AN ABSTRACT OF THE THESIS OF

David L. Baughman for the Master of Science in Mathematics/Computer Science presented on 1 October 1991.

Title: An Early History of Logarithms.

Abstract Approved: Robert W Kenfield

A large number of reference works on the history of mathematics seem to suggest either that an in-depth discussion of the ideas which led up to the simultaneous discovery of logarithms by Napier and Burgi would be too arcane to be of interest, or that not much did lead up to the achievement of these two discoverers and that the idea of logarithms simply spontaneously created itself in Napier and Burgi, as if of nothing.

Much the same attitude seems prevalent in the area of what the discovery of logarithms meant in subsequent developments in mathematical thinking. Ease of calculations is of course mentioned, but subsequent contributions to the development of calculus and the considerable scientific importance of the discovery of the number *e*, which occurred as a direct result of the discovery of logarithms, most often receives scant attention.

In fact, however, the history of logarithms stretches from Babylon to Newton, and a considerable

number of interesting problems and ingenious solutions can be encountered along the way.

The purpose of this thesis has been to explore in considerable detail the development of the idea of logarithms and of logarithms themselves from the first logarithmic-like tables of the Babylonians through the work of Isaac Newton.

No known study of the history of logarithms in such detail either exists or is currently available. This work attempts to fill that void. Additionally, the history of logarithms runs parallel to, is influenced by, or is in turn influential in a number of other significant developments in the history of mathematics and science, not the least of which is the development of trigonometry. These parallels and influences are an interesting source of study in their own right.

AN EARLY HISTORY

OF LOGARITHMS

A Thesis

Presented to

the Division of Mathematics and Computer Science

EMPORIA STATE UNIVERSITY

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

David L. Baughman

July 1992

Approved for the Division of
Mathematics/Computer Science

Approved for the Graduate Council

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Introduction. A large number of reference works on the history of mathematics seem to assume either that an in-depth discussion of the ideas which led up to the nearly simultaneous discovery of logarithms by Napier and Burgi would be too arcane to be of interest, or that not much did lead up to the achievement of these two discoverers and that the idea of logarithms simply spontaneously created itself in them, as if of nothing.

Much the same attitude seems prevalent in the area of what the discovery of logarithms meant in subsequent developments in mathematical thinking. Ease of calculations is of course mentioned, but subsequent contributions to the development of calculus and the considerable scientific importance of the discovery of the number e, which occurred as a direct result of the discovery of logarithms, most often receives scant attention.

In fact, however, the history of logarithms stretches from Babylon to Newton, and a considerable number of interesting problems and ingenious solutions can be encountered along the way.

<u>1.2 Statement of the Problem</u>. The purpose of this thesis is to explore in considerable detail the development of the idea of logarithms and of logarithms themselves from the first logarithmic-like tables of the Babylonians through the work of Isaac Newton.

<u>1.3 Importance of the Study</u>. No known study of the history of logarithms in such detail either exists or is currently available. This work attempts to fill that void. Additionally, the history of logarithms runs parallel to, is influenced by, or is in turn influential in a number of other significant developments in the history of mathematics and science, not the least of which is the development of trigonometry. These parallels and influences are an interesting source of study in their own right.

<u>1.4 Sources of Information</u>. Libraries of the Kansas Regents' universities have been used to complete the study.

<u>1.5 Organization</u>. The thesis has been divided into six major parts. Chapter I introduces the subject. Chapter II explores early Babylonian work with tables which resemble logarithmic tables as we know them today, and additionally explores Archimedes' work on extremely large numbers and his discovery, in consequence, of logarithmic-like laws. Chapter III discusses parallel developments in trigonometry, especially during the 16th Century and most especially the work of Viete. Such

work influenced the staff of Tycho Brahe and their work was in turn known to Napier. Chapter IV examines the period in which logarithms were actually discovered: their nearly simultaneous discovery by John Napier and Jobst Burgi, and the introduction of the concepts of base number and exponent. Chapter V discusses the subsequent discovery of the number e, and the later work of Newton which expanded the use and mathematical interest in logarithms. Finally, Chapter VI summarizes and draws conclusions based on the study.

CHAPTER II

THE IMPETUS

The Babylonians and Archimedes

2.1 Introduction. If we might say that the problem of the computation of very large and very small numbers, often over and over again, was not truly solved until the inventions of fully-functional and efficient calculators and computers, we may also say that a major step in the improvement of such calculations was made in the Seventeenth Century with the invention of logarithms. For what made such complicated calculations far simpler as was then becoming increasingly necessary was the new-found ability to reduce multiplication to the vastly simpler act of addition. Division thus became a process of subtraction, finding powers simple multiplication, and that of finding roots equally simple division.

But these processes, though infrequent in antiquity, were neither especially new nor unknown to those who may rightly claim for themselves the invention of logarithms. They are found upon Babylonian tablets dating as far back as 2400 B.C.[1] and in Archimedes' *The Sand Reckoner* we find the rule students of logarithms have learned for nearly four centuries, $a^m a^n = a^{m+n}$.[2]

The proper beginning for a study of the history of logarithms, then, takes us into the past more than 4000 years.

2.2 The Babylonians. What is commonly considered or termed 'Babylonia' was a series of Mesopotamian civilizations which existed between the Tigris and Euphrates rivers from about 2000 to 600 B.C., which neither at the beginning nor at the end were entirely dominated by Babylonia itself. Indeed, Babylonia's fall to Cyrus of Persia in 538 B.C. brought an end to the Babylonian empire, but what historians continue to call Babylonian mathematics continued until nearly the birth of Christ,[3] and going all the way back to the fourth millennium before the present era we find a remarkable period of cultural development and a high order of civilization, which included the use of writing, the wheel and metals.[4] Throughout this entire period we find a people who were highly skilled computationally, makers of sophisticated mathematical tables, and sophisticated algebraists.[5] Babylonian mathematicians skillfully developed algorithms for mathematical procedures of considerable complexity, one of which was for "a square-root process often ascribed to later men."[6]

This process was, it is claimed, highly efficient, but Babylonians seemed to prefer what we may consider the much more modern approach by resorting to tables, which were seemingly in abundance.[7] Of some 300 mathematical tablets recovered to date by archeologists fully 200 of these are table tablets covering such topics as multiplication, reciprocals, squares and cubes, and exponentials. In combination with interpolation these latter tables seem to have been used in problems relating to compound interest.[8] In the opinion of mathematics historian Carl B. Boyer, "There is a clear instance [in the Babylonian texts] of the use of interpolation within exponential tables wherein the scribe also uses the compound interest formula $a = P(1+r)^n$."[9]

If such a degree of sophistication seems surprising, historian Howard Whitley Eves gives the following description of to just what extent the Babylonians had developed a financial system which could include the processes of compund interest:

> Even the oldest tablets show a high level of computational ability and make it clear that the sexagesimal positional system was already long established. There are many texts of this early period dealing with farm deliveries and with arithmetical calculations based on these transactions. The tablets show that these ancient Sumerians were familiar with all kinds of legal and domestic contracts, like bills, receipts, promissory notes, accounts, both simple and compound interest, mortgages, deeds of sale, and guaranties.[10]

And such tablets are said to have existed around 2400 B.C.[11]

The floating-point process used in such tables is a familiar one to those of us who have used slide rules or worked with logarithms, and hence belongs in our discussion of the earliest history of logarithms, however no single number was used as a base as in our present system.[12] In the table below the sexagesimal system is used, but the numbers have been written in a form we can more easily understand. The number 23, for example, will stand as 23, while the number 63 will be written 1:3, and 147 will be written 2:27 (60 X 2 + 27). Fractions were a continual problem to Babylonian mathematicians, such that, as an example, 2 9/20 (2 9/20 = 49/20 = 147/60) would also be written as 2:27. The context would presumably determine whether one meant 2 9/20 or 147.[13]

TABLE 1

Sexagesimal Numbers and Their Inverses

Number	Inverse
2	30
3	20
4	15
5	12
6	10
8	7:30
9	6:40
10	6
12	5
15	4
16	3:45
18	3:20
20	3
24	2:30
25	2:24
27	2:13:20
30	2
32	1:52:30
36	1:40
40	1:30
45	1:20
48	1:15
50	1:12

54	1:6:40
1:4	56:15
1:12	50
1:15	48
1:20	45
1:21	44:26:40

To understand how the table works we might notice that the products of the first and second column for the first five rows is sixty, or 1:0. In the sixth row 7:30 is equivalent to 60 X 7 + 30, or 450, and the product of 8 and 450 is 3600, which is 1:0:0 in sexagesimal form. The product of the two numbers is the same in the seventh row and then returns to 60 in the next three rows. In fact, the products of the numbers in each of the rows is a power of 60 (see, for example, 27 X 2:13:20 = 26 X 8000 = 216,000 = 60^3 = 1:0:0:0), which would mean, given the Babylonian practice of dropping terminal zeros, that the products of the numbers in each of the rows is 1.

Now, to use modern terminology, since the product of a number and its multiplicative inverse is one, the consequence of the preceding observations is that the numbers in the second column are the multiplicative inverses of the numbers in the first column, and since it is possible to perform division, which was a lengthy and difficult process for the Babylonians, by multiplying the dividend by the multiplicative inverse of the divisor, we see in the second column of the above table the number to use as a factor if we wish to divide by the number in the first column.

Suppose, for example, we wish to divide 506 by 18 (28 1/9). This converts into 8;26 divided by 18 in sexagesimal notation. In the table above we see that 3;20 is the multiplicative inverse of 18, and so we now have a problem which converts to 8;26 X 3;20, and is performed as follows:

8;26 X 3 = 24 + 78 = 24 + 1;18 = 25;18

8;26 X 20 = 160 + 520 = 160 + 8;40 = 168;40 = 2;48;40

Adding:

25;18 + 2;48;40 = 28;6;40.

We know from context that 28 is the whole number and 6;40 is the fraction. 60 X 6 + 40 = 400 is the numerator in our own terms and 3600 is the denominator, yielding 28 1/9, the correct quotient.

It should be noted that all integers in the table are always factorable into powers of two, three and five, as these always have terminating multiplicative inverses in base 60,[14] but this did not eliminate the possibility of using such tables for other numbers. By proportional parts the Babylonians were able to interpolate intermediate values. Says Boyer, "Linear interpolation seems to have been a commonplace procedure in ancient Mesopotamia."[15]

Such, then, was the first example of procedures which foredate the use of logarithms. Impressive as such calculations were, they were not applied generally to all problems in which large numbers were to be multiplied or divided. That was to wait until Napier, but in the ancient world there was but one more impetus to logarithms, and to that one we may now turn.

2.3 Archimedes. To place Archimedes among the greats of Greek mathematics is perhaps to be a bit misleading if we think only of the classical Greek period. He was, it is true, born in Syracuse, a Greek settlement on Sicily, and it was there to which he returned to live out the rest of his life upon the completion of his education, but he was educated in Alexandria,[16] and that makes all the difference in the kind of mathematics he did. As Morris Kline puts it:

> Of course mathematics had a most important place in the Alexandrian world, but it was not the mathematics that the classical Greek scholars knew. No matter what some mathematicians may say about the purity of their thoughts and their indifference to, or elevation above, their environment, the fact of the matter is that the Hellenistic civilization of Alexandria produced a kind of mathematics almost opposite in character to that produced by the classical Greek age. The new mathematics was practical, the earlier entirely unrelated to application. The new mathematics measure the number of grains of sand in the universe and the distance to the farthest stars; the older one refused to measure.[17]

It turns out that he who measured the number of grains of sand which could be fitted into the universe insofar as he could determine its size was Archimedes,[18] and while the breadth of his work covers ten entire known treatises, as well as traces of lost works[19] we must content ourselves with that which counts the grains of sand in the universe: the *Sand Reckoner*; and in which, it might be noted, we find that Aristarchus, a contemporary mathematician and friend of Archimedes, advocated the Copernican system of the universe.[20] As Archimedes himself noted, most astronomers of the time rejected such theories.[21]

The *Sand Reckoner* itself was an essay addressed to the king of Syracuse to argue the premise that whomever would argue that grains of sand cannot be counted are in error, and that such large numbers as would be necessary are possible in the Greek system of numeration, in spite of the fact that the Greeks "never possessed the boon of a clear, comprehensive symbolism."[22]

The first task, of course, was to determine the size of the universe. This he did by using the accepted estimates of the size of the earth, moon, sun, and the distances from each to the other and to the stars. His determination was that the "diameter of the ordinary universe as far as the sun is less than 10^{10} stades," wherein one stade is about one tenth of a mile.[23]

The next task, of course, is to estimate the size of one grain of sand. For the sake of the argument his estimates throughout the treatise were over-sized, and thus was it that he considered a poppy seed no larger than 10,000 grains of sand, that a poppy seed is at least one fortieth of the width of a finger breadth, and on and on until it is estimated that the known universe can be no larger than would contain 10^{51} grains of sand. To then allow for the size of the universe advocated by Aristarchus, Archimedes increased his estimate to 10^{63}.[24]

It should be pointed out here that such numbers did not appear in the forms rendered above. In fact, what we accept as exponents in modern notations were for Archimedes, Eutocius and Diophantes how denominators of fractions were expressed.[25]

Archimedes instead introduced the concept of period (or order), wherein the first period began with one and ended at ten to the eighth itself raised to the ten to the eighth power less one. The second period began at ten to the eighth power raised to the ten to the eighth power, and there were successive periods through the 10 to the eighth period.[26] Some sense of the size of such numbers may be gathered from the observation that the second period begins, in modern notation, with the number 1 followed by 800,000,000 zeroes, and that his system continued up to "that which we should write down with 1 followed by 80,000 million million ciphers."[27]

In spite of the absence of exponents, the above suggests a system of writing large numbers which uses a base number, in the case of Archimedes, 100,000,000 or 10^8. From the above we know that such a base would have periods (or orders) functioning not unlike our own exponents, and in connection with this Archimedes mentions, "all too incidentally," that to multiply numbers expressible in such a base and by such periods, one need only add the orders.[28] The tremendous size of such a base number as Archimedes' may seem strange to modern readers, and there is nothing in his work which resembles modern exponential notation. Still, in the *Sand Reckoner* Archimedes created a system of notation and expressed an idea which we now associate with $a^m a^n = a^{m+n}$, and, which we shall subsequently see, led directly to the invention of logarithms nearly 2000 years later.

2.4 Summary and Conclusions. The contributions of the Babylonians and of Archimedes himself toward the ultimate invention of logarithms are impressive, even if frustrating. They would seem to come so close to what to us must seem so obvious. The Babylonians ingeniously developed tablets which used shorthand methods for the calculation of interest, but seemed either not interested in or unable to extend their methodology to more general problems and thus to general principles. Archimedes as well acknowledged the possibility of performing multiplication by the device of adding "orders," but there the notion dies.

Still, we see the impetus from which logarithms came into being - to somehow simplify the complicated calculations which are the inevitable by-products of increasingly sophisticated societies. The influence of the Babylonian scribes and Archimedes shall be demonstrated shortly.

CHAPTER III

WORKING UP TO IT

From Antiquity to Francois Viete

3.1 Introduction. Mathematical progression toward the invention of logarithms from Archimedes to John Napier begins slowly and proceeds painfully through the interference of the Roman hegemony and the Middle Ages to the renewal of learning in the Renaissance and in the late sixteenth century hastens to its conclusion.

Three distinct but not separable elements each contributed to Napier's discovery - trigonometry, map-making and astronomy, and each of these deserve attention in order to understand the magnitude of the discovery of logarithms. Trigonometry had, of course, been a source of interest throughout the period to which we refer, though so dormant one could hardly see it breathing. Map-making became a matter of economic as well as intellectual interest in the wake of Columbus' discovery of an entirely new world where none was supposed to exist. And the virtual revolution in observational astronomy brought about by the work of Tycho Brahe and Johannes Kepler had not only severely tested the computational abilities of those who tried to make sense of the discoveries they were making, but brought about a technique of making complicated and difficult computations called 'prosthaphaeresis,' a procedure known to Napier and influential upon his work.

This chapter will seek to examine these influences with consideration of their chronological order.

3.2 Early Significant Work in Trigonometry. Of course the term trigonometry was not coined until not long before the invention of logarithms, but the *Elements* of Euclid is said to have contained "all the essential ingredients for the development of plane trigonometry to the level attained before the co-ordinate method made it possible to deal with direction as opposed to static magnitude.[29] By the second half of the second century B.C. the astronomer Hipparchus of Nicea compiled the first trigonometric table and has thus earned the right to be called the father of trigonometry.[30]

Ptolemy's *Almagest*, a text which is said to have "enjoyed a prestige on all fours with that of Euclid's *Elements*" from 800 A.D to 1500[31] contained tables of sines which "remained an indispensable tool of astronomers for more than a thousand years."[32] To this, subsequent Arab mathematicians added Hindu notations and "added new functions and formulas."[33]

3.3 The Middle Ages. Corresponding developments in mathematical thinking as well began slowly to contribute to the ultimate development of mathematics. Thomas

Bradwardine (1290?-1349), for example, challenged Aristotelian assumptions about the determination of velocity and theorized that

> To double a velocity that arises from some ratio or proportion F/R, he said, it was necessary to square the ratio F/R; to triple the velocity, one must cube the 'proportion' or ratio F/R. This is tantamount to asserting that velocity is given in our notation, by the relationship $V = K \log F/R$, for $\log(F/R)^n = n \log(F/R)$[34]

Extending Bradwardine's work, as well as making considerable contributions of his own, was one Nicole Oresme (1323?-1382), a Parisian who ultimately became Bishop of Lisieux.[35] He generalized Bradwardine's theory to include any rational fractional power and to give rules for combining proportions which are equivalent to our laws of exponents, i.e., as had Archimedes.[36] Additionally, Oresme "was the first person to grasp the significance of a fractional index and the relevance of mapping to the notion of an algebraic fraction."[37] Of final note, but perhaps of greatest significance in the history of logarithms, Oresme was the first to conceive of the outline of a figure as a locus of points.[38]

<u>3.4 The Renaissance</u>. Though we know him usually by the name of Regiomontanus, John Mueller (1436-1476), a German, is important to our study, even if briefly, for it is to him we owe the revival of trigonometry, which shall become more important anon. Mueller was a student of George Peurbach, a Viennese scholar, in both

astronomy and trigonometry, and he extended the work of his master, who had in turn perceived the errors in the Latin translations of the *Almagest* and as well had argued that neither had Arabic authors remained true to the Greek originals.[39]

But more to the immediate point was the work of one Nicole Chuquet (1445?-1500?), of whom little is known save that he was French. Curiously, while both Chuquet and Stifel, some fifty years later, regarded negative numbers as absurdities,[40] it was Chuquet who first recognized positive and negative integral exponents and used them in his notations. Indeed, they are nearly recognizable for what they are to our own eyes. $5x^{-1}$, for example, would be written by Chuquet as .5.^{1m},[41] where the m stands for the French word *moins* - his own word for subtraction.[42] Having then developed exponential notation it was not difficult for Chuquet to restate the laws of exponents and then to further observe the relationships between the powers of the number two, which he did in a table from 0 to 20. Of course the products of the powers correspond to the sum of the exponents, which Chuquet demonstrated and his work, despite the gaps, is perhaps the first near table of logarithms since those of the Babylonian scribes. Stifel (ca. 1487-1567) carried Chuquet's work with tables of base two to the use of negative exponents, i.e., $2^{-1} = 1/2$, $2^{-2} = 1/4$. $2^{-3}= 1/8$, etc., but he did not use Chuquet's exponential notation.[43]

Map-making became a particularly important concern to mathematicians in the latter half of the sixteenth century, but although logarithmic-like tables were developed by at least two men, the influence of such work on Napier is unknown.

The problem for map makers is, of course, that spheres cannot be laid out on flat surfaces without distorting either distances, angles or both between points. Gerhard Kremer (1512-1594), sometimes known as Mercator, devoted his entire life to the development of trigonometric means of solving the problem,[44] however two British mathematicians are credited with developing logarithmic tables to aid in the processes of calculations - Edward Wright (1558-1615) published his in 1610 (four years before Napier) and, later, Thomas Harriot (1560-1621).[45] Both men, however, seem to have been unconscious of the significance of their work.[46]

3.5 Prosthaphaeresis and Francois Viete. The vast number of calculations needed for map-making and the new work by Tycho Brahe (1546-1601) in observational astronomy in Denmark and Prague had left a glaring need for efficient methods of doing such calculations with speed and accuracy, one which was in general use as opposed to the somewhat private and specialized methods of Wright and Harriot, and such a method, before logarithms, became known as Prosthaphaeresis, a Greek term meaning adding/ subtracting.[47] More to the point for our study, prosthaphaeresis permitted mathematicians and other scientists to perform complicated multiplications and divisions through, respectively, addition and subtraction, as did, of course, the subsequent invention of logarithms. While it is generally recognized that Brahe's only major contribution to the history of science is that he was the first to see the necessity for "precise and continuous" observational data in astronomy,[48] it is also true that Brahe's use of prosthaphaeresis and his methodology for it had come to be known by Napier through one Dr. John Craig as early as 1590.[49] But such a methodology first required the contributions of one Francois Viete (1540-1603), considered to be the greatest French mathematician of the sixteenth century.[50] While his most famous work was *In artem*, which did much for the development of symbolic algebra,[51] he is important to us in our work for the development of sum-to-product and product-to-sum trigonometric identities.

If we begin with the formulas for the cosines of the sum and difference of two angles, which were known at the time of Viete, we have:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y, \text{ and}$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Adding the two identities together gives us:

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y,$$

wherein the sin x sin y terms have dropped out. All that is necessary to arrive at the formula for which prosthaphaeresis was used is to divide both sides of the identity by two, resulting in:

$$\cos x \cos y = [\cos(x + y) + \cos(x - y)] / 2.$$ [52]

An example cited by historian Carl Boyer perhaps best illustrates how prosthaphaeresis worked:

> If, for example, one wished to multiply 98,436 by 79,253, one could let cos A = 49,218 (that is, 98,436/2) and cos B = 79,253. (In modern notation we would place a decimal point, temporarily, before each of the numbers and adjust the decimal point in the answer.) Then, from the table of trigonometric functions one reads off angles A and B, and from the table one looks up cos(A + B) and cos(A - B), the sum of these being the product desired. Note that the product is found without any multiplication having been performed.[53]

To illustrate the validity of the method, and not coincidentally, the accuracy of Boyer's description thereof, the author has written a program in Turbo Pascal which essentially duplicates the method, and it does work! A description of the program and the program itself can be found in Appendix A at the end of this study. Quotients were handled in the same manner by using tables of secants and cosecants.[54]

It is important to note, too, that for the process to work accurate tables were required, and by 1596 such tables had been worked out correctly to seven decimal places at intervals of ten seconds.[55]

<u>3.6 Summary and Conclusions</u>. Clearly, then, the

growing sophistication of trigonometric methods, the pressures of map-making and astronomy, and the development of prosthaphaeresis had set the stage for the final assault on what was to be the invention of logarithms. Some had come close and some had made significant discoveries which aided Napier and his contemporaries in their final development of what Napier himself called "the marvelous rule of logarithms." It is to such work that we may now turn.

CHAPTER IV

FULFILLMENT

John Napier and the Invention of Logarithms

4.1 Introduction. "One of the great diseases of this age," wrote Barnaby Rich (1540?-1617) in 1600, "is the multitude of books that doth so overcharge the world that it is not able to digest the abundance of idle matter that is every day hatched and brought into the world."[56] Rich himself not only complained of the problem, he contributed to it. A self-educated soldier and writer of innumerable light romances, one of which is said to have inspired Shakespeare's "Twelfth Night," he brought into the world much of what he so eloquently complains.[57] And of the sheer volume of work we can hardly dispute him. But even if we take only the twenty years, 1594-1614, in which John Napier is said to have worked on logarithms, we encounter, beyond the "idle matter" of Barnaby Rich, what must be one of the most productive, tumultuous and intellectually important periods in history.

It was the time of Shakespeare, Elizabeth I, Sir Walter Raleigh, Sir Francis Drake, Francis Bacon and the Spanish Armada (1597). Oliver Cromwell was born in 1599 and Elizabeth herself died in 1603, not before, however, the Earl of Essex in 1601 would lead a revolt against her and was executed for treason that same year for his trouble. James I succeeded Elizabeth to the throne, arousing Raleigh against him, for which Raleigh was himself sent to prison that same year. Peace between England and Spain was declared in 1604, but the next year saw Guy Fawkes try to blow up the House of Lords, for which he is remembered in England to this day.

Shakespeare wrote and produced some of his greatest works during this period: "The Two Gentlemen of Verona," "Love's Labor Lost," and "Romeo and Juliet" appeared in 1594. "A Midsummer Night's Dream," and "Richard II" followed in the next year. "The Merchant of Venice" appeared in 1596, "Henry IV" appeared in both parts the following year, "Much Ado about Nothing" the year after that, "Julius Caesar," "As You Like It," and "Twelfth Night" in 1599; "Hamlet" and "The Merry Wives of Windsor" in 1600; "King Lear" and "MacBeth" in 1605, and other works until fire destroyed the Globe Theatre in 1613, and the great man himself died in 1616.

Sir Francis Bacon published his influential *The Advancement of Learning* in 1605. The authorized version of the *King James Bible* was published in 1611, and in the following year, coincidentally one assumes, there occurred the last recorded burning of heretics in England. Passionate disputations regarding the evils of Protestantism by Catholics and of Catholicism by

Protestants, too many to even begin to list (but one by our own John Napier), were published during the period, but amidst all that Raleigh found the time to write and publish his great *A History of the World* in 1614.

Some evidence of the extraordinary work in science and mathematics during the period has already been seen. There is more. Galileo invented the thermometer, Johannes Kepler published *De admirabli proportione coelestium orbium*, and the *Trigonometric Tables* of G. D. Rheticus were published posthumously, all in the year 1596. In 1598 Tycho Brahe published an account of his work and a description of his instruments, already noted, and in 1600 he and Kepler began their brief, stormy but infinitely important work together. Dutch opticians invented the telescope that year as well. In 1604 Kepler published his *Optics*. Galileo invented the proportional compass in 1606 and in the following year constructed an astronomical telescope. In 1606 Kepler published his greatest work, *De motibus stellea Martis*. And while Napier himself is often given much of the credit for popularizing the decimal point, it was Bartholomew Pitiscus, a German mathematician, who first used them in trigonometric tables in 1612.

In 1614 John Napier published *Mirifici logarithmorum canonis descriptio.*[58]

The publication of this book certifies for most historians the invention of logarithms by John Napier in

1614. "There is evidence," writes J. M. Dubbey, "that he established the theoretical principles twenty years earlier at a time when indices were not generally used, when there was little idea of functional relationships, and even decimals were hardly respectable."[59] Lord Moulton is even more impressed:

> The invention of logarithms came on the world as a bolt from the blue. No previous work had led up to it; nothing had foreshadowed it or heralded its arrival. It stands isolated, breaking upon human thought abruptly without borrowing from the work of other intellects or following known lines of mathematical thought.[60]

The purpose of this study has been, of course, to argue otherwise and the rest of Lord Moulton's address, upon the occasion of Napier's tercentennial and cited above, seems to argue as much in support of the thesis of this study as his own. Yet still another controversy arises - that of the priority of the invention of logarithms. Florian Cajori cites evidence from Kepler that Jobst Burgi (1552-1632)[61] "had a complete set of logarithmic tables sometime between 1603 and 1611, but had declined to publish them before Napier did his in 1614."[62] And Carl Boyer contributes the possibility that "the idea of logarithms had occurred to Burgi as early as 1588, which would be half-a-dozen years before Napier began work in the same direction."[63]

Still, priority for the invention of logarithms is generally given to Napier, and even to Cajori, "Few inventors have a clearer title to priority than has

Napier to the invention of logarithms."[64] His argument is a fairly simple one: First, Napier clearly published first. Second, that Burgi is entitled to the honor of independent invention. And finally, though he offers no evidence in support of this contention, Napier had begun his work in 1594 and that this was, therefore, "probably much earlier than Burgi."[65]

That Napier did conceive of, develop and then publish a table of logarithms independently of anyone else does seem beyond contention, and that he was the first to publish his results, regardless of any claims to the prior existence of tables by Burgi, does seem to assure Napier the unassailable right of priority. As a result, this chapter is entirely devoted to Napier and his invention of logarithms, with the following chapter devoted to Burgi, Briggs, Vlacq and others up to Newton whose work with or independently of Napier so enriched his invention. After a brief look at what is known of the man Napier himself, the derivation of Napier's logarithms will be discussed in detail, followed by an examination of what is speculated about how he came to such invention.

<u>4.2 Napier, the Man</u>. Little is known of Napier, "but," as to what is known says Lord Moulton, "it suffices to show him strong and self-reliant, a man of solitary habits of thought and untiring industry, and I am content with such a delineation of the man of whom I

am about to speak."[66] That may be, but it is also true that Napier was a nobleman, Lord of Merchiston, and he was not in the slightest either a professional mathematician, or one who considered his mathematics the source of his ultimate fame. Even at the time of his birth the issue of whether Protestantism or Catholicism would prevail among the Scottish people had begun.[67] Napier himself was the son of his father, Protestant to the core, as had been his father before him,[68] and "It came to be," wrote P. Hume Brown, "his burning conviction that the salvation of mankind was bound up with the overthrow of the Papacy."[69] To argue to this end he wrote *A Plaine Discovery of the Whole Revelation of St. John*,[70] which brought him far more fame in his own lifetime than ever did the invention of logarithms.[71]

Napier's motivation for beginning the work which led ultimately to his discovery is not especially obscure. Carl Boyer points to the considerable interest in trigonometry during the period, a time, however, "primarily of synthesis and textbooks" on the subject.[72] Pitiscus, for example, first published his book, already cited, in 1595, which seems to have given the subject its name,[73] and Lord Moulton suggests that

> his original idea was only to construct tables that would enable the product of two sines to be readily ascertained. If I am right in this, the suggestion may well have come to him from his familiarity with the well-known trigonometrical formula

$$\text{Sin A Sin B} = 1/2[\text{Cos}(A - B) - \text{Cos}(A + B)],$$

> which expresses the product of two sines in terms of the cosines of the sum and difference of the angles.[74]

J. W. L. Glaisher seemingly concurs "...that Napier's table was one of sines, connecting him with the great table-makers of the previous century. His principal object was to facilitate the multiplication and division of sines, and this was effected by his table."[75] How he proceeded to such effect can and shall be examined, but to greater effect, perhaps, if we examine first what he invented and how it works. And to that we may now turn.

4.3 The Logarithms of John Napier. It should be pointed out at the outset that Napier's logarithms, as he first invented them, will not appear familiar to a reader who is familiar only with such versions as have since evolved. First of all, they will be without decimal points at their beginning. "Napier's object," Joseph Frederick Scott reminds us, "was to simplify trigonometrical computation. Since it was still customary to regard sines as lines drawn in a circle of suitable radius, the necessary accuracy could be obtained by making the radius very large, say of the order of 10,000,000. Napier followed the practice."[76] And as his logarithmic tables were tables of the logarithms of sines, we shall find therein sines from 0 to 10,000,000. Another difference is that as sines become smaller the logarithms themselves become larger. For example, the sine 2327 has a logarithm of 6063128 in Napier's tables published in 1616, while the sine of 999998 has the logarithm of 2.2 in the same table.[77]

But whatever the seeming oddities of Napier's tables, Napier understood that, in the words of Lord Moulton,

> In order to create tables which would enable numbers to be multiplied together without actually performing the operation, they must not be represented as resulting from continuous addition [but] as resulting from continuous multiplication.[78]

The first step, then, in creating a table of Naperian logarithms is to create a geometrical sequence, in which, by definition each succeeding element of the sequence is the product of a single factor used one more time than in the preceding element. For this Napier used

$$r,\ r(1-1/r),\ r(1-1/r)^2, \ldots, r(1-1/r)^n,$$

where r = 10,000,000.[79] The logarithm, then, of each succeeding sine is the exponent of the factor 1-1/r which was used to reach that element. And this pattern can be observed from the output of a Turbo Pascal computer program in Appendix B, written to generate Napierian logarithms by this method. In that program the angles corresponding to the sine of the angles are also given and they correspond to Napier's own.

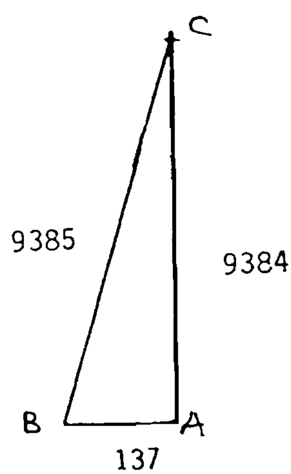
The result is that there is a correspondence between each element in an arithmetic sequence and another element in a geometric sequence, giving rise to what is Napier's own definition and explanation of a logarithm:

> The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which the radius began to decrease geometrically, and in the same time as the radius has decreased to the given sine.
>
> ```
> T        d                    S
> ---------¦--------------------
> g        g
>
> b                    c          i
> ---------------------¦-----------
> a                    a
> ```
>
> Let the line *TS* be the radius, and *dS* a given sine in the same line; let *g* move geometrically from *T* to *d* in certain determinate moments of time. Again, let *bi* be another line, infinite towards *i*, along which, from *b*, let *a* move arithmetically with the same velocity as *g* had at first when at *T*; and from the fixed point *b* in the direction of *i* let *a* advance in just the same moments of time up to the point *c*. The number measuring the line *bc* is called the logarithm of the given sine *dS*.[80]

Napier is understandably proud of his achievement and seems not to tire of providing examples of its trigonometric properties and their usefulness. An example of a theorem in a section devoted to right-angle triangles may prove interesting. "In a right-angle triangle the Logarithm of the leg is equal to both the Logarithm of the angle opposite thereto, and the Logarithm of the Hypotenuse."[81] Using the triangle in Figure 1, the right angle is at A, with a hypotenuse of 9385 and legs of 9384 and 137. Suppose we know the hypotenuse and the leg AB but wish to determine the angles at B and C. We look in the tables for the log of 9385, 63587, and the log for 9384, 63480. The difference is 107. This is the log of 89 degrees, 9 3/4 minutes, which is the Angle at C. The angle at B is the complement, or 0 degrees, 50 1/4 minutes. Corresponding examples are given for finding each of the other two elements if the other two are given.[82]

FIGURE 1

Napier Right-Angle Triangle



4.4 Methods of Discovery. While Napier's system, its derivation and the uses of logarithms seem simple enough, the process of discovery, lasting, as has been noted, some twenty years, is one of some controversy among his most respectful admirers. Most explanations of Napier's system begin with a two-line drawing such as that provided by Napier himself in the explanation cited above, thus giving rise to the supposition that, as David Eugene Smith has put it, "...Napier approached logarithms through considerations seemingly quite geometrical or mechanical."[83] But Lord Moulton argues forcibly that the process began quite differently and followed three successive stages.

Moulton seems to feel that Napier's first problem was to deal with the question of how to create a geometric sequence of sines in descending order. "Hence it came to pass," he states, "that Napier's first idea was to start from this large figure of 10,000,000 and to multiply repeatedly by a factor slightly less than 1."[84] He did so by the method which has already been described. Having then found such a series, says Moulton, Napier then discovered that he had found a second series, what Napier called logarithms, which was arithmetic in ascending order.

> In this way [writes Moulton] every number less than 10,000,000, that is to say every sine, would be very close to some number produced by a known number of these repeated operations of multiplication by the chosen factor, and might without appreciable error be taken to be the actual result of that number of operations.[85]

The first stage, then, was to discover the arithmetic means by which a radius, or sine, could be reduced by means of continued multiplication and then to assign a number which represented the number of times such an operation had to be performed in order to reach a given radius or sine. The second stage of discovery, according to Moulton, was to begin representing the continual reduction of the radius by a line. Thus was the line representing the radius successively cut to represent the new, shorter radius, while at the same time a second line grew by a single element <u>at the same time</u>.[86]

This, according to Moulton, "naturally gave rise to the idea of a moving point whose velocity is proportional to its distance from the other end of the line."[87] While Moulton agrees, however, that this new description differs little from what has already been described, i.e., it merely gives a geometrical representation to an arithmetic operation, he states that at this point Napier took a critical, though certainly obvious, step: He states that "the logarithms of proportionals are 'equally differing,'"[88] and it was this step which permitted him to affirm that his system of logarithms permitted its users to "find continued and mean proportionals of all kinds, to extract roots, to calculate powers."[89] Further,

> His representation of the single operations by equal distances on the logarithmic line adapted itself perfectly to the new line of thought. Equal groups were represented by equal lengths on the logarithmic line, and thus he came to view the addition of a certain length to the logarithms of numbers as giving you the logarithms of those numbers after they had been reduced in a given proportion.[90]

This concentration on groups of operations, states Moulton, now led Napier to what Moulton considers the third and, to Moulton, the most interesting stage of development - the notion of changes of 'determinant moments' so small as to make the reduction by each single operation infinitely small, "and the number of operations infinitely great."[91] To Moulton this represents a jump from "discontinuity to continuous motion,"[92] and he seems happily stunned that Napier ignored such theoretical difficulties as presented themselves thusly for the simple reason that his system worked anyway. "Napier," he states, "saw that his work must be true of continuous motion if it was true of all discontinuous motion, and that he was not going to be delayed in his great and practical task by any metaphysical difficulties that he foresaw could not affect his results."[93]

Whatever Lord Moulton's enthusiasm, however, Julian Coolidge remains unimpressed. Referring to Moulton's earlier assertion that Napier might have been influenced by his awareness of a product-to-sum formula in trigonometry to develop logarithms, Coolidge states, "I cannot feel that this point is well taken. We shall see that he began with a comparison of arithmetical and geometrical series, even though his first logarithms were merely those of sines."[94] Yet in seeming contradiction of at least a part of the above, Coolidge later asserts that Napier, "Instead of following the natural arithmetical route suggested by the relation of arithmetical and geometrical progressions, he introduced geometrical considerations which seem to me to have complicated the matter enormously."[95] Coolidge then goes on to describe moving points on lines such as what have already been considered, and as a final swipe at Moulton, and without offering evidence, Coolidge states that he is "not convinced that Napier had really the difficult idea of an instantaneous velocity which was so baffling to Newton and the other early writers on the calculus."[96]

4.5 Summary and Conclusions. By whatever the means of discovery, John Napier, Lord of Merchiston, inventor of logarithms, died in 1617, a scant three years after his invention was completed, though not before he could himself make considerable contributions, as shall be seen, to its continued development. It is probable that he knew little of the controversy that would eventually surround his invention, and it is pleasant to read his near boyish enthusiasm as he describes its wonders.

But perhaps it is good that he died when he did. The Thirty-Years War would begin in 1618 and Sir Walter Raleigh would be executed that same year. England and Scotland had already begun to nearly tear themselves apart in religious and political strife, spurred on by the rise of Puritanism. Galileo had faced the Inquisition for the first time in 1615 and was prohibited from further scientific work the following year. Dark clouds seemed to be gathering.

The Age of Reason, however, had only just begun, and dark prejudices could not stem the flood of books. "Already," wrote Robert Burton (1577-1640) in 1626, appropriately enough the author of *The Art of Melancholy*,[37] "we shall have a vast chaos and confusion of books; we are oppressed with them, our eyes ache with reading, our fingers with turning."[38]

attributes to Burgi the inventions of decimal fractions and logarithms, though Burgi did not publish his work until six years after Napier had published his and had become famous throughout Europe through the influence of Kepler.[100] As has been observed earlier herein, there seems little question that Napier deserves priority in the invention of logarithms for the reasons already given, a more contemporary historian has offered the additional observation that Burgi's logarithms were "not nearly so sophisticated or useful" as Napier's.[101]

Burgi's work was not unlike that of Napier's. He conceived of a correspondence between arithmetic and geometrical series, and his method of calculation was very similar to Napier's. That is, there is a common multiplier of 108, a common ratio, and an exponent which indicated the number of times an operation needed to be performed and which had correspondence to the number for which we wish to find the logarithm. A sample of Burgi's logarithmic tables may help to clarify his methods.

TABLE 2

Sample of Logarithmic Tables
of Jobst Burgi

#	Log.
0	1 0000 00000
10	1 0001 0000
20	1 0002 0001
30	1 0003 0003
- - -	- - - - - - - - - -
990	1 0099 4967

The common ratio, then, is 1.0001, and each individual element is determined by the following formula:

$$10 \text{ X } n = 10^{8} \text{ X } (1.0001)^{n}$$ [102]

Boyer seems more impressed with Burgi's tables than with Napier's. He argues, for example, that Burgi's tables are very nearly those of the natural logs as we know them today. For one thing, if we were to divide all the numbers in the left-hand column (Burgi called these the red numbers and had them printed in red) by 10^{5}, and all the numbers in the right-hand column (the 'black' numbers) by 10^{8}, we would have "virtually a system of natural logarithms." Further, he points out that $(1 + 10^{-4})^{10^{-4}}$ is approximately equal to *e* correct to four decimal places.[103]

All of this was published in 1620, presumably at the great urging of Kepler, but by that time, three years after Napier's death, great changes were being undertaken in Napier's system. Logarithms to base ten had been discussed by Napier and Briggs and the long, hard work of constructing those tables was underway. They would first appear in 1624 and occupy the energies of more than one outstanding mathematician. It is to such work that we may now turn.

5.3 Common Logarithms. In the remainder of the seventeenth century, and before Newton, the world of mathematics would encounter Descartes, Fermat and Pascal. But before them, says Lancelot Hogben, "the invention of logarithms was by far the most outstanding mathematical innovation made by Western Christendom,"[104] though Boyer cautions us that while logarithms had ultimately

> a tremendous impact on the structure of mathematics . . . at the time it could not be compared in theoretical significance with the work, say, of Viete. Logarithms were hailed gladly by Kepler not as a contribution to thought, but because they vastly increased the computation power of the astronomer.[105]

No one, however, was probably more excited by the invention of logarithms than one Henry Briggs, who has been heretofore only just mentioned. Thomas Smith in his 1707 biography of Briggs, characteristically for the time entitled "A Memoir on the Life and Work of that Most Famous and Learned Man, Henry Briggs," refers to the enthusiastic response of the scientific and mathematical world to Napier's *Descriptio* in 1614 and continues:

> but none did so more than our Briggs, who took the canon, which cleverly and ingeniously condensed such great matters into so few small pages, and studied it in every aspect; and furthermore, just as if he himself discovered

> by his own efforts, he penetrated into the deeper secrets of it. This book gave him the keenest delight. He carried it about in his hands or in his pocket, and read it through over and over again with most eager eyes and the closest attention.[106]

Quite apparently the feelings of admiration were reciprocated. Briggs had offered to visit Napier, a journey in those days of considerable distance and discomfort. When delayed Napier is said to have grown despondent that Briggs would not appear, but when at last Briggs did present himself, notes the variously described astrologer and astronomer William Lilly, "almost *one quarter of an hour was spent*, each beholding the other with admiration, *before one word was spoke.*"[107]

But Henry Briggs brought with him no uncritical admiration. Napier's logarithms were cumbersome by modern standards as a boon to calculation. The working rules for multiplication, division, powers and roots were complicated, and multiplication or division by ten necessitated, in Napier's logarithms the addition or subtraction of the ponderous number 23025842.[108] Cajori offers this description of what happened next:

> Briggs suggested to Napier the advantage of what would result from retaining zero for the logarithm of the tenth part of the same sine, i.e., $5°44'22''$. Napier said that he had already thought of that change and he pointed out a slight improvement on Briggs's idea; viz. that zero should be the logarithm of one, and 10^7 that of the whole sine, thereby making the characteristic of numbers greater than unity positive and not negative, as suggested by Briggs.[109]

Cajori's description confirms the level of cooperation between the two men, which seems to have been a matter of some controversy some little time after a publication of Briggs's logarithms in 1624. It seems clear, in other words, that Briggs not only accepted but readily concurred in Napier's suggestions as "'by far the most convenient.'"[110]

The first clear indication of the 'convenience' of such a system can be found in the following table, wherein the logarithms of the various portions of 'the whole sine' have been easily calculated.

TABLE 3

Logarithms of 'Whole Sines'

Number	Briggs Common Logarithm	Modern Common Logarithm
1	10^0	0
10	10^1	1
100	10^2	2
1,000	10^3	3
10,000	10^4	4
100,000	10^5	5
1,000,000	10^6	6
10,000,000	10^7	7

The question now remains how Briggs and subsequent mathematicians calculated the values for the numbers between these powers of ten. While some descriptions of the process vary in detail, the general process is called "proportional parts."[111] An examination of the numbers and logs to base 2 will provide an understanding of this simple process.

log	0	1	2	3	4	5
number	1	2	4	8	16	32

It might be observed that in the first row each number is the arithmetic mean of the two before and after it. For example, 3 is the arithmetic mean of 2 and 4. Each of the numbers in the second row, however, is the geometric mean of the two before and after it. Thus we may conclude that the number having the logarithm base two of three is the geometric mean of the numbers whose logarithms base two are 2 and 4 respectively. To find the geometric mean of any two numbers we find their product and take the square root of it. Hence we may find the number which has the logarithm base two of three by taking the product of 4 and 16, which is 64, and finding the square root of that, 8. And, of course, eight is the number which has the logarithm base two of 3, as can be seen above.

By exactly the same process did Briggs and others find the common logarithms of the numbers between powers of ten. Suppose, for example, we wished to find the missing number in the second row below which has the common logarithm 1.5.

log	0	1	1.5	2
number	1	10		100

total of 47 times."[113] The technique, however, was more complicated than that. We need first to understand that Briggs began by taking a large number of geometric means between unity and given primes - in this case two, though that is not immediately apparent in what follows - and then halving repeatedly the numbers corresponding those for which the roots are logarithms. A sample of a table of these results follows.

TABLE 4

Logarithms by Geometric Mean

NUMBERS	LOGARITHMS
10	1
3.162277...	0.5
1.778279...	0.25
1.333521...	0.125
1.154781...	0.0625

Briggs used 30 decimal places in his calculations and was eventually able to extract 54 roots of 10! At that point he was able to determine that the logarithm of 1.000000000000000012781914932003235 was 0.0000000000000000551115123125782702. Briggs was additionally able to determine that numbers of the above form, i.e., 1 followed by 15 zeros and seventeen or fewer significant digits, had logarithms proportional to those significant digits. By proportions he then concluded that the log of 1.0000000000000001 was 0.0000000000000000434294481903251804. This was significant because it permitted him to determine the logarithm of any number of the form 1.00000000000000x simply by multiplying the x by 4.34294481903251804.

But this only sets the stage for the calculation of the logarithm of 2. The first step in the actual calculation is to raise 2 to the tenth power, i.e., 1024. Dividing that by 1000 gives us a number, 1.024, between 2 and unity. The taking of the square root of that number 47 times then provides us with a number of the form above, 1.00000000000000016851605705394977, to be exact, and the multiplication of 4.342... provides the logarithm of that quantity, or 0.0000000000000-00731855936906239336, which when multiplied by 2 raised to the 47th power gives the logarithm of 1.024, which is then 0.01029995663981195265277444. All that remains, after all that, is to add the characteristic 3 and divide by 10 to get the logarithm of two: 0.301029995-663981195. Clearly such was a labor of love.[114]

Henry Briggs published the first common logarithms in 1617, the year of Napier's death. It contained the logarithms of the numbers 1 to 1000 to fourteen decimal places, of which a portion appear below. It might be noted, however, that no decimal points appeared as the 'whole sine' was still considered 10^7 or 10^8

TABLE 5

Partial List of Logarithms
by Henry Briggs

NUMBER	LOGARITHMI
1	00000,00000,0000
2	30102,99956,6398
3	47712,12547,1966
34	15314,78917,04226
35	15440,68044,35028
67	18260,74802,70083[115]

Briggs later published his *Arithmetica Logarithmica, sive Logarithmorum Chiliades Triginta* in 1624, which contained the logarithms of the first 20,000 numbers and those from 90,000 to 100,000. Adrian Vlacq, a Dutch bookseller, published the logarithms of the numbers 20,000 to 90,000 to ten decimal places, along with Briggs's prior work, in 1628.[116] Vlacq made considerable contributions to the understanding of the principles underlying the computation and use of logarithms in his book *Trigonometria Artificialis*, published around the year 1633. In it and subsequent works Vlacq introduced the word characteristic as follows:

> Here you will note that the first figure of the logarithm, which is called the characteristic, is always less by unity than the number of figures in the number whose logarithm is taken. For example, because log 3567894 is 6.5524118, it follows that
>
> log 3.567894 = 0.5524118
> log 35.67894 = 1.5524118
> log 356.7894 = 2.5524118
>
> and so on.[117]

Considerable work on the refinement of common logarithms to ever-greater accuracy occurred throughout the three centuries which have passed since the development of common logarithms, but except for one notable exception to be encountered a little later, these are beyond the scope of this study. What we have seen so far is that common logarithms arose from flaws in Napier's logarithms obvious even to him, and that while he clearly understood what needed to be done, he was blessed by the friendship of one who would carry his great work beyond him.

But neither Briggs nor those who followed after him completed the full invention of logarithms as we know them today. John Gunter came up with the idea of inscribing logarithms on linear scales and the result was what came to be known as Gunter's Scale, a precursor to William Oughtred's invention of the slide rule not long afterwards.[118] Hyperbolic or natural logarithms were to flow out of Napier's work as well, and that is what shall be examined next.

5.4 Natural Logarithms. Previous descriptions of Napier's tables of logarithms lead easily to the conclusion that while the concepts of base and exponent were entirely foreign to him, Napier's logarithms, despite the absence of decimal points, would in modern terminology be described as having a base of $1/e$.[119]

Thus, the transition to base e not long after Napier's logarithms should hardly come as a major surprise. In fact, John Speidell, a contemporary of Briggs's, published in 1619 the first known natural logarithms of the trigonometric functions. A subsequent publication by Speidell in 1622 expanded his work to include the numbers from one to 1,000.[120] The source of Speidell's development of natural logarithms is obscure and deserves far more study than it has seemed to receive, obviously, but we are aware that the use of infinite series to explore such problems as the square root of 2 had been undertaken first by one Pietro Antonio Cataldi (1548-1626) of Bologna. Gregory St. Vincent was the first, at about 1668, to examine the curve y = ln(x), but his influence was of little regard and much of what he accomplished was subsequently to be rediscovered much later by others.[121] His work with quadratures, however, is fascinating, and in 1647, in the words of Florian Cajori, St. Vincent

> found the grand property of the equilateral hyperbola which connected the hyperbolic space between the asymptotes with the natural logarithms, and led to these logarithms being called 'hyperbolic.' By this property Nicolas Mercator in 1668 arrived at the logarithmic series, and showed how the construction of logarithmic tables could be reduced by series to the quadrature of hyperbolic spaces.[122]

Mercator (1620-1687) was Danish but he lived in London for some considerable time and then moved to France in 1683 to design the fountains at Versailles.

The second part of his book of 1668, *Logarithmotechnia*, contains a number of approximation formulas for logarithms. It had been known from Gregory St. Vincent that the area under the hyperbola $y = 1/(1 + x)$ from $x = 0$ to $x = x$ is $\ln(1 + x)$. And long division followed by integration does give the infinite series

$$x/1 - x^2/2 + x^3/3 - x^4/4 + . \quad . \quad .$$ [123]

Still another approximation of the natural logarithms can be found in Appendix C. The computer program found therein uses Simpson's Rule to numerically integrate the function $f(x) = 1/x$.

In spite of the presence and widespread acceptance of the system of natural logarithms, the concept of base *e* was little understood, if at all. Work by Oughtred, Halley and Euler on the compound interest law ultimately brought about the discovery of $e = 2.718...$,[124] but it was not until Leonhard Euler in 1727 or 1728 that *e* was formally presented as the base of the system of natural logarithms.

In spite of the rapid development of logarithms in both ease of computation and theoretical development, some historians chafe at what one calls "the incredibly tortuous approach to the construction of tables of logarithms by Napier and Briggs" - and, one presumes, others.[125] The influence of Wallis and Newton, he continues, "universalized the use of negative and fractional exponents," and so, it would seem, placed the final brick into the structure we now know as logarithms.

5.5 Summary and Conclusions. With or without 'Napier's Bones,' Gunter's Scales, the slide rule and even negative and fractional exponents, the construction of logarithm tables, indeed of tables of every kind, took on vast proportions from the time of the first tables by Napier to the invention of calculators more than three centuries later. The standardization of logarithmic tables to base 10 and base *e* made possible vast improvements in the means, speed and accuracy of computations, in addition to theoretical implications of logarithmic curves and infinite series.

Yet the story of logarithms is not complete without reference to a gentleman of our own century who seemed to beckon back to a time when life was simpler and, it would seem, a lot more tedious. His name was W. E. Mansell and he was a London accountant. He retired at the age of 47, became a recluse, and devoted the next twelve years of his life to the construction of natural and common logarithms of the numbers one to 1,000 to 110 decimal places, entirely without any means of assistance in his calculations! His work was finally published in 1964 and discovered to have been without a single error.[126] That such an achievement was considerable is obvious, that it was useful is doubtful.

Yet it is, somehow, a fitting end.

CHAPTER VI

OVERVIEW

Summary and Conclusions

6.1 Introduction. The chronological scope of this study has been a period of something over 4,000 years - from the table-makers of Babylonia around 2400 B.C. to the standardization of notation by Wallis and Newton in the late seventeenth century. It has touched virtually every significant period of mathematical development during that time, and it has discussed the work, even if only incidentally at times, of more mathematicians than one would care to count.

This study has been an attempt to write as complete as possible a history of a subject - logarithms - which has not, by itself, seemed to have absorbed the interest of historians of mathematics or science, save as incidental to what has seemed more pressing to the curious intellect, done in the hope, one might add, of showing that the history of that subject is more interesting than might have been presupposed.

While one hopes that that has been achieved, it must be admitted that whatever might have been hoped for in regard to completeness, there remain large gaps in the fabric of this study, beyond the present resources of this author to repair, and not, one hopes, for lack of effort or intellect to comprehend what might be all too clear to others. The purpose of this chapter is to condense what has been found into a whole, and to point out what one can only wish had been discovered.

6.2 Summary. The story is essentially one of how to make complex computations more quickly, easily and accurately, and it began nearly as it ended - with table-makers. The Babylonians constructed them of all kinds, including ones for the computation of compound interest, the very topic central to the discovery of the number e four thousand years later. One of them was explored for how it used reciprocals for simplifying division.

Archimedes in the Alexandrian Greek period added the significant contribution in *The Sand Reckoner* of developing large numbers by what amounted to the use of the laws of exponents to facilitate multiplication and division. Save for the fact, however, that neither the Babylonians nor Archimedes, typical of their time in the history of mathematics, looked beyond the particular application to general theory, we might speculate that such discoveries might easily have brought about the invention of logarithms far earlier than what did in fact occur.

European mathematics was slow to regain the initiative toward discoveries which would unfold the mysteries of logarithms. There was, of course, the steady development of trigonometry even before the Renaissance, including the product-to-sum formulas which would figure so prominently in the sixteenth century. Too, there was the rediscovery of the laws of exponents, which with Stifel's arrangement of the table of base-two indices seems to us in retrospect to teeter on the brink of logarithms.

But that would wait for map-making and astronomy. Edward Wright compiled the first table of logarithms, but he, like his ancient forbearers, looked no further than his own particular application. And astronomers, eager to simplify their massive computational problems ultimately developed a method for multiplying by addition, which we have discussed under the name of prosthaphaeresis.

At last, in 1594 John Napier began his twenty-year search for a system of logarithms which could be used to simplify all of the basic computational needs of the user of mathematics. He saw the need to match an arithmetic sequence with a geometric one, such that the multiplication, division, taking of roots and raising to powers of the numbers in the geometric sequence would require only the respective operations of addition, subtraction, division and multiplication of the corresponding elements in the arithmetic sequence. His system, as first published in 1614, did not work well, but incurred the passionate support of his friend and intellectual benefactor, Henry Briggs.

Jobst Burgi as well published a table of logarithms, his in 1620, though some have speculated he had begun his work even earlier than had Napier, but Burgi seems not to have been one to assert himself, and besides, he did not have a Briggs to follow after him. Briggs offered such suggestions as, with a few suggestions of Napier's own, ultimately led to what we now know as Briggs or common logarithms, and most often as base-10. The compilation of the successive tables by Briggs, Vlacq and others who followed must have been a massive one, but it met with much success and acceptance by those whose work was now correspondingly less taxing.

While the basic work of the invention of logarithms was by 1624 virtually completed, much more work was to follow. John Speidell published the first table of natural logarithms in 1620, and others followed with significant and related studies on logarithmic curves and the relationship of the natural logarithms to the hyperbolic curves. Finally, Wallis's and Newton's use of fractional and negative exponents rendered not only respectability to both these forms of notation, but finally opened the theory and function of logarithms to the understanding of the entire scientific and mathematical community.

Yet while all this may seem complete enough, as has already been mentioned, gaps remain in the history of logarithms and seem worthy of further study.

6.3 Need for Further Study. Occasional reference has been made in various parts of the study to possible controvery between various historians on one topic or another, and some mention of these should be made. Still other gaps in our knowledge, or in the sources available for this study, should be included as well. Many of these would seem to require examination of primary sources, not available to this study and possibly not even in English.

One such of the latter type concerns the apparent lack of research on just how the Babylonians used their tables to perform calculation, especially exponential and compound-interest tables. While it may be stretching credulity to speculate that such uses were logarithmic-like, one cannot know for sure without further study.

A second area of concern would seem to have little relevance to any further understanding of the development of logarithms, but would, it seems, be interesting nonetheless, i.e., the origin and development of the computational device known a prosthaephaeresis. There are, of course, historians who strive to be relevant, but noted historian Barbara Tuchman has suggested that those who are, are probably doing something other than history.

By way of controversy, Jobst Burgi's claim to priority in the invention of logarithms has not as yet, in the opinion of this author, been sufficiently dismissed. While one must admit, as has been mentioned in an earlier chapter, that Napier's prior publication gives him a virtual strangle-hold on the question of priority in the invention of logarithms, there still remains whether Burgi began working before Napier and whether there had been a working model in the hands of a mathematician - Johannes Kepler - prior to Napier's first publication in 1614. And, even if none of the answers to these questions even slightly dent Napier's claim to priority, there is the historical question, entirely ignored in the sources available to this researcher as to what prompted Burgi and how he proceeded.

And finally, there is little or no information in the sources available to the researcher on the questions of why and how John Speidell developed his system of natural logarithms, a development of such importance in the history of science and mathematics as to deserve, it seems, far more attention than it has apparently received.

6.4 Conclusions. There is, then, much to be done in the study of the history of logarithms, some interesting only for its own sake, others of it which shall possibly help us to understand the history of such areas of mathematics as have been deemed more interesting. If this study has helped to promote an understanding of the history of logarithms and to advance its candidacy as one of the areas worthy of interest to the serious scholar, then it has more than accomplished its purpose.

ENDNOTES

[1]David Eugene Smith, *History of Mathematics* (New York: Dover Publications, 1958), 38.

[2]Smith, 113.

[3]Carl B. Boyer, *A History of Mathematics* (New York: John Wiley and Sons, 1989), 30.

[4]Boyer, 29.

[5]Howard Whitley Eves, *An Introduction to the History of Mathematics* (New York: Holt, Rinehart and Winston, 1969), 33.

[6]Boyer, 34.

[7]Boyer, 34.

[8]Eves, 30.

[9]Boyer, 36.

[10]Eves, 30.

[11]Smith, 38.

[12]Boyer, 35-36.

[13]R. C. Buck, "Sherlock Holmes in Babylon," *American Mathematical Monthly*, 87 (1980): 337-338. I am indebted to Mr. Buck for the table which follows and the algorithms for its use.

[14]Buck, 337-338.

[15]Boyer, 36.

[16]Morris Kline, *Mathematics in Western Culture* (London: Oxford University Press, 1964), 64.

[17]Kline, 63.

[18]Sir Thomas Heath, *A History of Greek Mathematics* (Oxford: Clarendon Press, 1960), 81.

[19]Eves, 85.

[20]Heath, 81.

[21]B. L. van der Waerden, *Science Awakening* (New York: Oxford University Press, 1961), 202.

[22]Florian Cajori, *A History of Mathematics* (New York: Chelsea Publishing Company, 1980), 54-55.

[23]Boyer, 141-142.

[24]Boyer, 141-142.

[25]Florian Cajori, *A History of Mathematical Notations* (Chicago: The Open Court Publishing Co., 1928), 26-27.

[26]Boyer, 141-142.

[27]Heath, 20.

[28]Boyer, 141-142.

[29]Lancelot Thomas Hogben, *Mathematics in the Making*, (Garden City: Doubleday and Company, 1960), 127.

[30]Boyer, 183.

[31]Hogben, 127.

[32]Boyer, 189-190.

[33]Boyer, 268.

[34]Boyer, 293.

[35]*Boyer, 293.*

[36]Boyer, 294.

[37]Hogben, 166.

[38]Hogben, 190.

[39]Cajori, *A History of Mathematics*, 131.

[40]Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press,

1972), 252.

[41]Kline, *Mathematical Thought*, 260.

[42]Boyer, 311.

[43]Boyer, 316.

[44]Kline, *Mathematical Thought*, 235.

[45]J. M. Dubbey, *Development of Modern Mathematics*, (New York: Crane Russak and Co., 1970), 29-30.

[46]Dubbey, 30.

[47]Boyer, 348-349.

[48]Arthur Koestler, *The Watershed: A Biography of Johannes Kepler*, (Garden City, New York: Anchor Books, 1960), 88.

[49]Boyer, 348-349.

[50]Eves, 221-222.

[51]Eves, 223.

[52]I am grateful to R. David Gustafson and Peter D. Frisk, *Plane Trigonometry*, (Monteray, California: Brooks/Cole Publishing Co., 1985), 154 for help in the derivation of this identity.

[53]Boyer, 346.

[54]Boyer, 346.

[55]Dubbey, 30.

[56]Quoted in Will and Ariel Durant, *The Age of Reason Begins* (New York: Simon and Schuster, 1961), 65.

[57]*Dictionary of National Biography*, Volume XVI (Oxford: Oxford University Press, 1968), 991-992.

[58]The chronology offered in this section and at the end of the chapter was selected from Bernard Grun, *The Timetables of History* (New York: Simon and Schuster, 1979).

[59]Dubbey, 30.

[60]Joseph Frederick Scott, *A History of Mathematics*, (London: Taylor and Francis Ltd., 1960), 129.

[61]Boyer, 340.

[62]Florian Cajori, "Algebra in Napier's Day and Alleged Prior Inventions of Logarithms," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 102-104.

[63]Boyer, 351.

[64]Cajori, "Algebra in Napier's Day and Alleged Prior Inventions of Logarithms," 93.

[65]Cajori, "Algebra in Napier's Day and Alleged Prior Inventions of Logarithms," 104-105.

[66]Lord Moulton, "The Invention of Logarithms, Its Genesis and Growth," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 2.

[67]P. Hume Brown, "John Napier of Merchiston," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 36.

[68]Brown, "John Napier of Merchiston," 38.

[69]Brown, "John Napier of Merchiston," 37.

[70]Brown, "John Napier of Merchiston," 48.

[71]Brown, "John Napier of Merchiston," 38.

[72]Boyer, 348.

[73]Boyer, 348.

[74]Moulton, 6.

[75]J. W. L. Glaisher, "Logarithms and Computation," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 65.

[76]Scott, 131.

[77]These figures are taken directly from Napier's own tables in John Nepair [Napier], *A Description of the Admirable Table of Logarithms* [London, 1616], Da Capo Press, Amsterdam, 1969.

[78]Moulton, 7.

[79]Julian L. Coolidge, *The Mathematics of Great Amateurs.* (Oxford: The Clarendon Press, 1950), p. 76.

[80]David Eugene Smith, "The Law of Exponents in the Works of the Sixteenth Century," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 152.

[81]Nepair, 31.

[82]Nepair, 32-33.

[83]Smith, "The Law of Exponents in the Works of the Sixteenth Century," 90.

[84]Moulton, 8.

[85]Moulton, 9.

[86]Moulton, 12.

[87]Moulton, 13.

[88]Moulton, 13.

[89]Moulton, 14.

[90]Moulton, 14.

[91]Moulton, 15.

[92]Moulton, 15.

[93]Moulton, 16.

[94]Coolidge, 71.

[95]Coolidge, 73-74.

[96]Coolidge, 75.

[97]*Dictionary of National Biography*, Volume III, pp. 464-466.

[98]Durant, 65.

[99]Boyer, 352-353.

[100]Florian Cajori, *History of Elementary Mathematics* (New York: The MacMillan Co., 1917), 166-167.

[101]N. T. Gridgeman, "John Napier and the History of

Logarithms," *Scripta Mathematica* 29(1973), 61.

[102]Scott, 134.

[103]Boyer, 352.

[104]Hogben, 186.

[105]Boyer, 352.

[106]Quoted in Alexander John Thompson, *Logarithmetica Britannica*, Part II (Cambridge: Cambridge University Press, 1952), lxxi.

[107]E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1937), 526.

[108]G. A. Gibson, "Napier's Logarithms and the Change to Briggs's Logarithms," *Napier Tercentenary Memorial Volume.* Ed. Cargill, Gilston, Knott (London: Longmans, Green and Company, 1915), 125.

[109]Cajori, *History of Elementary Mathematics*, 162.

[110]Gibson, 127.

[111]The explanation of proportional parts as used in the construction of common logarithms has been compiled from the following sources: Boyer, 351; Hogben, 179; Hogben, 180; and Scott, 133-134.

[112]Scott, 133.

[113]Gridgeman, 61.

[114]This explanation of the computation of the logarithm of two is taken from Glaisher, J.W.L., "Logarithms," *Encyclopedia Britannica*, 11th Edition, 875-876, though the reader will note some changes in the placement of decimal places. However, the accuracy of the computations described herein has been confirmed by computer.

[115]Scott, 133.

[116]Scott, 133.

[117]Scott, 136.

[118]Gridgeman, 60.

[119]Glaisher, 65.

[120]Scott, 135-136.

[121]Boyer, 430-431.

[122]Cajori, *History of Elementary Mathematics*, 167.

[123]Boyer, 429-430.

[124]Robert L. Oblander, *The History, Calculation and Application of the Number* e, Master's Thesis (Emporia: Kansas State Teachers College, 1971), 4.

[125]Hogben, 177.

[126]Gridgeman, 64.

BIBLIOGRAPHY

Bell, E. T. *Men of Mathematics.* New York: Simon and Schuster, 1937.

Boyer, Carl B. and Merzbach, Uta C. *A History of Mathematics.* New York: John Wiley and Sons, 1989.

Brown, P. Hume. "John Napier of Merchiston." *Napier Tercentenary Memorial Volume.* Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Buck, R. C. "Sherlock Holmes in Babylon." *American Mathematical Monthly,* 87(1980):335-345.

Cajori, Florian. *A History of Mathematical Notations.* Chicago: The Open Court Publishing Co., 1928.

Cajori, Florian. *A History of Mathematics.* New York: Chelsea Publishing Co., 1980.

Cajori, Florian. "Algebra in Napier's Day and Alleged Prior Inventions of Logarithms." *Napier Tercentenary Memorial Volume.* Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Cajori, Florian. *History of Elementary Mathematics.* New York: The MacMillan Company, 1917.

Coolidge, Julian L. *The Mathematics of Great Amateurs.* Oxford: The Clarendon Press, 1950.

Dictionary of National Biography. Oxford: Oxford University Press, 1968.

Dubbey, J. M. *Development of Modern Mathematics.* New York: Crane, Russak and Co., 1970.

Durant, Will and Ariel. *The Age of Reason Begins.* New York: Simon and Schuster, 1961.

Eves, Howard Whitley. *An Introduction to the History of Mathematics.* New York: Holt, Rinehart and Winston, 1969.

Gibson, G. A. "Napier's Logarithms and the Change to Brigg's Logarithms." *Napier Tercentenary Memorial Volume.* Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Glaisher, J. W. L. "Logarithms." *Encyclopedia Britannica*, 11th Edition, London, 1911.

Glaisher, J. W. L. "Logarithms and Computation." *Napier Tercentenary Memorial Volume*. Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Gridgeman, N. T. "John Napier and the History of Logarithms." *Scripta Mathematica* 29(1973): 49-65.

Grun, Bernard. *The Timetables of History*. New York: Simon and Schuster, 1979.

Gustafson, R. David and Frisk, Peter D. *Plane Trigonometry*. Monterey, California: Brooks/Cole Publishing Co., 1985.

Heath, Sir Thomas. *A History of Greek Mathematics.* Oxford: Clarendon Press, 1960.

Hogben, Lancelot Thomas. *Mathematics in the Making.* Garden City: Doubleday and Company, 1960.

Kline, Morris. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press, 1972.

Kline, Morris. *Mathematics in Western Culture.* London: Oxford University Press, 1964.

Koestler, Arthur. *The Watershed: A Biography of Johannes Kepler.* Garden City: Anchor Books, 1960.

Moulton, Lord. "The Invention of Logarithms, Its Genesis and Growth." *Napier Tercentenary Memorial Volume.* Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Nepair [Napier], John. *A Description of the Admirable Table of Logarithms* [London, 1616]. Amsterdam: Da Capo Press, 1969.

Oblander, Robert L. *The History, Calculation and Application of the Number* e. Emporia, Kansas: Unpublished Thesis, Kansas State Teachers College, 1971.

Scott, Joseph Frederick. *A History of Mathematics.* London: Taylor and Francis Ltd., 1960.

Smith, David Eugene. *History of Mathematics.* New York: Dover Publications, 1958.

Smith, David Eugene, "The Law of Exponents in the Works of the Sixteenth Century," *Napier Tercentenary Memorial Volume*. Ed. Cargill Gilston Knott. London: Longmans, Green and Company, 1915.

Thompson, Alexander John. *Logarithmetica Britannica*, Part II. Cambridge: Cambridge University Press, 1952.

van der Waerden, B. L. *Science Awakening*. New York: Oxford University Press, 1961.

APPENDICES

APPENDIX A

A COMPUTER SIMULATION OF PROSTHAPHAERESIS

The following program in Turbo Pascal is an illustration by the author of this study of the method of phosthaphaeresis in use by astronomers in the late sixteenth and early seventeenth centuries before the invention of logarithms in 1614 by John Napier. The reader will note, however, three deviations from the strict form described by Carl Boyer and quoted in this study on page 23. First, the periodic divisions and multiplications by 100,000 are the author's means of dealing with positioning of decimals parenthetically discussed by Carl Boyer in the text cited above. Second, as Turbo Pascal's only inverse trigonometric function is the arctangent function, on two separate occasions the cosines of the angles had to be translated into the tangents of those angles, and from these, of course, each of the corresponding angles was determined. Finally, since Mr. Boyer's description of the process used five-digit integers as illustrations, and to allow for the use of integers not in that range would have unnecessarily complicated a program designed only to illustrate prosthaphaeresis, this program is designed only for use with five-digit integers.

A number of tests of the program yielded results from prosthaphaeresis exactly the same as regular multiplication, except one which showed a difference of three one-hundredths. The author assumes that a difference that small was due to a rounding error and could have occurred in determining the tangent of one or both of the angles from their respective cosines.

```
program prosthaphaeresis (input,output);

var
   a       : real;
   b       : real;
   halfa   : real;
   sum     : real;
   product : real;
   cosa    : real;
   cosb    : real;
   tana    : real;
   tanb    : real;
   anglea  : real;
   angleb  : real;

begin
   write('   Please enter a 5-digit number: ');
   readln(a);
   halfa := a/2;
   cosa := halfa/100000;
   tana := (sqrt(1 - cosa * cosa)) / cosa;
   anglea := arctan(tana);
   write('   Please enter a second 5-digit number:
');
   readln(b);
   cosb := b/100000;
   tanb := (sqrt(1 - cosb * cosb)) / cosb;
   angleb := arctan(tanb);
   sum := ((cos(anglea + angleb) + cos(anglea -
   angleb)) * 100000 * 100000);
   product := a * b;
   writeln('   The regular product is:
',product:14:2);
   writeln('   The product by prosthaphaeretic
   multiplication is: ',sum:14:2);
   readln
end.
```

PROGRAM OUTPUT:

```
se enter a 5-digit number: 27659.00
se enter a second 5-digit number: 95304.00
regular product is:  2636013336.00
product by prosthaphaeretic multiplication is:  2636013336.00
```

APPENDIX B

A PROGRAM FOR CONSTRUCTING NAPIER LOGARITHMS

The program which follows constructs a very abbreviated table of Napier logarithms in the manner described in Chapter IV of this study and following the guidance of Napier's first book on the subject of logarithms, *A Description of the Admirable Table of Logarithms*, published in London in 1616. The results, published below, correspond to Napier's own tables.

The program begins by calculating the various sines according the the formula $r * (1 - 1/r)^n$. For each sine, then, the program shifts to a procedure to determine the angle for that sine, and finally all three figures are printed in tabular form. As was true of the first program, there were certain extra steps which had to be taken because Turbo Pascal provides for no other inverse trigonometric function than arctangent. The cosine function had to be determined, and the tangent was then determined by dividing the sine, as determined earlier, by the cosine. Angles then were converted from radians into degrees and minutes. One additional note: $r = 10^6$ was used in the program instead of 10^7 because the tables in Napier's book were so constructed.

The program follows, and on the following page one may find a printed copy from running the program.

```
program napierlogs (input, output);

uses
   crt;
   {printer;}

const
   r = 1000000;
   y = 10;

type
   logarithms = array[0..10] of integer;

var
   distance : real;
   rlog     : real;
   factor   : real;
   degree   : real;
   minute   : real;
   expt     : integer;
   t        : integer;
   napier   : logarithms;

procedure angleconstruct(distance : real; var
degree, minute : real);

var
   tana  : real;
   sina  : real;
   cosa  : real;
   angle : real;

begin
   sina := distance / r;
   cosa := sqrt(1 - sina * sina);
   if (cosa = 0.0) then
      begin
         degree := 0;
         minute := 0
      end
   else
      begin
         tana := sina / cosa;
         angle := arctan(tana) * (180 / pi);
         degree := int(angle);
         minute := frac(angle) * 60
```

```
      end
end;

begin
   clrscr;
   write('    PRESS ENTER TO BEGIN:    ');
   readln;
   factor := 1 - 1/r;
   writeln;writeln;writeln;
   writeln('           Angle');
   writeln('   degrees  minutes         Napier
   Logarithm Sine');
   writeln('==========================================
   ===== ===== ===========');
   for expt := 0 to y do
      begin
         if (expt = 0) then
            distance := r
         else
            if (expt = 1) then
               begin
                  rlog := 0;
                  distance := r * factor;
               end
            else
               if (expt >= 2) then
                  begin
                     distance := r;
                     for t := 1 to (expt) do
                        begin
                           distance := distance *
factor
                        end
                  end;
         napier[expt] := expt;
         angleconstruct(distance,degree,minute);
         writeln('
      ',degree:3:0,minute:10:3,expt:17,distance:25:
      2)
      end;
      writeln;writeln;writeln;
   write('    PRESS ENTER TO CONTINUE:    ');
   readln
end.
```

PROGRAM RESULTS:

Angle degrees	minutes	Napier Logarithm	Sine
0	0.000	0	1000000.00
89	55.138	1	999999.00
89	53.125	2	999998.00
89	51.579	3	999997.00
89	50.277	4	999996.00
89	49.129	5	999995.00
89	48.091	6	999994.00
89	47.137	7	999993.00
89	46.249	8	999992.00
89	45.415	9	999991.00
89	44.626	10	999990.00

APPENDIX C

THE NUMERICAL APPROXIMATION OF f(x) = 1/x USING SIMPSON'S RULE

The following is a program in Turbo Pascal which approximates the integration of the function f(x) = 1/x using Simpson's Rule. More accurate approximations could have been obtained by the use of a larger n, but the value of n was kept low so that differences between the approximations using Simpson's and the actual values could be observed. The program follows:

```
program SimpsonsRule(input,output);

uses
   crt;

const
   a = 1;
   n = 100;
   x = 1;

var
   i          : integer;
   b          : integer;
   c          : integer;
   f          : real;
   mfx        : real;
   factor     : real;
   sum        : real;
   approx     : real;
   actual     : real;
   difference : real;

begin
   clrscr;
   writeln;writeln;writeln;
   writeln('   THIS PROGRAM DETERMINES BY NUMERICAL
```

```
METHODS');
writeln('   THE NATURAL LOG OF INTEGERS FROM ONE
TO THE NUMBER ');
writeln('   SELECTED BY THE USER.  IN FIRST
COLUMN BELOW EACH');
writeln('   INTEGER WILL APPEAR.  IN THE SECOND
COLUMN IS THE NATURAL');
writeln('   LOG OF THE INTEGER DETERMINED BY
USING SIMPSONS RULE,');
writeln('   WITH N = 100.  IN THE THIRD COLUMN
IS THE COMPUTER VALUE');
writeln('   OF THE NATURAL LOG OF THE INTEGER,
AND THE FOURTH COLUMN');
writeln('   IS THE DIFFERENCE BETWEEN THE TWO
VALUES.');
writeln;
write('   PLEASE ENTER THE NUMBER OF INTEGERS
YOU WISH TO EXAMINE:  ');
readln(c);
clrscr;
writeln('                    Numerical Computer');
writeln('   Integer          Method Statement
Difference');
writeln('=========================================
===== ===== =================');
for b := 1 to c do
    begin
        factor := (b - a) / (3 * n);
        sum := 0;
        for i := 0 to n do
            begin
                f := frac(i / 2);
                if (i = 0) then mfx := 1
                else
                    if (i = n) then mfx := 1/b
                    else
                        if (f = 0) then mfx := 2 * (1
                        / (x + ((b - a) * i / n)))
                        else mfx := 4 * ( 1 / (x +
                        ((b -a) * i / n)));
                    sum := sum + mfx
            end;
        approx := factor * sum;
        actual := ln(b);
        difference := approx - actual;
        writeln('
        ',b,approx:18:5,actual:20:5,difference:20:
        5);
    end;
 writeln;
 write('   PRESS ENTER TO CONTINUE:    ');
 readln
end.
```

PROGRAM OUTPUT:

SIMPSONS RULE

THIS PROGRAM DETERMINES BY NUMERICAL METHODS THE NATURAL LOG OF INTEGERS FROM ONE TO THE NUMBER SELECTED BY THE USER. IN FIRST COLUMN BELOW EACH INTEGER WILL APPEAR. IN THE SECOND COLUMN IS THE NATURAL LOG OF THE INTEGER DETERMINED BY USING SIMPSONS RULE, WITH N = 100. IN THE THIRD COLUMN IS THE COMPUTER VALUE OF THE NATURAL LOG OF THE INTEGER, AND THE FOURTH COLUMN IS THE DIFFERENCE BETWEEN THE TWO VALUES.

PLEASE ENTER THE NUMBER OF INTEGERS YOU WISH TO EXAMINE: 20

Integer	Numerical Method	Computer Statement	Difference
1	0.00000	0.00000	0.00000
2	0.69315	0.69315	0.00000
3	1.09861	1.09861	0.00000
4	1.38629	1.38629	0.00000
5	1.60944	1.60944	0.00000
6	1.79176	1.79176	0.00000
7	1.94591	1.94591	0.00000
8	2.07944	2.07944	0.00000
9	2.19723	2.19722	0.00000
10	2.30259	2.30259	0.00000
11	2.39790	2.39790	0.00000
12	2.48491	2.48491	0.00000
13	2.56496	2.56495	0.00001
14	2.63907	2.63906	0.00001
15	2.70806	2.70805	0.00001
16	2.77260	2.77259	0.00002
17	2.83323	2.83321	0.00002
18	2.89040	2.89037	0.00003
19	2.94447	2.94444	0.00003
20	2.99577	2.99573	0.00004

I, David L. Baughman, hereby submit this thesis to Emporia State University as partial fulfillment of the requirements for an advanced degree. I agree that the Library of the University may make it available for use in accordance with its regulations governing materials of this type. I further agree that quoting, photocopying, or other reproduction of this document is allowed for private study, scholarship (including teaching) and research purposes of a nonprofit nature. No copying which involves potential gain will be allowed without written permission of the author.

David L. Baughman

Signature of Author

31 July 1992

Date

An Early History of Logarithms

Title of Thesis Project

Doug Cooper

Signature of Graduate Office Staff Member

July 30, 1992

Date Received