AN ABSTRACT OF THE THESIS OF

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In Chapter I the statement of the problem and the scope of the thesis were given. Historical background and the geometrical foundation of the Golden Section were presented in Chapter II. The general additive sequence and the Fibonacci sequence were introduced and a survey of their properties, ranging from simple sums to a sequence which converges to the Golden Ratio, were established in Chapter III. The Golden Ratio and its properties were investigated in Chapter IV. This investigation included some special sequences, Binet's Formula, convergence of some infinite geometric series and a graph associated with the cyclic group generated by the Golden Ratio. Chapter V summarizes the preceding chapters, suggests possible uses for the information contained in the thesis and concludes the paper.

# THE GOLDEN SECTION AND RELATED TOPICS

A Thesis

Presented to

the Division of Mathematics and Computer Science

EMPORIA STATE UNIVERSITY

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

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Approved for the Major Division

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# CHAPTER I

# INTRODUCTION

This chapter is an introductory one to establish a basis for the study that the writer undertook. It includes a section on the purpose of the study and one on the importance of the study to teachers of mathematics. In addition the writer includes information regarding the organization of this thesis as well as limitations imposed on the study.

## Purpose of the Study

The purpose of this study was to present a brief account of the ancient history of the Golden Section followed by properties of varying complexity relating to the Golden Section, which will allow mathematics teachers to choose related topics suited to their courses and to their students' abilities.

## Importance of the Study

Most teachers are concerned with questions of motivation. They want to know how they can motivate students to learn the material to be presented. Motivation is usually defined as an awareness that causes one to choose a goal and work toward achieving that goal. How can a teacherof mathematics influence students to accept the learning of mathematics as a personal goal? This is an important question since many students believe they cannot learn mathematics.

Covington, Beery and Olemich suggested there are three kinds of students: mastery-oriented, failure-avoiding and failure-accepting. Mastery-oriented students are high in achievement motivation and are not likely to pose a motivational problem for teachers. Failure-avoiding students have a strong need to protect themselves from failure. They may set goals far below their abilities in order to assure success. Some failure-avoiding students will set excessively high goals; then, when they fail, their sense of self-worth is preserved since they believe no one could have met such an unreasonable goal. Failure-accepting students have decided they are incompetent and helpless to do anything about it (14 : 323).

All students need to experience meaningful successes. Mastery-oriented students will make their own successes. When they experience failure, they will simply work harder to succeed next time. Failure-avoiding and failure-accepting students must be guided to opportunities to succeed in ways they will perceive to be meaningful. In practice, it is often difficult to select topics which will meet this need.

The Golden Section and topics related to the Golden Section can provide suitable material for many different mathematics courses. Teachers may select related projects requiring nothing more than simple arithmetic or they may select projects requiring varying levels of ability in areas such as algebra, geometry or calculus. With a minimum of introduction, teachers can encourage students to experiment with, develop and test, or prove, hypotheses. Students' discoveries could represent a significant success, thus encouraging them to accept the learning of mathematics as a reasonable goal.

## Organization of the Thesis

Chapter I is the introduction of the study.

Historical background is presented in Chapter II. Pythagorean discoveries are emphasized as they appear in Euclid's Elements.

Additive sequences are introduced in Chapter III. Basic properties, depending only on the recursive nature of additive sequences are developed. Then the Fibonacci Sequence and its properties are explored.

Properties of the Golden Ratio, including properties which demonstrate its relationship to the Fibonacci sequence, are presented in Chapter IV.

Chapter V summarizes the preceding chapters and concludes the paper.

## Limitations of the Study

Many topics related to the Golden Section are omitted from this study. Some are mentioned, though not developed in the paper, such as the relationship between the Golden Section and subjects in botany, biology, entomology and music. This study has been limited to the mathematical properties of the Golden Section and those topics, such as additive sequences, which are mathematically related to the Golden Section. Within these guidelines, published material is still quite extensive. The writer has chosen to limit this paper to an introduction to topics which will be useful enrichment material in those courses where motivation is most likely to be a problem, such as secondary and college undergraduate mathematics courses for non-mathematics majors.

# CHAPTER II

# THE SECTION IN ANCIENT HISTORY

## Brief Historical References to the Divine Proportion

Johann Kepler (1571 - 1630), the German astronomer said "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel" (6 : 23). Through Kepler we know that people of his time called this jewel "sectio divina" and "propotio divina" (1 : 45n). The Divine Proportion, the Golden Section, and the Golden Ratio are the names commonly used today.

The Divine Proportion is said to be the most aesthetically pleasing of all proportions. It appears in nature and in art, including sculpture, painting, music and architecture.

In nature we can observe the Divine Proportion in the shell of the chambered nautilus, a bee's honeycomb, the way leaves grow out the stems of plants and the way florets are arranged on the heads of the flower.

Luca Pacioli wrote De Divina Proportione in 1509 and Leonardo da Vinci illustrated the work. It was important for renewing interest in the study of solid figures. Many of the artists of the 15th and 16th centuries were amateur mathematicians. More than that, some claimed they created art for the sole purpose of studying geometry. It was during this period that painters were studying and practicing perspectivity. They studied Euclid's Elements. Among the problems in which we know they were interested was Euclid's Book IV, Proposition 11: "In a given circle to inscribe an equilateral and equiangular pentagon" (10 : 100). Solid figures are the topic of Euclid's Book XIII and Pacioli's De Divina Proportione. Construction of the regular pentagon and many of the constructions related to solid figures require cutting a line in the Divine Proportion. The artistic geometers of the Renaissance are largely responsible for popularizing the study of geometry in Europe.

Antonio Stradivari (born about 1644), the Italian violinmaker is said to have used the Divine Proportion in the construction of his instruments because this proportion produces the most pleasing tone.

## The Golden Ratio and the Pyramid of Gizeh

Earlier artists were inspired by the Divine Proportion. The best surviving evidence is in architecture. Herodotus (485 - 430 B.C.) reported that the area of each triangular face of the Great Pyramid at Gizeh (built about 2600 B.C.) is equal to the square of the vertical height (2 : 58). Consider figure 1. The height (h) of the pyramid is one leg of a right triangle whose hypotenuse is the altitude (a) of a triangular face. Let b denote the other leg of the right triangle, then $h^2 + b^2 = a^2$. Notice the base of the triangular face is twice the measure of b, thus $\frac{1}{2}a(2b)$ is the area to be considered. Herodotus claimed $ab = h^2$, but $h^2 = a^2 - b^2$ which gives $a^2 - b^2 = ab$, and

(2.1) $$a^2 - ab - b^2 = 0.$$

By the quadratic formula

$$\begin{aligned} a &= [b \pm \sqrt{b^2 - 4(1)(-b^2)}]/2 \\ &= (b \pm \sqrt{b^2 + 4b^2})/2 \\ &= (b \pm b\sqrt{5})/2 \\ &= b(1 \pm \sqrt{5})/2. \end{aligned}$$

The positive solution to this quadratic equation yields

(2.2) $$a/b = (1 + \sqrt{5})/2.$$

This is the ratio called the Golden Ratio.

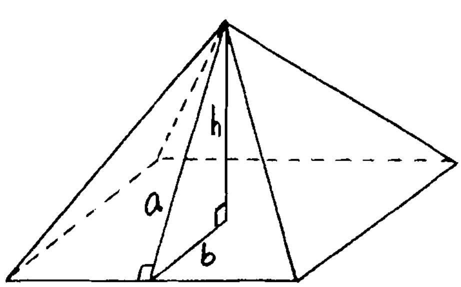


Figure 1
Geometry of The Great Pyramid at Gizeh

## The Golden Rectangle

The Divine Proportion can also be found in ancient Greek architecture. The face of the Parthenon is a Golden Rectangle. That is to say that the sum of the width and the height compares to the width alone as the width compares to the height. If a is the width of the rectangle and b is its height then $(a + b)/a = a/b$. This is the Divine Proportion. It implies $a^2 - ab - b^2 = 0$ and the positive solution to this quadratic equation, given in (2.2), is called the Golden Ratio. The Golden Rectangle has a special property. Cut a square of side b from the original rectangle and the remaining rectangle is another Golden Rectangle. It can be shown that this procedure may be repeated indefinitely, with the remaining rectangle always a Golden Rectangle.

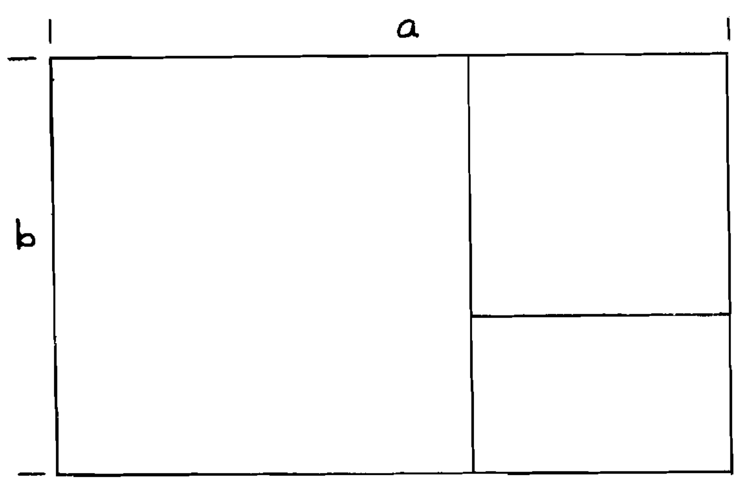


Figure 2
The Golden Rectangle

The Parthenon may have been purposefully constructed in Divine Proportion. Evidence suggests that Greek mathematicians and philosophers were well acquainted with this proportion.

## The Pythagoreans and the Golden Ratio

Pythagoras was born about 580 B.C. Very little is known about his personal life. Apparently he was a charismatic person. He significantly influenced the development of Greek mathematics and philosophy. He had a large and loyal following during his lifetime and for centuries after his death. Many historians and fans have written about him, but much of that has been fanciful.

Numerous reliable sources suggest we can accept these facts: Pythagoras was a student of Thales, often called the father of geometry. Thales encouraged him to travel to Egypt to study geometry and astronomy with the priests there. From Egypt, Pythagoras traveled to Babylonia, where he seems to have resided for several years. Eventually he returned from his travels to establish a school at Croton in the Greek colonies of southern Italy. This school became very important.

Followers of Pythagoras called themselves the Pythagorean Brotherhood. They studied religion, philosophy, music, astronomy, medicine and mathematics. Unfortunately, they did not publish their results. Indeed, some say they took an oath, vowing not to divulge the teachings of the Brotherhood.

The pentagram star became the badge by which members recognized each other. They called it "health." Its construction is given in Euclid's Elements. Two previous constructions of the Elements are crucial to this one. They are attributed to the Pythagoreans and will be reproduced here. The first of these constructions is found in Euclid's second book.

Book II, Proposition 11

> To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

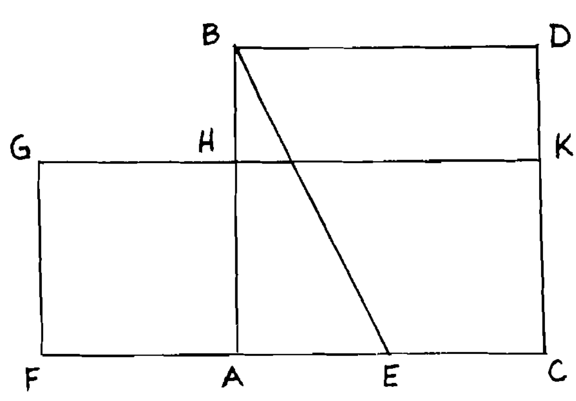


Figure 3
Construction for Book II, Proposition 11

Let AB be the given straight line; thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square ABDC be described on AB; let AC be bisected at the point E, and let BE be joined; let CA be drawn through to F, and let EF be made equal to BE; let the square FH be described on AF, and let GH be drawn through to K.

I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH (9 : 402).

Euclid's proof for this construction was essentially the geometric equivalent of the proof that follows, in which notations of algebra are utilized.

Let the length of $\overline{AB}$ be denoted x. ABDC is a square, thus CA = AB = x and E bisects $\overline{CA}$, so EA = $\frac{1}{2}x$. Let HA be denoted b.

It is necessary to prove $AB(HB) = (HA)^2$.

Since ABDC is a square, $\angle$ EAB is a right angle and $(EA)^2 + (AB)^2 = (BE)^2$, but BE = EF = $\frac{1}{2}x + b$, thus

$$(\tfrac{1}{2}x)^2 + x^2 = (\tfrac{1}{2}x + b)^2 = (\tfrac{1}{2}x)^2 + bx + b^2$$

$$x^2 = bx + b^2$$

$$x^2 - bx = b^2$$

$$x(x - b) = b^2$$

Therefore $AB(HB) = (HA)^2$.

Notice AB/HA = HA/BH and AB has been cut in mean and extreme ratio at the point H, but Euclid could not make this claim until he presented Eudoxus's Theory of Proportion in Book V. Instead he presented the proposition twice, with the second appearing in Book VI.

The next Proposition needed is found in Book IV.

Book IV, Proposition 10

To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA; with centre A and distance AB let the circle BDE be described, and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE. Let AD, DC be joined, and let the circle ACD be circumscribed about the triangle ACD (10 : 96).

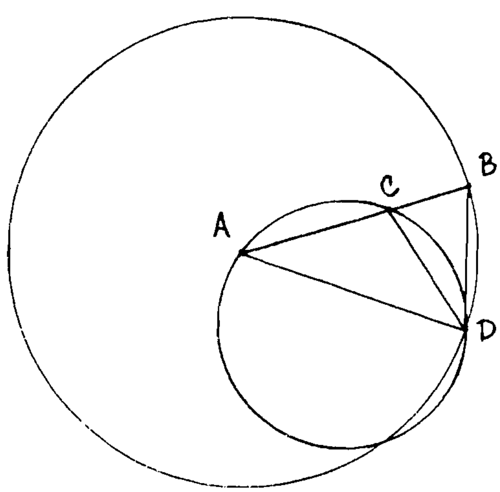


Figure 4
Construction for Book IV, Proposition 10

It must be proved that $\angle DBA = \angle BDA = 2\angle BAD$.

A is the center of circle BDE, thus AB = AD, which makes $\angle DBA = \angle BDA$. $\angle BCD$ is an exterior angle of the triangle ACD, thus $\angle CAD + \angle CDA = \angle BCD$.

B is an exterior point of the circle ACD, $\overline{BA}$ and $\overline{BD}$ extend from B, falling on the circle with $\overline{BA}$ cutting the circle. Since $BA(BC) = (CA)^2$ and BD was made equal to CA, making $BA(BC) = (BD)^2$, $\overline{BD}$ is tangent to the circle ACD. Therefore $\angle BDC = \angle CAD$. $\angle BDC + \angle CDA = \angle BDA = \angle BCD$. Then $\angle CBD = \angle BCD$ and DC = BD = CA. Therefore $\angle CAD = \angle CDA$. But $\angle CAD = \angle BDC$ Thus $2\angle CAD = 2\angle BAD = \angle BDC + \angle CDA = \angle BDA$.

Therefore $\angle DBA = \angle BDA = 2\angle BAD$.

The scholiast to the <u>Clouds</u> of Aristophanes called the pentagram the "triple interwoven triangle" (8 : 99). This, along with the presentation of these propositions in Book IV, which has been attributed wholly to the Pythagoreans, suggests that the next construction, Proposition 11 of Book IV, is an authentic demonstration of the technique used by the Pythagoreans to construct their badge of brotherhood.

Book IV, Proposition 11

In a given circle to inscribe an equilateral and equiangular pentagon.

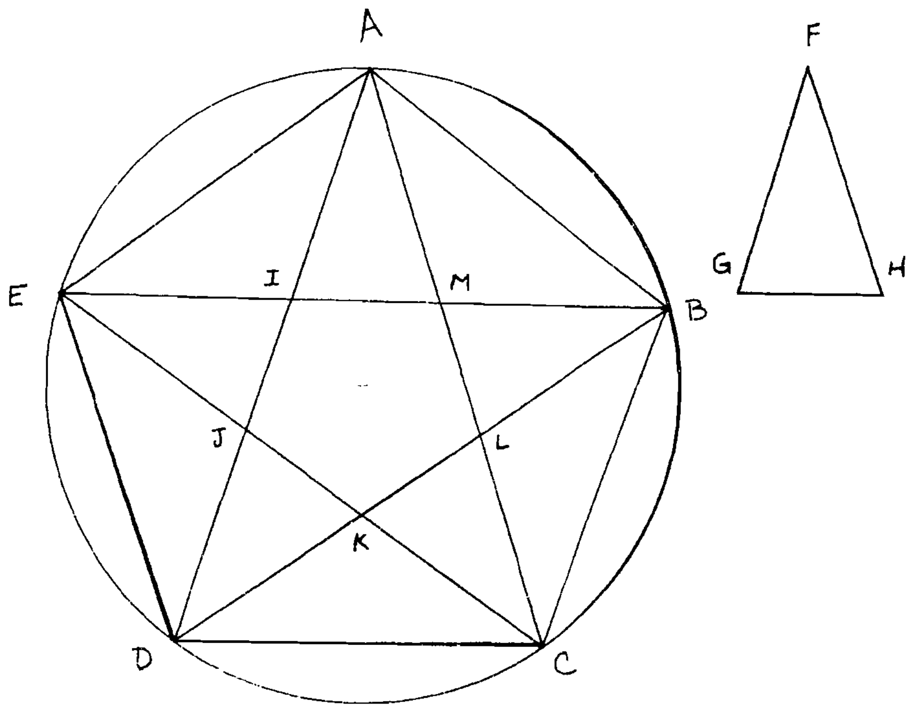


Figure 5
Construction for Book IV, Proposition 11
(The Pentagram Star)

Let ABCDE be the given circle; thus it is required to inscribe in the circle ABCDE an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G, H double of the angle at F; let there be inscribed in the circle ABCDE the triangle ACD equiangular with the triangle FGH, so that the angle CAD is equal to the angle at F and the angles at G, H respectively equal to the angles ACD, CDA; therefore each of the angles ACD, CDA is also double of the angle CAD. Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB and let AB, BC, DE, EA be joined (10 : 100).

It is required to prove the figure thus constructed is both equilateral and equiangular.

To show the sides of the pentagon are equal in length:

$\Delta ACD$ was constructed equiangular with $\Delta FGH$, thus $m\angle ACD = m\angle CDA = 2m\angle DAC$. $\overline{CE}$ bisects $\angle ACD$ and $\overline{DB}$ bisects $\angle CDA$, thus $m\angle ACE = m\angle ECD = m\angle CDB = m\angle BDA = m\angle DAC$. Equal angles intercept equal arcs, thus the arcs AE, ED, CB, BA and DC are equal. Equal arcs are subtended by equal chords, thus AE = ED = CB = BA = DC.

Therefore the pentagon is equilateral.

To show that the angles of the pentagon have equal measures:

Since the arcs AE, ED, CB, BA, DC are equal, the arcs AEDC, EDCB, DCBA, CBAE, BAED are equal. Standing on these equal arcs are the angles CBA, BAE, AED, EDC, DCB, respectively. Equal arcs are intercepted by equal angles. Thus, CBA, BAE, AED, EDC, DCB are equal angles.

Therefore the pentagon is equiangular.

This construction produced the pentagram star except for one line. $\overline{BE}$ must be joined to complete the star. It is a figure rich in examples of the Golden Ratio.

Some interesting properties of the pentagram star are found in the following examination of figure 5.

Label the intersections of the diagonals of the pentagon as follows: let $\overline{AD}$ intersect $\overline{BE}$ in point I, $\overline{EC}$ intersect $\overline{AD}$ in point J, $\overline{DB}$ intersect $\overline{EC}$ in point K, $\overline{CA}$ intersect $\overline{DB}$ in point L and $\overline{BE}$ intersect $\overline{AC}$ in point M.

Given the arcs BC, CD and DE are congruent to one another, the angles BAC, CAD and DAE are also congruent to each other. But these three angles taken together make the angle BAE, thus angle BAE is trisected by the diagonals which meet in the vertex A.

Likewise, each of the vertices of the pentagon are trisected by the diagonals which meet in those vertices, and the pentagon is equiangular, thus the fifteen angles on the vertices are congruent.

Now, since $\angle$ MBA = $\angle$ EBA $\simeq$ $\angle$ BAC = $\angle$ BAM, $\Delta$MBA is isosceles and $\overline{MB} \simeq \overline{AM}$. Also $\angle$ IAE = $\angle$ DAE $\simeq$ $\angle$ AEB = $\angle$ AEI, thus $\Delta$IAE is isosceles and $\overline{IA} \simeq \overline{EI}$. However $\overline{BA} \simeq \overline{AE}$ and $\angle$ MBA $\simeq$ $\angle$ BAM $\simeq$ $\angle$ IAE $\simeq$ $\angle$ AEI, thus, $\Delta$MBA $\simeq$ $\Delta$IAE and $\overline{MB} \simeq \overline{IA}$ $\simeq$ $\overline{EI}$ $\simeq$ $\overline{AM}$ and $\Delta$AIM is also isosceles. Then, as $\angle$ MEA = $\angle$ IEA $\simeq$ $\angle$ IAM and $\angle$ AME = $\angle$ AMI, $\Delta$EMA $\sim$ $\Delta$AIM and $\Delta$EMA is also isosceles. Therefore $\overline{AE} \simeq \overline{EM}$.

Furthermore, EM/AI = MA/IM = AE/MA. Let a denote the length of the line segments MA = AI and b denote the length of the line segment IM, then we see that $(a + b)/a = a/b$. This is the Divine Proportion, thus a/b is the Golden Ratio (equation 2.2). Therefore, MA/IM = $(1 + \sqrt{5})/2$.

All of the triangles in the pentagram star turn out to be similar to either $\Delta AIM$ or $\Delta MBA$. Where $\Delta AIM$ has base angles of $2(36°) = 72°$ and its third angle is $36°$, $\Delta MBA$ has base angles of $36°$ and its third angle is $3(36°) = 108°$. Whether a given triangle is similar to $\Delta AIM$ or to $\Delta MBA$ the length of the base of the triangle is in Golden Ratio to the length of a leg of the triangle. Additionally, extending the alternating sides of the pentagon to their intersections, forms another star. This process may be repeated as many times as one wishes. It can be shown that the similarities and the Golden Ratios are preserved and multiplied.

## The Pythagoreans and Irrational Numbers

A Pythagorean, Theodorus of Cyrene, born about 470 B.C., proved $\sqrt{5}$ is irrational. Plato reported that Theodorus, his teacher, proved irrational the irrational square roots from $\sqrt{3}$ to $\sqrt{17}$. He does not mention $\sqrt{2}$ in this passage. Elsewhere Plato has discussed $\sqrt{2}$ as if it is well known by his time. This indicated, according to Sir T. L. Heath, that the early Pythagoreans should be credited with the proof that $\sqrt{2}$ is irrational (9 : 412).

The Pythagorean discovery of the irrational is often cited as their greatest contribution to science. At the time it could not have been considered great. It was a tragedy. By then the Pythagoreans had already developed their theories of proportion and similar figures, but these theories were invalid for incommensurable magnitudes. Book II of Euclid's _Elements_ presents the algebraic geometry of the Pythagoreans. Proclus, commenting on this book, says theorems "about sections like those in Euclid's second book are common to both [arithmetic and geometry] _except that in which the straight line is cut in extreme and mean ratio_" (9 : 137).

## _Eudoxus and the Golden Section_

Proclus also reported that Eudoxus "greatly added to the theorems which Plato originated regarding _the section_." Heath inferred

> It is obvious that 'the section' was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the 'golden section'. . . . Secondly, as Cantor points out, Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations with proportions not expressible by whole numbers that he came to realize the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The 'golden section' would furnish such a case (9 : 137).

Eudoxus's Theory of Proportion restored the confidence of the Greek geometers. With this theory they could now look upon the existence of irrationals as an opportunity rather than a tragedy.

The following proposition may be attributable to Eudoxus. It is the second version of Book II, Proposition 11. This is clearly the construction of the Golden Section for its own sake.

Book VI, Proposition 30

To cut a given finite straight line in extreme and mean ratio.
Let AB be the given finite straight line; thus it is required to cut AB in extreme and mean ratio.

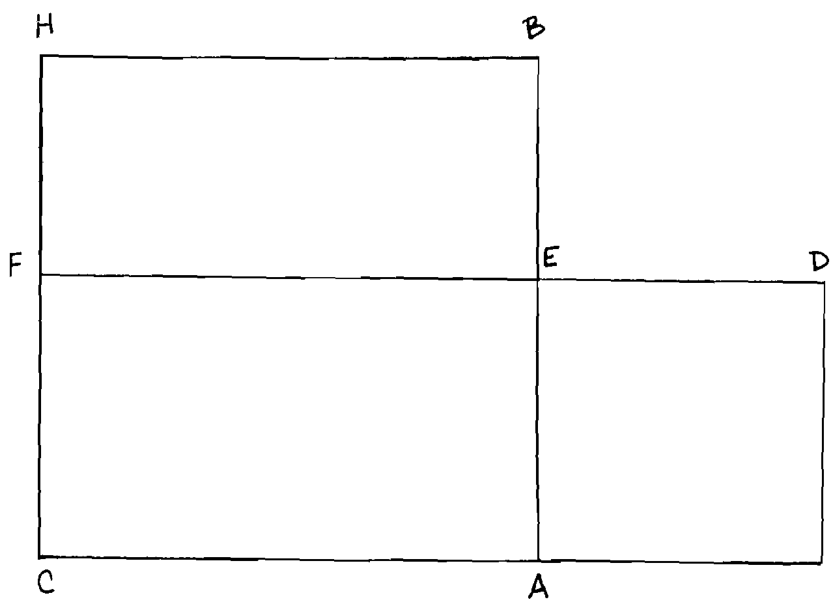


Figure 6
Construction for Book VI, Proposition 30
(The Golden Section)

> On AB let the square BC be described; and let there be applied to AC the parallelogram CD equal to BC and exceeding by the figure AD similar to BC. Now BC is a square. And, since BC is equal to CD, let CE be subtracted from each; therefore the remainder BF is equal to the remainder AD. But it is also equiangular with it; therefore in BF, AD the sides about the equal angles are reciprocally proportional; therefore, as FE is to ED, so is AE to EB. But FE is equal to AB, and ED to AE.
>
> Therefore, as BA is to AE, so is AE to EB. And AB is greater than AE; therefore AE is also greater than EB.
>
> Therefore the straight line AB has been cut in extreme and mean ratio at E, and the greater segment of it is AE (10 : 267).

The ratio of the longer segment, AE, to the shorter segment, EB, is the Golden Ratio. Let a = AE and b = EB, then it was proved above that $(a + b)/a = a/b$. This relationship yields the quadratic equation of (2.1) which was shown to have the solution $a/b = (1 + \sqrt{5})/2$, the Golden Ratio.

Pythagoreans held that "the four elements earth, air, fire and water have characteristic shapes: the cube is appropriated to earth, the octahedron to air, the sharp pyramid or tetrahedron to fire, and the blunter icosahedron to water, while the Creator used the fifth, the dodecahedron, for the Universe itself" (8 : 96).

Some say Hippasus of Metapontum, a Pythagorean, was drowned at sea for revealing the secret of the construction of the dodecahedron in the sphere. Others say he died for revealing the existence of irrationals. This could be two ways of correctly interpreting the same event.

Proposition 17 of Euclid's Book XIII is stated: "To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome" (11 : 493). Proposition 17 and many of the other propositions of Book XIII require the cutting of a line segment in mean and extreme ratio. All of the propositions of this book deal with the five regular solids. The constructions of the dodecahedron and the icosahedron both require the construction of regular pentagons. For the dodecahedron, Euclid described the construction of a regular pentagon on each of the twelve edges of a cube. To construct the icosahedron, he began by inscribing two regular pentagons in parallel circular sections of a sphere. Other propositions of Book XIII prove the irrationality of the segments which result when a line is cut in mean and extreme ratio.

## Irrationals and a Glimpse at the Fibonacci Sequence

H. W. Turnbull suggested a method the Pythagoreans may have used to approach an irrational number arithmetically. He set out a ladder; each rung consisting of two numbers. Descending the ladder, the ratio of the numbers on a rung comes closer and closer to the desired irrational. Turnbull went on to show that a ladder of this type, suggested by Professor D'arcy Thompson, approaches the Golden Ratio.

> Here the right member of each rung is the sum of the pair on the preceding rung, so that the ladder may be extended with the greatest ease.
>
> | | |
> |---|---|
> | 1 | 1 |
> | 1 | 2 |
> | 2 | 3 |
> | 3 | 5 |
> | 5 | 8 |
>
> etc.
>
> In this case the ratios approximate, again by the little more and the little less, to the limit $(\sqrt{5} + 1) : 2$. It is found that they provide the arithmetical approach to the golden section of a line AB . . . (8 : 97).

There is no evidence that the ancients constructed this ladder. However it is pleasant to imagine they did, since the side rails of the ladder form the Fibonacci sequence.

# CHAPTER III

## ADDITIVE SEQUENCES

### Introduction

The sequence of numbers, which Eduardo Lucas in 1877, named the Fibonacci sequence appeared in the manuscript Liber Abaci, dated 1202, by Leonardo of Pisa (also known as Fibonacci, son of Bonacci). The Liber Abaci was important for encouraging the use of Hindu-Arabic numerals and algebraic procedures. However it is best known today for the problem whose solution produced the sequence we now call the Fibonacci sequence. This problem is stated as follows:

> A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive (2 : 263)?

To solve the problem note that in the first month there is one pair of adult rabbits which produce a pair of infant rabbits, so that by the end of the first month there are two pairs present. In the second month, the adult pair produces a pair of infants, while last month's infants do not produce and we shall call them adolescents. At the end of the second month there are three pairs of rabbits. In the third month there are two pairs of adults, each produce a pair of infants, and there is a fifth pair which have become adolescents. At this point it is convenient to prepare a table.

TABLE I

RABBIT TOTALS BY MONTH

| MONTH | ADULTS | ADOLESCENTS | INFANTS | TOTAL |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 2 |
| 2 | 1 | 1 | 1 | 3 |
| 3 | 2 | 1 | 2 | 5 |
| 4 | 3 | 2 | 3 | 8 |
| 5 | 5 | 3 | 5 | 13 |
| 6 | 8 | 5 | 8 | 21 |
| 7 | 13 | 8 | 13 | 34 |
| 8 | 21 | 13 | 21 | 55 |
| 9 | 34 | 21 | 34 | 89 |
| 10 | 55 | 34 | 55 | 144 |
| 11 | 89 | 55 | 89 | 233 |
| 12 | 144 | 89 | 144 | 377 |

Thus at the end of the year there are a total of 377 pairs of rabbits. Inspecting the columns of the table, we see that the entry for the current month is the sum of the entries for the two previous months. In this paper, sequences having that characteristic are called "additive sequences."

Definition 3.1 (Additive Sequence)

$T = (t_{n+2} : t_{n+2} = t_{n+1} + t_n,\ n = 0, 1, 2, \ldots$ with some specified $t_0$ and $t_1$).

General Properties of Additive Sequences

To generate an additive sequence simply select any two numbers for $t_0$ and $t_1$. Suppose $t_0 = 1$ and $t_1 = 4$. The sequence generated is

1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, . . . .

Experimentation with this sequence leads to results applicable to additive sequences in general. A natural property to investigate might be the sums of the first m+1 terms of the sequence. Table II lists these sums.

TABLE II

SUMS OF SEQUENCE TERMS

| m | $t_m$ | sum of 1st m+1 terms |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 4 | 5 |
| 2 | 5 | 10 |
| 3 | 9 | 19 |
| 4 | 14 | 33 |
| 5 | 23 | 56 |
| 6 | 37 | 93 |
| 7 | 60 | 153 |
| 8 | 97 | 250 |
| 9 | 157 | 407 |
| 10 | 254 | 661 |

Observe the sums in the third column are each 4 less than the second sequence term following the last one included in the sum. Furthermore, in this sequence, $4 = t_1$. Additional experimentation, with different sequences, would lead to the conclusion that $t_1$ is the required difference, regardless of its value.

Theorem 3.1

$$t_0 + t_1 + t_2 + t_3 + . . . + t_n = t_{n+2} - t_1.$$

Proof: The proof follows from definition 3.1. Since

$$t_{n+2} = t_{n+1} + t_n,\ t_n = t_{n+2} - t_{n+1},$$

$$t_0 = t_2 - t_1$$

$$t_1 = t_3 - t_2$$

$$t_2 = t_4 - t_3$$

$$t_3 = t_5 - t_4$$

$$. . .$$

$$t_n = t_{n+2} - t_{n+1}$$

Adding both sides of the above equations, all terms on the right side cancel except $-t_1$ from the expression for $t_0$ and $t_{n+2}$ from the expression for $t_n$. Therefore

$$t_0 + t_1 + t_2 + t_3 + . . . + t_n = t_{n+2} - t_1.$$

Creating a table of terms is helpful in discovering many properties of additive sequences. In table III sums of even-subscripted terms and sums of odd-subscripted terms are considered. The sums of the odd-subscripted terms are 1 less than the next even-subscripted term and $t_0 = 1$. The sums of even-subscripted terms are $4 = t_1$ less than the next odd-subscripted term.

TABLE III

SUMS OF ODD-SUBSCRIPTED TERMS AND SUMS OF EVEN-SUBSCRIPTED TERMS

| $k$ | $t_{2k-1}$ | sums of the first k odd-subscripted terms | $t_{2k}$ | sums of the first k even-subscripted terms |
|---|---|---|---|---|
| 1 | 4 | 4 | 5 | 5 |
| 2 | 9 | 13 | 14 | 19 |
| 3 | 23 | 36 | 37 | 56 |
| 4 | 60 | 96 | 97 | 153 |
| 5 | 157 | 253 | 254 | 407 |

Theorem 3.2

(1) $t_1 + t_3 + t_5 + \ldots + t_{2k-1} = t_{2k} - t_0$

(2) $t_2 + t_4 + t_6 + \ldots + t_{2k} = t_{2k+1} - t_1$

Proof: To prove (1) note that $t_{n+2} = t_{n+1} + t_n$ implies

$$t_{n+1} = t_{n+2} - t_n.$$

Then,
$$t_1 = t_2 - t_0$$
$$t_3 = t_4 - t_2$$
$$t_5 = t_6 - t_4$$
$$\ldots$$
$$t_{2k-1} = t_{2k} - t_{2k-2}$$

Adding both sides of these equations produces the desired result.

To prove (2) subtract the sum of the first k odd-subscripted terms, $t_{2k} - t_0$, given in part (1) and $t_0$ from the sum of the first 2k terms, $t_{2k+2} - t_1$, given in Theorem 3.1.

This gives $(t_{2k+2} - t_1) - (t_{2k} - t_0) - t_0$

$$= t_{2k+2} - t_{2k} - t_1 + t_0 - t_0$$

$$= t_{2k+1} - t_1, \text{ as required.}$$

Corollary

$$t_1 - t_2 + t_3 - t_4 + \ldots + (-1)^{n-1}t_n = (-1)^{n-1}t_{n-1} - t_0 + t_1$$

Proof: From Theorem 3.2, if n is even, then

$$\begin{aligned} t_1 - t_2 + t_3 - t_4 + \ldots - t_n &= (t_1 + t_3 + t_5 + \ldots + t_{n-1}) \\ &\quad - (t_2 + t_4 + t_6 + \ldots + t_n) \\ &= (t_n - t_0) - (t_{n+1} - t_1) \\ &= -(t_{n+1} - t_n) - t_0 + t_1 \\ &= -t_{n-1} - t_0 + t_1. \end{aligned}$$

Similarly, if n is odd,

$$\begin{aligned} t_1 - t_2 + t_3 - t_4 + \ldots + t_n &= (t_1 + t_3 + t_5 + \ldots + t_n) \\ &\quad - (t_2 + t_4 + t_6 + \ldots + t_{n-1}) \\ &= (t_{n+1} - t_0) - (t_n - t_1) \\ &= (t_{n+1} - t_n) - t_0 + t_1 \\ &= t_{n-1} - t_0 + t_1. \end{aligned}$$

Thus the equation holds for all positive n.

Properties of squared terms of additive sequences are given in the following theorem.

Theorem 3.3

(1) $t_n^2 + t_{n+1}^2 = t_n t_{n+2} + t_{n+1} t_{n-1}$

(2) $t_{n+1}^2 - t_n^2 = t_{n-1} t_{n+2}$

(3) $2(t_n^2 + t_{n+1}^2) = t_{n-1}^2 + t_{n+2}^2$

(4) $t_0^2 + t_1^2 + t_2^2 + \ldots + t_n^2$

$$= t_0^2 - t_0 t_1 + t_n t_{n+1}$$

Proof: (1) $t_n^2 + t_{n+1}^2 = t_n^2 + t_n t_{n+1} - t_n t_{n+1} + t_{n+1}^2$

$$= t_n(t_n + t_{n+1}) + t_{n+1}(t_{n+1} - t_n)$$

$$= t_n t_{n+2} + t_{n+1} t_{n-1}$$

(2) $t_{n+1}^2 - t_n^2 = (t_{n+1} - t_n)(t_{n+1} + t_n)$

$$= t_{n-1} t_{n+2}$$

(3) $2(t_n^2 + t_{n+1}^2)$

$$= 2(t_n^2 + t_n t_{n+1} - t_n t_{n+1} + t_{n+1}^2)$$

$$= (t_{n+1} - t_n)^2 + (t_n + t_{n+1})^2$$

$$= t_{n-1}^2 + t_{n+2}^2$$

(4) Write $t_n^2 = t_n t_{n+1} - t_n t_{n-1}$, for $n > 0$.

Then, $t_1^2 = t_1 t_2 - t_1 t_0$

$$t_2^2 = t_2 t_3 - t_2 t_1$$

$$t_3^2 = t_3 t_4 - t_3 t_2$$

$$\ldots$$

$t_n^2 = t_n t_{n+1} - t_n t_{n-1}$ and

$$t_0^2 + t_1^2 + t_2^2 + \ldots + t_n^2 = t_0^2 - t_1 t_0 + t_n t_{n+1}$$

It is interesting to note that many additive sequences can be used to generate Pythagorean Triples.

Definition 3.2 (Pythagorean Triple)

A Pythagorean Triple is any three positive integers, (a, b, c), such that $a^2 + b^2 = c^2$.

Theorem 3.4

Let $t_n$, $t_{n+1}$, $t_{n+2}$ and $t_{n+3}$ be any four consecutive, terms of an additive sequence then

$$(t_n t_{n+3})^2 + (2t_{n+1}t_{n+2})^2 = (t_{n+1}^2 + t_{n+2}^2)^2$$

Proof: $(t_n t_{n+3})^2 + (2t_{n+1}t_{n+2})^2$

$$= [(t_{n+2} - t_{n+1})(t_{n+2} + t_{n+1})]^2 + 4t_{n+1}^2 t_{n+2}^2$$

$$= (t_{n+2}^2 - t_{n+1}^2)^2 + 4t_{n+1}^2 t_{n+2}^2$$

$$= (t_{n+2}^2)^2 + 2t_{n+1}^2 t_{n+2}^2 + (t_{n+1}^2)^2$$

$$= (t_{n+1}^2 + t_{n+2}^2)^2$$

Corollary 1

$$(t_n t_{n+3})^2 + (2t_{n+1}t_{n+2})^2 = (t_{n+1}t_{n+3} + t_n t_{n+2})^2$$

This identity follows immediately from Theorem 3.4 and Theorem 3.3 (1).

Corollary 2

If $t_n$ and $t_{n+1}$ are both positive or both negative consecutive integral terms of an additive sequence then $(t_n t_{n+3},\ 2t_{n+1}t_{n+2},\ t_{n+1}^2 + t_{n+2}^2)$ is a Pythagorean triple.

Proof: Since $t_n$, $t_{n+1}$, $t_{n+2}$ and $t_{n+3}$ are all integers, $t_{n+1}^2 + t_{n+2}^2$ is a positive integer and because $t_n$,

$t_{n+1}$, $t_{n+2}$ and $t_{n+3}$ all have the same sign, $t_n t_{n+3}$ and $2t_{n+1}t_{n+2}$ are also positive integers. Thus, it follows from Theorem 3.4 and definition 3.2 that the given terms constitute a Pythagorean triple.

This provides an easy method of finding Pythagorean triples. Select any pair of integers having the same sign, such as 3 and 7, then the sequence terms to be used are 3, 7, 10 and 17. By corollary 2, (51, 140, 149) is a Pythagorean triple. Reversing the first and second terms produces a new sequence, 7, 3, 10, 13, which generates a different Pythagorean triple, (91, 60, 109). For an example with negative terms, -5, -2, -7, -9, might be used to produce (45, 28, 53) and, reversing the first two terms, (24, 70, 74). However, it is not required that the first and second terms be different. If they are both 1, for instance, the result is the (3, 4, 5) Pythagorean triple.

The sequence which produces the (3, 4, 5) Pythagorean triple is a special additive sequence. It includes the terms 1, 1, 2, 3. Refer back to Table I. If in month 1 there are $f_1$ adult rabbits and in month 2 there are $f_2$ adult rabbits, then $f_1$, $f_2$, $f_3$ and $f_4$ are the sequence terms used to produce the (3, 4, 5) Pythagorean triple. Since $f_1 = f_2$, $f_0$ must be 0. Sequences of this type have several special properties and properties previously shown for additive sequences can be simplified since $f_0 = 0$. When $f_1 = f_2 = 1$, the sequence has many fascinating properties and it is known as the Fibonacci sequence. A partial listing of terms of this sequence appears in Table IV.

Definition 3.3 (Fibonacci Sequence)

The Fibonacci sequence is an additive sequence with $f_0 = 0$, $f_1 = 1$.

## Properties of the Fibonacci Sequence

Theorem 3.5 is a restatement of those properties of general additive sequences which can be simplified using definition 3.3.

Theorem 3.5

(1) $f_0 + f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$

(2) $f_1 + f_3 + f_5 + \ldots + f_{2k-1} = f_{2k}$

(3) $f_2 + f_4 + f_6 + \ldots + f_{2k} = f_{2k+1} - 1$

(4) $f_1 - f_2 + f_3 - f_4 + \ldots + (-1)^{n-1}f_n = (-1)^{n-1}f_{n-1} + 1$

(5) $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$

Many properties of the Fibonacci sequence can be proved easily using the Principle of Mathematical Induction. Some of these properties are listed in Theorem 3.6.

TABLE IV

PARTIAL LISTING OF FIBONACCI NUMBERS*

| n | $f_n$ | n | $f_n$ |
|---|---|---|---|
| 0 | 0 | 24 | 46368 |
| 1 | 1 | 25 | 75025 |
| 2 | 1 | 26 | 121393 |
| 3 | 2 | 27 | 196418 |
| 4 | 3 | 28 | 317811 |
| 5 | 5 | 29 | 514229 |
| 6 | 8 | 30 | 832040 |
| 7 | 13 | 31 | 1346269 |
| 8 | 21 | 32 | 2178309 |
| 9 | 34 | 33 | 3524578 |
| 10 | 55 | 34 | 5702887 |
| 11 | 89 | 35 | 9227465 |
| 12 | 144 | 36 | 14930352 |
| 13 | 233 | 37 | 21157817 |
| 14 | 377 | 38 | 39088169 |
| 15 | 610 | 39 | 63245986 |
| 16 | 987 | 40 | 102334155 |
| 17 | 1597 | 50 | 12586269025 |
| 18 | 2584 | 60 | 1548008755920 |
| 19 | 4181 | 70 | 190392490709135 |
| 20 | 6765 | 80 | 23416728348467685 |
| 21 | 10946 | 90 | 2880067194370816120 |
| 22 | 17711 | 100 | 354224848179261915075 |
| 23 | 28657 | | |

*Vickrey, Thomas Loren, "Fibonacci Numbers" (unpublished Doctoral dissertation, Oklahoma State University, 1968), p. 9.

Theorem 3.6

(1) $f_1f_2 + f_2f_3 + f_3f_4 + \ldots + f_{2k-1}f_{2k} = f_{2k}^2$

(2) $f_{n+1}^2 - f_nf_{n+2} = (-1)^n$

(3) $f_{n+2}f_n - f_{n+3}f_{n-1} = 2(-1)^{n+1}$

Proof: (1) When $k = 1$, $f_1f_2 = 1(1) = 1^2 = f_2^2$. This is a true statement, thus the equation holds when $k = 1$.

Assume the equation is true when $k = m$, then

$$f_1f_2 + f_2f_3 + \ldots + f_{2m-1}f_{2m} = f_{2m}^2.$$

Add $f_{2m}f_{2m+1} + f_{2m+1}f_{2m+2}$ to both sides of the equation, then $f_{2m}^2 + f_{2m}f_{2m+1} + f_{2m+1}f_{2m+2}$

$$= f_{2m}(f_{2m} + f_{2m+1}) + f_{2m+1}f_{2m+2}$$

$$= f_{2m}f_{2m+2} + f_{2m+1}f_{2m+2}$$

$$= f_{2m+2}(f_{2m} + f_{2m+1})$$

$$= f_{2m+2}^2$$

$$= f_{2(m+1)}^2,$$

which is the required result when $k = m + 1$.

Therefore (1) holds for all integers $k > 0$.

(2) When $n = 1$, $f_2^2 - f_1f_3 = 1 - 2 = (-1)^1$. Thus the case is clearly true for $n = 1$.

Assume the equation holds when $n = k$. Then $f_{k+1}^2 - f_kf_{k+2} = (-1)^k$. Add and subtract $f_{k+2}^2$ on the left side of the equation. Then

$$f_{k+1}^2 + f_{k+2}^2 - f_{k+2}^2 - f_kf_{k+2} = (-1)^k$$

Apply Theorem 3.3 (1) to obtain

$$f_{k+1}f_{k+3} + f_k f_{k+2} - f_{k+2}^2 - f_k f_{k+2} = (-1)^k.$$

Then, $-(f_{k+2}^2 - f_{k+1}f_{k+3}) = (-1)^k$ and

$$f_{(k+1)+1}^2 - f_{k+1}f_{(k+1)+2} = (-1)^{k+1},$$

This was the result needed for $n = k + 1$.

Therefore (2) is true for all $n > 0$.

(3) When $n = 1$, $f_3 f_1 - f_4 f_0 = 2 = 2(-1)^2$ and the statement is clearly true when $n = 1$.

Assume the statement holds for $n = k$. Then,

$$f_{k+2}f_k - f_{k+3}f_{k-1} = 2(-1)^{k+1}.$$

Add and subtract $f_{k+3}f_k$ on the left side of the equation to obtain

$$f_{k+2}f_k + f_{k+3}f_k - f_{k+3}f_k - f_{k+3}f_{k-1} = 2(-1)^{k+1}.$$

Then $(f_{k+2} + f_{k+3})f_k - f_{k+3}(f_k + f_{k-1}) = 2(-1)^{k+1}$ and

$$-(f_{k+3}f_{k+1} - f_{k+4}f_k) = 2(-1)^{k+1}.$$

$$f_{(k+1)+2}f_{k+1} - f_{(k+1)+3}f_{(k+1)-1} = 2(-1)^{(k+1)+1}$$

is the result desired.

Therefore (3) holds for all $n > 0$.

The following property of Fibonacci numbers will turn out to be important later and it is quite interesting in itself. Select any pair of adjacent Fibonacci numbers and any second pair, as distant from the first pair as you like. Multiply the first terms of each pair and the second terms of each pair. The sum of these two products is a Fibonacci number and its subscript bears a certain relationship to the subscripts of the original numbers. For example, consider $f_3 = 2$, $f_4 = 3$ and $f_7 = 13$, $f_8 = 21$. $f_3f_7 + f_4f_8 = 26 + 63 = 89 = f_{11} = f_{3+8} = f_{7+4}$. This formula also works when the pairs are the same, such as $f_6f_6 + f_7f_7 = 64 + 169 = 233 = f_{13} = f_{6+7}$. Theorem 3.6 is proved by the Principle of Strong Induction.

Theorem 3.6

$$f_{m+n} = f_{m-1}f_n + f_mf_{n+1}.$$

Proof: The proof is by induction on n, holding m fixed.

When $n = 1$, $f_{m+1} = f_{m-1}f_1 + f_mf_2 = f_{m-1}(1) + f_m(1) = f_{m+1}$ and the statement holds for $n = 1$.

Assume $f_{m+(k-1)} = f_{m-1}f_{k-1} + f_mf_k$ and $f_{m+k} = f_{m-1}f_k + f_mf_{k+1}$. Add these two equation together to obtain

$f_{m+(k-1)} + f_{m+k} = f_{m-1}f_{k-1} + f_mf_k + f_{m-1}f_k + f_mf_{k+1}$.

Then $f_{m+k+1} = f_{m-1}f_{k+1} + f_mf_{(k+1)+1}$, as was needed.

Therefore the statement holds for all $m$, $n > 0$.

Corollary

(1) $f_{2n} = f_{n-1}f_n + f_nf_{n+1}$ and $f_{2n}/f_n = f_{n-1} + f_{n+1}$

(2) $f_{2n+1} = f_n^2 + f_{n+1}^2$ and $f_{2n+1} = f_{n-1}f_{n+1} + f_nf_{n+2}$

(3) $f_{2n-1} = f_{n-1}^2 + f_n^2$ and $f_{2n-1} = f_{n-2}f_n + f_{n-1}f_{n+1}$

Proof: (1) $f_{2n} = f_{(n+n)} = f_{n-1}f_n + f_nf_{n+1}$ and $f_{2n}/f_n = f_{n-1} + f_{n+1}$ is simply the previous equation with $f_n$ dividing both sides. This is significant because it shows that $f_{2n}$ is divisible by $f_n$.

(2) and (3) follow by rewriting the subscripts and applying Theorem 3.6.

$f_{2n+1} = f_{[(n+1)+n]} = f_n f_n + f_{n+1} f_{n+1} = f_n^{\,2} + f_{n+1}^{\,2}$ and

$f_{2n+1} = f_{[n+(n+1)]} = f_{n-1} f_{n+1} + f_n f_{n+2}$.

$f_{2n-1} = f_{[n+(n-1)]} = f_{n-1} f_{n-1} + f_n f_n = f_{n-1}^{\,2} + f_n^{\,2}$ and

$f_{2n-1} = f_{[(n-1)+n]} = f_{n-2} f_n + f_{n-1} f_{n+1}$.

Inspecting early terms of the Fibonacci sequence (listed in Table IV), adjacent terms appear to be relatively prime. It turns out, all adjacent Fibonacci terms are relatively prime. The following Lemma will be helpful in proving Theorem 3.7.

Lemma

If $a = qb + r$, with $a$, $q$, $b$, $r$ integers, then the greatest common divisor of $a$ and $b$ is also the greatest common divisor of $b$ and $r$.

This is denoted by $(a, b) = (b, r)$.

Proof: Let $d = (a, b)$ and $c = (b, r)$.

$d = (a, b)$ implies $ds = a$ and $dt = b$ for some integers $s$, $t$. Hence $ds = qdt + r$ and $d(s - qt) = r$. Thus, $d|r$ and $d|b$, while $c = (b, r)$. Therefore $d \leq c$. $c = (b, r)$ implies $cu = b$ and $cv = r$ for some integers $b$, $r$.

Hence $a = qcu + cv$ and $a = c(qu + v)$. Thus, $c|a$ and $c|b$, while $d = (a, b)$. Therefore $c \leq d$, but $d \leq c$

which requires $d = c$ and the lemma is proved.

Theorem 3.7

Any two consecutive Fibonacci terms are relatively prime. This is denoted by $(f_n, f_{n+1}) = 1$.

Proof: For $n = 1$, $(f_1, f_2) = (1, 1) = 1$ is a true statement.

Assume $(f_k, f_{k+1}) = 1$. From the Lemma above, $f_{k+2} = f_{k+1} + f_k$ implies $(f_{k+2}, f_{k+1}) = (f_{(k+1)+1}, f_{k+1}) = (f_{k+1}, f_k) = 1$.

Therefore $(f_n, f_{n+1}) = 1$ for all $n > 0$.

Since it has now been shown that $f_n$ divides $f_{2n}$ but does not divide $f_{n+1}$, it seems natural to wonder what other Fibonacci terms are divisible by $f_n$. Inspection of Table IV reveals that $2 = f_3$ divides $8 = f_6$, as expected, but 2 also divides 34, 144, 610, 2584, 10946 and 46368, which are $f_9$, $f_{12}$, $f_{15}$, $f_{18}$, $f_{21}$ and $f_{24}$, respectively. It appears that Fibonacci terms which are divisible by 2 all have subscripts which are divisible by 3. This statement will be confirmed in the following theorem.

Theorem 3.8

If $n, m > 2$, $f_n | f_m$ if and only if $n | m$.

Proof: $n|m$ implies $nr = m$ and $f_m = f_{nr}$. $f_{nr}$ will be shown to be divisible by $f_n$ by induction on $r$.

When $r = 1$, the statement is obviously true. Assume $f_n | f_{nk}$. $f_{n(k+1)} = f_{nk+n} = f_{nk-1}f_n + f_{nk}f_{n+1}$, by Theorem 3.6 and since $f_n | f_{nk}$, $f_n | (f_{nk-1}f_n + f_{nk}f_{n+1})$ and thus, $f_n | f_{nk+n}$. Therefore $f_n | f_m$ if $n | m$.

To show that $f_n | f_m$ implies $n | m$ when $n, m > 2$, assume $n$ does not divide $m$. Then $m = nq + r$, for some positive integers $q$, $r$, $0 < r < n$ and $f_n | f_m$ implies $f_n | f_{nq+r}$. Thus $f_n | (f_{nq-1}f_r + f_{nq}f_{r+1})$, while $n | nq$. As previously shown, $f_n | f_{nq}$. Thus $f_n$ must divide $f_{nq-1}f_r$. $r < n$, so $f_n > f_r$ and $f_n$ does not divide $f_r$. Thus $f_n | f_{nq-1}$ and $f_{nq+1} = f_{nq} + f_{nq-1}$, but $f_n | f_{nq}$ implies $f_n s = f_{nq}$, for some integer $s$. Therefore $(f_n s, f_{nq-1}) = (f_{nq}, f_{nq+1}) = 1$ by theorem 3.7. This is a contradiction, for if $f_n | f_{nq-1}$, $(f_n s, f_{nq-1}) \geqq f_n$. But $n > 2$, so $f_n > 1$. Therefore $f_n | f_m$ implies $n | m$ when $n, m > 2$.

Therefore, if $n, m > 2$, $f_n | f_m$ if and only if $n | m$.

The following Lemma and Theorem were proved by David M. Burton in Elementary Number Theory. The theorem is a very nice extension of theorem 3.8.

Lemma

If $m = qn + r$, then $(f_m, f_n) = (f_r, f_n)$.

Proof: $(f_m, f_n) = (f_{qn+r}, f_n)$

$$= (f_{qn-1}f_r + f_{qn}f_{r+1}, f_n),$$

by theorem 3.6.

Since $f_n | f_{qn}$ (theorem 3.8) and $(a + c, b) = (a, b)$, whenever $b|c$

$(f_{qn-1}f_r + f_{qn}f_{r+1}, f_n) = (f_{qn-1}f_r, f_n)$.

Let $d = (f_{qn-1}, f_n)$. $d|f_n$ and $f_n|f_{qn}$ implies $d|f_{qn}$, but $d|f_{qn-1}$, while $(f_{qn-1}, f_{qn}) = 1$ from theorem 3.7. Thus $d = 1$ and whenever $(a, b) = 1$ then $(a, bc) = (a, c)$. Therefore $(f_{qn-1}f_r, f_n) = (f_r, f_n)$ and the lemma is proved (3 : 334).

Theorem 3.9

The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; specifically,

$$(f_m, f_n) = f_{(m, n)}.$$

Proof: Assume $m \geqq n$. Applying the Euclidean Algorithm to m and n, produces the following system of equations:

$$\begin{array}{ll}
m = q_1 n + r_1, & 0 < r_1 < n \\
n = q_2 r_1 + r_2, & 0 < r_2 < r_1 \\
r_1 = q_3 r_2 + r_3, & 0 < r_3 < r_2 \\
\cdot & \\
\cdot & \\
\cdot & \\
r_{n-2} = q_n r_{n-1} + r_n, & 0 < r_n < r_{n-1} \\
r_{n-1} = q_{n+1} r_n + 0. &
\end{array}$$

In accordance with the previous lemma,

$$(f_m, f_n) = (f_{r_1}, f_n) = (f_{r_1}, f_{r_2})$$
$$= \ldots = (f_{r_{n-1}}, f_{r_n}).$$

Since $r_n | r_{n-1}$, $f_{r_n} | f_{r_{n-1}}$ by theorem 3.8.

Thus $(f_{r_{n-1}}, f_{r_n}) = f_{r_n}$. But $r_n$, being the last nonzero remainder in the Euclidean Algorithm for m and n, is equal to (m, n).

Therefore $(f_m, f_n) = f_{(m, n)}$ (3 : 335).

Knowing something about the divisibility of Fibonacci numbers, it is natural to ask how it might be determined which of the Fibonacci numbers will be prime. This is an unsolved problem. Also unsolved is the question of whether the number of prime Fibonacci numbers is finite or infinite.

Knowing that two consecutive Fibonacci numbers are relatively prime, the ratio $f_{n+1}/f_n$ will not be an integer when n > 2. This ratio was mentioned at the end of Chapter II, where it was claimed that, as n gets large, these ratios approach the Golden Ratio, $(1 + \sqrt{5})/2 = 1.61803 \ldots$. The next topic to be considered is the sequence of ratios of Fibonacci numbers.

Definition 3.4

$$S = (s_n : s_n = f_{n+1}/f_n, n = 1, 2, 3 \ldots).$$

Table V lists the first few terms of the sequence S.

TABLE V

RATIOS OF FIBONACCI NUMBERS

| n | $s_n$ | n | $s_n$ |
|---|---|---|---|
| 1 | 1 | 6 | 1.625 |
| 2 | 2 | 7 | 1.615385 |
| 3 | 1.5 | 8 | 1.619048 |
| 4 | 1.666667 | 9 | 1.617647 |
| 5 | 1.6 | 10 | 1.618182 |

Theorem 3.10

$\lim_{n\to\infty} s_n = (1 + \sqrt{5})/2.$

Proof: To prove theorem 3.10, two subsequences, which both converge to the same limit, will be considered.

(1) $1 \leqq s_n \leqq 2$ for all $s_n$ in S. $f_n \leqq f_{n+1}$ for all n, thus $f_{n+1}/f_n = s_n \geqq 1$ and $f_{n-1} \leqq f_n$ for all $n > 0$ implies $f_n + f_{n-1} \leqq f_n + f_n$ and $f_{n+1} \leqq 2f_n$, thus $f_{n+1}/f_n = s_n \leqq 2$ for all $n > 0$. Thus S is a bounded sequence of real numbers.

(2) $S' = (s_{2k-1} : s_{2k-1}$ is an element of S and $k = 1, 2, 3, \ldots)$ is monotonically increasing. It must be shown that $s_{2k-1} < s_{2(k+1)-1} = s_{2k+1}$, for all $k > 0$. This will be done with induction on k. When $k = 1$, $s_1$ must be less than $s_3$. From Table V, $s_1 = 1$, $s_3 = 1.5$, thus the claim holds for $k = 1$. Assume $s_{2m-1} < s_{2m+1}$.

Then $f_{2m}/f_{2m-1} < f_{2m+2}/f_{2m+1}$, which implies $f_{2m-1}/f_{2m} > f_{2m+1}/f_{2m+2}$. Add 1 to both sides and

$f_{2m+1}/f_{2m} > f_{2m+3}/f_{2m+2}$. Then $f_{2m}/f_{2m+1} < f_{2m+2}/f_{2m+3}$. Adding 1 to both sides a second time yields $f_{2m}/f_{2m+1} + 1 = f_{2m+2}/f_{2m+1} < f_{2m+2}/f_{2m+3} + 1 = f_{2m+4}/f_{2m+3}$ and $s_{2(m+1)-1} = f_{2m+2}/f_{2m+1} < f_{2m+4}/f_{2m+3} = s_{2(m+1)+1}$, the desired result.

Therefore S' is monotonically increasing.

(3) $S'' = (s_{2k} : s_{2k}$ is an element of S and k = 1, 2, 3, . . . ) is monotonically decreasing. It must be shown that $s_{2k} > s_{2(k+1)} = s_{2k+2}$, for all $k > 0$. This will be done with induction on k. When k = 1, $s_2$ must be greater than $s_4$. $s_2 = 2$, $s_4 = 5/3$, thus the claim holds for k = 1. Assume $s_{2m} > s_{2m+2}$.

Then $f_{2m+1}/f_{2m} > f_{2m+3}/f_{2m+2}$, which implies $f_{2m}/f_{2m+1} < f_{2m+2}/f_{2m+3}$. Add 1 to both sides, and $f_{2m+2}/f_{2m+1} < f_{2m+4}/f_{2m+3}$. Taking the reciprocals and adding 1 to both sides a second time yields the desired result: $s_{2(m+1)} = f_{2m+3}/f_{2m+2} > f_{2m+5}/f_{2m+4} = s_{2(m+1)+2}$.

Therefore S" is monotonically decreasing.

S' and S" are monotone bounded sequences of real numbers. Therefore S' and S" converge.

(4) Let $x = \lim_{k\to\infty} S'$ and $y = \lim_{k\to\infty} S''$.

$$\begin{aligned} f_{2k}/f_{2k-1} &= (f_{2k-1} + f_{2k-2})/f_{2k-1} \\ &= 1 + 1/(f_{2k-1}/f_{2k-2}), \text{ thus} \end{aligned}$$

$$x = 1 + 1/y.$$

$$f_{2k+1}/f_{2k} = (f_{2k} + f_{2k-1})/f_{2k}$$

$$= 1 + 1/(f_{2k}/f_{2k-1}), \text{ thus}$$

$$y = 1 + 1/x.$$

Therefore $x = 1 + 1/(1 + 1/x)$, thus

$x + 1 = 2 + 1/x$ and $x^2 - x - 1 = 0$.

Applying the quadratic formula, $x = (1 \pm \sqrt{5})/2$, but $1 \leqq x \leqq 2$, thus $x = (1 + \sqrt{5})/2$.

While $y = 1 + 1/(1 + 1/y)$, so $y = x$.

(5) s' and s" converge to x, thus

$|s_{2k-1} - x| < \varepsilon$, $2k-1 \geqq m_1$ and

$|s_{2k} - x| < \varepsilon$, $2k \geqq m_2$.

Let $m = \max\{m_1, m_2\}$.

Need to show that $|s_n - x| < \varepsilon$, $n \geqq m$.

Case 1:

Assume n is odd and $n \geqq m$.

Since n is odd, $n = 2k - 1 \geqq m \geqq m_1$ and

$|s_n - x| = |s_{2k-1} - x| < \varepsilon$.

Case 2:

Assume n is even and $n \geqq m$.

Since n is odd, $n = 2k \geqq m \geqq m_2$ and

$|s_n - x| = |s_{2k} - x| < \varepsilon$.

Therefore the sequence S converges to x, the Golden Ratio.

The limit of the sequence of ratios of consecutive Fibonacci terms, $(1 + \sqrt{5})/2 = 1.61803 \ldots$, is the ratio, found repeatedly in Chapter II, known as the Golden Ratio. The additive sequence whose first two terms are 1 and the Golden Ratio will be considered in Chapter IV.

# CHAPTER IV

# THE GOLDEN RATIO

## Introduction

There is a close relationship between the Golden Ratio and Fibonacci's sequence. In theorem 3.9 the Golden Ratio was found to be the limit of the sequence of ratios of consecutive Fibonacci terms. Properties of the Golden Ratio will be presented in this Chapter and many of these properties will involve Fibonacci numbers.

Consider an additive sequence $R = (r_n : r_0 = 1$ and $r_1 = x$, where $x$ is any real number.) Table VI is a list of the first few terms of the sequence R.

TABLE VI

TERMS OF THE R SEQUENCE

| $n$ | $r_n$ | $n$ | $r_n$ | $n$ | $r_n$ |
|---|---|---|---|---|---|
| 0 | $0x + 1$ | 6 | $8x + 5$ | 12 | $144x + 89$ |
| 1 | $1x + 0$ | 7 | $13x + 8$ | 13 | $233x + 144$ |
| 2 | $1x + 1$ | 8 | $21x + 13$ | 14 | $377x + 233$ |
| 3 | $2x + 1$ | 9 | $34x + 21$ | 15 | $610x + 377$ |
| 4 | $3x + 2$ | 10 | $55x + 34$ | 16 | $987x + 610$ |
| 5 | $5x + 3$ | 11 | $89x + 55$ | 17 | $1597x + 987$ |

Inspection of the terms in Table VI reveals the Fibonacci sequence in both, the coefficients of $x$ and the constant terms. Theorem 4.1 establishes this relationship.

Theorem 4.1

$r_n = f_n x + f_{n-1}$, for $n > 0$, when $(r_n)$ is any additive sequence with $r_0 = 1$ and $r_1 = x$, where x is any real number.

Proof: $r_1 = 1x + 0 = f_1 x + f_0$, thus the statement is true when $n = 1$.

Assume $r_{k-1} = f_{k-1}x + f_{k-2}$ and $r_k = f_k x + f_{k-1}$.

$$r_{k+1} = r_{k-1} + r_k = (f_{k-1}x + f_{k-2}) + (f_k x + f_{k-1})$$
$$= f_{k+1}x + f_k.$$

Therefore the statement is shown to be true for all $n > 0$ using strong induction on n.

Theorem 4.1 holds for any value x. When x is a root of the equation $x^2 - x - 1 = 0$, it happens that the terms of this sequence are powers of x.

## The Golden Sequences

Definition 4.1 (Golden Sequence)

A Golden Sequence is an additive sequence with $g_0 = 1$ and $g_1 = x$, where x is a root of the equation $x^2 - x - 1 = 0$.

A Golden Sequence is an additive sequence with $g_0 = 1$ and $g_1$ is a real number. Theorem 4.2 follows immediately.

Theorem 4.2

$g_n = f_n x + f_{n-1}$, for all $n > 0$.

Theorem 4.3 will establish the fact that the nth term of a Golden sequence is the nth power of x and that Golden sequences are the only additive sequences to have this property.

Theorem 4.3

Golden sequences are the only additive sequences for which the nth term is also the nth power of a fixed real number x, for all $n \geq 0$.

Proof: It must be shown that $g_n = x^n$, for all $n \geq 0$ if and only if $x^2 = x + 1$.

(1) $g_n = x^n$ if $x^2 = x + 1$ is proven by induction on n. The statement is obviously true when n is 0 and when n is 1. Assume $g_k = x^k$ and multiply both sides by x, then $g_k x = x^{k+1}$, and $g_k x = (f_k x + f_{k-1})x = f_k x^2 + f_{k-1}x$, but $x^2 = x + 1$. Therefore $f_k x^2 + f_{k-1} x = f_k(x + 1) + f_{k-1}x = f_{k+1}x + f_k = g_{k+1}$.

(2) $x^2 = x + 1$ if $g_n = x^n$ since $g_{n+2} = g_{n+1} + g_n$, $x^{n+2} = x^{n+1} + x^n$. Factoring $x^n$ from both sides, $x^2 = x + 1$, and the proof is complete.

The corollary combines theorems 4.2 and 4.3.

Corollary

$$x^n = f_n x + f_{n-1}, \text{ for all } n > 0 \text{ and } x^2 = x + 1.$$

Theorem 4.3 and its corollary require only that x be a root of the equation $x^2 - x - 1 = 0$, so there are exactly two numbers for which these results will hold true. One of these, the positive root, is the Golden Ratio. The other is the negative root, $(1 - \sqrt{5})/2 = -0.61803 \ldots$.

Let $x = (1 + \sqrt{5})/2$ and $x' = (1 - \sqrt{5})/2$. Then

(4.1) $$x' = 1 - x,$$

(4.2) $$x' = x - \sqrt{5} \text{ and}$$

(4.3) $$x' = -1/x$$

Identities (4.1) and (4.2) are obvious. (4.3) follows from simplifying the expression $-2/(1 + \sqrt{5})$.

A Golden sequence is an additive sequence, thus properties of additive sequences hold for Golden Sequences. Writing $g_n$ as $x^n$, theorem 4.4 is a restatement of properties of general additive sequences.

Theorem 4.4

(1) $1 + x + x^2 + x^3 + \ldots + x^n = x^{n+2} - x$

(2) $x + x^3 + x^5 + \ldots + x^{2k-1} = x^{2k} - 1$

(3) $x^2 + x^4 + x^6 + \ldots + x^{2k} = x^{2k+1} - x$

(4) $x - x^2 + x^3 - x^4 + \ldots + (-1)^{n-1}x^n =$

$$(-1)^{n-1}x^{n-1} - 1 + x$$

The properties of theorem 4.4 also hold for x' and thus, for $1 - x$, $x - \sqrt{5}$ and $-1/x$.

Since $|x'| < 1$ the infinite geometric series composed of powers of x' converges.

Theorem 4.5

$$1 + x' + x'^2 + x'^3 + \ldots = -x'$$

Proof: $1 + x' + x'^2 + x'^3 + \ldots$ is a geometric series with the common ratio $x'$, $|x'| < 1$ and thus converges to $1/(1 - x') = 1/(1 - 1 + x) = 1/x = -x'$.

Theorem 4.5 confirms that powers of x' approach 0 as n gets large. Table VII lists some powers of x' rounded to five decimal places. Inspecting this table reveals that the powers of x' are approaching 0 even when n is relatively small.

TABLE VII

POWERS OF X'

| n | $x'^n$ | n | $x'^n$ | n | $x'^n$ |
|---|---|---|---|---|---|
| 1 | -0.61803 | 5 | -0.09017 | 9 | -0.01316 |
| 2 | 0.38197 | 6 | 0.05573 | 10 | 0.00813 |
| 3 | -0.23607 | 7 | -0.03444 | 15 | -0.00073 |
| 4 | 0.14590 | 8 | 0.02129 | 20 | 0.00007 |
| | | | | 25 | -0.00001 |

## Binet's Formula

Jacques P. M. Binet (1786 - 1856) set out a formula, called the Binet formula, which finds the nth term of the Fibonacci sequence in the difference between powers of x and x' divided by $\sqrt{5}$. This, once again demonstrates the connection between the Fibonacci sequence and the Golden Ratio. Since even the somewhat small powers of x' approach 0, the key to Binet's formula must be in the powers of x. In this formula the powers of x and x' are divided by $\sqrt{5}$, so the x' term will approach 0 even more rapidly than shown in Table VII. Table VIII lists some early powers of x divided by $\sqrt{5}$ rounded to five decimal places. A comparison of the terms in Table VIII to those of Table IV shows that the values of $x^n/\sqrt{5}$ are very close to $f_n$.

TABLE VIII

POWERS OF THE GOLDEN RATIO DIVIDED BY $\sqrt{5}$

| n | $x^n/\sqrt{5}$ | n | $x^n/\sqrt{5}$ | n | $x^n/\sqrt{5}$ |
|---|---|---|---|---|---|
| 1 | 0.72361 | 5 | 4.95967 | 9 | 33.99412 |
| 2 | 1.17082 | 6 | 8.02492 | 10 | 55.00364 |
| 3 | 1.89443 | 7 | 12.98460 | 15 | 609.99967 |
| 4 | 3.06525 | 8 | 21.00952 | 20 | 6765.00003 |
| | | | | 25 | 75024.99999 |

Theorem 4.6 (Binet's formula)

$f_n = [(1 + \sqrt{5})/2]^n/\sqrt{5} - [(1 - \sqrt{5})/2]^n/\sqrt{5}$, for any non-negative integer n.

Proof: To prove $f_n = x^n/\sqrt{5} - x'^n/\sqrt{5}$:

(1) when n = 0, Binet's formula has

$$0 = x^0/\sqrt{5} - x'^0/\sqrt{5}$$
$$= 1/\sqrt{5} - 1/\sqrt{5} = 0$$

When $n > 0$, the corollary to theorem 4.3 has

$$f_n = x^n/\sqrt{5} - x'^n/\sqrt{5} = (f_n x + f_{n-1})/\sqrt{5} - (f_n x' + f_{n-1})/\sqrt{5}$$
$$= (f_n x - f_n x' + f_{n-1} - f_{n-1})/\sqrt{5}$$
$$= f_n(x - x')/\sqrt{5}$$
$$= f_n\sqrt{5}/\sqrt{5}$$
$$= f_n$$

Therefore Binet's formula holds for all $n \geqq 0$.

## The Inverse of the Golden Ratio

In Chapter II, the Golden Ratio was shown to be the relation between the larger segment and the smaller segment when the original line had been cut in extreme and mean ratio. It was proven that if the length of the longer segment is a and the length of the shorter segment is b, then $(a + b)/a = a/b$. This proportion led to the conclusion that $a/b$ is the Golden Ratio. Compare the shorter segment to the longer segment to obtain the inverse of the Golden Ratio. That is $a/(a + b) = b/a$. This proportion leads to the quadratic equation $b^2 + ba - a^2 = 0$. The solutions are $b/a = (-1 \pm \sqrt{5})/2$. Of course the inverse of the Golden Ratio is the positive solution $(-1 + \sqrt{5})/2 = 0.61803 \ldots$.

Let $y = (-1 + \sqrt{5})/2$ and $y' = (-1 - \sqrt{5})/2$. Four of the most obvious of the relationships between $x$, $x'$ and $y$, $y'$ are stated below.

(4.4) $$y = 1/x$$

(4.5) $$y = x - 1$$

(4.6) $$y = -x'$$

(4.7) $$y' = 1/x'$$

(4.8) $$y' = x' - 1$$

(4.9) $$y' = -x$$

It was shown in theorem 4.3 that Golden Sequences are the only additive sequences for which the nth term is also the nth power of a real number $x$. Therefore the sequence of powers of the inverse of the Golden Ratio cannot be an additive sequence. However, since $1/y = x$ and $-y = x'$, these could be substituted in the previous theorems. For example: $(1/y)^{n+2} = (1/y)^{n+1} + (1/y)^n$. Such substitutions may yield meaningful results. Some algebraic manipulation in the given example leads to theorem 4.7.

Theorem 4.7

$y^{n+2} = y^n - y^{n+1}$, for all $n \geq 0$ if and only if $y$ is a root of the equation $y^2 + y - 1 = 0$.

Proof: (1) $y^{n+2} = y^n - y^{n+1}$ if $y^2 + y - 1 = 0$. $y^2 + y - 1 = 0$ implies $y^2 = 1 - y$. Multiply both sides of the equation $y^2 = 1 - y$ by $y^n$ to obtain the desired result.

(2) $y^2 + y - 1 = 0$ if $y^{n+2} = y^n - y^{n+1}$.

Factor $y^n$ from the given equation to obtain $y^2 = 1 - y$.

Theorem 4.7 provides the formula for a recursive sequence for the powers of y. However it would be pleasing if this sequence involved Fibonacci numbers. The corollary to theorem 4.3 holds for x' as well as x and $-y = x'$, which implies $(-y)^n = f_n(-y) + f_{n-1}$ and yields the following.

Theorem 4.8

$$y^n = (-1)^n(f_{n-1} - f_n y) \text{ for all } n > 0.$$

The first fifteen terms of this recursive sequence for the powers of the inverse of the Golden Ratio are listed in Table IX.

TABLE IX

POWERS OF THE INVERSE OF THE GOLDEN RATIO

| n | $y^n$ | n | $y^n$ | n | $y^n$ |
|---|---|---|---|---|---|
| 0 | $1 - 0y$ | 5 | $5y - 3$ | 10 | $34 - 55y$ |
| 1 | $1y - 0$ | 6 | $5 - 8y$ | 11 | $89y - 55$ |
| 2 | $1 - 1y$ | 7 | $13y - 8$ | 12 | $89 - 144y$ |
| 3 | $2y - 1$ | 8 | $13 - 21y$ | 13 | $233y - 144$ |
| 4 | $2 - 3y$ | 9 | $34y - 21$ | 14 | $233 - 377y$ |

Substitution of $-y$ for x in theorem 4.4 leads to similar properties for this inverse.

Theorem 4.9

(1) $1 - y + y^2 - y^3 + \ldots + (-1)^n y^n = (-1)^n y^{n+2} + y$

(2) $y + y^3 + y^5 + \ldots + y^{2k-1} = 1 - y^{2k}$

(3) $y^2 + y^4 + y^6 + \ldots + y^{2k} = y - y^{2k+1}$

(4) $1 + y + y^2 + y^3 + \ldots + y^n = 2 + y - y^{n-1}$

Theorem 4.10 deals with two infinite series

Theorem 4.10

(1) $1 - y + y^2 - y^3 + \ldots = y$

(2) $1 + y + y^2 + y^3 + \ldots = x^2$

Proof: (1) $|-y| < 1$, therefore the series converges to $1/(1 + y) = 1/x = y$.

(2) $|y| < 1$, therefore the series converges to $1/(1 - y) = (1/y)^2 = x^2$.

Theorem 4.10 (1) is just a restatement of theorem 4.5 with x' replaced by -y. Similarly, -y' and -y could be used in place of x and x', respectively, to generate Fibonacci numbers in Binet's formula.

The last topic to be considered is a graphic interpretation of the group generated by the Golden Ratio.

<u>A Golden Group</u>

The cyclic group, G, generated by the Golden Ratio, under regular multiplication, can be graphically interpreted on the real number line. This might be a tedious task if it were necessary to calculate each power of x and then plot the corresponding point on the number line. Since $x^n = x^{n-2} + x^{n-1}$, for every $n > 1$, this graph is particularly easy to plot. Label $x^0 = 1$, $x^1 = (1 + \sqrt{5})/2$, $x^2 = x + 1$, $x^3 = x^2 + x$, . . . . Plotting $x^{-k}$ is no more difficult for $x^{-1} = y = x - 1$ and $x^{-k} = y^k = y^{k-2} - y^{k-1}$, when $k > 1$. Thus label $x^{-1} = x - 1$, $x^{-2} = 1 - x^{-1}$, $x^{-3} = x^{-1} - x^{-2}$, . . . .

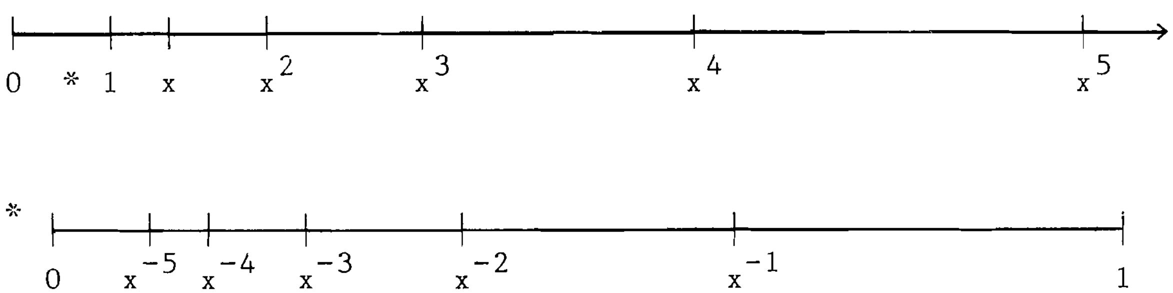


Figure 7
Graphs of Powers of x

# CHAPTER IV

# CONCLUSION

Mathematics teachers will find a wide variety of enrichment material, suitable for many different mathematics courses, presented in this paper. Non-threatening problems can readily be selected to intrigue and challenge the most fearful students as well as those who are more confident.

## Summary

In Chapter I the statement of the problem and the scope of the thesis were given. Historical background and the geometrical foundation of the Golden Section were presented in Chapter II. The general additive sequence and the Fibonacci sequence were introduced and a survey of their properties, ranging from simple sums to a sequence which converges to the Golden Ratio, were established in Chapter III. The Golden Ratio and its properties were investigated in Chapter IV. This investigation exposed some special sequences, Binet's Formula, a nonrecursive function for generating Fibonacci numbers, convergence of the infinite geometric series whose addends have the common ratio, the inverse of the Golden Ratio, and a graphical interpretation of the cyclic group generated by the Golden Ratio.

## Suggestions for Implementation

Problem-solving skills are an important part of every mathematics course and are appropriately stressed as an objective in most math courses. A unit on pattern recognition could be effectively introduced with the rabbit problem of Chapter III. Students should be encouraged to discover the patterns in the table in order to solve the problem without having to complete the table for all twelve months. Then, they could create their own additive sequences and be led to discovering the properties that are common to all of the additive sequences they created. Children enjoy this activity as early as the third grade; as soon as they are able to add multiple addends of two and three digits.

Once students have been introduced to the quadratic formula, they can discover the unique positive real number whose powers are the sum of the two previous powers and the unique positive real number whose powers are the difference of the two previous powers.

Geometry students benefit from the historical presentation of Chapter II. They can do the constructions and investigate relationships in the regular pentagon and the pentagram star.

With an appropriate introduction, more advanced students will enjoy finding the limit of the sequence of ratios of Fibonacci terms as seen in Chapter III and the sum of the infinite series presented in Chapter IV. They might also appreciate the graph of the Golden Group.

Suggestions for Further Study

Many topics related to the Golden Section have not been considered here and those which have been investigated have many known properties which have not been developed in this paper. Interested students may wish to explore the relationship between the Golden Section and topics in the natural sciences and fine arts. The Divine Proportion: A Study in Mathematical Beauty (6) and Fibonacci Numbers by N. N. Vorob'ev (12) are excellent sources for additional information.

Students interested in modern applications may wish to investigate the uses for the Golden Ratio and Fibonacci numbers in computer optimization algorithms. It can be shown that the rate of convergence of the secant method for approximating roots of functions is the Golden Ratio. Another algorithm in numerical analysis, called the Golden Section search, uses the Golden Ratio to reduce intervals over which the search for roots will continue (5 : 642).

Michael L. Fredman and Robert Endre Tarjan developed a new data structure, Fibonacci heaps, with which they were able to improve the running times of several network optimization algorithms. The close relationship between the Fibonacci sequence and the Golden Ratio has been demonstrated. Fredman and Tarjan make use of the relationship to prove the efficiency of Fibonacci heaps (4 : 604).

Another topic worthy of additional research is the relationship between the Golden Ratio and logarithms. The shell of the chambered nautilus grows in an equiangular spiral which is also known as the Golden spiral and the logarithmic spiral. This spiral occurs in other growth patterns as well. This suggests the question of whether the Golden Ratio and logarithms are related in other ways.

Certainly many topics related to the Golden Ratio remain to be investigated. The often surprising occurrence of this number in a wide variety of applications will surely inspire future students of mathematics.

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