)=n=dim(F^{\perp})=s$. By the theorem, σ cannot be factored into less than n symmetries; that is, σ is the product of at least n symmetries. From Cartan's Theorem, we know σ is the product of at most n symmetries. Therefore, σ is the product of exactly n symmetries.

Corollary 2:

If the dimension n of E is odd, then every rotation ρ of E has a nonzero fixed vector. If the dimension n of E is even, then every reflection σ of E has a nonzero fixed vector.

Proof:

Let ρ be a rotation of E. By Cartan's Theorem, $\rho=\tau_1\tau_2...\tau_s$, where τ_i is a symmetry for $i=1,2,...s$ and $s\leq n$. Then $det(\rho)=1=(-1)^s$. Hence s is even. Since s is even and n is odd then $s\neq n$. By the theorem, the dimension of the fixed space of ρ is at least $n-s>0$, and therefore, ρ has a nonzero fixed vector. The proof of the second part of the corollary is similar to the argument used above.

Now we present a proof of Euler's Theorem.

Theorem 3.17 (Euler):

Let σ be a rotation of a 3-dimensional Euclidean space E and $\sigma \neq 1_E$. Then σ leaves one and only one line L in E pointwise fixed. This unique line L is called the axis of rotation for σ.

Proof:

By Corollary (2) above, σ has a nonzero fixed vector, say x. Let $L=\langle x\rangle=\{ax | a\in\mathbb{R}\}$. Then $\sigma(ax)=a\sigma(x)=ax$. Thus σ leaves L fixed pointwise. Now we are going to show L is unique. Let $y\notin L$. We must show $\sigma(y)\neq y$. Assume the contrary, namely $\sigma(y)=y$. Then σ would leave the plane $H=\langle x,y\rangle$ pointwise fixed. But as a consequence of Theorem 3.3, the only rotation leaving a hyperplane pointwise fixed is 1_E. This contradicts our assumption that $\sigma\neq 1_E$. So $\sigma(y)\neq y$. This completes the proof.

Let E be a 3-dimensional Euclidean space, and let L be any line in E. Let $O(E;L)$ be the set of orthogonal transformations of E that leave L pointwise fixed. That is, $O(E;L) =\{\sigma\in O(E) | \sigma(x)=x, \forall x\in L\}$. Let $O^{+}(E;L)$ be the set of rotations of E that leave L pointwise fixed.

Theorem 3.18:

$O(E;L)$ is a subgroup of $O(E)$, and $O^{+}(E;L)$ is a subgroup of $O^{+}(E)$.

Proof:

First, $O(E;L)\neq\emptyset$ since $1_E\in O(E;L)$, and also $O(E;L)\subseteq O(E)$. We need to show that if $\sigma,\tau\in O(E;L)$ then $\sigma\tau\in O(E;L)$. Let $x\in L$, then $\sigma\tau(x)=\sigma(\tau(x))=x$. Thus $\sigma\tau$ leaves L pointwise fixed. Hence $\sigma\tau\in O(E;L)$. Also since $\sigma(x)=x$, $\sigma^{-1}(\sigma(x))=\sigma^{-1}(x)$. Hence $x=\sigma^{-1}(x)$, and thus $\sigma^{-1}\in O(E;L)$. Therefore, $O(E;L)\leq O(E)$. Similarly, we can show that $O^{+}(E;L)\leq O^{+}(E)$.

Let $L^{\perp}$ be the orthogonal complement of L. Then $dim(L^{\perp})=2$ and $E=L\oplus L^{\perp}$. Suppose that $\sigma\in O(E;L)$. Since $\sigma(L)=L$, then by Lemma 2.8, $\sigma(L^{\perp})=L^{\perp}$, and hence $\sigma=1_L\oplus\sigma'$, where σ' is the restriction of σ to the hyperplane $L^{\perp}$. So we have a mapping $\phi:O(E;L)\to O(L^{\perp})$ given by $\phi(\sigma)=\sigma'=\sigma|_{L^{\perp}}$.

Theorem 3.19:

The mapping $\phi:O(E;L)\to O(L^{\perp})$ is an isomorphism carrying the subgroup $O^{+}(E;L)$ onto the rotation group $O^{+}(L^{\perp})$ of the 2-dimensional Euclidean plane $L^{\perp}$.

Proof:

Let $\sigma,\tau\in O(E;L)$, and let $x\in E$. Then $x=a+b$, where $a\in L$ and $b\in L^{\perp}$. So $(\sigma\tau)(x)=(\sigma\tau)(a+b)=\sigma(\tau(a)+\tau(b))=\sigma(a+\tau'(b))$ $=\sigma(a)+\sigma\tau'(b)=a+\sigma'\tau'(b)$. This implies $(\sigma\tau)|_{L^{\perp}}=\sigma'\tau'$, hence $\phi(\sigma\tau)=\sigma'\tau'=\phi(\sigma)\phi(\tau)$; therefore, ϕ is a homomorphism. To show ϕ is one-to-one, assume $\phi(\sigma)=\phi(\tau)$. Then $\sigma|_{L^{\perp}}=\tau|_{L^{\perp}}$. Hence by the Corollary to Theorem 3.3, $\sigma=\tau$. To show ϕ is onto, let

$\sigma' \in O(L^\perp)$ and $\sigma = 1_L \oplus \sigma'$. Then $\sigma' = \sigma|_{L^\perp} = \phi(\sigma) \in O(E;L)$. Therefore, ϕ is an isomorphism.

From this theorem, it follows that for any line L in $\mathbb{R}^3$, the group $O(\mathbb{R}^3;L) \cong O(\mathbb{R}^2)$ and $O^+(\mathbb{R}^3;L) \cong O^+(\mathbb{R}^2)$. That is, the structure of the orthogonal group and the rotation group of $\mathbb{R}^3$ does not depend on the line L. In particular if $O^+(\mathbb{R}^3;L)$ and $O^+(\mathbb{R}^3;L')$ are the rotation groups of $\mathbb{R}^3$ about the axes L and L' respectively, then $O^+(\mathbb{R}^3;L) \cong O^+(\mathbb{R}^3;L')$. Thus in studying the rotation group of $\mathbb{R}^3$, it will be assumed that a rotation $\rho_\theta \in O^+(\mathbb{R}^3)$ is about a line passing through the origin through an angle θ radians. In this case the rotation ρ_θ has a matrix representation of the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Thus any rotation in $\mathbb{R}^3$ is completely determined by its axis of rotation L and the angle of rotation around the axis L. The phrase "angle of rotation around the axis L" could be replaced by the phrase "the rotation of the plane $L^\perp$ induced by σ".

B. Reflections in 3-Space

We now turn to the investigation of reflections in a 3-dimensional Euclidean space E.

Theorem 3.20:

An orthogonal transformation of E leaves a nonzero vector fixed if and only if $\sigma=1_E$, σ is a symmetry or σ is the product of two symmetries. That is, σ is a rotation or σ is a symmetry.

Proof:

Let σ be an orthogonal transformation that leaves a nonzero vector $A \in E$ fixed. We need to show σ is the product of two or less symmetries. Let $L=\{tA / t \in \mathbb{R}\}$, then σ fixes L pointwise since $\sigma(tA)=t\sigma(A)=tA$. Thus $\sigma \in O(E;L)$. Since $O(E;L) \cong O(L^\perp)$, the image of σ in the orthogonal group of the plane $L^\perp$ is the product of at most two symmetries (by Cartan's Theorem). Thus σ is the product of two or less symmetries.

Corollary 1:

An orthogonal transformation σ of E leaves only the origin fixed if and only if σ is the product of exactly 3 symmetries.

Corollary 2:

The reflections of E consist of the symmetries of E, together with the orthogonal transformations of E that leave only the origin fixed.

According to corollary 2 above, the problem of characterizing the reflections of E is reduced to determining the orthogonal transformations of E which leave only the origin fixed. These are given in the next lemma.

Lemma 3.21:

The reflections of E which leave only the origin fixed are given by: *$S=\{-\rho / \rho \in O^{+}(E)$, and ρ inverts no nonzero vectors$\}$.*

Proof:

Let ρ be a rotation of E which inverts no nonzero vectors. There are two cases to consider.

Case 1: $\rho=1_E$.

Then $-\rho=-1_E$ is a reflection of E which leaves only the origin fixed.

Case 2: $\rho\neq 1_E$.

By Euler's Theorem 3.17, there exists a nonzero vector A fixed by ρ. Then $-\rho$ reverses the vector A. Now we proceed by contradiction. Let us assume that there is another nonzero vector B such that $-\rho(B)=B$. Clearly the vectors A and B are linearly independent. Moreover, ρ^2 is a rotation which fixes every vector of the hyperplane $H=\langle A,B\rangle$. Thus by Theorem 3.3, ρ^2 is either 1_E or a symmetry. But since $\rho\neq 1_E$, then ρ^2 is a symmetry, a contradiction. Conversely assume that σ is a reflection which leaves only the origin fixed. We must show $\sigma=-\rho$, where $\rho\in O^{+}(E)$ and $\rho(A)\neq -A$ for any nonzero vector $A\in E$. Clearly $\rho=-\sigma$ is a rotation. Moreover, if $\rho(A)=(-\sigma)(A)=-A$

for a nonzero vector $A \in E$, then $\sigma(A)=-1((-\sigma)(A))=-1(-A)=A$. Thus σ fixes a nonzero vector, a contradiction.

Now we must find the rotations of E which invert no nonzero vectors.

Lemma 3.22:

The involutions of a 3-dimensional Euclidean space E consist of $1_E, -1_E$, the symmetries of E, and the 180° rotations.

Proof:

Let E be a 3-dimensional Euclidean space, and let σ be an involution of E. Then $\sigma=-1_U \oplus 1_W$, where U and W are subspaces of E and $E=U \oplus W$. The possible dimensions of the subspaces U and W are $dim(U)=3$ and $dim(W)=0$, in which case, $\sigma=-1_E$; or $dim(U)=2$ and $dim(W)=1$, in which case σ is an involution of *type 2*, that is a 180° rotation; or $dim(U)=1$ and $dim(W)=2$, in which case, σ is a symmetry; or finally, $dim(U)=0$ and $dim(W)=3$, in which case, $\sigma=1_E$.

The only involutions of E of *type 0* and *type 3* are 1_E and -1_E respectively. The next theorem describe the involutions of *type 1* and *type 2*.

Lemma 3.23:

Let σ be an orthogonal transformation of the 3-dimensional Euclidean space E. Then σ leaves a nonzero vector

A in E fixed and reverse a nonzero vector B in E if and only if σ is an involution of *type 1* or *type 2*.

Proof:

Let σ be an involution of *type 1*. Then σ is a symmetry. By Theorem 3.2, σ leaves a nonzero vector fixed and reverses a nonzero vector. If σ is an involution of *type 2*, then $\sigma=-1_U \oplus 1_W$ with $dim(U)=2$. Thus σ reverses the vectors in U and fixes the vectors in W. Conversely, assume that σ is an orthogonal transformation of E such that $\sigma(A)=A$ and $\sigma(B)=-B$ for some nonzero vectors $A,B \in E$. Then $\sigma^2(A)=\sigma(\sigma(A))=\sigma(A)=A$, and $\sigma^2(B)=\sigma(\sigma(B))=\sigma(-B)=-\sigma(B)=-(-B)=B$; therefore, σ^2 leaves the two vectors A and B fixed. But A and B must be linearly independent because if $A=\alpha B$ for some nonzero scalar α, then $\sigma(A)=\alpha\sigma(B)$ would imply $A=\alpha B=\alpha(-B)=-\alpha A=-A$ and this is impossible since $A \neq 0$. Thus the vectors A and B determine a hyperplane $H=\langle A,B\rangle$, and σ^2 fixes H pointwise. Thus by Theorem 3.3, $\sigma^2=1_E$. Therefore, σ is an involution. By Lemma 3.22, the only involutions of E are $\pm 1_E$, the symmetries of E, and 180° rotations. But 1_E inverts no nonzero vector, and -1_E does not fix a nonzero vector; therefore, σ must be either a symmetry or a 180° rotation.

Corollary:

The only rotations that invert a nonzero vector are the 180° rotations.

Proof:

A 180° rotation inverts a whole plane of vectors. Conversely, let ρ be a rotation of E which inverts a nonzero vector. By Euler's Theorem 3.17, ρ leaves a nonzero vector fixed. Thus by the Lemma above, ρ is a 180° rotation.

Now we are going to describe the 180° rotations of E. By Euler's Theorem 3.17, every 180° rotation of E has a unique axis of rotation. Conversely, for every line in E, there is precisely one 180° rotation with that line as axis. Let L be any line in E, and let $O^+(E;L)$ be the set of rotations of E that leave L pointwise fixed. We know that if $\rho \in O^+(E;L)$, then $\rho = 1_L \oplus \rho'$, where $\rho' \in O^+(L^\perp)$. Thus a rotation ρ of $O^+(E;L)$ inverts no nonzero vector if the rotation ρ' is not a 180° rotation of the hyperplane $L^\perp$; that is, if $\rho' \neq -1_{L^\perp}$. So the rotations of E which invert no nonzero vector are given in the following theorem.

Theorem 3.24:

The rotations of E which invert no nonzero vectors are $\Lambda = \{O^+(E;L) - \{-1_{L^\perp}\} \mid L \textit{ is a line in } E\}$.

By Lemma 3.21, we obtain the reflections of E which leave only the origin fixed. These are $S = \{-\rho \mid \rho \in \Lambda\}$. Thus $\sigma \in S$ if and only if $\sigma=-\rho=-1_L \oplus \sigma'$, where σ' is any rotation of the plane $L^\perp$ different from $1_{L^\perp}$, and L is a line in E.

Summarizing the above discussion concerning the orthogonal transformations of a 3-dimensional Euclidean space, we may state: an orthogonal transformation of a 3-dimensional Euclidean space E is either

a. 1_E: Identity (Rotation)

b. rotation about a line. In this case, the set of all such rotations (together with 1_E) is denoted by $O^+(E;L)$ and $O^+(E;L) \cong O^+(E^2)$, where E^2 is a 2-dimensional Euclidean space.

c. the reflections of a 3-dimensional Euclidean space are of two types:

   Type (1) Symmetries (i.e., hyperplane reflections).

   Type (2) reflections that are product of 3-symmetries these are of the form $\sigma=-1_L \oplus \sigma'$ where L is a line in E and $\sigma' \in \{O^+(E;L)-\{1_{L^\perp}\} / L \text{ is a line in } E\}$

   $\cong O^+(E^2)-\{1_{E^2}\}$.

Now we are going to combine the results of this section concerning the classification of orthogonal transformations of a 3-dimensional Euclidean space E and Theorem 2.13 to give a matrix representation of the different types of the orthogonal transformations of E.

From Theorem 2.13, it follows that a matrix representation relative to an orthonormal basis B of an orthogonal transformation σ of E has either one of the following forms:

$$A_1(\theta) = [\sigma]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \ 0 \le \theta \le 2\pi$$

or

$$A_2(\theta) = [\sigma]_B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \ 0 \le \theta \le 2\pi .$$

The matrix A_1 corresponds to the rotations and the matrix A_2 to the reflections.

1. For $\theta=0$, we obtain

$$A_1(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ which corresponds to the identity

$$A_2(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ *180°* rotation about a line, and this is

the same as a reflection (or symmetry) through a plane.

2. For $\theta=\pi$, we obtain

$$A_1(\pi)=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix},$$ a reflection through a line and this is

the same as 180° rotation about a line.

$$A_2(\pi)=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix},$$ a reflection through a point, namely

$-I_3$.

3.3 The Classification of Euclidean Motions in Dimensions 2 and 3

This section describes the Euclidean motions of a *2*-dimensional Euclidean plane and a *3*-dimensional Euclidean space E.

A. The *2*-Dimensional Euclidean Plane E^2

First, recall that every Euclidean motion of a space E can be written in the form $T_A \circ \sigma$, where T_A is a translation of E and σ is an orthogonal transformation of E. In particular, since the orthogonal transformations of a *2*-dimensional Euclidean plane are either rotations about a point in E^2 through an angle θ or reflections through a line, it follows that every Euclidean motion of a *2*-dimensional plane is either a translation, a rotation, a product of a translation and a rotation, or a product of a translation and a reflection. Now we are going to describe each type.

1. Translations

These are given by $T_A : E^2 \to E^2$, where $A \in E^2$ and $T_A(x) = x + A$. Geometrically, T_A translates the whole plane along the vector A. If $A \neq O$, T_A has no fixed points.

2. Rotations about a point

Geometrically these rotate the plane through an angle θ about a point $A \in E^2$, and they are denoted by $\rho_{\theta,A}$. In particular, if $A=0$, we write ρ_θ to mean $\rho_{\theta,0}$. Note that $\rho_{\theta,A}=T_A\rho_\theta T_A^{-1}$. The rotation ρ_θ has matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

Thus $\rho_\theta(x) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1\cos\theta - x_2\sin\theta\\ x_1\sin\theta + x_2\cos\theta\end{bmatrix}$.

Thus
$$
\begin{aligned}
\rho_{\theta,A} &= T_A\rho_\theta T_A^{-1}(x)\\
&= T_A\rho_\theta(T_{-A}(x))\\
&= T_A\rho_\theta\left(\begin{bmatrix}x_1-a_1\\ x_2-a_2\end{bmatrix}\right)\\
&= T_A\left(\begin{bmatrix}(x_1-a_1)\cos\theta-(x_2-a_2)\sin\theta\\ (x_1-a_1)\sin\theta+(x_2-a_2)\cos\theta\end{bmatrix}\right)\\
&= T_A\left(\begin{bmatrix}x_1\cos\theta-x_2\sin\theta\\ x_1\sin\theta+x_2\cos\theta\end{bmatrix}+\begin{bmatrix}-(a_1\cos\theta-a_2\sin\theta)\\ -(a_2\sin\theta+a_2\cos\theta)\end{bmatrix}\right)\\
&= T_A\left(\begin{bmatrix}x_1\cos\theta-x_2\sin\theta\\ x_1\sin\theta+x_2\cos\theta\end{bmatrix}-\rho_\theta(A)\right)\\
&= T_{A-\rho_\theta(A)}\left(\begin{bmatrix}x_1\cos\theta-x_2\sin\theta\\ x_1\sin\theta+x_2\cos\theta\end{bmatrix}\right)\\
&= T_{A-\rho_\theta(A)}\,\rho_\theta.
\end{aligned}
$$

Thus the rotation $\rho_{\theta,A}$ may be obtained by first rotating the plane through an angle θ about the origin and then translating along the vector $A-\rho_\theta(A)$.

3. Reflections about a line

Let L be any line in E^2. The reflection of E^2 through the line L is denoted by R_L. Let L' be the line parallel to L and passes through the origin. There is a unique vector A that is perpendicular to L and passes through the origin. Now we claim that $R_L=T_A S_A T_A^{-1}$, where S_A is the symmetry of the plane E^2 determined by A. The proof of the claim follows by examining the figure below.

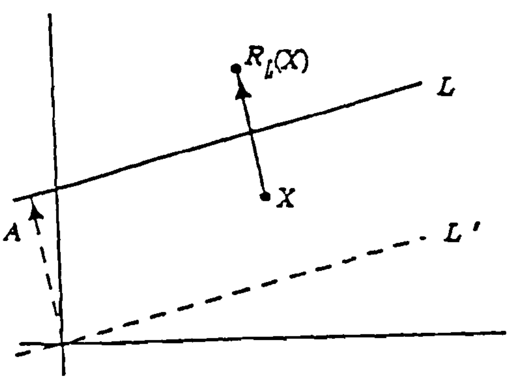


Figure 3.1

$$
\begin{aligned}
R_L(x)=(T_A S_A T_A^{-1})(x)=T_A S_A(x-A) \\
&=T_A\left((x-A)-2\frac{\langle x-A,A\rangle}{\langle A,A\rangle}A\right) \\
&=\left((x-A)-2\frac{\langle x-A,A\rangle}{\langle A,A\rangle}A\right)+A \\
&=x-2\frac{\langle X,A\rangle-\langle A,A\rangle}{\langle A,A\rangle}A \\
&=x-2\frac{\langle x,A\rangle}{\langle A,A\rangle}A+2A \\
&=T_{2A}(S_A(x)).
\end{aligned}
$$

4. Glide reflections

A glide reflection of E^2 is any translation of E^2 by a nonzero vector followed by a reflection about any line parallel to the translating vector. If A is the nonzero translating vector and L stands for any line parallel to A, then the glide reflection determined by A and L is the map $g_{A,L}:E^2 \to E^2$, given by $g_{A,L}=R_L T_A$.

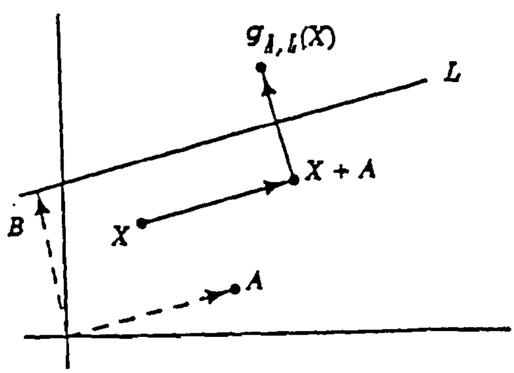


Figure 3.2

Now we are going to show that a Euclidean motion of E^2 that is a product of a rotation and a translation is a rotation of E^2.

Let $m=T_B\rho_\theta$. We claim that $m=T_B\rho_\theta=\rho_{\theta,B}$ for some vector $B\in E^2$. Consider the map $1_{E^2}-\rho_\theta$. Since ρ_θ fixes only the zero vector, then $1_{E^2}-\rho_\theta$ fixes only the zero vector. Thus $Ker(1_{E^2}-\rho_\theta)=\{0\}$.

Hence the linear map $1_{E^2}-\rho_\theta$ is one-to-one and hence it is onto. Therefore, there exists a vector $B\in E^2$ such that $(1_{E^2}-\rho_\theta)(B)=A$, or $A=B-\rho_\theta(B)$. So $T_A\rho_\theta=T_{B-\rho_\theta(B)}\rho_\theta=T_BT^{-1}_{\rho_\theta(B)}\rho_\theta=T_B(\rho_\theta T_B^{-1}\rho_\theta^{-1})\rho_\theta=T_B\rho_\theta T_B^{-1}$ $=\rho_{\theta,B}$. Thus $m=T_A\rho_\theta$ is a rotation of E^2 about the point B through an angle θ.

Finally, consider a Euclidean motion of the form $m=T_AS_B$, where S_B is a symmetry with respect to $\langle B\rangle^\perp$. If $A=0$, then $m=S_B$ and hence m is a symmetry (or reflection). Therefore, assume $A\neq 0$, and let $X=A-\dfrac{\langle A,B\rangle}{\langle B,B\rangle}B$ and $Y=A-\dfrac{\langle A,B\rangle}{2\langle B,B\rangle}B$. If $X=0$, then B is a scalar multiple of A, and in this case $S_B=S_A$; therefore, $T_AS_A=T_{2(\frac{1}{2}A)}S_{\frac{1}{2}A}$ and from (3) above, this is a reflection of the plane about the line $L=\frac{1}{2}A+\langle A\rangle^\perp$. If, on the other hand, $X\neq 0$, then X is orthogonal to Y since

$$\begin{aligned}\langle X,Y\rangle&=\left\langle A-\frac{\langle A,B\rangle}{\langle B,B\rangle}B,\frac{\langle A,B\rangle}{2\langle B,B\rangle}B\right\rangle\\&=\left\langle A,\frac{\langle A,B\rangle}{2\langle B,B\rangle}B\right\rangle-\left\langle\frac{\langle A,B\rangle}{\langle B,B\rangle}B,\frac{\langle A,B\rangle}{2\langle B,B\rangle}B\right\rangle\\&=\frac{\langle A,B\rangle}{2\langle B,B\rangle}\langle A,B\rangle-\frac{1}{2}\left(\frac{\langle A,B\rangle}{\langle B,B\rangle}\right)^2\langle B,B\rangle\\&=0.\end{aligned}$$

Let $L=Y+\langle X\rangle$. Then L is a line through Y and parallel to X. In this case $T_AS_B=g_{X,L}$, to see this, recall from part (4) above $g_{X,L}=T_{2Y+S_Y(X)}S_Y$. But $2Y+S_Y(X)=A-X+X=A$ since $X\perp Y$ and $S_Y=S_B$ since Y and B are linearly dependent. Therefore $g_{X,L}=T_AS_B$. We conclude that every Euclidean motion of E^2 is either a translation, a rotation, a reflection, or a glide reflection. Hence we have proved the following theorem.

Theorem 3.25:

Any Euclidean motion of E^2 is a translation, a rotation, a reflection, or a glide reflection.

B. The *3*-Dimensional Euclidean Space E^3

In a *3*-dimensional Euclidean space E^3, there are six types of Euclidean motions:

1. **Translation.**
2. **Rotation about a line L in E^3.**
3. **reflection in a hyperplane H (or symmetry with respect H).**
4. **Glide reflection.**

If H is a hyperplane in E^3 and T_A is a translation that leave H fixed then the rigid motion $T_A R_H$ is called a *glide reflection.*

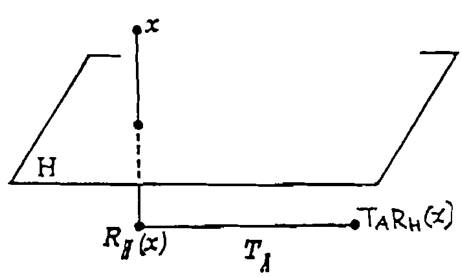


Figure 3.3

5. Rotatory inversion (or reflection)

Let ρ be a rotation with axis of rotation L, and let P be a point on L. A *rotatory inversion* is the composition of 180° rotation and ρ. Note that a rotatory inversion is also the composition of a reflection R_H and a rotation see the figure below.

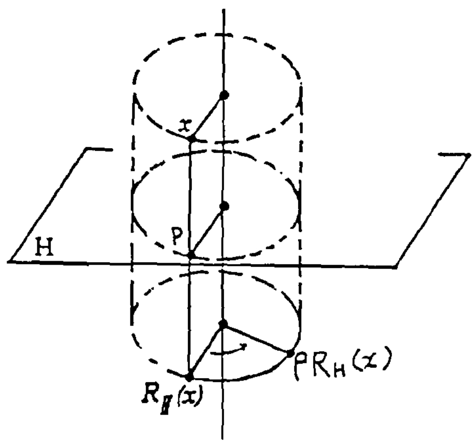


Figure 3.4

6. Screw displacement

Let ρ be a rotation with axis L and T_A a translation determined by A in the direction of L. The composition ρT_A is called a *screw displacement* with axis L.

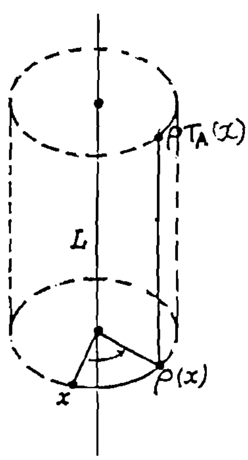


Figure 3.5

Theorem 3.26:

Every Euclidean motion of E^3 is one of the above six types.

Proof:

The proof of this theorem is similar to the proof of Theorem 3.25.

CHAPTER 4

Symmetry and the Orthogonal Group

This chapter deals with the study of symmetry groups of bounded sets in the *2*- and *3*-dimensional Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. We are interested in classifying all finite symmetry groups of bounded sets in $\mathbb{R}^2$ and $\mathbb{R}^3$. These symmetry groups are particularly important in science. For example, they are used in chemistry to describe the symmetry of molecules, and they are important in the study of the symmetry properties of crystals. It turns out that the problem of classifying the finite symmetry groups of bounded sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ reduces to the problem of classifying all finite subgroups of the orthogonal groups $O(\mathbb{R}^2)$ and $O(\mathbb{R}^3)$ respectively, so our main objective in this chapter is to classify all finite subgroups of $O(\mathbb{R}^2)$ and $O(\mathbb{R}^3)$. In the applications of group theory in science, finite subgroups of $O(\mathbb{R}^3)$ are called *point groups* since they always have a fixed point. These point groups are of two types: point groups of the *first kind*, which contain only rotations (that is finite subgroups of $O^+(\mathbb{R}^3)$), and point groups of the *second kind*, which also contain reflections.

4.1 Symmetry groups

Definition 4.1:

Let E be an n-dimensional Euclidean space. Let S be a nonempty subset of E. A rigid motion (or isometry) σ of E is said to be a *symmetry* of S if $\sigma(S)=S$; that is, if σ maps every point in S to a point in S.

Theorem 4.1:

Let S be a nonempty subset of E. The set of all symmetries of S forms a subgroup of the Euclidean group $M(E)$ of E. That is, $G_S=\{\sigma\in M(E) | \sigma(S)=S\}$ is a subgroup of $M(E)$.

Proof:

Clearly we have $G_S\subseteq M(E)$ and $G_S\neq\emptyset$ since $1_E\in G_S$. Let $\sigma,\tau\in G_S$. Then $\sigma(S)=S$ and $\tau(S)=S$. Thus $(\sigma\tau)(S)=\sigma(\tau(S))=\sigma(S)=S$, and hence $\sigma\tau\in G_S$. Also $\sigma^{-1}\sigma(S)=\sigma^{-1}(S)$; that is, $S=\sigma^{-1}(S)$. This implies $\sigma^{-1}\in G_S$. Therefore, $G_S\leq M(E)$.

Definition 4.2:

The group of symmetries G_S of S is called the *symmetry group* of S.

Example 1:

Let $S=\{A\}$, where $A\in E$. Then the symmetry group G_S of S consists of all isometries of E that fix the point A. Thus the symmetry group of $S=\{A\}$ is $M(E)_A$.

Example 2:

Let S be an equilateral triangle in $\mathbb{R}^2$. Then the symmetry group of S is $G_S=\{1_{\mathbb{R}^2},\rho_{p,\frac{2\pi}{3}},\rho_{p,\frac{4\pi}{3}},\sigma_{L_1},\sigma_{L_2},\sigma_{L_3}\}$, where $\rho_{p,\frac{2\pi}{3}}$ and $\rho_{p,\frac{4\pi}{3}}$ are rotation about the point p through angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians respectively. σ_{L_1}, σ_{L_2} and σ_{L_3} are reflections in the lines L_1, L_2 and L_3 respectively.

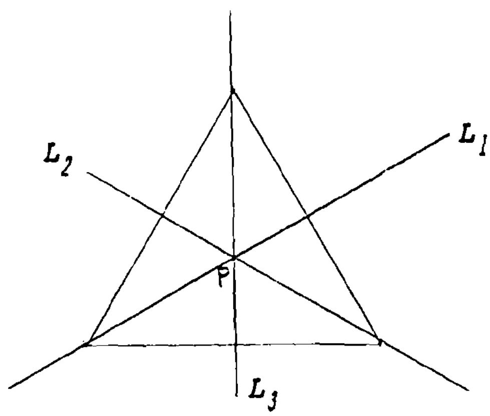


Figure 4.1

Example 3:

Let S be a regular n-gon ($n\geq 3$) in $\mathbb{R}^2$. We need to find the symmetry group of S. First, we are going to show that the symmetry group of S has order $2n$. Consider a regular n-gon whose vertices are numbered $0,1,2,\ldots,(n-1)$. Since a symmetry of S preserves distance between the locations of the vertices

0 and 1, the locations of the vertices 0 and 1 determine the entire symmetry. There are n choices for the vertex 0; having made any of these n choices for vertex 0, there are two choices for the vertex 1. Hence there are at most $2n$ possible symmetries. Now we are going to show that all these possibilities do arise. Let ρ be the counterclockwise rotation of $\mathbb{R}^2$ about the center p of the n-gon through an angle of $2\pi/n$ radians, and let σ be the reflection of $\mathbb{R}^2$ about the line L. We claim that G_S is generated by ρ and σ. The rotation ρ takes $0\to 1, 1\to 2, 2\to 3, \ldots, (n-1)\to 0$, and ρ^2 takes $0\to 2, 1\to 3, \ldots, i\to i+2, \ldots, (n-1)\to 1$ and in general for $1\le k\le n$, ρ^k takes $0\to k, 1\to k+1, 2\to k+2, \ldots, (n-1)\to (k-1)$.

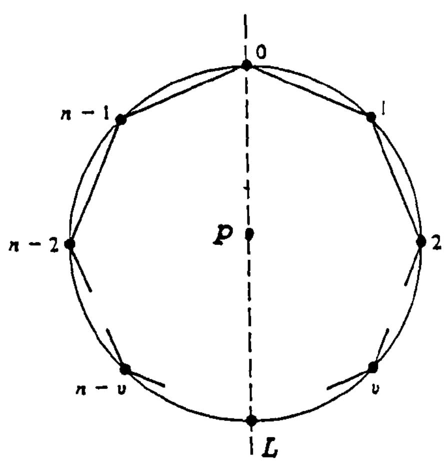


Figure 4.2

Thus ρ generates a subgroup $\langle\rho\rangle$ of G_S whose elements are the n rotations $\{\rho, \rho^2, \rho^3, \ldots, \rho^n = 1_{\mathbb{R}^2}\}$. Since the reflection σ is not an element of the subgroup $\langle\rho\rangle$, the right coset $\langle\rho\rangle\sigma \ne \langle\rho\rangle$

and in fact $\langle\rho\rangle\sigma$ gives the other n elements of G_S, namely $\{\rho^k\sigma \mid k=1,\dots,n\}$. Clearly for $k=1,\dots,n$, $\rho^k\sigma$ are distinct; to see why $\{\rho^k\sigma \mid k=1,\dots,n\}$ are distinct, assume that $\rho^i\sigma=\rho^j\sigma$ for some $1\le i<j\le n$. Then $\rho^{j-i}=1$, where $1\le j-i<n$ which is impossible. Clearly, $\rho^n=1$ and $\sigma^2=1$. We claim that $\sigma\rho=\rho^{-1}\sigma$. Since a symmetry in G_S is determined by the locations of 0 and 1, we consider $\sigma\rho\sigma^{-1}(0)=\sigma\rho(0)=\sigma(1)=n-1$. Also $\rho^{-1}(0)=n-1$. Thus $\sigma\rho\sigma^{-1}(0)=\rho^{-1}(0)$. Similarly, $\sigma\rho\sigma^{-1}(1)=\sigma\rho(n-1)=\sigma(0)=0$, and $\rho^{-1}(1)=0$, thus $\sigma\rho\sigma^{-1}(1)=\rho^{-1}(1)$. Therefore, $\sigma\rho\sigma^{-1}=\rho^{-1}$ or $\sigma\rho=\rho^{-1}\sigma$. We conclude, therefore, that G_S is a group of order $2n$ generated by the two elements ρ and σ that satisfy the relations $\rho^n=\sigma^2=1$ and $\sigma\rho=\rho^{-1}\sigma$. Thus G_S is the dihedral group D_{2n} generated by ρ and σ.

Example 4:

Finally, let us find the symmetry group of a regular tetrahedron in $\mathbb{R}^3$.

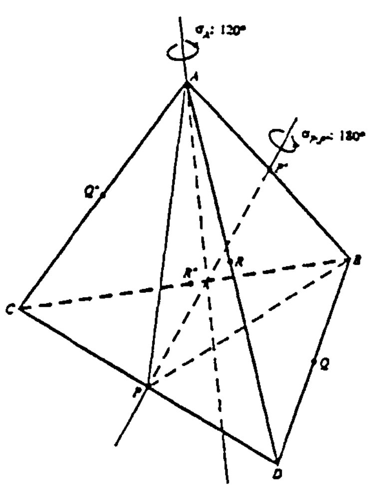


Figure 4.3

Let us begin by finding the rotations of $\mathbb{R}^3$ fixing a vertex and leaving the tetrahedron invariant. Let σ_V stand for a 120° counterclockwise rotation off the tetrahedron about an axis passing through vertex V and the center of the face opposite to V. For example, the effect of σ_A on the vertices of the tetrahedron is given by $\sigma_A : \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}$. Also, the σ_A^2 effect on the vertices is given by $\sigma_A^2 : \begin{pmatrix} A & B & C & D \\ A & D & B & C \end{pmatrix}$. The σ_A^3 effect on the vertices is given by $\sigma_A^3 : \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$; that is, $\sigma_A^3 = 1_{\mathbb{R}^3}$. Thus we conclude that there are eight rotations that fix one vertex of the tetrahedron: σ_A, σ_A^2, σ_B, σ_B^2, σ_C, σ_C^2, and σ_D, σ_D^2. Now we find the 180° rotations of $\mathbb{R}^3$ that leaves the tetrahedron invariant. These are the 180° rotations about an axis passing through the midpoints of opposite edges. There are three such pairs of opposite edges, namely (P,P'), (Q,Q') and (R,R'); these produce the 180° rotations: $\sigma_{(P,P')} : \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$, $\sigma_{(Q,Q')} : \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}$ and $\sigma_{(R,R')} : \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$. Next we find the hyperplane reflections of $\mathbb{R}^3$ that leave the tetrahedron invariant. These are reflections through the planes bisecting the tetrahedron. A plane is determined by an edge and the midpoint of the opposite edge, for example, the edge AB and the point P. Let τ_P denote the reflection of $\mathbb{R}^3$ through the plane determined by P and AB. Then the effect of τ_P on the vertices of the tetrahedron is given by: $\tau_P:\begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix}$. There are another five reflections given by: $\tau_Q:\begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$,

$\tau_R:\begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}$, $\tau_{P'}:\begin{pmatrix} A & B & C & D \\ B & A & C & D \end{pmatrix}$, $\tau_{Q'}:\begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$ and $\tau_{R'}:\begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$.

The remaining six orthogonal transformations that leave the tetrahedron invariant are the ones that are products of three symmetries. These are given below with the effect of each on the vertices of the tetrahedron.

$\tau_{Q'}\tau_{P'}\tau_R:\begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix}$, $\tau_R\tau_{P'}\tau_{Q'}:\begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix}$, $\tau_{P'}\tau_{Q'}\tau_R:\begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$,

$\tau_R\tau_{Q'}\tau_{P'}:\begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$, $\tau_{Q'}\tau_R\tau_{P'}:\begin{pmatrix} A & B & C & D \\ B & D & A & C \end{pmatrix}$, $\tau_{P'}\tau_{R'}\tau_{Q'}:\begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}$.

The table below is the list of the *24* elements of the symmetry group of the tetrahedron.

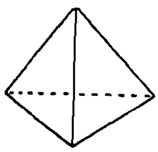
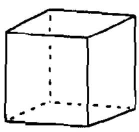
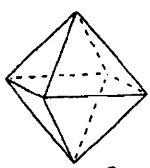
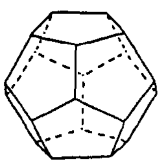
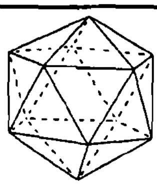
Table 4.1

Rigid motion	Description	Vertex permutation
$1_{\mathbb{R}^3}$	Identity transformation	$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$
$\sigma_A,\ \sigma_A^2$ $\sigma_B,\ \sigma_B^2$ $\sigma_C,\ \sigma_C^2$ $\sigma_D,\ \sigma_D^2$	*120°*, *240°* rotations	*(BCD)*, *(DBC)* *(ADC)*, *(ACD)* *(ABD)*, *(ADB)* *(ACB)*, *(ABC)*
$\sigma_{P,P'}$ $\sigma_{Q,Q'}$ $\sigma_{R,R'}$	*180°* rotations	*(AB)(CD)* *(AC)(BD)* *(BC)(AD)*
$\tau_P,\ \tau_{P'}$ $\tau_Q,\ \tau_{Q'}$ $\tau_R,\ \tau_{R'}$	Symmetries, i.e., hyperplane reflections	*(CD)*, *(AB)* *(BD)*, *(AC)* *(AD)*, *(BC)*
$\tau_{Q'}\tau_{P'}\tau_R$ $\tau_R\tau_{P'}\tau_{Q'}$ $\tau_{P'}\tau_{Q'}\tau_R$ $\tau_R\tau_{Q'}\tau_{P'}$ $\tau_{Q'}\tau_R\tau_{P'}$ $\tau_{P'}\tau_R\tau_{Q'}$	Transformations that are products of *3* symmetries	*(ADBC)* *(ACBD)* *(ADCB)* *(ABCD)* *(ABDC)* *(ACDB)*

From the table, it is not hard to see that the symmetry group of the tetrahedron G_S is isomorphic to the symmetric group S_4. Moreover, the rotation subgroup of the tetrahedron $G^+_S=\{1_{\mathbb{R}^3}, \sigma_A, \sigma_A^2, \sigma_B, \sigma_B^2, \sigma_C, \sigma_C^2, \sigma_D, \sigma_D^2, \sigma_{P,P'}, \sigma_{Q,Q'}, \sigma_{R,R'}\}$ is isomorphic to the alternating group A_4.

Similarly, one can find the rotational symmetry groups of the other platonic solids. The list of the five platonic solids and their rotational symmetry groups are given below together with the number of their vertices, edges, faces and the shape of their faces.

Table 4.2

Name of platonic solid	Number of vertices	Number of edges	Number of faces	Shape of faces	rotational symmetry group
 Tetrahedron	*4*	*6*	*4*	Triangles	$\cong A_4$
 Cube	*8*	*12*	*4*	Squares	$\cong S_4$
 Octahedron	*6*	*12*	*8*	Triangles	$\cong S_4$
 Dodecahedron	*20*	*30*	*12*	Pentagons	$\cong A_5$
 Icosahedron	*12*	*30*	*20*	Triangles	$\cong A_5$

Definition 4.3:

Let S and S' be two nonempty subsets of E. We say S and S' are congruent if there is an isometry $\sigma : E \to E$ such that $S' = \sigma(S)$.

Theorem 4.2:

Let S and S' be congruent subsets of E. Then their symmetry groups are isomorphic; that is, $G_S \cong G_{S'}$.

Proof:

Since S and S' are congruent, there exists an isometry σ such that $\sigma(S)=S'$. Let $\tau \in G$, and let $\tau'=\sigma\tau\sigma^{-1}$. Then $\tau'(S') =\sigma\tau\sigma^{-1}(S')=\sigma\tau(S)=\sigma(S)=S'$, and hence $\tau' \in G_{S'}$. Define a map $\phi: G_S \to G_{S'}$ by $\phi(\tau)=\tau'=\sigma\tau\sigma^{-1}$. Let $\tau_1, \tau_2 \in G_S$. Then $\phi(\tau_1\tau_2)=\sigma\tau_1\tau_2\sigma^{-1} =\sigma\tau_1\sigma^{-1}\sigma\tau_2\sigma^{-1} =\phi(\tau_1)\phi(\tau_2)$. Thus ϕ is an homomorphism. To show ϕ is onto, let $\tau' \in G_{S'}$, and let $\tau=\sigma^{-1}\tau\sigma$. Then $\tau(S)=\sigma^{-1}\tau'\sigma(S)=\sigma^{-1}\tau'(S')=\sigma^{-1}(S')=S$, hence $\tau \in G_S$ and $\phi(\tau)=\sigma\tau'\sigma^{-1}=\sigma(\sigma^{-1}\tau'\sigma)\sigma^{-1}=\tau'$. To show ϕ is one-to-one, assume $\phi(\tau_1)=\phi(\tau_2)$. Then $\sigma\tau_1\sigma^{-1}=\sigma\tau_2\sigma^{-1}$, hence $\tau_1=\tau_2$. Therefore, ϕ is an isomorphism.

It follows from this theorem that a symmetry group G_S of a subset S of E does not depend upon the location of the set S in the space E. However, it is important to realize that the symmetry group of a set S does depend on the space in which we view it. For example, the symmetry group of a line segment in $\mathbb{R}^1$ is $\{1_{\mathbb{R}}, -1_{\mathbb{R}}\}$. However, the symmetry group of a line segment considered as a set of points in $\mathbb{R}^2$ is $\{1_{\mathbb{R}^2}, \sigma_y, \sigma_x, \rho_\pi\}$, where σ_x, σ_y are reflections in the x and y axis and ρ_π is a rotation about the origin through $180°$.

The next theorem shows that the problem of classifying the (finite) symmetry groups of bounded sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

reduces to the problem of classifying all (finite) subgroups of $O(\mathbb{R}^2)$ and $O(\mathbb{R}^3)$.

Theorem 4.3:

Every rigid motion in a symmetry group of a bounded set in $\mathbb{R}^2$ or $\mathbb{R}^3$ has at least one fixed point.

Proof:

Let S be a bounded set in $\mathbb{R}^2$ or $\mathbb{R}^3$. Since S is bounded, S can be contained inside some sufficiently large sphere. A rigid motion in G_S cannot be a nontrivial translation, a glide reflection, or a screw displacement since one of these rigid motions indefinitely repeated would map S outside of the bounding sphere. Thus the only rigid motions in G_S are rotations, reflections, and rotatory inversion. Hence every element of G_S has at least one fixed point.

Corollary:

1. A symmetry group of a bounded set in $\mathbb{R}^2$ is isomorphic to a subgroup of $O(\mathbb{R}^2)$.
2. A symmetry group of a bounded set in $\mathbb{R}^3$ is isomorphic to a subgroup of $O(\mathbb{R}^3)$.

Proof:

Let G_S be a symmetry group of a bounded set in $\mathbb{R}^2$ ($\mathbb{R}^3$). But each element of G_S must have a fixed point. Thus the only rigid motions in G_S are rotations, reflections, and rotatory inversions (in the case of $\mathbb{R}^3$). Thus G_S is isomorphic to a subgroup of $O(\mathbb{R}^2)$ (or $O(\mathbb{R}^3)$).

4.2 Classification of Finite Subgroups of $O(\mathbb{R}^2)$

Before we start the classification of the finite subgroups of $O(\mathbb{R}^2)$, we recall a few basic facts from group theory.

Definition 4.4:

Let G be a group and S be a (finite) set. To say that G *acts* (or *operates*) on S means that, for every $g \in G$ and every $s \in S$, there is defined a unique element $gs \in S$ such that;

1. $es=s$ for all $s \in S$ (e is the identity of G).
2. $(gg')s=g(g's)$, for all $g,g' \in G$ and $s \in S$.

A set S on which a group G acts is often called a *G-set*.

For example, let $G=O(\mathbb{R}^2)$. Then G operates on the set of all points of the plane $\mathbb{R}^2$, the set of triangles in $\mathbb{R}^2$, and so on.

Let S be a finite *G-set* and let $g \in G$ be a fixed element of G. Thus we have a map $m_g : S \to S$ defined by $m_g(s)=gs$. This map describes the way the fixed element $g \in G$ operates on S.

Theorem 4.4:

The map $m_g : S \to S$ defined by $m_g(s)=gs$ is a permutation of S; that is, m_g is one-to-one and onto.

Proof:

First, we show m_g is one-to-one. Assume $m_g(s_1)=m_g(s_2)$. Then $gs_1=gs_2$. Therefore, $g^{-1}gs_1=g^{-1}gs_2$; that is, $s_1=s_2$, hence m_g is one-to-one. Next, we show m_g is onto. Let $s'\epsilon S$. Let $s=g^{-1}s'$. Therefore, $gs=s'$; that is, $m_g(s)=s'$. Hence m_g is onto.

Theorem 4.5:

Let S be a *G-set*. The relation ~ on S defined by $s\sim s'$ if and only if $s'=gs$ for some $g\epsilon G$ is an equivalence relation on S.

Proof:

To show ~ is reflexive, let $s\epsilon S$. Then $s=es$ where $e\epsilon G$ is the identity element. Therefore, $s\sim s$. To show ~ is symmetric, assume $s\sim s'$. Then $s'=gs$ for some $g\epsilon G$. Since $g^{-1}\epsilon G$, $g^{-1}s'=g^{-1}gs=s$. This implies $s'\sim s$. To show ~ is transitive, assume $s\sim s'$ and $s'\sim s''$. Then $s'=g_1s$ and $s''=g_2s'$ for some $g_1,g_2\epsilon G$. Then $s''=g_2(g_1s)=(g_1g_2)s$. Since $g_1g_2\epsilon G$, $s\sim s''$. Since ~ is reflexive, symmetric, and transitive relation, ~ is an equivalence relation.

Definition 4.5:

Let S be a *G-set* and $s\epsilon S$. The equivalence class of the relation ~ determined by s is called the *orbit* of s in S. Thus the orbit of s in S is the set $O_s=\{gs|g\epsilon G\}$.

Being equivalence classes, the orbits of the elements of S partition S into disjoint subsets. And the group G operates on S by operating independently on each orbit. In other words, an element $g \in G$ permutes the elements of each orbit and dose not carry elements of one orbit to another orbit.

Definition 4.6:

Let S be a *G-set*. We say G operates transitively on S if there is just one orbit in S; that is, if $S=O_s$ for some $s \in S$.

Note that if G operates on a set S transitively then every element of S is carried to every other one by some element of the group.

Definition 4.7:

Let S be a *G-set*. The *stabilizer* of an element $s \in S$ is denoted by $stab(s)$ and is defined by $stab(s)=\{g \in G \mid gs=s\}$. That is, $stab(s)$ is the set of elements in G that leaves s fixed.

Theorem 4.6:

$stab(s)$ is a subgroup of G.

Proof:

Note that $stab(s) \subseteq G$, and $stab(s) \neq \emptyset$ (since $e \in stab(s)$). Let $g_1, g_2 \in stab(s)$. Then $(g_1 g_2)s=g_1(g_2 s)=g_1(s)=s$; therefore, $g_1 g_2 \in stab(s)$. If $g \in stab(s)$, then $gs=s$, and $g^{-1}gs=g^{-1}s$; that is, $s=g^{-1}s$, so $g^{-1} \in stab(s)$. Therefore, $stab(s) \leq G$.

Let H be a subgroup of a group G. Let G/H denote the set of all left cosets of H in G; that is, $G/H=\{gH|g\in G\}$. We shall call G/H the left coset space of G relative to H. Note that G/H is not a group unless the subgroup H is normal in G. However, G operates on the left coset space G/H in a natural way. The action of G on G/H is specified by $g'(gH)=(g'g)H$, where $g'\in G$ and $gH\in G/H$. The orbit of the coset $H=1H$ is given by $O_H=\{gH|g\in G\}=G/H$. Thus G operates transitively on G/H. The stabilizer of a coset gH is given by $stab(gH)=\{g'\in G|g'(gH)=gH\}$ $=\{g'\in G|g^{-1}g'g\in H\}$. In particular $stab(H)=H$.

Theorem 4.7:

Let S be a G-set, and let $s\in S$. Then $|O_s|=[G:stab(s)]$. That is the order of the orbit of s is equal to the index of the stabilizer subgroup of s.

Proof:

We define a one-to-one map ϕ from O_s onto the left coset space $G/stab(s)$. Let $s_1\in O_s$. Then there exists $g_1\in G$ such that $s_1=g_1s$. Define $\phi:O_s\to G/stab(s)$ as follows: $\phi(s_1)=g_1stab(s)$. First, we need to show that ϕ is well defined; that is, it is independent of the choice of $g_1\in G$. Suppose that $g_2s=s_1$ for some $g_2\in G$. Then $g_1s=g_2s$, so $g_1^{-1}(g_1s)=g_1^{-1}(g_2s)$, and thus $s=(g_1^{-1}g_2)(s)$. Therefore, $g_1^{-1}g_2\in stab(s)$, so $g_2\in stab(s)g_1$ and $g_1stab(s)$ $=g_2stab(s)$. Thus the map ϕ is well defined. To show the map ϕ is one-to-one, suppose $s_1,s_2\in O_s$ and $\phi(s_1)=\phi(s_2)$. Then there exist $g_1,g_2\in G$ such that $s_1=g_1s$, $s_2=g_2s$, and $g_2\in g_1stab(s)$. Then

$g_2=g_1g$ for some $g\in stab(s)$, so $s_2=g_2s=g_1g(s)=g_1(gs)=g_1(s)=s_1$. Thus ϕ is one-to-one. Finally, we show that ϕ is onto. Let $g_1stab(s)$ be an element in $G/stab(s)$. Then if $g_1s=s_1$, we have $g_1stab(s)=\phi(s_1)$. Thus ϕ is onto. Hence $|O_s|=|G/stab(s)|$. By Lagrange's Theorem, $|G/stab(s)|=|G|/|stab(s)|=[G:stab(s)]$.

Corollary (Counting Formula):

$|G|=|stab(s)||O_s|$.

To classify the symmetry groups of bounded sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, we need to establish some preliminary results.

Lemma 4.8:

Let E be either $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $S=\{s_i|s_i\in E, i=1,...,n\}$ be a finite set of points of E, and let p be the center of gravity of S, defined by $p=\frac{1}{n}(s_1+...+s_n)$, where the right side is computed by vector addition. Let m be a rigid motion, and let $s_i'=m(s_i)$ for all i and $p'=m(p)$. Then $p'=\frac{1}{n}(s_1'+...+s_n')$; in other words, a rigid motion carries a center of gravity to a center of gravity.

Proof:

Case 1: m is a translation, say $m=T_a$.

Then $p'=m(P)=p+a$ and $s_i'=m(s_i')=s_i+a$. Therefore,

$$\begin{aligned}\frac{1}{n}(s_1'+\ldots+s_n') &= \frac{1}{n}[(s_1+a)+\ldots+(s_n+a)]\\ &= \frac{1}{n}(s_1+\ldots+s_n)+a\\ &= p+a\\ &= p'.\end{aligned}$$

Case 2: m is a rotation or a reflection.

Then m is a linear transformation; therefore,

$$\begin{aligned}p' &= m\left(\frac{1}{n}(s_1+\ldots+s_n)\right)\\ &= \frac{1}{n}(m(s_1)+\ldots+m(s_n))\\ &= \frac{1}{n}(s_1'+\ldots+s_n').\end{aligned}$$

Lemma 4.9 (Fixed Point Theorem):

Let G be a finite subgroup of the group of rigid motions $M(E)$. Then there is a point p in E such that for all $g \in G$, $g(p)=p$; that is, G has a common fixed point.

Proof:

Note that E is a G-set. Let $s \in E$, and let $O_s=\{g(s) \mid g \in G\}$ be the orbit of s. Note that G operates on O_s transitively; that is, G will permute the elements of O_s. Let p be the center of gravity of the orbit O_s. Then $p=\frac{1}{|O_s|}\sum_{g\in G} g(s)$. Let $g'\in G$; then

$$\begin{aligned} g'(p) &= g'\left(\frac{1}{|O_s|}\sum_{g\in G} g(s)\right) \\ &= \frac{1}{|O_s|}\sum_{g\in G} g'g(s) \\ &= \frac{1}{|O_s|}\sum_{g^*\in G} g^*(s) \\ &= p. \end{aligned}$$

Corollary:

The elements of a finite subgroup of $O(E)$ have a common fixed point.

We are now ready to state the main Theorem; Hermann Weyl [30] credits the discovery of this theorem, in essence, to Leonardo da Vinci, who wanted to determine the possible ways to attach chapels and niches to a central building without destroying the symmetry of the nucleus.

Theorem 4.10:

Let G be a finite subgroup of the group $O(\mathbb{R}^2)$ which fixes the origin. Then G is one of following groups:

1. $G=C_n$: the cyclic group of order n, generated by the rotation ρ_θ, where $\theta=2\pi/n$.

2. $G=D_{2n}$: the dihedral group of order $2n$, generated by two elements, the rotation ρ_θ, where $\theta=2\pi/n$, and a reflection σ about a line through the origin.

Proof:

There are two cases to consider.

Case 1: All elements of G are rotations.

We need to show G is cyclic in this case. If $G=\{1_E\}$, then $G=C_1$. Otherwise G contains a nontrivial rotation ρ_θ. Let θ be the smallest positive angle of rotation among the elements of G. Then G is generated by ρ_θ. Let $\rho_\alpha \in G$, where $\alpha \in \mathbb{R}$. Let $n\theta$ be the greatest integer multiple of θ which is less than α, so that $\alpha=n\theta+\beta$, with $0\le\beta<\theta$. Since G is a group and since $\rho_\alpha, \rho_\theta \in G$, $\rho_\beta=\rho_\alpha\rho_{-n\theta}\in G$. But θ is the smallest positive angle of rotation in G. Therefore, $\beta=0$ and $\alpha=n\theta$. Hence $\rho_\alpha=\rho_\theta^n$. This shows that G is cyclic. Let $n\theta$ be the smallest multiple of θ which is $\ge 2\pi$, so that $2\pi\le n\theta<2\pi+\theta$. Since θ is the smallest positive angle of rotation in G, $n\theta=2\pi$. Thus $\theta=2\pi/n$ for some integer n.

Case 2 : G contains a reflection σ.

Let $\sigma\in G$ be a reflection in a line through the origin. Let H be the subgroup of rotations in G. We can apply what has been proved in case 1 to the group H, to conclude that it is a cyclic group generated by ρ_θ (*i.e.*, $H=C_n$). Then the $2n$ elements of the set $\{\rho_\theta^i, \rho_\theta^i\sigma|$

$0 \le i \le n-1\}$, are in G. It is not hard to see that $\rho_\theta^n = 1_{\mathbb{R}^2}$, $\sigma^2 = 1_{\mathbb{R}^2}$ and $\sigma\rho_\theta = \rho_\theta^{-1}\sigma$. Therefore, G contains the dihedral group D_{2n} generated by ρ_θ and σ. We need to show in fact $G = D_{2n}$. Now if $g \in G$ is a rotation, then $g \in H$; hence g is one of the elements of D_{2n}. If g is a reflection, then $g\sigma$ is a rotation because it is the product of two reflections. Therefore, $g\sigma = \rho_\theta^k$ for some k so that $g = \rho_\theta^k \sigma^{-1} = \rho_\theta^k \sigma$ since $\sigma^2 = 1$. Thus $g \in D_{2k}$ and hence $G = D_{2n}$.

Corollary 1:

Let G be a finite subgroup of $O(\mathbb{R}^2)_p$, which fixes the point p. Then G is either $T_p C_n T_p^{-1}$ or $T_p D_{2n} T_p^{-1}$.

Proof:

Let O be the origin. Note that $T_p(O) = p$. Then $T_p^{-1} G T_p(O) = T_p^{-1} G(p) = T_p^{-1}(p) = O$, hence $T_p^{-1} G T_p$ fixes the origin. Therefore, by the theorem, $T_p^{-1} G T_p = C_n$ or $T_p^{-1} G T_p = D_{2n}$, thus G is either $T_p C_n T_p^{-1}$ or $T_p D_{2n} T_p^{-1}$.

Corollary 2:

Let G be a finite subgroup of the Euclidean group $M(\mathbb{R}^2)$. If coordinates are introduced suitably, then G becomes one of the groups C_n or D_{2n}, where C_n is generated by ρ_θ, $\theta = 2\pi/n$ and D_{2n} is generated by ρ_θ and a reflection σ.

Proof:

Since G is a finite subgroup of motions $M(\mathbb{R}^2)$, G has a fixed point p by Theorem 4.9. Then introducing a coordinate system with the origin at p, G is either C_n or $G=D_{2n}$.

4.3 Classification of the Finite Subgroups of $O(\mathbb{R}^3)$

In this section, we turn to the classification of the finite subgroups of the orthogonal group of the 3-dimensional Euclidean space $O(\mathbb{R}^3)$.

First, we are going to show that the problem of classifying the finite subgroups of $O(\mathbb{R}^3)$ can be reduced to a great extent to the problem of classifying only the finite subgroups of the rotation group $O^+(\mathbb{R}^3)$.

Let G be a finite subgroup of $O(\mathbb{R}^3)$. By Lemma 4.9, G has a fixed point p. Thus G contains only rotations and rotatory-inversions with axes through the point p. If G consists entirely of rotations then G is a subgroup of $O^+(\mathbb{R}^3)$. If not, G contains a rotatory-inversion, say, $\sigma \in G$. Let $\rho_1, \rho_2, \ldots, \rho_n$ be all rotations in G, where $\rho_1 = 1_{\mathbb{R}^3}$. The set $H=\{\rho_1, \rho_2, \ldots, \rho_n\}$ is a subgroup of G, called the rotation subgroup of G. Consider the elements in G given by $\rho_1\sigma, \rho_2\sigma, \ldots, \rho_n\sigma$. Clearly, $\rho_i\sigma$ are rotatory-inversions for each $i=1,2,\ldots,n$. Also these elements are distinct. Moreover, any rotatory-inversion in G is one of these elements. To see this, let σ' be any rotatory-inversion in G. Then $\sigma'\sigma^{-1}$ is a rotation in G. Thus $\sigma'\sigma^{-1}=\rho_i$ for some $1 \le i \le n$. Hence $\sigma'=\rho_i\sigma$. Thus $G=\{\rho_1, \rho_2, \ldots, \rho_n, \rho_1\sigma, \rho_2\sigma, \ldots, \rho_n\sigma\}$. Hence the order of G, $O(G)=2n$ and $[G:H]=2$, and hence H is a normal subgroup of G. There are two possibilities; either G contains the inversion $\tau=-1_{\mathbb{R}^3} \in G$ or it does not.

Case 1: If $\tau=-1_{\mathbb{R}^3}\in G$, then $\tau\in H\sigma$, so $H\tau=H\sigma$ and we can write $G=H\cup H\tau$, or $G=\{\rho_1,\rho_2,\ldots,\rho_n,\rho_1\tau,\rho_2\tau,\ldots,\rho_n\tau\}$. Since $\tau^2=1_{\mathbb{R}^3}$ and $\rho_i\tau=\tau\rho_i$ for all i, then $K=\{\rho_1,\tau\}$ forms a subgroup of G and $G=HK$. Moreover, H and K are normal subgroup in G and $H\cap K=\{\rho_1\}$. Thus G is a direct product of H and K; that is, $G=H\times K$.

Case 2: If $\tau=-1_{\mathbb{R}^3}\notin G$, then the description of G becomes a little more complicated. $G=\{\rho_1,\rho_2,\ldots,\rho_n,\rho_1\sigma,\rho_2\sigma,\ldots,\rho_n\sigma\}$. Note that $\rho_i\sigma=(-\rho_i\sigma)(-1_{\mathbb{R}^3})=(-\rho_i\sigma)\tau=t_i\tau$, where $-\rho_i\sigma=t_i$ is a rotation. Hence $G=\{\rho_1,\rho_2,\ldots,\rho_n,t_1\tau,t_2\tau,\ldots,t_n\tau\}$. Claim: all the t_i are different from all the ρ_j. Assume the contrary, namely, $t_i=\rho_j$ for some j. Then $t_i\tau=\rho_j\tau$ implies $\rho_i\sigma=\rho_j\tau$, thus $\rho_j^{-1}\rho_i\sigma=\tau$ and hence $\tau=\rho_j^{-1}\rho_i\sigma\in G$, a contradiction. Hence all the t_i are different from all the ρ_j.

Let $G^+=\{\rho_1,\rho_2,\ldots,\rho_n,t_1,t_2,\ldots,t_n\}$. Then G^+ is a subgroup of $O^+(\mathbb{R}^3)$ of order $2n$ with $H=\{\rho_1,\ldots,\rho_n\}$ a normal subgroup of G^+ of index 2. Conversely, if $G^+=\{\rho_1,\rho_2,\ldots,\rho_n,t_1,t_2,\ldots,t_n\}$ is a subgroup of $O^+(\mathbb{R}^3)$ of order $2n$ with $H=\{\rho_1,\ldots,\rho_n\}$ a normal subgroup of G^+ of index 2, then $G=\{\rho_1,\rho_2,\ldots,\rho_n,t_1\tau,t_2\tau,\ldots,t_n\tau\}$ is a finite subgroup of $O(\mathbb{R}^3)$. Moreover, G^+ is isomorphic to G. To show that G^+ is isomorphic to G, define a map $\varphi:G\to G^+$ by $\varphi(\rho_i)=\rho_i$ for $i=1,2,\ldots,n$ and $\varphi(t_i\tau)=t_i$ for $i=1,2,\ldots,n$. Clearly φ is one-to-one and onto. Thus it remains to show that φ is a homomorphism. Let $\rho_i\rho_j=\rho_k$, then $\varphi(\rho_i\rho_j)=\varphi(\rho_k)=\rho_k=\rho_i\rho_j$ $=\varphi(\rho_i)\varphi(\rho_j)$. Let $t_i\tau$ and $t_j\tau$ be two rotatory inversions in G.

Then the product $(t_i\tau)(t_j\tau)$ is a rotation, say $(t_i\tau)(t_j\tau)=t_i\tau^2t_j=t_it_j=\rho_k$. $\varphi((t_i\tau)(t_j\tau))=\varphi(\rho_k)=\rho_k$. On the other hand, $\varphi(t_i\tau)\varphi(t_j\tau)=t_it_j=\rho_k$. Thus $\varphi((t_i\tau)(t_j\tau))=\varphi(t_i\tau)\varphi(t_j\tau)$. Finally $\varphi(\rho_i(t_i\tau))=\varphi(\rho_i(\rho_j\sigma))=\varphi(\rho_k\sigma)=\varphi(t_k\tau)=t_k$. On the other hand, $\varphi(\rho_i)\varphi(\rho_j)=(\rho_j)(t_j)=\rho_i(\rho_j\sigma)=\rho_k\sigma=t_k$. Thus $\varphi(\rho_i(t_j\tau))=\varphi(\rho_i)\varphi(t_j\tau)$. Therefore, φ is an isomorphism and hence $G\cong G^{+}$. Thus in this case, G is isomorphic to the rotation group G^{+}.

Thus we have proved the following theorem.

Theorem 4.11:

Let G be a finite subgroup of $O(\mathbb{R}^3)$, and let $H=G\cap O^{+}(\mathbb{R}^3)$ be the subgroup of rotations in G. Then there are exactly 3 possibilities:

1. $G=H$ if and only if G is a subgroup of $O^{+}(\mathbb{R}^3)$.
2. G is a direct product of the rotation subgroup H and the cyclic group $K=\{1_{\mathbb{R}^3},-1_{\mathbb{R}^3}\}$ if and only if $-1_{\mathbb{R}^3}\in G$.
3. $G=H\cup\{t\tau / t\in G^{+}-H\}\cong G^{+}$, where $\tau=-1_{\mathbb{R}^3}$ and G^{+} is a finite rotation subgroup of $O^{+}(\mathbb{R}^3)$ containing H with $[G^{+}:H]=2$. G is of this form if and only if $G\neq H$ and $-1_{\mathbb{R}^3}\notin G$.

Remark:

Note that G and G^{+} are as abstract groups isomorphic, *i.e.*, algebraically they are the same. However, they are geometrically different since G has rotatory-inversions but G^{+} has only rotations.

From the theorem above, it follows that in order to find all finite subgroup of $O(\mathbb{R}^3)$, we need only find all finite subgroups of $O^+(\mathbb{R}^3)$ and look for all subgroup of these groups with index 2. Thus we turn now to the classification of all the finite subgroups of $O^+(\mathbb{R}^3)$. The method given below for the classifications of finite subgroups of $O^+(\mathbb{R}^3)$ is essentially due to Felix Klein [15]. Recently, Marjorie Senechal [27] extended Klein's method to include the classification of all finite subgroups of the orthogonal group $O(\mathbb{R}^3)$.

Let G be a finite subgroup of $O^+(\mathbb{R}^3)$. By Euler's Theorem 3.17, every element $g \in G$ except the identity is a rotation about a unique line L. Let $S=\{x \in \mathbb{R}^3 / \| x-x_0 \| =1\}$ be the unit sphere with center at the common fixed point of G, namely x_0. G acts on S, for every $g \in G$, $gx \in S$ for every $x \in S$ and g is completely determined by its action on the elements of S because S contains a basis for $\mathbb{R}^3$. Thus every $g \in G$ such that $g \neq 1_{\mathbb{R}^3}$ fixes exactly two points of the unit sphere S, namely the two antipodal points of the intersection $S \cap L$. We call these points the poles of g. Thus we have the following definition.

Definition 4.8:

Let G be a finite subgroup of $O^+(\mathbb{R}^3)$. A point p on S is a *pole* if there exists an element $g \in G$ such that $g \neq 1_{\mathbb{R}^3}$ and $g(p)=p$.

Note that it is possible that the same point $p \in S \cap L$ may be a pole for more than one group element.

Since the group G is finite, the set of all poles of all elements $g \in G$ such that $g \neq 1_{\mathbb{R}^3}$ is a finite set of points on the sphere S; we denote this set by P. Next we are going to examine this set in great detail.

Lemma 4.12:

The set of poles P is a *G-set*. That is, the set P is carried to itself by the action of G on the sphere.

Proof:

Let $p \in P$, then p is a pole for some element $1_{\mathbb{R}^3} \neq g \in G$. Let ρ be any element in G. We must show $\rho(p) \in P$; that is, $\rho(p)$ is left fixed by some element $\rho' \in G$ where $\rho' \neq 1_{\mathbb{R}^3}$. Let $\rho' = \rho g \rho^{-1}$. Then $\rho'(\rho(p)) = (\rho g \rho^{-1})(\rho(p)) = \rho g(p) = \rho(p)$ since $g(p)=p$; also $\rho g \rho^{-1} \neq 1_{\mathbb{R}^3}$ because if $\rho g \rho^{-1} = 1_{\mathbb{R}^3}$ then $\rho g = \rho$ and hence $g = 1_{\mathbb{R}^3}$, a contradiction.

Now we are going to apply the counting formula to the *G-set* P and show that the set of poles P has to be a particularly "nice" configuration of points.

Lemma 4.13:

Let G be a nontrivial finite subgroup of $O^+(\mathbb{R}^3)$. Then the number of *G-orbits* of poles is either 2 or 3.

Proof:

Let $O(P)=\{O_1,O_2,\ldots,O_k\}$ denote the distinct *G-orbits* of the set of poles P. We need to show that $k=2$ or 3. Choose a pole from each orbit, $x_i \in O_i$, $i=1,2,\ldots,k$. For each x_i, let $stab(x_i)$ be the stabilizer of x_i, and let r_i be the order of $stab(x_i)$. Since x_i is a pole, the stabilizer $stab(x_i)$ contains an element besides the identity element $1_{\mathbb{R}^3}$ of G. Then $r_i \geq 2$ for every $i=1,2,\ldots,k$. Let n_i be the number of poles in the ith *G-orbit* O_i. By the counting formula, $r_i n_i=|G|$ for $i=1,2,\ldots,k$. Each element $g \in G$, where $g \neq 1_{\mathbb{R}^3}$ has two poles; thus the total number of poles, counting repetition is $2(|G|-1)$. The set of elements of G with a given x_i is the stabilizer $stab(x_i)$ minus the identity element $1_{\mathbb{R}^3}$; that is, $\{g \in G | g(x_i)=x_i\}=stab(x_i)-\{1_{\mathbb{R}^3}\}$. Thus the pole x_i occurs as a pole of an element $g \in G$, when $g \neq 1_{\mathbb{R}^3}$, (r_i-1)-times. Now if p and p' are in the same orbit then the stabilizers $stab(p)$ and $stab(p')$ have the same order. To see this, we apply the counting formula, since p and p' are in the same orbit, then

$O_p=O_{p'}$, but $|stab(p)|=|G|/|O_p|=|G|/|O_{p'}|=|stab(p')|$. Since there are n_i poles in the ith orbit O_i, the total number of poles counting repetitions is $\sum_{i=1}^{k} n_i(r_i-1)$. Thus

$$\sum_{i=1}^{k} n_i(r_i-1)=2(|G|-1) \quad \ldots(1).$$

By the counting formula, we have $|G|=n_i r_i$. Dividing both sides of equation (1) by $|G|$, we obtain

$$2-\frac{2}{|G|}=\sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right) \quad \ldots(2).$$

Since G is nontrivial, the left side of equation (2) satisfies the inequality $1\leq 2-\frac{2}{|G|}<2$. On the other hand, since $r_i\geq 2$, each term on the right hand side satisfies the inequality $\frac{1}{2}\leq 1-\frac{1}{r_i}<1$. It follows form equation (2) that there can be at most three orbits; because if the number of orbits $k\geq 4$, then $2-\frac{2}{|G|}=\sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right)\geq\frac{4}{2}=2$ hence $\frac{2}{|G|}\leq 0$ and this is impossible. Thus the possible number of orbits are $k=1$, 2 or 3. If there is one orbit, then $2-\frac{2}{|G|}=1-\frac{1}{r_1} \Rightarrow \frac{1}{2}\leq 2-\frac{2}{|G|}<1$ which is impossible because $2-\frac{2}{|G|}\geq 1$. Hence the number of orbits is either *2* or *3*.

Now we are going to examine these two possibilities separately.

Theorem 4.14:

If G is a nontrivial finite subgroup of $O^{+}(\mathbb{R}^{3})$ and if there are only two *G-orbits* of poles, then G is cyclic and is generated by a rotation through the angle $2\pi/n$, where $n=|G|$.

Proof:

Let O_i, x_i, n_i, and r_i, $i=1,2,...,k$ be as in the proof above. Then the equation $2(|G|-1)=\sum_{i=1}^{k}n_i(r_i-1)$ becomes

$$2(|G|-1)=n_1(r_1-1)+n_2(r_2-1)$$
$$=n_1r_1-n_1+n_2r_2-n_2$$
$$=2|G|-(n_1+n_2).$$

Thus $2=n_1+n_2$. Since n_1 and n_2 are positive integers, then $n_1=n_2=1$, and hence $r_1=r_2=|G|=n$. This means that there are two poles, $x_1\in O_1$ and $x_2\in O_2$, and the two poles must lie on a diameter of the sphere S , and hence the rotations in G are rotations about this diameter, say L. Thus by Theorem 3.18, G is a subgroup of $O^{+}(\mathbb{R}^{2})$. Hence by Theorem 4.10, G is a cyclic group generated by the rotation ρ_θ about L through the angle $2\pi/n$.

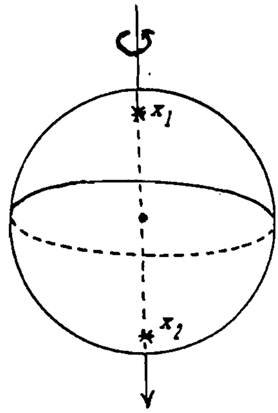


Figure 4.4

Next we consider the case when the number of *G-orbits* $k=3$. In this case, the situation is more complicated.

Theorem 4.15:

If G is a nontrivial subgroup of $O^+(\mathbb{R}^3)$ whose order is $n<\infty$ and if there are three *G-orbits* of poles, then G is one of the following groups:

1. The dihedral group, D_{2r} of the symmetry group of a regular r-gon.
2. The group T of rotations of the tetrahedral symmetry group.
3. The group O of rotations of a cube or regular octahedron symmetry group.
4. The group I of rotations of a regular dodecahedron or a regular icosahedron symmetry group.

Proof:

Let O_i, x_i, n_i, and r_i, $i=1,2,...,k$ be as in the proof of the previous Lemma 4.13. When $k=3$, equation (2) of the Lemma 4.13 becomes $2-\frac{2}{|G|}=\sum_{i=1}^{3}\left(1-\frac{1}{r_i}\right)=\left(1-\frac{1}{r_1}\right)+\left(1-\frac{1}{r_2}\right)+\left(1-\frac{1}{r_3}\right)$ and hence

$1+\frac{2}{n}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$. Without loss of generality assume that

$r_1\le r_2\le r_3$. If $r_1\ge 3$, we would have $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\le\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$, and

this implies $1+\frac{2}{|G|}\le 1$ hence $\frac{2}{|G|}\le 0$, a contradiction. Thus

$2\le r_1<3$ and hence $r_1=2$. Now $1+\frac{2}{n}=\frac{1}{2}+\frac{1}{r_2}+\frac{1}{r_3}\Rightarrow\frac{1}{2}+\frac{2}{n}=\frac{1}{r_2}+\frac{1}{r_3}$. If

$r_2\ge 4$, we would have $\frac{1}{r_2}+\frac{1}{r_3}\le\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$; then $\frac{1}{2}+\frac{2}{n}\le\frac{1}{2}\Rightarrow\frac{2}{n}\le 0$, a

contradiction. Hence r_2 is 2 or 3. Thus there are four possibilities:

Case 1: $r_1=2$, $r_2=2$, $r_3=n/2$, n is even and $n\ge 4$.

In this case, $n_1=n/2$, $n_2=n/2$ and $n_3=2$.

Case 2: $r_1=2$, $r_2=3$, $r_3=3$.

In this case, $n=12$, $n_1=6$, $n_2=4$ and $n_3=4$.

Case 3: $r_1=2$, $r_2=3$, $r_3=4$.

In this case, $n=24$, $n_1=12$, $n_2=8$ and $n_3=6$.

Case 4: $r_1=2$, $r_2=3$, $r_3=5$.

In this case, $n=60$, $n_1=30$, $n_2=20$ and $n_3=12$.

These four cases are the only possibilities. For if $r_1=2$ and $r_3\geq 6$, then $\frac{1}{r_2}+\frac{1}{r_3}\leq\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$, which implies $\frac{2}{n}\leq 0$, a contradiction.

Now we are going to consider each of these four cases in turn.

Case 1: $r_1=r_2=2$ and $r_3=r\geq 3$.

Then $n=2r$. Since $r_3=n/2$ then $n_3=2$ and hence there is one pair of poles $\{x_3,x_3'\}$ making the G-orbit O_3. Thus every element $\rho\in G$ either fixes x_3 and x_3' or interchanges them. Hence the elements of G are rotation about the diameter D passing through the poles x_3 and x_3', or else they are 180° rotation about a line D' perpendicular to D. Every rotation in the stabilizer $stab(x_3)$ fixes D, thus $stab(x_3)$ is a subgroup of $O^+(\mathbb{R}^3,D)\cong O^+(\mathbb{R}^2)$. Therefore, $stab(x_3)$ is a cyclic subgroup of order r generated by ρ_θ, where $\theta=2\pi/r$. Let $x_1\in O_1$ and consider the points x_1, $\rho_\theta(x_1)$, $\rho_\theta^2(x_1),\ldots,$ $\rho_\theta^{n-1}(x_1)$ on the sphere. We claim that they are distinct and they are the vertices of a regular r-gon. To see this, suppose $\rho_\theta^s(x_1)=\rho_\theta^t(x_1)$, where $t>s$. Then $\rho_\theta^{t-s}(x_1)=x_1$. But x_3 and x_3' are the only poles which are left fixed by ρ_θ^{t-s}, and x_1 cannot be x_3 or x_3' since $O_1\cap O_2=\{1_{\mathbb{R}^3}\}$. To see why these points are the vertices of a regular r-gon, recall that ρ_θ preserves distance, and thus $d(x_1,\rho_\theta(x_1))$ $=d(\rho_\theta(x_1),\rho_\theta^2(x_1))=\ldots=d(\rho_\theta^{r-1}(x_1),x_1)$. Therefore,

$x_1, \rho_\theta(x_1), \ldots, \rho_\theta^{r-1}(x_1)$ are the vertices of a regular r-gon. Since G consists of $2r$ rotations each of which leave the regular r-gon invariant, then G must be the symmetry group of the r-gon. Hence, G is the dihedral group D_{2r}.

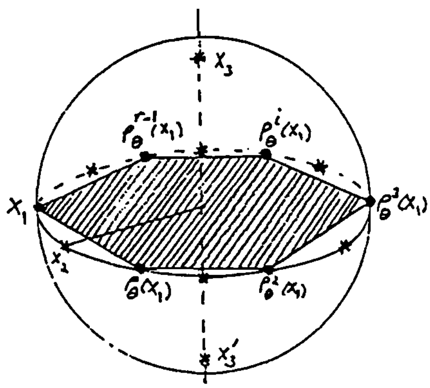


Figure 4.5

Case 2: $r_1=2,\ r_2=r_3=3$

Then G is a group of order 12. The orbit of x_3 consists of four points. Choose one, say u, and choose a generator ρ for the $stab(x_3)$. Then the points u, $\rho(u)$, $\rho^2(u)$ are distinct points of the sphere. Since ρ preserves distance, they are equidistant from x_3, and they are the vertices of an equilateral triangle; see the figure below. Note that the points x_3, $\rho(u)$, and $\rho^2(u)$ are equidistant form u. Therefore, x_3, u, $\rho(u)$, $\rho^2(u)$ are the vertices of a regular tetrahedron T, which is left invariant by the rotations of G. Since the order of G is 12, it must be the rotational symmetry group of T.

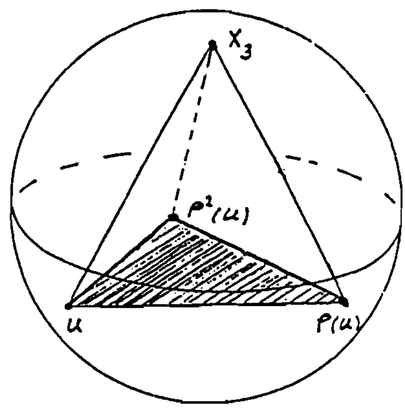


Figure 4.6

Case 3: $r_1=2$, $r_2=3$, $r_3=4$.

Then the order of G is 24. There are six points in the orbit of x_3. Choose one, say u, and let ρ generate $stab(x_3)$. Then u, $\rho(u)$, $\rho^2(u)$, and $\rho^3(u)$ are equidistant from x_3 and are the vertices of a square. Also the points x_3 and x_3' are equidistant from u. Therefore, x_3, u, $\rho(u)$, $\rho^2(u)$, $\rho^3(u)$, x_3' are the vertices of a regular octahedron O and O is left invariant by G, hence G is the rotational symmetry group of O.

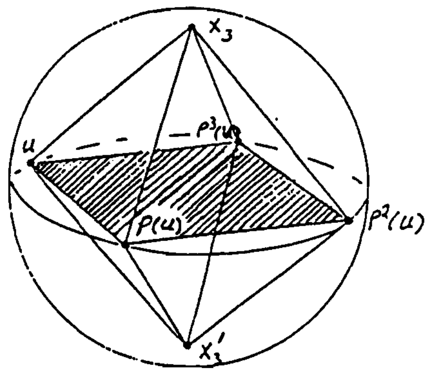


Figure 4.7

Case 4: $r_1=2$, $r_2=3$, $r_3=5$.

Then G is a group of order 60. There are 12 points in the orbit of x_3. Choose two, say u and v. If ρ is a generator of $stab(x_3)$, then u, $\rho(u)$, $\rho^2(u)$, $\rho^3(u)$, and $\rho^4(u)$ are all distinct and equidistant from x_3, and they are the vertices of a regular pentagon; see the figure below. Also v, $\rho(v)$, $\rho^2(v)$, $\rho^3(v)$, and $\rho^4(v)$ are all distinct and equidistant from x_3 and form the vertices of a regular pentagon. The 12 points x_3, x_3', u, $\rho(u)$, $\rho^2(u)$, $\rho^3(u)$, $\rho^4(u)$, v, $\rho(v)$, $\rho^2(v)$, $\rho^3(v)$, and $\rho^4(v)$ are the vertices of a regular icosahedron I, and hence G is the rotational symmetry group of I.

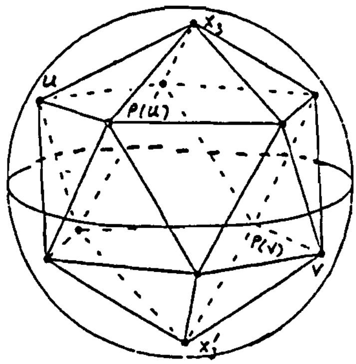


Figure 4.8

To complete the proof of the Theorem, it remains to show that the rotation group of a regular octahedron is also a rotation group of a cube; and the rotation group of a regular icosahedron is the also a rotation group of a regular dodecahedron. For this purpose we introduce the concept of dual polyhedron. Given a polyhedron, we can construct its dual polyhedron as follows: The vertices of the dual are the centers of the faces of the original polyhedron. Two centers are joined by an edge if the corresponding faces meet in an edge. The dual of a platonic solid is again a platonic solid. If this process in done twice one recovers the original platonic solid (or smaller version of it). The dual of a tetrahedron is a tetrahedron; the dual of the regular octahedron is a cube; and the dual of an icosahedron is a dodecahedron. Since any symmetry of a platonic solid will induce a symmetry on its corresponding dual and vice versa, the proof is complete.

Summarizing these results, we find that any finite subgroup of $O^+(\mathbb{R}^3)$ is one of the groups in the list below:

1 C_n, a cyclic group of order n generated by a rotation through an angle $2\pi/n$, $n=1,2,\ldots$; $C_n \cong Z_n$.

2 D_{2r}, a dihedral group of order $2r$ generated by a rotation through an angle $2\pi/r$, $r=1,2,\ldots$ and 180° rotations.

3 T, the rotational symmetry group of a tetrahedron; $T \cong A_4$.

4 O, the rotational symmetry group of a cube or an octahedron; $O \cong S_4$.

5 I, the rotational symmetry group of an icosahedron or a dodecahedron; $I \cong A_5$.

As an immediate consequence of this result and Theorem 4.11, we can give a complete list of all possible finite subgroup of $O(\mathbb{R}^3)$. In the theorem below, we are going to use the same symbols as in Theorem 4.11. Let G be a finite subgroup of $O(\mathbb{R}^3)$, and let $H=G \cap O^+(\mathbb{R}^3)$, $K=\{1_{\mathbb{R}^3}, -1_{\mathbb{R}^3}\}$, and $\tau=-1_{\mathbb{R}^3}$. Also, G^+ is a finite rotation subgroup of $O(\mathbb{R}^3)$ containing H with $[G^+:H]=2$.

Theorem 4.16:

If G is a finite subgroup of $O(\mathbb{R}^3)$ then G is one of the groups in the following two tables.

Table 4.3

if G contains $\tau=-1_{\mathbb{R}^3}$				
	H	G	Order of G $\lvert G\rvert=2\lvert H\rvert$	Remarks
1	C_n	$C_n\times K$ or $C_n\cup\tau C_n$	$2n$	$G\cong C_{2n}$ (if n is odd) $G\cong D_4^{2n}$ (if $n=2$)
2	D_{2n}	$D_{2n}\times K$ or $D_{2n}\cup\tau D_{2n}$	$4n$	$G\cong D_{2(2n)}$ if n is odd, $n\neq 1$
3	T	$T\times K$ or $T\cup\tau T$	24	G is not the complete symmetry group of tetrahedron
4	O	$O\times K$ or $O\cup\tau O$	48	Complete symmetry group of octahedron
5	I	$I\times K$ or $I\cup\tau I$	120	Complete symmetry group of icosahedron

Table 4.4

if G does not contain $\tau=-1_{\mathbb{R}^3}$			
H	$G^{+}=H\cup H^{+}$	$G\cong G^{+}$	Order of G
C_n $n=1,2,...$	C_{2n}	C_{2n}	$2n$
C_n $n=2,3,...$	D_{2n}	D_{2n}	$2n$
D_{2n} $n=1,2,...$	D_{4n}	D_{4n}	$4n$
T	O	O	24

Table 4.3

if G contains $\tau=-1_{\mathbb{R}^3}$				
	H	G	Order of G $\|G\|=2\|H\|$	Remarks
1	C_n	$C_n\times K$ or $C_n\cup\tau C_n$	$2n$	$G\cong C_{2n}$ (if n is odd) $G\cong D_4$ (if $n=2$)
2	D_{2n}	$D_{2n}\times K$ or $D_{2n}\cup\tau D_{2n}$	$4n$	$G\cong D_{2(2n)}$ if n is odd, $n\neq 1$
3	T	$T\times K$ or $T\cup\tau T$	24	G is not the complete symmetry group of tetrahedron
4	O	$O\times K$ or $O\cup\tau O$	48	Complete symmetry group of octahedron
5	I	$I\times K$ or $I\cup\tau I$	120	Complete symmetry group of icosahedron

Table 4.4

if G does not contain $\tau=-1_{\mathbb{R}^3}$			
H	$G^+=H\cup H^+$	$G\cong G^+$	Order of G
C_n $n=1,2,...$	C_{2n}	C_{2n}	$2n$
C_n $n=2,3,...$	D_{2n}	D_{2n}	$2n$
D_{2n} $n=1,2,...$	D_{4n}	D_{4n}	$4n$
T	O	O	24

CHAPTER 5

Applications

Geometric groups play a very important role in science, in particular, in physics and chemistry and their importance continue to grow rapidly. In this chapter, we confine our selves with two applications. First, we use the rotation group of a *2*-dimensional Euclidean space to develop plane trigonometry. Second, we use the finite symmetric groups of $O(\mathbb{R}^2)$ and $O(\mathbb{R}^3)$ via Pólya's enumeration theory to isomer enumeration.

5.1 Plane Trigonometry

Elementary courses in plane trigonometry are concerned with the study of the six trigonometric functions sine, cosine, tangent, cosecant, secant, and cotangent. These six trigonometric functions are defined either for angles (or their measurement) or as circular functions via the winding function. In Chapter 2, we have shown that every rotation of a Euclidean plane E is completely determined by the angle of rotation around the origin. In this section, we are going to demonstrate how the six trigonometric functions can be defined as functions from the rotation group $O^+(E)$ to the field of real numbers $\mathbb{R}$, where E is a *2*-dimensional real Euclidean plane. Moreover, we are going to demonstrate how this approach to trigonometry makes the derivations of the properties of the trigonometric functions natural and straight forward. Throughout the discussion, a comparison between the results we obtain using the rotation group and the corresponding classical ones will be made.

First we need the following lemma.

Lemma 5.1:

Let e_1 and e_2 be two unit vectors in E. Then there exists a unique element $\rho \in O^+(E)$ such that $\rho(e_1)=e_2$.

Proof:

If e_1 and e_2 are linearly dependent then $e_2=re_1$, where $r=\pm 1$. If $r=1$, let $\rho=1_E$, then $\rho(e_1)=e_2$. If $r=-1$, let $\rho=-1_E$, then $-1_E(e_1)=1_E(-e_1)=-e_1=e_2$. Clearly ρ in both cases is unique. On the other hand, if e_1 and e_2 are linearly independent then $B=\{e_1,e_2\}$ is a basis for E. Define $\rho:E\rightarrow E$ by $\rho(e_1)=e_2$ and $\rho(e_2)=e\in E$, e being a nonzero vector. Thus it remains to show that ρ is unique. Let $\rho' \in O^+(E)$ where $\rho'(e_1)=e_2$.

Then $\rho(e_2)=\rho(\rho'(e_1))$

$=\rho'(\rho(e_1))$ (since $O^+(E)$ is commutative)

$=\rho'(e_2)$.

Thus $\rho(e_1)=\rho'(e_1)$ and $\rho(e_2)=\rho'(e_2)$. But since $\{e_1,e_2\}$ is a basis for E then $\rho=\rho'$.

Corollary:

Let A and B be nonzero vectors in E, then there exists a unique rotation $\sigma \in O^+(E)$ such that $\sigma(A)=cB$ for some positive scalar c.

Proof:

Since A and B are nonzero vectors, than $\frac{A}{|A|}$ and $\frac{B}{|B|}$ are unit vectors, and by Lemma 5.1, there exists a unique element $\sigma \in O^+(E)$ such that $\sigma\left(\frac{A}{|A|}\right)=\frac{B}{|B|}$ and hence $\sigma(A)=\frac{|A|}{|B|}B$.

We begin by defining the cosine function as a function whose domain is the rotation group $O^+(E)$ and range a subset of the field of real numbers $\mathbb{R}$.

$$cos : O^+(E) \to \mathbb{R}.$$

The definition is based on the following lemma.

Lemma 5.2:

Let ρ be any rotation of E, and let A and B be any two nonzero elements in E. Then $\frac{\langle A,\rho(A)\rangle}{|A|^2}=\frac{\langle B,\rho(B)\rangle}{|B|^2}$. That is, the scalar $\frac{\langle A,\rho(A)\rangle}{|A|^2}$ is the same for all nonzero vectors $A \in E$.

Proof:

By the corollary above, let $\sigma \in O^{+}(E)$ be such that $\sigma(A)=cB$ for a nonzero scalar c. Then

$$\frac{\langle A,\rho(A)\rangle}{|A|^2}=\frac{\langle\sigma(A),\sigma(\rho(A))\rangle}{|\sigma(A)|^2} \text{ (since } \sigma \text{ is an isometry)}$$

$$=\frac{\langle\sigma(A),\rho(\sigma(A))\rangle}{|\sigma(A)|^2} \text{ (since } O^{+}(E) \text{ is commutative)}$$

$$=\frac{\langle cB,\rho(cB)\rangle}{|cB|^2} \text{ (since } \sigma(A)=cB)$$

$$=\frac{c^2\langle B,\rho(B)\rangle}{c^2|B|^2} \text{ (since } \langle\ ,\ \rangle \text{ is bilinear)}$$

$$=\frac{\langle B,\rho(B)\rangle}{|B|^2}.$$

Definition 5.1:

The *cosine* of a rotation ρ of E is defined by

$$\cos\rho=\frac{\langle A,\rho(A)\rangle}{|A|^2}$$

where A is any nonzero vector of E.

As an immediate consequence of this definition, we have

Theorem 5.3:

1. $\cos\rho=0$ if and only if A is orthogonal to $\rho(A)$ for a nonzero vector $A \in E$.

2. $\cos 1_E = 1$.

3. $\cos(-1_E) = -1$.

The corresponding results for angles are:

1'. $\cos(\pi/2) = 0$.

2'. $\cos 0 = 1$.

3'. $\cos\pi = -1$.

Theorem 5.4:

For any $\rho \in O^+(E)$, $\cos(\rho^{-1}) = \cos\rho$.

Proof:

Let A be any nonzero vector of E. Since $\rho \in O^+(E)$, then $\rho^{-1} \in O^+(E)$. $\langle A, \rho^{-1}(A)\rangle = \langle \rho(A), \rho(\rho^{-1}(A))\rangle$. Thus $\frac{\langle A, \rho^{-1}(A)\rangle}{\|A\|^2} = \frac{\langle \rho(A), A\rangle}{\|A\|^2}$ and this implies $\cos(\rho^{-1}) = \cos(\rho)$.

The corresponding result in terms of angles is $\cos(-\alpha) = \cos\alpha$; that is, $\cos$ is an even function. Note that if ρ is a rotation represented by the angle α then ρ^{-1} is the rotation represented by the angle $-\alpha$.

Let $\rho \in O^+(E)$. Then $-\rho$ denotes the composition $-1_E \rho \in O^+(E)$.

Theorem 5.5:

Let $A \in E$ be a nonzero vector. Then

$$\cos(-\rho) = \frac{\langle A, -\rho(A)\rangle}{\|A\|^2}$$

$$= \frac{\langle A, (-1_E \circ \rho)(A)\rangle}{\|A\|^2}$$

$$= \frac{\langle A, \rho(-1_E(A))\rangle}{\|A\|^2} \text{ (since } O^+(E) \text{ is commutative)}$$

$$= \frac{\langle A, \rho(-A)\rangle}{\|A\|^2}$$

$$= \frac{\langle A, (-1)\rho(A)\rangle}{\|A\|^2} \text{ (since } \rho \text{ is a linear transformation)}$$

$$= -\left(\frac{\langle A, \rho(A)\rangle}{\|A\|^2}\right) = -\cos\rho.$$

If ρ is a rotation represented by the angle α, then the rotation $-\rho$ is represented by the angle $\pi+\alpha$; see the figure below.

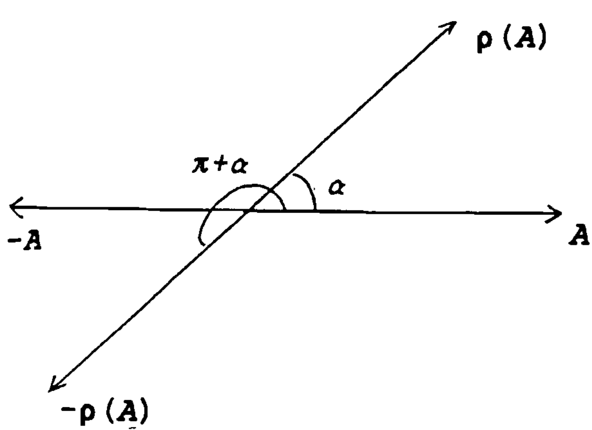


Figure 5.1

Thus in terms of angles, the above theorem states $cos(\pi+\alpha) = -cos\alpha$.

Corollary:

$cos(-\rho^{-1}) = -cos\rho$.

In terms of angles, this states $cos(\pi-\alpha) = -cos\alpha$.

We know from linear algebra if A and B are nonzero vectors in an inner product space V, then the angle θ between A and B is obtained from $\cos\theta = \frac{\langle A,B\rangle}{\|A\|\|B\|}$. Now we need to examine this result in terms of rotations.

Let A and B be nonzero vectors in E. Then by the corollary to Lemma 5.1, there exists a unique rotation $\rho \in O^{+}(E)$ such that $\rho(A)=cB$ for some positive scalar c. Thus the "angle" between A and B is the angle represented by the rotation ρ.

$$\cos\rho = \frac{\langle A, \rho(A)\rangle}{\|A\|^2} = \frac{\langle A, cB\rangle}{\|A\|^2} = \frac{c\langle A, B\rangle}{\|A\|^2}.$$

Since $\rho(A)=cB$

$$\begin{aligned}\|A\| &= \sqrt{\|A\|^2} = \sqrt{\langle A,A\rangle} = \sqrt{\langle \rho(A), \rho(A)\rangle} \\ &= \sqrt{\langle cB, cB\rangle} = \sqrt{c^2\langle B,B\rangle} = c\sqrt{\langle B,B\rangle} \\ &= c\|B\|.\end{aligned}$$

Thus $\cos\rho = \frac{\langle A,B\rangle}{\|A\|\|B\|}$.

The next theorem is in part a restatement of Lemma 2.7.

Theorem 5.6:

Let $\rho \in O^{+}(E)$, and let $[\rho]_B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the matrix of ρ relative to an orthonormal basis $B=\{e_1, e_2\}$. Then

1. $a=d=\cos\rho$.
2. $b^2=c^2=1-\cos^2\rho$.
3. $c=-b$.

Proof:

1. $\rho(e_1)=ae_1+be_2$. Since e_1 is a nonzero unit vector in E, then

$$\cos\rho = \frac{\langle e_1, \rho(e_1)\rangle}{\|e_1\|^2} = \langle e_1, ae_1+be_2\rangle = a\langle e_1, e_1\rangle = a.$$

Similarly if e_2 is used to compute $\cos\rho$, we obtain $\cos\rho=d$ and thus $a=d=\cos\rho$.

2. Since $\|\rho(e_1)\|=\|\rho(e_2)\|=1$, then $a^2+c^2=1$ and $b^2+d^2=1$, and this implies $c^2=1-a^2=1-\cos^2\rho$ and $b^2=1-\cos^2\rho$.
3. $\{\rho(e_1), \rho e_2)\}$ is an orthonormal set. Thus $\rho(e_2) \in \langle\rho(e_1)\rangle^{\perp}$, but $\dim(\langle\rho(e_1)\rangle^{\perp})=1$ and $(-c,a) \in \langle\rho(e_1)\rangle^{\perp}$. Thus $\rho(e_2)=r(-c,a)=(-rc,ra)$ for some scalar r. But $\|\rho(e_1)\|=\|\rho(e_2)\|=1$ implies $a^2+c^2=1$ and $r^2a^2+r^2c^2=1$, and thus $r=\pm 1$. $\rho(e_2)=(b,d)=\pm(-c,a)$ implies $b=\pm c$ and $d=\pm a$. Since $\rho \in O^{+}(E)$, then $\det\rho=ac-bd=\cos^2\rho-bc=1$, and this implies

$-bc=1-cos^2\rho=c^2$ (by part 2). If $c\neq 0$ then $-b=c$, and if $c=0$ then since $b=\pm c$, we have $b=0$.

Remark:

The scalars a and $|c|$ do not depend on the choice of basis $b=\{e_1,e_2\}$ for E. That is, the a and $|c|$ are unique for a given rotation $\rho\in O^+(E)$. In fact, $a=cos\rho$ and $|c|=\sqrt{1-\cos^2\rho}$.

Corollary:

If $\rho\in O^+(E)$ then $-1\leq cos\rho\leq 1$.

Proof:

By Theorem 5.6 (2), we know $1-cos^2\rho$ is a square. Thus $1-cos^2\rho\geq 0$ and this is equivalent to $-1\leq cos\rho\leq 1$.

Our next objective is to define the sine of a rotation ρ. Of course, we want the identity $sin^2\rho+cos^2\rho=1$ to hold for any $\rho\in O^+(E)$. The identity $sin^2\rho+cos^2\rho=1$ is equivalent to $sin^2\rho=1-cos^2\rho$. We know from Theorem 5.6, that $1-cos^2\rho$ is the square of a scalar b; however, the trouble is that $1-cos^2\rho$ is also the square of $-b$. The question is should we define $sin\rho=b$ or $sin\rho=-b$? The choice depends on the "orientation" of E.

Now we are going to develop the concept of orientation in a vector space V. Let $B=\{e_1,e_2,\ldots,e_n\}$ and $B'=\{e_1',e_2',\ldots,e_n'\}$ be two bases for a real vector space V. Let $P=(p_{ij})$ be the transition matrix from B to B'. Recall that p_{ij} is given by $e'_i = \sum_{j=1}^{n} e_j p_{ji}$. Also the matrix P is unique and invertible, and thus $det(P) \neq 0$.

Definition 5.2:

The two bases B and B' of V are said to be *similarly oriented* (or have the same *orientation*) if $det(P)>0$ and they are said to have *opposite orientation* if $det(P)<0$. If B and B' have the same orientation, we denote this by $B \sim B'$; otherwise we denote this by $B \nsim B'$.

The proof of the next theorem is straightforward.

Theorem 5.7:

The relation $\sim$ is an equivalence relation on the set of all bases of a vector space V.

Since $\sim$ is an equivalence relation on the set of bases for V, it partitions the bases for V into two equivalence classes. To "orient" a vector space V means to decide which class is designated as the positively oriented class and which class the negatively oriented one. More precisely, we have the following definition.

Definition 5.3:

An *oriented vector space* is a vector space V where one basis (and hence the whole class) is designated as being positively oriented.

In the case of the 2-dimensional real vector space $\mathbb{R}^2$, the space is oriented so that the standard basis $\{(1,0),(0,1)\}$ has a positive orientation. Note that if $\{\overline{e_1},\overline{e_2}\}$ has the same orientation as $\{(1,0),(0,1)\}$ then $\overline{e_1}$ can be rotated about the origin to coincide with $\overline{e_2}$ in a counterclockwise direction through an angle θ with measure such that $0<\theta<\pi$. This observation leads to the following definition.

Definition 5.4:

Let ρ be a rotation in $O^+(E)$ where $\rho\neq\pm 1_E$, and let A be a nonzero vector. ρ is called a *counterclockwise rotation* if $\{A,\rho(A)\}$ is positively oriented and ρ is called a *clockwise rotation* if $\{A,\rho(A)\}$ is negatively oriented.

Theorem 5.8:

Let $\rho\in O^+(E)$ when $\rho\neq\pm 1_E$. If A is a nonzero vector of E, the vectors A and $\rho(A)$ are linearly independent. Further, the orientation of $\{A,\rho(A)\}$ is the same for all nonzero vectors $A\in E$.

Proof:

Assume that $A \in E$ and $A \neq 0$ but $\{A, \rho(A)\}$ is linearly dependent. Then $\rho(A)=cA$ for some scalar c. Thus $\|\rho(A)\|=\|A\|$ implies $c^2\|A\|^2=\|A\|^2$. Since $\|A\|^2 \neq 0$, then $c=\pm 1$, and hence $\rho=\pm 1_E$ contrary to the hypothesis.

Let A and B be nonzero vectors in E. We need to show that $\{A, \rho(A)\}$ and $\{B, \rho(B)\}$ have the same orientation. By the corollary to the Lemma 5.1, there exists a unique rotation $\sigma \in O^+(E)$ such that $\sigma(A)=cB$ for some scalar c. Let $[\sigma]_B$ be the matrix of σ relative to the basis $\{A, \rho(A)\}$. Thus $[\sigma(A), \sigma(\rho(A))]=[A, \rho(A)][\sigma]_B$. Since σ is a rotation, $det[\sigma]_B=1$.

$[\sigma(A), \sigma(\rho(A))]=[\sigma(A), \rho(\sigma(A))]$ since $O^+(E)$ is commutative

$$=[cB, \rho(cB)]$$

$$=[cB, c\rho(B)]$$

$$=c^2[B, \rho(B)].$$

Thus $1/c^2[\sigma(A), \sigma(\rho(A))]=[B, \rho(B)]$, or $[B, \rho(B)]$

$$=[c^{-1}\sigma(A), c^{-1}(\sigma(\rho(A))]=[A, \rho(A)]\begin{bmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{bmatrix}[\sigma]_B.$$

Since $det\left(\begin{bmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{bmatrix}[\sigma]_B\right)=(c^{-1})^2>0$, it follows that $\{A, \rho(A)\}$ and $\{B, \rho(B)\}$ have the same orientation.

Now we are in a position to define the sine function $sin: O^+(E) \to \mathbb{R}$. Recall that we want the identity $sin^2\rho=1-cos^2\rho$ to hold for all rotations ρ. Since $cos 1_E=1$ and $cos(-1_E)=-1$, we define $\sin 1_E = \sin(-1_E) = 0$. If $\rho \neq \pm 1_E$, $1-\cos^2\rho$ is a nonzero square real number. As usual $\sqrt{1-\cos^2\rho}$ denotes the positive real number b such that $b^2 = 1-\cos^2\rho$.

Definition 5.5:

Let $\rho \in O^+(E)$. The *sine* of the rotation ρ is defined as follows:

$$\sin\rho = \begin{cases} \sqrt{1-\cos^2\rho} & \text{if } \rho \text{ is a counterclockwise rotation} \\ -\sqrt{1-\cos^2\rho} & \text{if } \rho \text{ is a clockwise rotation} \\ 0 & \text{if } \rho = \pm 1_E. \end{cases}$$

Note that the definition of $\sin\rho$ does not depend on the choice of a basis for E. It depends only on the orientation of ρ.

Theorem 5.9:

Let $\rho \in O^+(E)$. Then

1. $\sin(\rho^{-1}) = -\sin\rho$.
2. $\sin(-\rho) = -\sin\rho$.

Proof:

1. Clearly if $\rho = \pm 1_E$, then $\sin(\rho^{-1}) = 0$, and the identity holds. If $\rho \neq \pm 1_E$, then the rotations ρ and ρ^{-1} have opposite orientations. To see this, let A be a nonzero vector in E. Then by Theorem 5.8, $B = \{A, \rho(A)\}$ is a basis of E. Let

$P = [\rho^{-1}]_B = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ be the matrix of ρ^{-1} relative to the basis B.

Since $\rho^{-1} \in O^+(E)$, then $\det(P) = 1$. $(\rho^{-1}(A), \rho^{-1}(\rho(A)) = (A, \rho(A))P$, from which it follows that $(\rho^{-1}(A), A) = (A, \rho(A))P = (A, \rho(A))\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$.

Hence $(A, \rho^{-1}(A)) = (A, \rho(A))\begin{bmatrix} p_2 & p_1 \\ p_4 & p_3 \end{bmatrix}$. Since $det\begin{bmatrix} p_2 & p_1 \\ p_4 & p_3 \end{bmatrix} = -1 < 0$, it follows that $\{A, \rho(A)\}$ and $\{A, \rho^{-1}(A)\}$ have opposite orientations. Therefore, ρ and ρ^{-1} have opposite orientations. Without loss of generality, we may assume ρ is a counterclockwise rotation, and hence ρ^{-1} is a clockwise rotation. Since ρ is a counterclockwise rotation, $\sin\rho = \sqrt{1-\cos^2\rho}$, and since ρ^{-1} is a clockwise rotation, $\sin\rho^{-1} = -\sqrt{1-\cos^2\rho^{-1}} = -\sqrt{1-\cos^2\rho}$; since by Theorem 5.4, $\cos\rho^{-1} = \cos\rho$. Thus $\sin\rho^{-1} = -\sin\rho$.

2. The proof is similar to part 1.

Theorem 5.10:

Let $A \neq 0$ be a vector in E. Then there exists a unique counterclockwise rotation $\rho \in O^+(E)$ such that the vectors A and $\rho(A)$ are orthogonal. The rotation ρ is called the "$90°$ rotation". In this case, $\sin\rho = 1$.

Proof:

Let B be a vector in E orthogonal to A. Then by the corollary to Lemma 5.1, there exists a unique rotation $\sigma \in O^+(E)$

such that $\sigma(A)=cB$, where $c\neq 0$ is a scalar. Hence, $B=\frac{1}{c}\sigma(A)$. Since A and B are orthogonal, then $0=\langle A,B\rangle=\langle A,\frac{1}{c}\sigma(A)\rangle$ $=\frac{1}{c}\langle A,\sigma(A)\rangle$. Thus $\langle A,\sigma(A)\rangle=0$ and hence $\cos\sigma=0$. If σ is a counterclockwise rotation, we take $\sigma=\rho$. If σ is a clockwise rotation then σ^{-1} is a counterclockwise rotation and $\sigma^{-1}(\frac{1}{c}A)=B$. Moreover, $0=\langle A,B\rangle=\langle A,\sigma^{-1}(\frac{1}{c}A)\rangle$ $=\frac{1}{c}\langle A,\sigma^{-1}(A)\rangle$. Thus $\langle A,\sigma^{-1}(A)\rangle=0$ and hence $\cos\sigma^{-1}=0$. In this case, we take $\rho=\sigma^{-1}$. In either case, $\sin\rho=1$.

Corollary:

There exists a unique clockwise rotation τ such that A is orthogonal to $\tau(A)$ for all vectors $A\in E$. We call τ the "270° rotation", and $\sin\tau=-1$.

Proof:

Take $\tau=\rho^{-1}$, where ρ is the unique "90° rotation" of Theorem 5.10.

In Theorem 5.6, we have shown if $\rho\in O^{+}(E)$ and $B=\{e_1,e_2\}$ is an orthonormal basis of E, then the matrix of ρ relative to B is given by $[\rho]_B=\begin{bmatrix}\cos\rho & -c\\ c & \cos\rho\end{bmatrix}$, where c is a scalar and

$|c|=\sqrt{1-\cos^2\rho}$. Thus if the basis $B=\{e_1,e_2\}$ is positively oriented, then $c=\sin\rho$; if $B=\{e_1,e_2\}$ is negatively oriented, $c=-\sin\rho$. Thus we have proved the following theorem.

Theorem 5.11:

Let ρ be a rotation of E, and let $B=\{e_1,e_2\}$ be an orthonormal basis of E. Then the matrix of ρ relative to B has one of the following forms:

1. $[\rho]_B=\begin{bmatrix}\cos\rho & -\sin\rho\\ \sin\rho & \cos\rho\end{bmatrix}$ if and only if B is positively oriented.

2. $[\rho]_B=\begin{bmatrix}\cos\rho & \sin\rho\\ -\sin\rho & \cos\rho\end{bmatrix}$ if and only if B is negatively oriented.

Now we are going to establish the formulas that correspond to the usual sum and difference of angles formulas in classical trigonometry. For the angles α and β, these formulas are:

1. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.
2. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.
3. $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.
4. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

Note that if an angle α represents the rotation ρ and an angle β represents the rotation σ, then the angle $\alpha+\beta$ represents the rotation $\rho\sigma$ and the angle $\alpha-\beta$ represents the rotation $\rho\sigma^{-1}$.

Theorem 5.12:

If ρ and $\sigma \in O^{+}(E)$, then

1. $\cos\rho\sigma = \cos\rho\cos\sigma - \sin\rho\sin\sigma$.

2. $\sin\rho\sigma = \sin\rho\cos\sigma + \cos\rho\sin\sigma$.

Proof:

Let $B=\{e_1, e_2\}$ be a positively oriented basis of E. From Theorem 5.11, it follows that $[\rho]_B = \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix}$ and $[\sigma]_B = \begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix}$. Thus the matrix of the rotation $\rho\sigma$ relative to the basis B is given by:

$$[\rho\sigma]_B = [\rho]_B[\sigma]_B = \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix}\begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix}$$

$$= \begin{bmatrix} \cos\rho\cos\sigma - \sin\rho\sin\sigma & -\cos\rho\sin\sigma - \sin\rho\cos\sigma \\ \cos\rho\sin\sigma + \sin\rho\cos\sigma & \cos\rho\cos\sigma - \sin\rho\sin\sigma \end{bmatrix}.$$

Since $[\rho\sigma]_B = \begin{bmatrix} \cos(\rho\sigma) & -\sin(\rho\sigma) \\ \sin(\rho\sigma) & \cos(\rho\sigma) \end{bmatrix}$, this proves the two formulas.

Corollary:

The following identities hold for any rotation $\rho, \sigma \in O^{+}(E)$.

1. $\cos\rho\sigma^{-1} = \cos\rho\cos\sigma + \sin\rho\sin\sigma$.

2. $\cos\rho^2 = \cos^2\rho - \sin^2\rho$.

3. $\cos^2\rho = \frac{1}{2}(1+\cos\rho^2)$.

4. $\sin\rho\sigma^{-1} = \sin\rho\cos\sigma - \cos\rho\sin\sigma$.

5. $\sin\rho^2 = 2\sin\rho\cos\rho$.

6. $\sin^2\rho = \frac{1}{2}(1-\cos\rho^2)$.

The proof of the corollary is straightforward.
Here, we stop and leave to the reader to define the other four trigonometric functions, tangent, cosecant, secant, and cotangent, and to develop the remainder of trigonometry. Our objective in this section was not only to demonstrate how the group structure of the rotation group $O^+(E)$ simplifies the derivation of various trigonometric formulas but it makes the study of trigonometry more attractive and general.

Before we close this section, we need to make a final comment concerning the relationship between orientation of an n-dimensional real Euclidean space E and the rotations and reflections of E.

In the 2-dimensional Euclidean plane $\mathbb{R}^2$, we are accustomed to the fact that rotations of the plane "preserve orientation" and reflections of the plane about a line "reverse orientation". That is, if $\Delta = \rightarrow ABC$ is a directed triangle in $\mathbb{R}^2$ and ρ is a rotation of $\mathbb{R}^2$, then its image under ρ is the directed triangle $\Delta' = \rightarrow\rho(A)\rho(B)\rho(C)$ which agrees with Δ in orientation (see the Figure 5.2 below).

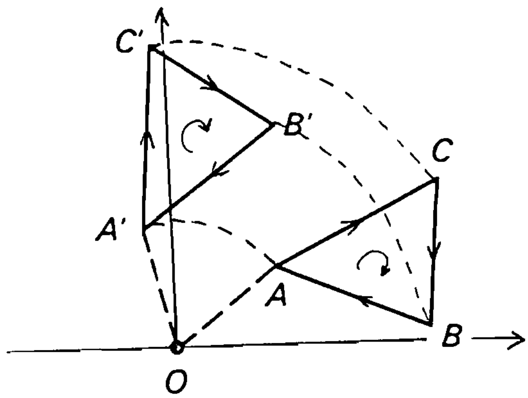


Figure 5.2

On the other hand, if σ is a reflection of $\mathbb{R}^2$ about line L, then the image of the triangle $\blacktriangle$ under σ is the directed triangle $\blacktriangle^{t} = \to\sigma(A)\sigma(B)\sigma(C)$, which disagrees with $\blacktriangle$ in orientation (see Figure 5.3 below).

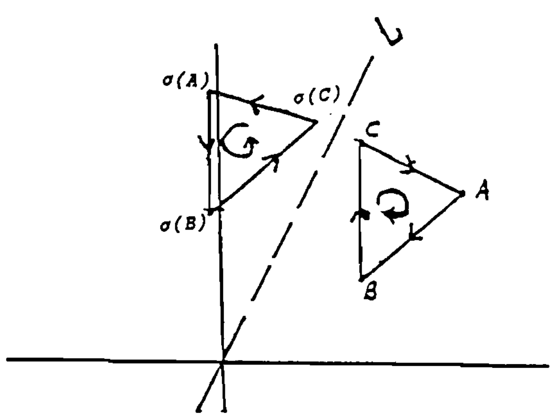


Figure 5.3

In general, it turns out any rotation of an n-dimensional Euclidean space E preserves orientation and any reflection of E reverses orientation. For this reason, rotations of E are sometimes called orientation-preserving (direct or proper) isometries, and reflections of E are called orientation-reversing (opposite or improper) isometries.

Theorem 5.13:

Let E be an n-dimensional real Euclidean space. Let $B=\{e_1,e_2,\ldots,e_n\}$ be a basis for E. If ρ is a rotation of E, then the bases B and $B'=\{\rho(e_1),\rho(e_2),\ldots,\rho(e_n)\}$ have the same orientation. On the other hand, if σ is a reflection of E, then B and $B'=\{\sigma(e_1),\sigma(e_2),\ldots,\sigma(e_n)\}$ have opposite orientation. That is, rotations of E preserve orientation while reflections reverse orientation.

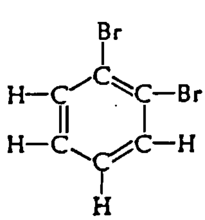
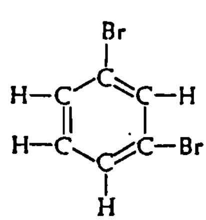
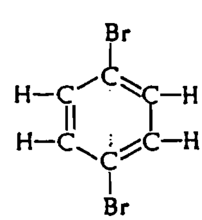
Proof:

Let ρ be a rotation of E, and let $P=[\rho]_B$ be the matrix of ρ relative to the basis B. Then $\rho(e_j)=\sum_i P_{ij}e_i$ for $j=1,2,\ldots,n$.

Thus, $B'=[\rho(e_1)\rho(e_2)\ldots\rho(e_n)]=[e_1e_2\ldots e_n]P$. But ρ is a rotation of E; thus, $det(P)=1>0$, and hence B and B' have the same orientation. On the other hand, if σ is a reflection of E, then $det(P')=-1<0$, where $p'=[\sigma]_B$. Thus, B and B' have opposite orientation.

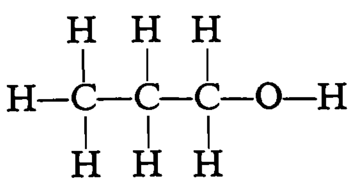
5.2 Isomer Enumeration and Pólya's Theorem

In this section, we are going to see how the finite symmetry groups of bounded sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ are employed via Pólya's Theorem of Enumeration to the study of the chemical enumeration problem. In particular, we will be interested in the study of the enumeration of isomers in organic chemistry. Chemical isomers are compounds having the same chemical (or molecular) formula, but different in their physical and chemical properties. These isomers remain stable for periods of time that are long in comparison with those during which measurements of their properties are made. The existence of isomers may be explained by assuming that the atoms in a molecule are arranged in a definite manner, i.e., two different isomers with the same chemical formula would have different arrangement of their atoms. To distinguish different isomers of the same chemical formula, chemists use what is called structural formulae or bond-diagram to represent the arrangement of the atoms in the molecules of these isomers. For example, dibromobenzene, $C_6H_4Br_2$, has three isomers and their structural formulae are given below.

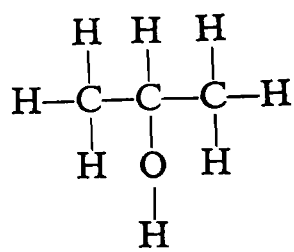




ortho-dibromobenzene meta-dibromobenzene para-dibromobenzene

Another example, propyl alcohol alcohol C_3H_7OH has two isomers, and their structural formulae are given below.



n-propyl alcohol



z-propyl alcohol

Traditionally, isomers have been classified as either structural isomers or stereoisomers. Structural (or constitutional) isomers differ in their structures; that is, in the manner the atoms are bonded in the molecule. Stereoisomers have identical structure but differ in configuration or conformation; that is, they differ only in the way the atoms are oriented in space but are like one another with respect to which atoms are joined to which other atoms. In other words, stereoisomers differ in the spatial architecture of the molecule.

There are a number of methods available for isomer enumerations [4], [7], and [29]. In this paper the enumeration methods of Burnside and Pólya-Redfield will be presented.

In this section, we are going to find the number of (theoretically) possible derivatives or isomers of chemical compounds. The first serious attempts at isomer enumeration by chemists were made over 100 years ago. Most of the early work on isomer enumeration was done on what are known as homologous series. These are series of chemical compounds which can be represented by one general formula such as alkanes (formally known as paraffins) whose molecular formula is C_nH_{2n+2} and aliphatic alcohols $C_nH_{2n+1}OH$. The best known early organic chemists to undertake such a study, were Berzelius, Couper, Vańt Hoff, Körner, Butlerov, Kekulń, Blair, and Henze. Among the first mathematicians to study this problem was Arthur Cayley who studied the problem over a 20-years period beginning in 1874. In 1937, [23] ([23] is an English translation of the 1937 paper), the Hungarian mathematician George Pólya presented to the combinatorial world a powerful theorem that can be applied to solve a wide range of counting or enumeration problems. In particular, Pólya's results had a profound influence on the enumeration of the chemical isomers. Ten years before Pólya published his paper many of the important aspects of Pólya's results were anticipated by J. H. Redfield [25]. Unfortunately, Redfield's paper went unnoticed until the 1960's.

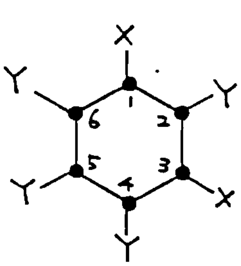
We now turn to the problem of chemical enumeration. The general problem of chemical enumeration can be stated as follows: given the number of atoms of each kind that occur in a molecule, determine the corresponding number of possible molecules, either as structural isomers or stereoisomers. In this general form the problem does not seem to admit any useful practical solutions [4]. For this reason, we are going to restrict ourself to compounds which have a common basic structure called the frame of the molecule. Thus given a frame of a molecule, and a set *A* of atoms (or radicals), attaching to each atom which, in the frame, has less than its proper valency, enough atoms form *A* to bring its valency up to the correct value. The problem is to determine how many isomers or substituted compounds there are.

Let us illustrate the problem with an example.

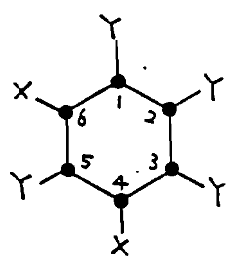
Example 1:

Let us consider the class of chemical compounds obtained from the benzene ring by attaching atoms *X* and *Y* in place of the six hydrogen atoms to the carbon ring. Here we assume that *X* and *Y* each has a valency of one. The problem we are interested in solving is how many different molecules can be obtained in this way? Altogether, there are 2^6 possibilities to attach either *X* or *Y* atoms to the carbon ring. But many of the resulting arrangements (or configurations) of atoms in a molecule represent the same chemical compound.

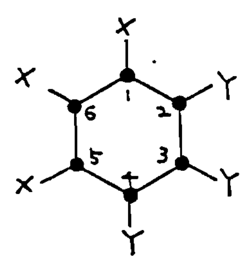
In order to solve this problem and other similar problems, we need to formulate the problem mathematically. First, note that each attachment of atoms *X* or *Y* to the carbon ring can be regarded as a function from the vertices of a regular hexagon (i.e., the carbon ring) to the set $A=\{X,Y\}$ consisting of the atoms *X* and *Y*. For example, the functions *f*, *g*, and *h*: $\{1,2,3,4,5,6\}\to\{X,Y\}$ defined by: $f(1)=f(3)=X$, $f(2)=f(4)=f(5)=f(6)=Y$, $g(4)=g(6)=X$, $g(1)=g(2)=g(3)=g(5)=Y$, $h(1)=h(5)=h(6)=X$, $h(2)=h(3)=h(4)=Y$.



Function *f*



Function *g*



Function *h*

Thus the number of chemical compounds that can be obtained is equal to the number of different functions from $\{1,2,3,4,5,6\}$ to $A=\{X,Y\}$. There are a total of 2^6 such functions, but many of them represent the same chemical compound. For example, the functions f and g above represent the same chemical compound since the function g can be obtained from the function f by a rotation of the carbon ring. In fact, let $\alpha=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 4 & 5 & 6 & 1 & 2 & 3\end{pmatrix}$ be the permutation representation of the rotation through 180° about the center of the carbon ring, then $f(\alpha(k))=g(k)$ for every $k=1,2,3,4,5,6$. On the other hand, f and h represent different compounds because for any rotation or reflection of the hexagon σ, we have $f(\sigma(k))\neq h(k)$ for some $k=1,2,\ldots,6$. Thus the symmetry of the hexagon completely determines the partition of the 2^6 functions into disjoint classes, where the functions in the same class represent the same compound. In general the symmetry group of a given frame of a molecule acts as a permutation group on the vertices (or on the set of atoms) in the frame of the molecule. Thus we can generalize the example as follows: given a frame of a molecule whose set of atoms is D, let G be a permutation group of D. Let A be the set of atoms to be attached to the vertices of the frame to bring its valency to the correct value. Let f be a function from D into A, that is; $f:D\to A$. In this context, the functions f denotes an attachment of atoms from A to the frame. Thus two functions $f,g:D\to A$ represent the same chemical compound if and only if there exists a permutation $\alpha\in G$ such that $f(\alpha(x))=g(x)$ for every $x\in D$. In this case, we write $f\sim g$. Clearly this define an equivalence relation on the set of all function $\{f:D\to A\}$ or equivalently this determines when two functions represent the same chemical compound. Thus the problem we are concerned with here is that of counting the number of equivalence classes of this relation. Mathematically the problem can be stated as follows: given a group of permutations G on a finite set D, and a finite set A, let $\Omega=\{f:D\to A\}$ be the set of all functions from D into A. Define a G-equivalence relation $\sim$ on Ω by: $f\sim g$ if and only if there exists a permutation $\pi\in G$ such that $f\circ\pi=g$, for every $f,g\in\Omega$. Let $\Omega/\sim$ be the set of all G-equivalence classes. The problem is to develop a formula for counting the number of G-equivalence classes $\Omega/\sim$. Clearly G-equivalence is an equivalence relation on the set Ω.

Lemma 5.14:

Ω is a *G-set* where the G action is given by $\pi(f)=f\circ\pi^{-1}$ for every $\pi\in G$ and $f\in\Omega$.

Proof:

Let $\pi_1,\pi_2\in G$, and let $f\in\Omega$. Let e be the identity of G. Then $e(f)=f\circ e^{-1}=fe=f$. Also $(\pi_1\pi_2)(f)=f\circ(\pi_1\pi_2)^{-1}=f\circ(\pi_2^{-1}\pi_1^{-1})$ $=(f\circ\pi_2^{-1})\circ\pi_1^{-1}=(\pi_1)(\pi_2(f))$. Therefore, Ω is a *G-set*.

Corollary:

The G-orbit of an element $f\in\Omega$ is the G-equivalence class of f.

Thus the number of G-equivalence classes in Ω is equal to the number of G-orbits of Ω.

Theorem 5.15 (Burnside's Lemma):

Let G be a finite group, and let S be a finite G-*set*. Then the number of G-orbits is equal to

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

where $Fix(g)$ denotes the set of elements in S that are left fixed (or invariant) by g; that is, $Fix(g)=\{s\in S|g(s)=s\}$.

Proof:

The idea of the proof is to count, in two different ways, the number of ordered pairs (g,s) satisfying $g(s)=s$, where $g\in G$ and $s\in S$, that is; we want to compute the cardinality of the set $\{(g,s)|g\in G,\ s\in S,\ \text{and } g(s)=s\}$ in two different ways and equate the results. On one hand, we have

$$|\{(g,s)|g\in G,\ s\in S,\ \text{and} g(s)=s\}|=\sum_{g\in G}|Fix(g)|.$$

On the other hand, since $stab(s)=\{g\in G|g(s)=s\}$, we have

$$\sum_{g\in G}|Fix(g)|=|\{(g,s)|g\in G,\ s\in S,\ \text{and} g(s)=s\}|=\sum_{s\in S}|stab(s)|.$$

If $S=O_1\cup O_2\cup\ldots\cup O_k$ is the partition of S into orbits, we have

$$\sum_{s\in S}|stab(s)|=\sum_{s\in O_1}|stab(s)|+\sum_{s\in O_2}|stab(s)|+\ldots+\sum_{s\in O_k}|stab(s)|.$$

Claim:

If $x, y \in O_i$ for some $i=1,2,\ldots,k$ then $|stab(x)|=|stab(y)|$. By the counting formula, we have

$$|stab(x)| = \frac{|G|}{[G:stab(x)]} = \frac{|G|}{|O_i|} = \frac{|G|}{[G:stab(y)]} = |stab(y)|.$$

Let us choose representatives $s_1, s_2, \ldots, s_k$ from the k orbits, $O_1, O_2, \ldots, O_k$. Thus

$$\begin{aligned}\sum_{s\in S}|stab(s)| &= |O_1||stab(s_1)| + |O_2||stab(s_2)| + \ldots + |O_k||stab(s_k)| \\ &= \sum_{i=1}^{k}|O_i||stab(s_i)| \\ &= \sum_{i=1}^{k}\frac{|G|}{|stab(s_i)|}|stab(s_i)| \\ &= k|G|.\end{aligned}$$

Thus $\sum_{g\in G}|Fix(g)| = k|G|$ and hence $k = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$.

Now we are going to apply Burnside's Lemma to find the number of G-equivalence classes of the set of functions Ω. We have shown that Ω is a G-*set*. Thus Burnside's Lemma implies the number of G-equivalence classes in Ω is given by

$$\frac{1}{|G|}\sum_{\pi\in G}|Fix(\pi)|.$$

To illustrate this result, let us consider the problem in example (1) of finding the number of chemically different molecules obtained from the benzene ring by attaching atoms X and Y in place of the hydrogen atoms. Let us denote the $2^6=64$ attachments of X or Y by $\Omega=\{f_1, f_2, \ldots, f_{64}\}$, where each f_i is a function from $\{1,2,3,4,5,6\} \rightarrow \{X,Y\}$. Two attachments f_i and f_j will yield the same molecule if and only if f_i can be obtained from f_j by means of an element of D_{12}, the dihedral group of order 12. Thus the number k of different chemical compounds is equal to the number of G-equivalence classes of the set Ω.

$$k=\frac{1}{|D_{12}|}\sum_{\pi\in D_{12}}|Fix(\pi)|=\frac{1}{12}\sum_{\pi\in D_{12}}|Fix(\pi)|.$$

The elements of D_{12} are listed below.

Table 5.1

Rotations	Corresponding permutations of the vertices
Identity	$(1)(2)(3)(4)(5)(6)=\pi_1$
Clockwise rotation about the center through $\theta=\pi/3$	$(1\ 2\ 3\ 4\ 5\ 6)=\pi_2$
Clockwise rotation about the center through $\theta=2\pi/3$	$(1\ 3\ 5)(2\ 4\ 6)=\pi_3$
Clockwise rotation about the center through $\theta=\pi$	$(1\ 4)(2\ 5)(3\ 6)=\pi_4$
Clockwise rotation about the center through $\theta=4\pi/3$	$(6\ 4\ 2)(5\ 3\ 1)=\pi_5$
Clockwise rotation about the center through $\theta=5\pi/3$	$(6\ 5\ 4\ 3\ 2\ 1)=\pi_6$

Table 5.2

Reflections	Corresponding permutations of the vertices
Reflection in the line through *1* and *4*	$(2\ 6)(3\ 5)=\pi_7$
Reflection in the line through *2* and *5*	$(4\ 6)(1\ 3)=\pi_8$
Reflection in the line through *3* and *6*	$(2\ 4)(1\ 5)=\pi_9$
Reflection in the line through the midpoints of *(1 2)* and *(4 5)*	$(1\ 2)(3\ 6)(4\ 5)=\pi_{10}$
Reflection in the line through the midpoints of *(2 3)* and *(5 6)*	$(1\ 4)(2\ 3)(5\ 6)=\pi_{11}$
Reflection in the line through the midpoints of *(3 4)* and *(6 1)*	$(3\ 4)(2\ 5)(1\ 6)=\pi_{12}$

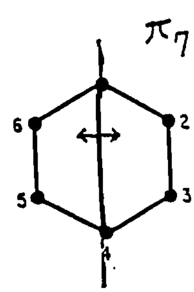


Figure 5.4

Now we wish to find $|Fix(\pi_i)|$ for all $i=1,2,...,12$. Clearly $|Fix(\pi_1)|=64$. Consider a reflection in a line through opposite vertices, say π_7. If an attachment of atoms *X* and *Y* is fixed by π_7, the atoms at vertex 2 must be the same as the atom at vertex 6, and the atoms at 3 and 5 must be the same, but the atoms at the vertices 1 and 4 are arbitrary.

Thus $|Fix(\pi_7)|$ = (the number of ways the atoms X or Y can be attached at the vertices 1,2,3 and 4) $=2^4=16$.

Similarly, $|Fix(\pi_8)|=|Fix(\pi_9)|=16$.

Now consider a reflection in a line through the midpoints of two opposite sides, say π_{11}. An attachment of atoms X and Y is fixed by π_{11} if the atoms at the vertices 2 and 3 are the same, the atoms at the vertices 1 and 4 are the same, and the atoms at the vertices 5 and 6 are the same. Thus $|Fix(\pi_{11})|=2^3=8$. Similarly $|Fix(\pi_{10})|=|Fix(\pi_{12})|=8$.

Next, we find the number of attachments fixed by the rotations. First, note that $|Fix(\pi_2)|=|Fix(\pi_6)|$. An attachment of atoms X and Y is fixed by π_2 if the atoms at the vertices are all X or all Y. Thus $|Fix(\pi_2)|=|Fix(\pi_6)|=2$. By similar analysis, we find $|Fix(\pi_3)|=|Fix(\pi_5)|=4$ and $|Fix(\pi_4)|=8$.

Therefore $\sum_{\pi\in D_{12}} |Fix(\pi)|=156$. Thus by Burnside's Lemma, the number of different chemical compounds obtained by attaching atoms X or Y to the carbon ring is equal to

$$k=\frac{1}{12}\sum_{\pi\in D_{12}} Fix(\pi)=\frac{1}{12}(156)=13.$$

The direct application of Burnside's Lemma becomes impractical for complex permutation groups especially when the set D is large. This is due to the fact that the computation of the number of elements fixed by the elements of the permutation group can be both difficult and tedious. Moreover, it is frequently necessary to have more information than merely the number of different chemical compounds. In the example above, one may wish to know the number of chemical compounds consisting of two X atoms and four Y atoms. Burnside's Lemma does not give a solution to this problem and does not provide a method of finding a representative from the different equivalent classes of chemical compounds. Pólya's theory of enumeration offers solutions to both of these problems.

Before developing the powerful enumeration Theorem of Pólya, we need to introduce the concept of the cycle index of a permutation group.

Let G be a group of permutations on a finite set D. There is no loss in generality to assume $D=\{1,2,...,n\}$. It is well-known that every permutation $\pi \in G$ can be expressed uniquely (except for order) as a product of disjoint cycles. Let $j_k(\pi)$ denote the number of cycles in π of length k. Then we have $\sum_{k=1}^{n} k \cdot j_k(\pi) = n$ for every $\pi \in G$.

Example 2:

The permutation $\pi=(3\ 5)(2\ 6)$ on the set $D=\{1,2,3,4,5,6\}$ has two cycles of length 2 and two cycles of length 1. Thus $j_1(\pi)=2$ and $j_2(\pi)=2$.

If $\pi\in G$ has n_1 cycles of length 1, and n_2 cycles of length 2, etc., then we say that π is of type $(n_1,n_2,...)$.

Definition 5.6:

Let G be a permutation group on D. Let $X_1,X_2,...,X_n$ be indeterminates. The *cycle index* of G in $X_1,X_2,...,X_n$ is the polynomial in the variables $X_1,X_2,...,X_n$ denoted by $Z(G;X_1,X_2,...,X_n)$, and given by

$$Z(G;X_1,X_2,\ldots,X_n)=\frac{1}{|G|}\sum_{\pi\in G}\prod_{k=1}^{n}X_k^{j_k(\pi)}.$$

When there is no danger of confusion, we denote the cycle index of a group G in the indeterminates $X_1,X_2,...,X_n$ simply by $Z(G)$.

Example 3:

Let $G=S_3$ be the symmetric group of degree 3. $S=\{id,(1\ 2),(1\ 3),(2\ 3),(1\ 2\ 3),(1\ 3\ 2)\}$. The identity permutation id is of type $(3,0,0)$. The permutations $(1\ 2)$, $(1\ 3)$, $(2\ 3)$ are of type $(1,1,0)$. The permutations $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are type $(0,0,1)$. Thus $Z(S_3)=\frac{1}{6}(X_1^3+3X_1X_2+2X_3)$.

Example 4:

Let $G=D_{12}$ be the dihedral group of order *12*. The identity permutation is of type *(6,0,0,0,0,0)*. The permutations *(1 2 3 4 5 6)* and *(6 5 4 3 2 1)* are of type *(0,0,0,0,0,1)*. The permutations *(1 3 5)(2 4 6)* and *(6 4 2)(5 3 1)* are of type *(0,0,2,0,0,0)*. The permutations *(1 4)(2 5)(3 6)*, *(1 2)(3 6)(4 5)*, *(2 3)(5 6)(1 4)*, and *(2 4)(2 5)(1 6)* are of type *(0,3,0,0,0,0)*. The permutations *(1)(4)(2 6)(3 5)*, *(2)(5)(1 3)(4 6)*, *(3)(6)(1 5)(2 4)* are of type *(2,2,0,0,0,0)*. Thus the cycle index of D_{12} is

$$Z(D_{12}) = \frac{1}{12}\left(X_1^6+2X_6+2X_3^2+4X_2^3+3X_1^2X_2^2\right) .$$

Example 5:

Let G be the group of rotations of a cube. $|G|=24$, and the elements of G can be divided into five categories:

1. The identity.
2. Three 180° rotations around lines connecting the centers of opposite faces.
3. Six 90° rotations around lines connecting the centers of opposite faces.
4. Six 180° rotations around lines connecting the midpoints of opposite edges.
5. Eight 120° rotations around lines connecting opposite vertices.

(a). Let V be the set of the vertices of the cube; then G acts as a permutation group on V. Since the cube has 8 vertices, then $|V|=8$. Now we need to determine the types of all of the permutations on V induced by the group G.

1. The identity permutation has type $(8,0,0,0)$.
2. A permutation of category (2) has type $(0,4,0,0)$.
3. A permutation of category (3) has type $(0,0,0,2)$.
4. A permutation of category (4) has type $(0,4,0,0)$.
5. A permutation of category (5) has type $(2,0,2,0)$.

Therefore the cycle index in this case is given by

$$Z_V(G;X_1,X_2,X_3,X_4) = \frac{1}{24}(X_1^8+9X_2^4+6X_4^2+8X_1^2X_3^2) .$$

(b). Now let E be the set of the edges of the cube. Then G acts as a permutation group on E. E has 12 elements. In this case, the types of the permutations on E induced by the group G are given below.

1. The identity permutation is of type $(12,0,0,0)$.
2. A permutation of category (2) has type $(0,6,0,0)$.
3. A permutation of category (3) has type $(0,0,0,3)$.
4. A permutation of category (4) has type $(2,5,0,0)$.
5. A permutation of category (5) has type $(0,0,4,0)$.

Therefore the cycle index in this case is given by

$$Z_E(G;X_1,X_2,X_3,X_4) = \frac{1}{24}(X_1^{12}+3X_2^2+6X_4^3+6X_1^2X_2^5+8X_3^4) .$$

(c). Let F be the set of all faces of the cube. G acts as a permutation group on F. Since the cube has six faces, $|F|=6$.

The five categories of rotations now produce permutations of the following types: *(6,0,0,0)*, *(2,2,0,0)*, *(2,0,0,1)*, *(0,3,0,0)*, and *(0,0,2,0)* respectively, and therefore

$$Z_F(G;X_1,X_2,X_3,X_4)=\frac{1}{24}(X_1^6+3X_1^2X_2^2+6X_1^2X_4+6X_2^3+8X_3^2)$$

The following theorem gives the cycle index of groups which appear as symmetry groups of sets in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem 5.16:

1. $$Z(S_n)=\sum_{j_1+2j_2+\dots+nj_n=n}\frac{1}{(1^{j_1}j_1!)(2^{j_2}j_2!)\dots(n^{j_n}j_n!)}X_1^{j_1}X_2^{j_2}\dots X_n^{j_n}.$$

2. $$\begin{aligned}Z(A_n)&=Z(S_n;X_1,X_2,\dots,X_n)+Z(S_n;X_1,-X_2,X_3,-X_4,\dots)\\&=\sum\frac{1+(-1)^{j_2+j_4+\dots}}{2(1^{j_1}j_1!)(2^{j_2}j_2!)\dots(n^{j_n}j_n!)}X_1^{j_1}\dots X_n^{j_n}.\end{aligned}$$

3. $Z(C_n)=\frac{1}{n}\sum_{d|n}\varphi(d)X_d^{\frac{n}{d}}$, where φ is Euler's φ-function and C_n is the cyclic group of degree n.

4. $$Z(D_{2n})=\begin{cases}\frac{1}{2}Z(C_n)+\frac{1}{2}X_1X_2^{\frac{(n-1)}{2}} & \text{if } n \text{ is odd}\\ \frac{1}{2}Z(C_n)+\frac{1}{4}(X_2^{\frac{n}{2}}+X_1^2X_2^{\frac{(n-2)}{2}}) & \text{if } n \text{ is even.}\end{cases}$$

5. Let G and H be permutation groups on two disjoint sets D_1 and D_2. The direct product of G and H is the permutation group on the set $D_1 \cup D_2$ denoted by $G \times H$, where the action of $(g,h) \in G \times H$ on $D_1 \cup D_2$ is given by

$$(g,h)(k) = \begin{cases} g(k) & \text{if } k \in D_1 \\ h(k) & \text{if } k \in D_2 \end{cases}.$$

Then the cycle index of $G \times H$ is given by

$Z(G \times H) = Z(G) Z(H)$.

The proof of this theorem is not difficult, but it is lengthy and it can be found in [17].

To formulate and prove Pólya's Fundamental Theorem in an abstract and concise manner, it is convenient to regard the objects to be counted as the set of all functions on a finite set D into a finite set A. Let $\Omega = \{f: D \rightarrow A\}$ be the set of all functions defined on D with values in A. Let G be a group of permutations on D. Every element $\pi \in G$ defines a mapping $\pi^{*}: \Omega \rightarrow \Omega$ as follows $\pi^{*}(f) = f \circ \pi^{-1}$. It is obvious that for a fixed $\pi \in G$, π^{*} is a one-to-one mapping of Ω onto itself, and thus π^{*} is a permutation on Ω. Let $G^{*} = \{\pi^{*} / \pi \in G\}$; then we have the following Lemma whose proof is straightforward.

Lemma 5.17:

The set G^{*} is a permutation group on Ω; moreover $|G^{*}| = |G|$.

The permutation group G^* partitions Ω into equivalence classes under the relation ~ defined on Ω by $f \sim g$ if and only if $\pi^*(f)=g$ for some $\pi^* \in G^*$. These G^*-equivalence classes are called patterns. Note that the G^*-equivalence classes of Ω are simply the G-equivalence classes introduced earlier.

Example 6:

Let G be the Klein 4-group considered as a permutation group on $D=\{1,2,3,4\}$. Then $G=\{\pi_1,\pi_2,\pi_3,\pi_4\}$ where

$\pi_1=id=(1)(2)(3)(4)$,

$\pi_2=(1\ 2)(3\ 4)$,

$\pi_3=(1\ 3)(2\ 4)$,

$\pi_4=(1\ 4)(2\ 3)$.

Let $A=\{X,Y\}$ and $\Omega=\{f_i:D\to A / i=1,2,\ldots,16\}$. $\Omega=\{f_1,f_2,\ldots,f_{16}\}$, where $f_1=\{(1,X),(2,X),(3,X),(4,X)\}$,

$f_2=\{(1,X),(2,X),(3,X),(4,Y)\}$,

$f_3=\{(1,X),(2,X),(3,Y),(4,X)\}$,

$f_4=\{(1,X),(2,X),(3,Y),(4,Y)\}$,

$f_5=\{(1,X),(2,Y),(3,X),(4,X)\}$,

$f_6=\{(1,X),(2,Y),(3,X),(4,Y)\}$,

$f_7=\{(1,X),(2,Y),(3,Y),(4,X)\}$,

$f_8=\{(1,X),(2,Y),(3,Y),(4,Y)\}$,

$f_9=\{(1,Y),(2,X),(3,X),(4,X)\}$,

$f_{10}=\{(1,Y),(2,X),(3,X),(4,Y)\}$,

$f_{11}=\{(1,Y),(2,X),(3,Y),(4,X)\}$,

$f_{12}=\{(1,Y),(2,X),(3,Y),(4,Y)\}$,

$f_{13}=\{(1,Y),(2,Y),(3,X),(4,X)\}$,

$f_{14}=\{(1,Y),(2,Y),(3,Y),(4,X)\}$,

$f_{15}=\{(1,Y),(2,Y),(3,Y),(4,Y)\}$,

$f_{16}=\{(1,Y),(2,Y),(3,X),(4,Y)\}$.

Then $G^t=\{\pi_1{}^t,\pi_2{}^t,\pi_3{}^t,\pi_4{}^t\}$, where

$\pi_1{}^t=id$

$\pi_2{}^t=(f_2\ f_3)(f_5\ f_9)(f_6\ f_{11})(f_7\ f_{10})(f_{12}\ f_8)(f_{14}\ f_{16})$,

$\pi_3{}^t=(f_2\ f_5)(f_3\ f_9)(f_4\ f_{13})(f_7\ f_{10})(f_8\ f_{16})(f_{12}\ f_{14})$,

$\pi_4{}^t=(f_2\ f_9)(f_3\ f_5)(f_4\ f_{13})(f_6\ f_{11})(f_8\ f_{14})(f_{12}\ f_{16})$.

Let us find the G^t-orbits of Ω.

$O_1=\{f_1\}$,

$O_2=\{f_2,f_3,f_5,f_9\}$,

$O_3=\{f_4,f_{13}\}$,

$O_4=\{f_6,f_{11}\}$,

$O_5=\{f_7,f_{10}\}$,

$O_6=\{f_8,f_{12},f_{16},f_{14}\}$,

$O_7=\{f_{15}\}$.

From this example, it follows that the number of G^t-equivalence classes of Ω is equal to the number of G^t-orbits of Ω. Now we are going to apply Burnside's Lemma to the group G^t. Let $\pi^t\in G^t$. We need to compute $|Fix(\pi^t)|$. Note that an element $f\in\Omega$ is left invariant by π^t if and only if the corresponding permutation π of D is such that $f(\pi(k))=f(k)$ for all k. Thus $f\in\Omega$ is left invariant by π^t if and only if all the elements of D in each cycle of π have the same function value. For instance, suppose that $\pi=\pi_2=(1\ \ 2)(3\ \ 4)$. If

$f(1)=f(2)$ and $f(3)=f(4)$ then $\pi^{*}(f)=f$. Hence $|Fix(\pi^{*})|=2^{cyc(\pi)}$, where $cyc(\pi)$ is the number of cycles in the unique cycle decomposition of the permutation π.

Thus $|Fix(\pi_1^{*})|=2^4=16$,

$|Fix(\pi_2^{*})|=2^2=4$,

$|Fix(\pi_3^{*})|=2^2=4$,

$|Fix(\pi_4^{*})|=2^2=4$.

By Burnside's Lemma, the number of G^{*}-orbits of Ω is equal to

$$\frac{1}{|G^{*}|}\sum_{\pi^{*}\in G^{*}}|Fix(\pi^{*})| = \frac{1}{4}(16+4+4+4) = \frac{28}{4} = 7 .$$

We are now ready to present another formula for counting the number of G^{*}-equivalence classes (or G^{*}-orbits) of Ω which leads to a special case of Pólya's Theorem of enumeration.

Theorem 5.18:

Let G be a group of permutations on a finite set D. Let $A=\{a_1,a_2,...,a_m\}$ and $\Omega=\{f:D\to A\}$ be the set of all functions defined on D with values on A. Then the number of G^{*}-equivalence classes of the set Ω is given by

$$K=\frac{1}{|G|}\sum_{\pi\in G} m^{cyc(\pi)}$$

where $cyc(\pi)$ is the number of cycles in the unique cycle decomposition of π.

Proof:

We apply Burnside's Lemma to G^t. Since $|G|=|G^t|$, it suffices to show that $m^{cyc(\pi)}=|Fix(\pi^t)|$ for every $\pi\in G$. Let $\pi\in G$ and π^t be the corresponding permutation in G^t. An element $f\in\Omega$ belongs to $Fix(\pi^t)$ if and only if the value of the function f is the same for the elements of D in each cycle of the corresponding permutation π of D. Now π has $cyc(\pi)$ different cycles in its cycle decomposition, and we have m choices for the common function value of each cycle. Hence, there are $m^{cyc(\pi)}$ different functions left invariant by π^t. That is $|Fix(\pi^t)|=m^{cyc(\pi)}$.

Corollary (Special case of Pólya's Theorem):

The number of G^t-equivalence classes is given by

$$K=Z(G;m,m,\dots,m)=\frac{1}{|G|}\sum_{\pi\in G}m^{j_1}m^{j_2}\dots m^{j_n}$$

where $Z(G;x_1,\dots,x_n)=\frac{1}{|G|}\sum_{\pi\in G}x_1^{j_1}\dots x_n^{j_n}$ is the cycle index of G.

Proof:

Let $\pi\in G$ be a permutation of type $(j_1,j_2,\dots,j_n)$. Then $cyc(\pi)=j_1+j_2+\dots+j_n$ and the corresponding term in the cycle index of G is $x_1^{j_1}x_2^{j_2}\dots x_n^{j_n}$. If we substitute for x_i by m for $i=1,2,\dots,n$ then $x_1^{j_1}x_2^{j_2}\dots x_n^{j_n}$ becomes $m^{j_1}\dots m^{j_n}=m^{j_1+\dots+j_n}=m^{cyc(\pi)}$. Thus $K=Z(G;m,\dots,m)=\frac{1}{|G|}\sum_{\pi\in G}m^{cyc(\pi)}$.

Example 1 (Revisited):

Let us apply the corollary to the carbon ring in Example 1. $G=D_{12}$ is the dihedral group of order *12* and $A=\{X,Y\}$. The cycle index of D_{12},

$$Z(D_{12})=\frac{1}{12}(X_1^6+2X_6+2X_3^2+4X_2^3+3X_1^2X_2^2).$$

Thus the number of different compounds is equal to

$$K=\frac{1}{12}(2^6+2\cdot2+2\cdot2^2+4\cdot2^3+3\cdot2^2\cdot2^2)$$
$$=\frac{1}{12}(64+4+8+32+48)=13.$$

Hence the corollary give a straightforward solution for counting the number of different chemical compounds obtained by attaching atoms *X* or *Y* to the carbon ring.

However, we may be interested in counting not just the number of different chemical compounds but the number of different compounds with the same number of atoms. For example, we might be interested in counting the number of chemical compounds that has two *X*-atoms and four *Y*-atoms. To answer questions of this type, we need to introduce the concept of the weight of a function. Each element $a\in A$ is assigned a weight $W(a)$ which is a number of a symbol. Sums, products and rational multiples of weights can be formed, and these operations satisfy the usual associative, commutative and distributive laws.

Definition 5.7:

The *weight of a function* $f \in \Omega$, denoted by $W(f)$, is defined by $W(f) = \prod_{d \in D} W(f(d))$.

Example 7:

Let $D=\{1,2,3\}$, $A=\{a_1,a_2\}$, $W(a_1)=x$ and $W(a_2)=y$. The weight of the function f defined by $f(1)=f(2)=x$ and $f(3)=y$ is $W(f)=x^2y$. The weight of the function g defined by $g(1)=g(2)=g(3)=y$ is $W(g)=y^3$.

Lemma 5.19:

If $f,g \in \Omega$ are G-equivalent then $W(f)=W(g)$.

Proof:

Since f and g are G-equivalent then there exists $\pi \in G$ such that $f \circ \pi = g$. $W(g) = \prod_{d \in D} W(g(d)) = \prod_{d \in D} W(f(\pi(d)))$. But $\prod_{d \in D} W(f(\pi(d)))$ and $\prod_{d \in D} W(f(d))$ contain the same factors, only in different orders. Thus $\prod_{d \in D} W(f(\pi(d))) = \prod_{d \in D} W(f(d))$. Therefore, $W(g) = \prod_{d \in D} W(f(d)) = W(f)$.

This Lemma justifies the following definition.

Definition 5.8:

The *weight of a pattern P* (or G-equivalence class), denoted by $W(P)$ is defined by $W(P)=W(f)$ where $f \in P$.

Definition 5.9:

The *inventory of a set* of functions $S \subseteq \Omega$ is defined as the sum of the weight of all the functions in S; that is, inventory of $S = \sum_{f \in S} W(f)$.

Example 8:

Let $D=\{1,2\}$, $A=\{a_1,a_2\}$, and let $W(a_1)=x$, $W(a_2)=y$.

$f_1=\{(1,a_1),(2,a_1)\}$ $\quad W(f_1)=W(a_1)W(a_2)=x^2$

$f_2=\{(1,a_1),(2,a_2)\}$ $\quad W(f_2)=W(a_1)W(a_2)=xy$

$f_3=\{(1,a_2),(2,a_1)\}$ $\quad W(f_1)=W(a_1)W(a_2)=xy$

$f_4=\{(1,a_1),(2,a_1)\}$ $\quad W(f_1)=W(a_1)W(a_2)=y^2$

Inventory of $(\Omega)=x^2+2xy+y^2=(x+y)^2$.

We are now in a position to state Pólya's Theorem of Enumeration.

Theorem 5.20 (Pólya's Theorem of Enumeration):

Suppose that G is a group of permutations on a finite set D. Let $\Omega=\{f:D \to A\}$. Then the pattern inventory (or the inventory) of equivalence classes in Ω is given by

$$P\left(G; \sum_{a\in A} W(a), \sum_{a\in A} [W(a)]^2, \ldots, \sum_{a\in A} [W(a)]^k\right)$$

where $P(G; X_1, X_2, \ldots, X_k)$ is the cycle index of the permutation group G.

Before we present a proof of Pólya's Theorem, we illustrate its use by examples.

Example 1 (Revisited):

The cycle index of D_{12} is given by

$$Z(D_{12}) = \frac{1}{12}(X_1^6 + 2X_6 + 2X_3^2 + 4X_2^3 + 3X_1^2X_2^2) .$$

Now let us assign a weight x to an X-atom and a weight y to a Y-atom. Then $\sum_{a\in A} W(a) = x+y$, $\sum_{a\in A} [W(a)]^2 = x^2+y^2$, $\sum_{a\in A} [W(a)]^3 = x^3+y^3$, ..., $\sum_{a\in A} [W(a)]^6 = x^6+y^6$. By Pólya's Theorem, the pattern inventory is given by taking $Z(D_{12}; x_1, x_2, \ldots, x_6)$ and substituting $\sum_{a\in A} W(a)$ for x_1, $\sum_{a\in A} [W(a)]^2$ for x_2, and so on. Thus the pattern inventory is

$$\frac{1}{12}[(x+y)^6 + 2(x^6+y^6) + 2(x^3+y^3)^2 + 4(x+y)^3 + 3(x+y)^2(x^2+y^2)^2]$$
$$= x^6 + x^5y + 3x^4y^2 + 3x^3y^3 + 3x^2y^4 + xy^5 + y^6 .$$

Hence the *13* different compounds are listed below:

One molecule that has only X-atoms,

One molecule that has five X-atoms and one Y-atom, Three molecules that have four *X*-atoms and two *Y*-atoms,

Three molecules that have three *X*-atoms and three *Y*-atoms,

Three molecules that have two *X*-atoms and four *Y*-atoms,

One molecule that has one *X*-atom and five *Y*-atoms,

One molecule that has only *Y*-atoms.

Example 9:

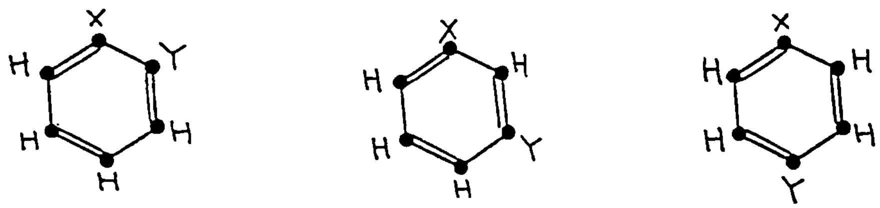
In this example, we are going to consider the class of compounds called "twice-substituted benzene" C_6H_4XY, where *X* and *Y* represents atoms that have taken the place of two hydrogen atoms. The question is how many twice-substituted benzene compounds are possible. As before, we let $D=\{1,2,3,4,5,6\}$ and $A=\{H,X,Y\}$. Let $W(H)=a$, $W(X)=b$, $W(Y)=c$.

$$Z(D_{12}) = \frac{1}{12}(X_1^6+2X_6+2X_3^2+4X_2^3+3X_1^2X_2^2)$$

By Pólya's Theorem, the pattern inventory is given by

$$\frac{1}{12}[(a+b+c)^6+2(a^6+b^6+c^6)+2(a^3+b^3+c^3)^2+4(a+b+c)^3$$
$$+3(a+b+c)^2(a^2+b^2+c^2)^2]$$

To find the number of twice-substituted benzene compounds, we count the number of terms in the pattern inventory of the form a^4bc. There are three such terms and thus there are three twice-substituted benzene compounds. The structural formulae for these three compounds are given in the figure below.



ortho compound   meta compound   para compound

In Example 1, at the beginning of this section, we assumed in the structural formula for benzene that the carbon atoms were arranged as to form the vertices of a regular hexagon. Kekulé (1865) was the first to suggest a hexagon structure for benzene. From this, he argued for the existence of only one mono-substituted derivative and three di-substituted derivatives of benzene. For this conclusions rigorous proof did not exist at the time. Of course our calculations supports Kekulé's conclusions. Shortly after Kekulé proposed this hexagon formula for benzene, several chemists criticized his formula and suggested alternative structures. Among the several suggestions for the structure of benzene, we mention the Thomsen's (1886) octahedral formula, Figure 5.4 below, and the Landenburg's (1869) prism formula, Figure 5.5 below.

Figure 5.5

In both of these models, it was assumed that the carbon atoms are positioned at the vertices. Our objective now is to see how Pólya's Theorem can be used to rule out these two models as possible structures of the benzene molecule. This is done by finding the number of twice-substituted benzene derivatives possible when these models are assumed. We will show that the number of possible derivatives of benzene does not agree with the experimental results.

First, suppose the structure of the benzene molecule is octahedron. Let $D=\{1,2,3,4,5,6\}$ be the set of vertices of the octahedron. The cycle index for the octahedron is given by

$$Z(G;X_1,X_2,X_3,X_4) = \frac{1}{24}[X_1^6+3X_1^2X_2^2+6X_1^2X_4+6X_2^3+8X_3^2] .$$

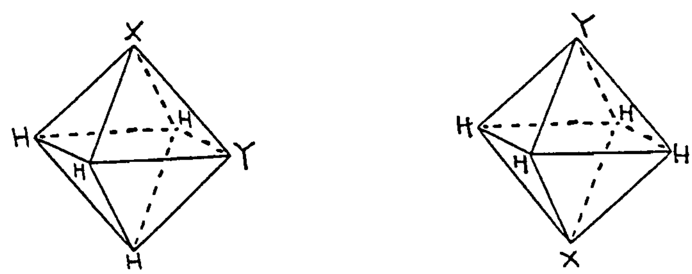
Let $A=\{H,X,Y\}$ and assign weights to the elements of A, $W(H)=a$, $W(X)=b$, and $W(Y)=c$. By Pólya's Theorem, the pattern inventory is given by

$$\frac{1}{24}[(a+b+c)^6+3(a+b+c)^2(a^2+b^2+c^2)^2+6(a+b+c)^2(a^4+b^4+c^4) +6(a^2+b^2+c^2)^3+8(a^3+b^3+c^3)^2] .$$

To find the number of twice-substituted benzene compounds, we count the number of terms in the pattern inventory of the form a^4bc. These terms are given by

$$\frac{1}{24}[\binom{6}{4\ 1\ 1}a^4bc+3(2bc)(a^4)+6(2bc)(a^4)]$$
$$=\frac{1}{24}[30a^4bc+6a^4bc+12a^4bc]=2a^4bc.$$

Thus there are two possible twice-substituted benzene compounds. The two molecules are given in the figure below.



Next, we assume that the structure of the benzene molecule is a triangular right prism. Let $D=\{1,2,3,4,5,6\}$ be the vertices of the prism. The cycle index for the triangular prism is given by

$$Z(G;X_1,X_2,X_3)=\frac{1}{6}(X_1^6+2X_3^2+3X_2^3).$$

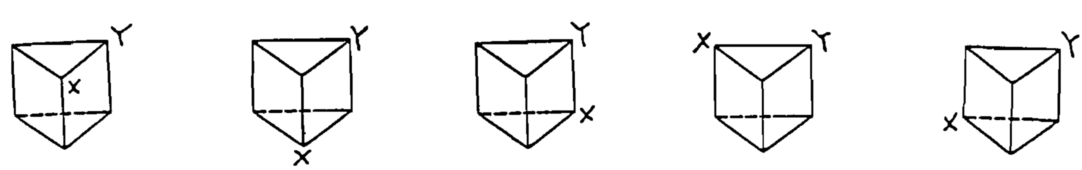
Let $A=\{H,X,Y\}$ and assign the weights $W(H)=a$, $W(X)=b$, $W(Y)=c$. By Pólya's Theorem, the pattern inventory is given by

$$\frac{1}{6}[(a+b+c)^6+2(a^3+b^3+c^3)^2+3(a^2+b^2+c^2)^2].$$

To find the number of twice-substituted benzene compounds, we count the number of terms in the pattern inventory of the form a^4bc. There are five terms,

$$\frac{1}{6}[\binom{6}{4\ 1\ 1}a^4bc]=\frac{1}{6}[\frac{6!}{4!}a^4bc]=5a^4bc.$$

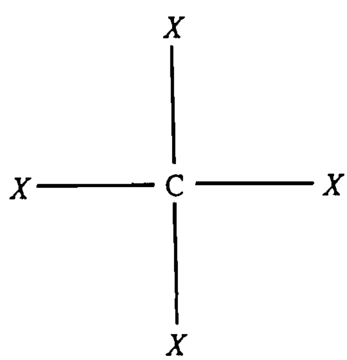
Thus there are five possible twice-substituted benzene compounds. These five molecules are given in the figure below.



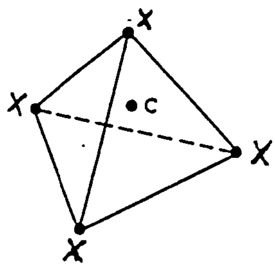
What have we established? Well, if the carbon atoms were arranged as in an octahedron, there would be only two possible isomers of C_6H_4XY. If they were arranged as a triangular prism, there would be five isomers. In reality, chemists were able to find only three; therefore, those two models must be wrong. We haven't actually proven that the hexagonal model is correct, but we have circumstantial evidence in its favor. (Various other methods have backed this up, and modern chemists are essentially certain that the carbon atoms of benzene do indeed form a regular hexagon, at least insofar as chemical bonds maintain any rigid shape.)

Example 10:

Consider the class of organic molecules of the form



where C is a carbon atom, and each *X* denotes any one of the components CH_3 (methyl), C_2H_5 (ethyl), H (hydrogen), or Cl (chlorine). Each molecule can be modeled as a regular tetrahedron with the carbon atom at the center and the components labeled X at the corners; see the figure below.



Recall that the symmetry group of the tetrahedron is isomorphic to the alternation group A_4. The cycle index of A_4 is given by

$$Z(A_4;X_1,X_2,X_3)=\frac{1}{12}(X_1^4+8X_1X_3+3X_2^2).$$

Therefore, the number of different molecules is

$$P(A_4;4,4,4)=\frac{1}{12}(4^4+8\cdot 4\cdot 4+3\cdot 4^2)=36.$$

Suppose we wish to find the number of molecules containing two hydrogen atoms and two chlorine atoms, or the number of molecules containing three hydrogen atoms. Assign weights of the elements of the set $A=\{CH_3,C_2H_5,H,Cl\}$ as follows; $W(CH_3)=a$, $W(C_2H_5)=b$, $W(H)=c$, $W(Cl)=d$. By Pólya's Theorem, the pattern inventory is given by evaluating

$$Z(A_4; \sum_{a \in A} W(a), \sum_{a \in A} [W(a)]^2, \sum_{a \in A} [W(a)]^3)$$

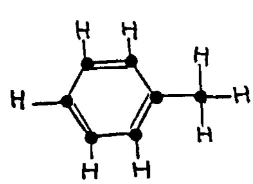
$$= \frac{1}{12} [(a+b+c+d)^4 + 8(a+b+c+d)(a^3+b^3+c^3+d^3) + 3(a^2+b^2+c^2+d^2)]$$

$$= a^4+b^4+c^4+d^4+a^3b+a^3c+a^3d+ab^3+b^3c+b^3d+ac^3+bc^3+c^2d+ad^3+bd^3+cd^3$$
$$+a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2+a^2bc+a^2cd+a^2bd+ab^2c+ab^2d+b^2cd$$
$$+abc^2+ac^2d+bc^2d+abd^2+acd^2+bcd^2+2abcd.$$

Thus there is only one molecule that contain two hydrogen atoms and two chlorine atoms. To find the number of molecules that contain three hydrogen atoms, we count the number of terms in the pattern inventory above of the form xc^3, where $x \in \{a,b,c\}$. There are three terms, namely ac^3, bc^3, and dc^3. Thus there are three molecules containing three hydrogen atoms.

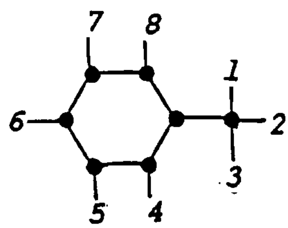
Example 11:

As a final illustration of Pólya's Theorem, we calculate the number of molecules that can be obtained by replacing one hydrogen atom in toluene by a chlorine atom.



(Toluene)

Let $D=\{1,2,3,4,5,6,7,8\}$ and $A=\{H,Cl\}$.



Let G be the symmetry group of the toluene molecule. Then G is the direct product of the symmetric group S_3 of the vertices $\{1,2,3\}$ on the group F generated by the reflection of the hexagon around the horizontal axis through the vertex 6 and the opposite vertex.

$S_3=\{id,(2\ 3),(1\ 3),(1\ 2),(1\ 2\ 3),(1\ 3\ 2)\}$

$F=\{id,(5\ 7),(4\ 8)\}$.

Thus $G=S_3 \times F$, and the cycle index of G is the product of the cycle indexes of S_3 and F. That is,

$$Z(G)=Z(S_3)Z(F)$$

$$=\frac{1}{6}(x_1^3+3x_1x_2+2x_3)\cdot\frac{1}{2}(x_1^5+x_1x_2^2)$$

$$=\frac{1}{12}(x_1^8+3x_1^6x_2+2x_1^5x_3+x_1^4x_2^2+3x_1^2x_2^3+2x_1x_2^2x_3).$$

Therefore, by Pólya's Theorem the number of mono-substituted toluene is equal to

$$\begin{aligned} Z(G;2,2,2) &= \frac{1}{12}(2^8+3\cdot2^6\cdot2+2\cdot2^5\cdot2+2^4\cdot2^2+3\cdot2^2\cdot2^3+2\cdot2\cdot2^2\cdot2) \\ &= \frac{1}{12}(256+384+128+64+96+32) \\ &= \frac{1}{12}(960) = 80. \end{aligned}$$

Thus there are *80* mono-substituted toluene compounds.

We now present a proof of Pólya's Enumeration Theorem. We begin by proving some preliminary Lemmas. Throughout this discussion and without loss of generality, we may assume $A=\{1,2,\ldots,m\}$.

Lemma 5.21:

If $D_1, D_2, \ldots, D_p$ form a partition of D, and S is the set of all functions of D into A which are constant on each subset D_i, for $i=1,2,\ldots,p$. Then the inventory of the set S is given by: $\mathit{inventory}(S) = \prod_{i=1}^{p}\sum_{j=1}^{m}[W(j)]^{|D_i|}$;

that is, $\sum_{f\in S} W(f) = \prod_{i=1}^{p}\left(\sum_{j=1}^{m}[W(j)]^{|D_i|}\right) \quad \ldots(*)$.

Proof:

A typical term on the right hand side is of the form

$$[W(j_1)]^{|D_1|}[W(j_2)]^{|D_2|}\dots[W(j_p)]^{|D_p|}$$

which is precisely the weight of a function $f\epsilon\Omega$ which assumes the value j_1 on D_1, j_2 on D_2, ..., j_p on D_p, and hence a term on the right hand side of the inventory *(S)*. Conversely, any such function has a weight just of the above form.

Lemma 5.22:

Suppose that $G^t=\{\pi_1^t,\pi_2^t,\dots\}$ is a group of permutations of Ω. For each $\pi_j^t\epsilon G^t$, let $\overline{W}(\pi_j^*)$ be the sum of the weights of all functions $f\epsilon\Omega$ left invariant by π^{jt}. Suppose that $C_1,C_2,\dots$ are the G^t-equivalence classes and $W(C_i)$ is the common weight of all f in C_i. Then

$$\sum_i W(C_i)=\frac{1}{|G^*|}\sum_j \overline{W}(\pi_j^*)\quad \dots(**).$$

Proof:

The sum $\sum_j \overline{W}(\pi_j^*)$ adds up for each π_j^t the weights of all functions $f\epsilon\Omega$ left fixed by π_j^t. Thus $W(f)$ is added in here exactly the number of times it is left invariant by some π^t. That is, $W(f)$ is added exactly $|stab(f)|$-times. By the counting formula $|stab(f)| = \frac{|G^*|}{|O_f|}$, where O_f is the orbit of f.

Therefore, if $\Omega = \{f_1, f_2, \ldots\}$, the right-hand side

$$\begin{aligned}\frac{1}{|G^*|}\sum_j \overline{W}(\pi_j^*) &= \frac{1}{|G^*|}\sum_j W(f_j)\,|stab(f_j)| \\ &= \frac{1}{|G^*|}\sum_j W(f_j)\,\frac{|G^*|}{|O_{f_j}|} \\ &= \sum_j \frac{W(f_j)}{|O_{f_j}|} = \sum_j \frac{W(f_j)}{|C(f_j)|}.\end{aligned}$$

Now we add up the terms $\frac{W(f_i)}{|C(f_j)|}$ for all f_i in equivalence

classes C_j. That is, we need to find $\sum_{f_i \in C_j} \frac{W(f_i)}{|C(f_i)|}$. Each $f_i \in C_j$

has the same weight, namely $W(f_i) = W(C_j)$. Moreover, $|C(f_i)| = |C_j|$. Thus $\sum_{f_i \in C_j} \frac{W(f_i)}{|C(f_i)|} = |C_j| \frac{W(C_j)}{|C_j|} = W(C_j)$. Therefore,

$\sum_j \frac{W(f_j)}{|C(f_j)|} = \sum_j W(C_j)$. Hence $\frac{1}{|G^*|}\sum_j \overline{W}(\pi_j^*) = \sum_j W(C_j)$.

We are ready to present a proof of Pólya's Theorem.

Proof (of Pólya's Enumeration Theorem):

The sum of the left hand side of equation (**) in Lemma 5.22, namely $\sum_j W(C_i)$ is the pattern inventory of the set of all functions Ω. On the other hand, $\overline{W}(\pi^*)$ is the sum of all weights of all functions $f \in \Omega$ left invariant by π^t. Let $D_1, D_2, \ldots, D_p$ be the sets containing the elements of the cycle decomposition of π. Then $F \in \Omega$ is left invariant by π^t if and only if $f(a)=f(b)$ where a and b are in the same D_i. Thus equation (*) in Lemma 5.21 gives the inventory or the sum of weight of the set of functions left invariant by π^t; that is, equation (*) is of the form

$[W(1)]^j+[W(2)]^j+\ldots+[W(m)]^j=\sum_{k\in A}[W(k)]^j$, where $j=|D_i|$. Thus a term in the above expression occurs in equation (*) as many times as $|D_i|$ equals j; that is, as many times as π has a cycle of length j. Hence if π has a cycle decomposition of type $(k_1, k_2, \ldots)$ then among the numbers $|D_1|, |D_2|, \ldots, |D_p|$ the number 1 occurs k_1 times, the number 2 occurs k_2 times, etc. Thus, $\overline{W}(\pi^*)$ or equation (*) can be rewritten as $[\sum_{k\in A}[W(k)]^1]^{k_1}[\sum_{k\in A}[W(k)]^2]^{k_2}\ldots$. Therefore, the right-hand side of (**) becomes $P(G; [\sum_{k\in A}[W(k)]^1]^{k_1}, [\sum_{k\in A}[W(k)]^2]^{k_2},\ldots)$. This completes the proof of Pólya's Theorem.

Corollary:

The number of patterns equals

$$P(G; m, m, \ldots, m),$$

where $m=|A|$.

Proof:

The proof follows immediately from the theorem by choosing all the weights of the elements of A to be one.

CHAPTER 6

Summary and Conclusion

Our objective in this paper was to discuss and illustrate some basic properties of geometric groups and some of their applications. These are groups having their origin in some branch of geometry. Geometric groups are useful in many applications of group theory in science. From a mathematical point of view, they provide a better understanding of the interaction between different branches of mathematics, in particular between group theory, linear algebra, and geometry.

In Chapter 1, we provided the readers with some basic concepts from linear algebra and abstract algebra that are needed in later chapters. We defined some important terms and stated theorems without proofs.

In Chapter 2, we studied two types of length (or distance) preserving transformations of a finite-dimensional Euclidean space namely, orthogonal transformations and Euclidean transformations.

In Chapter 3, we stated and proved Cartan's Theorem and applied it to the classification of orthogonal and Euclidean transformations on *2*- and *3*-dimensional Euclidean spaces.

In Chapter 4, we defined the symmetry group of a set in a Euclidean space and classified the finite symmetry groups of bounded sets in the *2*- and *3*-dimensional Euclidean spaces, $\mathbb{R}^2$ and $\mathbb{R}^3$.

In Chapter 5, we presented two applications of geometric groups, namely, the study of the trigonometric functions of a *2*-dimensional Euclidean space and isomer enumeration in organic chemistry by using Pólya's Theorem.

In conclusion, we suggest that the results of this paper could be extended in different ways. One way would be the classification of orthogonal and Euclidean transformations of n-dimensional Euclidean spaces for $n>3$. A second way would be the study of geometric groups in the context of algebraic topology. A third way would be to investigate the different applications of geometric groups in science.

BIBLIOGRAPHY

[1] Artin, E. *Geometric Algebra*. New York, N.Y.: Interscience Publishers, 1957.

[2] Artin, Michael. *Algebra*. Englewood Cliffs, New Jersey: Prentice Hall, 1991.

[3] Artzy, Rafael. *Linear Geometry*. Reading, Massachusetts: Addison-Wesley, 1965.

[4] Balaban, Alexandru T., Editor, *Chemical Applications of Graph Theory*. London: Academic Press, 1976.

[5] Beckenbach, Edowin F., Editor, *Applied Combinatorial Mathematics*. New York: John Wiley & Sons, 1964.

[6] Benson, C. T. and Grove, L. C. *Finite Reflection Groups*. Tarrytown-on-Hudson, N.Y.: Bogden & Quigley, 1971.

[7] Bonchev, Danail. and Rouvray, Dennis H., Editors, *Chemical Graph Theory: Introduction and Fundamentals*. New York, N.Y.: Gordon and Breach Science Publishers, 1991.

[8] Coxeter, H. S. M. *Introduction to Geometry*. New York: John Wiley & Sons, 1961.

[9] Curtis, Charles W. *Linear Algebra: An Introductory Approach*. 3rd ed. Boston: Allyn and Bacon, 1974.

[10] Dieudonné, Jean. *Linear Algebra and Geometry*. Paris: Hermann, 1969.

[11] Gardiner, Cyril F. *A First Course in Group Theory*. New York, N.Y.: Springer-Verlag, 1980.

[12] Hall, Lowell H. *Group Theory and Symmetry in Chemistry.* New York, N.Y.: McGraw-Hill Book, 1969.

[13] Harary, Frank. and Palmer, Edgar M. *Graphical Enumeration.* New York: Academic Press, 1973.

[14] Kettle, S. F. A. *Symmetry and Structure.* Chichester: John Wiley & Sons, 1985.

[15] Klein, F. *Lectures on Icosahedron.* N.Y.: Dover, 1957.

[16] Kletzing, Dennis. *Abstract Algebra.* San Diego: Harcourt Brace Jovanovich, 1991.

[17] Krishnamurthy, V. *Combinatorics: Theory and Applications.* West Sussex, England: Ellis Horwood, 1986.

[18] Lidl, Rudolf. and Pilz, Günter. *Applied Abstract Algebra.* New York: Springer-Verlag, 1984.

[19] Liu, Chung Lanng. *Introduction to Combinatorial Mathematics.* New York, N.Y.: McGraw-Hill Book, 1968.

[20] Miller, Willard, Jr. *Symmetry Groups and Their Applications.* New York, N.Y.: Academic Press, 1972.

[21] Nomizu, Katsumi. *Fundamentals of Linear Algebra.* 2nd ed. New York, N.Y.: Chelsea Publishing, 1979.

[22] Perkins, Peter G. *Elementary Molecular Bonding Theory.* London: Methuen & Co Ltd, 1969.

[23] Pólya, George. and Read, Ronald C. *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds.* Berlin Heidelberg, N.Y.: Springer-Verlag, 1987.

[24] Pólya, George. and Tarjan, Robert E. and Woods, Donald R. *Notes on Introductory Combinatorics*. Boston: Birkhäuser, 1983.

[25] Redfield, J. Howard. The Theory of Group-Reduced Distributions. *American Journal of Mathematics*. vol. XLIX, 1927. 433-455.

[26] Rees, Elmer G. *Notes on Geometry*. 2nd ed. Berlin Heidelberg, N.Y.: Springer-Verlag, 1988.

[27] Senechal, Marjorie. Finding the Finite Groups of Symmetries of the Sphere. *The American Mathematical Monthly*. vol. 97, Apr. 1990. 329-335.

[28] Snapper, Ernst. and Troyer, Robert J. *Metric Affine Geometry*. New York, N.Y.: Academic Press, 1971.

[29] Trinajstić, Nenad. *Chemical Graph Theory*. Boca Raton, Florida: CRC Press, 1992.

[30] Weyl, Hermann. *Symmetry*. Princeton, New Jersey: Princeton University Press, 1952.

[31] Yale, Paul B. *Geometry and Symmetry*. San Francisco, California: Holden-Day, 1986.

To: All Graduate Students Who Submit a Thesis or Research Problem/Project as Partial Fulfillment of the Requirements for an Advanced Degree

From: Emporia State University Graduate School

I, Hiroshi Shinoyama, hereby submit this thesis/report to Emporia State University as partial fulfillment of the requirements for an advanced degree. I agree that the Library of the University may make it available for use in accordance with its regulations governing materials of this type. I further agree that quoting, photocopying, or other reproduction of this document is allowed for private study, scholarship (including teaching) and research purposes of a nonprofit nature. No copying which involves potential gain will be allowed without written permission of the author.

Hiroshi Shinoyama
Signature of Author

4 / 15 / '93
Date

Geometric Groups
Title of Thesis/Research Project

Doug Cooper
Signature of Graduate Office Staff Member

4-15-93
Date Received

Distribution: Director, William Allen White Library
Graduate School Office
Author