A Thesis

Presented to
The Division of Mathematics and Computer Sciences

EMPORIA STATE UNIVERSITY

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Kindra Jo Wells

December, 1991

AN ABSTRACT OF THE THESIS OF

Kindra Jo Wells for the Master of Science in Mathematics

presented on December 2,1991.

Title: MEASURE THEORY IN THE RATIONALS

Abstract Approved: Joe Yanik

For the graduate math student with an interest though little background in measure theory, *Measure Theory in the Rationals* presents a simplified look at the theory's development. The idea, just like measure theory in the reals, is to develop complex measurement ideas using familiar, simple objects. These objects include rational numbers and lengths of intervals of rational numbers. We start with an agreement that we will work only with rational numbers as if irrationals do not exist. From here we seek a function that matches any set of rationals, not just intervals, to a unique number that describes the set's "size". This "size" is called the set's quasi-measure. This sought-after function should have special properties and we set out to find a function which best obtains the ideal properties we have in mind. It's all a matter of give and take as the most ideal properties prove to be impossible to attain at once. What is achieved is summarized and then compared to its parallel in the real numbers. Not only does the reader see similarities in measure theory's development between the rationals and the reals, but sees the contrast between the rationals and reals themselves.

Approved for the Major Division

Committee Member

Committee Member

Committee Member

Committee Chairman

Approved for the Graduate Council

Table of Contents

Chapter		Page
1.	INTRODUCTION	1
2.	MEASURE THEORY IN THE RATIONALS	3
3.	OUTER QUASI-MEASURE	5
4	QUASI-MEASURABLE SETS	13
5.	SUMMARY ON m_q^*	21
6.	MEASURE THEORY IN THE REALS	25

§1 Introduction

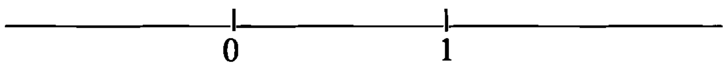
For the Pythagorean school in the sixth century B.C., the idea of number in arithmetic was limited to integers and rationals. Algebraically, they were aware that there is no rational number x which solves the simple equation $x^2 = 2$. Geometrically, due to the Pythagorean theorem applied to the diagonals of a unit square, they had to acknowledge a length x whose square is two. But logically, the existence of such "unutterable" numbers, as the irrationals were called, caused so much anguish that members were sworn to secrecy, forbidden to mention them to outsiders of the school ([Wi], pg. 7). Considering then that they had no real number to measure a very real length, it may be said that the first crisis in mathematics arose from a measurement problem.. As pointed out in the article, "How Good is Lebesgue Measure?" by Krysztof Ciesielski, "the problem of determining the distance between two points, the area of a region, and the volume of a solid are some of the oldest and most important problems in mathematics" ([Ci], pg. 54). It is the basis of what we call measure theory.

The idea is simple. Given a subset of $\mathbf{R}^n$ (where $\mathbf{R}$, as usual, stands for the set of real numbers) we want to assign some number that is the length (for $n = 1$), the area (for $n = 2$), the volume (for $n = 3$) or, more generally, the n-dimensional measure of that subset. The assigned number must describe the size of the set, and this function that associates with subsets of $\mathbf{R}^n$ their measure must have some "good" properties of measure theory (described later). It is from the construction of this function that the technical definition comes. "Mathematical measure theory," states mathematician Joseph Kupla. "is a branch of modern mathematics which deals with systematic techniques for measuring complicated or irregular objects when the measurements of simple objects are known" ([Ku], pg.47).

Though the idea is simple, like all mathematical branches, complicated studies arise. This paper, however, is aimed at the graduate student who is familiar with real analysis and is founded in basic set theory, though has little or no background in measure theory. We will be working strictly in the first dimension though much of what we find generalizes in higher dimensions.

We are going to begin our view of measurement with the same oath of secrecy as the Pythagorean school and deal exclusively in the rationals. We'll argue as the devil's advocate and say that since we can find rational numbers arbitrarily close to any given real number, we can dispense with the nonrationals and continue our study of measure. This study in the rationals achieves several things: one, it parallels yet simplifies H. L. Royden's development of the concept of measure theory in the reals (found in his book, *Real Analysis*); two, it offers evidence of the vast difference between the rationals and the reals; and three, it suggests a pattern for future study.

We shall make use of the geometric representation of the rational numbers as points on a line. Just as with the reals, let $|x| = x$ if $x \geq 0$. $|x| = -x$ if $x < 0$ and call $|x - y|$ the distance between points x and y. Again as usual, an origin is marked on the line to represent the number zero and a second distinct point is marked to represent the number one. Given these two markings, the representation is unique. In discussion of such a line it is natural to draw it across the page with the point one on the right of the origin.



We refer to this line as the rational line and denote it by **Q**, the same symbol used to denote the set of all rationals. Results aren't actually based on this representation since one can always speak in terms of the set **Q** and its elements x, but there are advantages to having both languages. The pictorial rational line is helpful for the intuition such as in arguments involving the ordering of the rationals and also suggests the following terms in a vivid way. Let a and b be rational numbers such that $a \leq b$. The points satisfying $a \leq x \leq b$ form the *closed interval* $[a,b]$, the points x satisfying $a < x < b$ form the *open interval* (a,b), and the points x satisfying $a \leq x < b$, $a < x \leq b$ form the *half-intervals* $[a,b)$, $(a,b]$, respectively. These are *bounded* intervals on **Q** of length $b - a$. We need to augment the rational line with two 'points' at infinity, ∞ and $-\infty$. It is convenient to write $[a, \infty)$, (a, ∞), $(-\infty,a]$, $(-\infty,a)$ for sets defined for $x \geq a$, $x > a$, $x \leq a$, $x < a$, respectively, and $(-\infty,\infty)$ for the whole rational line **Q**; these are *unbounded* intervals. It is now possible to define the extended rationals as the set of all rationals along with $\pm\infty$, denoted by **Q***. Similarly, the extended reals are denoted by **R***.

§2 Measure Theory in the Rationals

The term measure most likely brings to mind the idea of length. In terms of **Q**, the length of an interval is simply the difference of the endpoints. Thus, given the domain of the set of intervals, **I**, on **Q**, a function which assigns to each interval its length is well defined and is an example of a set function: a function that assigns to each set in the domain an extended real number. We want to extend this notion of length and the set function $\mathbb{L}:\mathbf{I}\rightarrow \mathbf{R}^*$ beyond the domain of intervals and look instead for a set function m_q that assigns to a set S of rational numbers a nonnegative extended real number called the quasi-measure of S. In the sense of Kupla's description of measure theory given previously, intervals are for us the simple objects whose measurements are known in advance used to measure more complicated sets. Thus a quasi-measure is defined as follows:

Definition 2.1: A *quasi-measure* of **Q** is a function m_q: $\mathbb{S} \rightarrow \mathbf{R}^*$, where $\mathbb{S} \subseteq \mathbb{P}(\mathbf{Q})$, the set of all subsets of **Q**.

Ideally we would like to see m_q satisfy the following properties:

i. $m_q S$ is defined for any set S of rational numbers. [$\mathbb{S} = \mathbb{P}(\mathbf{Q})$]
ii. $m_q I = \mathbb{L}(I)$ for any interval I.
iii. m_q is *countably additive*. That is, if $\{S_n\}$ is a sequence of disjoint sets, then,

$$m_q\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} m_q(S_n)$$

iv. m_q is *translation invariant*. That is, if S is a set for which m_q is defined and $S+y = \{x + y \mid x \in S\}$ is the set obtained by replacing each x of S by x + y, then $m_q(S+y) = m_q S$.

In addition to these ideal properties, under our oath we would like to say that $\mathbf{Q}^*$ is the range of m_q, but $\mathbf{R}^*$ covers all and is in fact necessary for us given our definition of a particular quasi-measure as we will see later. It will then be worth pointing out a certain generalization of this quasi-measure: the quasi-measure of *any* interval (a,b) as defined on **R** intersected with the rationals will be $b - a$ whether a and b are rationals or not. For now, unless specified otherwise, we will only consider intervals on **Q** where all have rational endpoints.

Again, these are ideal properties; maybe they are reachable, maybe not. But before going on, it might be helpful to look at a concrete example of a quasi-measure. Call it the counting quasi-measure n_q.

Definition 2.2: The *counting quasi-measure* n_q is the function n_q: $\mathbb{P}(\mathbf{Q}) \rightarrow \mathbf{R}^*$ defined by

$$n_q S = \begin{cases} \infty & \text{if S is infinite.} \\ \text{the number of elements in } S & \text{otherwise.} \end{cases}$$

Clearly, n_q is defined on all sets. Also, for a sequence $\{S_n\}$ of disjoint sets, $n_q(\cup S_n) = \Sigma\, n_q(S_n)$ therefore n is countably additive. Furthermore, given any set S, the set S+y, as defined in property (iv) of our ideals, has the same number of elements; thus, n_q is translation invariant. However, $n_q I \neq \mathbb{L}(I)$ for every interval I. For example, $n_q(0,1) = \infty$ while $\mathbb{L}(0,1) = 1$. Thus, n_q is a quasi-measure satisfying all but the property on interval length. The search for one satisfying all 4 properties continues.

§3 Outer Quasi-measure

Let's continue this search by creating a second quasi-measure, then checking its characteristics. For each set S on **Q** consider the finite collections $\{I_n\}$ of open intervals which cover S. The set S is then said to be contained in $\cup I_n$, denoted by $S \subset \cup I_n$. For each such collection consider the sum of the lengths of the intervals. Define m_q^* of S to be the greatest lower bound (inf for infinum) of all such sums. That is,

$$m_q^* S = \inf \left\{ \Sigma \, \ell(I_i) \;/\; S \subset \bigcup_{i=1}^{n} I_i \right\}$$

Call m_q^* the *outer quasi-measure*. By definition, we see several properties of m_q^*, one of which the next proposition shows.

Proposition 3.1: For sets A and B such that $A \subset B$, $m_q^* A \leq m_q^* B$. (This property is called *monotonicity*.)

Proof: Let $\{I_m\}$ be a finite cover for B. Since $A \subset B$, then $\{I_m\}$ is also a finite cover for A. Therefore,

$$\{ \{I_m\} / \; B \subset \bigcup I_m \} \subset \{ \{I_n'\} / A \subset \bigcup I_n' \} \text{ and}$$

$$\inf \{\Sigma \, \ell\{I_n'\} / A \subset \bigcup I_n' \} \leq \inf \{\Sigma \, \ell\{I_m\} / \; B \subset \bigcup I_m \}.$$

Hence, $m_q^* A \leq m_q^* B$ follows. ■

For another property, notice that the outer quasi-measure of the empty set is zero, as is the outer quasi-measure of any set containing just one element (such a set is called a singleton). In fact, as the next proposition shows this property generalizes to any finite set.

Proposition 3.2: If S is finite, $m_q^* S = 0$.

Proof: For $\varepsilon > 0$ and $S = \{s_1, s_2, \cdots s_n\}$ let $I_n' = (s_i - \frac{\varepsilon}{n}, s_i + \frac{\varepsilon}{n})$, $i = 1,2,\ldots,n$. Clearly $\bigcup\{I_n'\}$ is one particular cover of S. Thus,

$$m_q^* S \le \Sigma\, \ell(I_n') = n\left[\frac{\varepsilon}{n}\right] = \varepsilon$$

Since ε is arbitrary, $m_q^* S = 0$. ■

Let's continue to look at the properties of m_q^*. For a moment let's return to the reals in this next proposition and generalize the idea of intervals containing only rational numbers. Let a and b be arbitrary real numbers with $a \le b$ and define the interval (a,b) to be the set of all rational numbers between a and b (similarly, for $[a,b)$ and $(a,b]$). Even with this more general definition of interval, which allows for irrational endpoints, we have the following:

Proposition 3.3: The m_q^*-measure of an interval is its length.

Proof: Case 1. Show $m_q^*[a,b] = b - a$.
Given $\varepsilon > 0$, let a_1 be a rational number such that $a - \frac{\varepsilon}{2} < a_1 < a$ and b_1 a rational such that $b < b_1 < b + \frac{\varepsilon}{2}$. This is possible based on a corollary from the Axiom of Archimedes: between any two reals is a rational. Because the single interval (a_1,b_1) provides a covering for $[a,b]$, then $m_q^*[a,b] \le \ell\,(a_1, b_1) \le \ell\,(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) = b - a + \varepsilon$ for $[a,b]$. This is true for all ε, so $m_q^*[a,b] \le b - a$.

Let $\{I_1, I_2, \ldots I_n\}$ be a finite cover of $[a,b]$. Since a is in this cover, one of the I_i's for $i \in (1, 2, \ldots n)$ must contain a; call it (a_1,b_1). If $b_1 < b$ then, since $b_1 \notin (a_1,b_1)$, there must be a second I_i, (a_2,b_2), such that b_1 is an element and $a_2 \le b_1$. This argument continues until ending with $b \in (a_n,b_n)$. We have then

$$\sum_{i=1}^{n} \ell(I_i) = \sum_{i=1}^{n} \ell(a_i,b_i) = (b_n - a_n) + (b_{n-1} - a_{n-1}) + \cdots + (b_1 - a_1)$$

$$= b_n - (a_n - b_{n-1}) - (a_{n-1} - b_{n-2}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_n - a_1 \qquad \text{since } a_n \le b_{n-1}$$

$$> b - a \qquad \text{since } b_n > b \text{ and } a_1 < a.$$

It follows that since $\Sigma\, \mathbb{L}(I_n) > b - a$ for any cover,

$$m_q^*[a,b] = \inf_{[a,b] \subset \cup I_n} \Sigma\, \mathbb{L}(I_n) \geq b - a$$

and this finishes case 1.

Case 2. Show $m_q^* I = \mathbb{L}(I)$ for any bounded interval.
In this case, I is of the form (a,b), [a,b), (a,b], or [a,b]. Whichever the form, given $\varepsilon > 0$, there is a closed interval J with rational endpoints such that $J \subset I$ and $\mathbb{L}(J) > \mathbb{L}(I) - \varepsilon$. This again is based on the corollary from the Axiom of Archimedes. Thus, refer to a and b as the respective left and right hand endpoints of interval I. Let the left endpoint of J be between a and $a + \frac{\varepsilon}{2}$; likewise, let the right endpoint of J be between b and $b - \frac{\varepsilon}{2}$. Hence $J \subset I$ and $\mathbb{L}(J) > \mathbb{L}(I) - \varepsilon$. With this,

$$\mathbb{L}(I) - \varepsilon < \mathbb{L}(J) = m_q^* J \text{ by case 1 and } m_q^* J \leq m_q^* I \leq m_q^*[a,b] = b - a = \mathbb{L}(I)$$

by the monotonicity of m_q^*. Then $\mathbb{L}(I) - \varepsilon \leq m_q^* I \leq \mathbb{L}(I)$ for every $\varepsilon > 0$ and case 2 is done.

Case 3: Show $m_q^* I = \mathbb{L}(I)$ for any unbounded interval.
Given any rational number Δ, there is a closed interval $J \subset I$ such that $\mathbb{L}(J) = \Delta$. Then $m_q^* I \geq m_q^* J = \mathbb{L}(J) = \Delta$, and this holds for every Δ; thus, $m_q^* I = \infty = \mathbb{L}(I)$. ■

Indeed our quasi-measure has property (ii) of our idealproperties. Even though the endpoints of an interval as defined in the general way are not strictly rationals, we still have that $m_q^* I = \mathbb{L}(I)$ We will now return to intervals as defined before on **Q** with rational endpoints, but notice that this finding shows that our definition of a quasi-measure m_q: $\mathbb{S} \rightarrow$ **R*** rather than m_q: $\mathbb{S} \rightarrow$ **Q*** was not only convenient but necessary for m_q^*. As a consequence, we verify that the infinum function of m_q^* is, as normal, defined over the reals. There are more properties to check.

Proposition 3.4: The m_q^*-measure is translation invariant.

Proof: Let $\{I_n\}$ be any finite cover of a set S. For each n consider the translated interval I_n+y on $\mathbf{Q}$. We have that $\mathbb{L}(I_n) = \mathbb{L}(I_n+y)$ and if $s \in I_n$ then $s + y \in I_n+y$. Thus the set

$$S+y = \{x + y \mid x \in S\} \subset \bigcup_{i=1}^{n} (I_n+y). \text{ Then,}$$

$$m_q^*S = \inf_{S \subset \cup I_n} \Sigma\, \mathbb{L}(I_n) = \inf_{S \subset \cup I_n} \Sigma\, \mathbb{L}(I_n+y) = \inf_{S+y \subset \cup(I_n+y)} \Sigma\, \mathbb{L}(I_n+y) \geq \inf_{S+y \subset \cup(I_n+y)} \Sigma\, \mathbb{L}(I_n') = m_q^*(S+y)$$

where $\{I_n'\}$ represents a finite cover of S+y. So we have

$$m_q^*S \geq m_q^*(S+y)$$

and by making the substitution T = S+y,

$$m_q^*(S+y) = m_q^*(T) \geq m_q^*(T+[-y]) = m_q^*(S).$$

Therefore $m_q^*(S+y) = m_q^*S$. ■

So far we have constructed a quasi-measure which is defined for all sets of rationals, is translation invariant and has the property that the quasi-measure of an interval is the length of the interval. If m_q^* is countably additive, our quasi-measure satisfies the four desired ideals.

Consider the interval (0,1) on $\mathbf{Q}$. Recall that countable sets, like the rationals, are those sets which can be put into one-to-one correspondence with the positive integers. Therefore, we can conveniently use $\{r_i\}_{i=1}^{\infty}$ to enumerate those rationals in (0,1). If we assume countable additivity, we have

$$m_q^*(0,1) = m_q^*\left[\bigcup_{i=1}^{\infty} \{r_i\}\right] = m_q^*\{r_1\} + m_q^*\{r_2\} + \cdots = 0 \qquad (*)$$

It must be that this quasi-measure, although possessing 3 of the 4 ideal properties, is not countably additive. In fact, the generality of the (*) line argument suggests that any quasi-measure which satisfies (i), (ii), and (iv) must fail to be countably additive.

Indeed this is the case. There are three possibilities for the quasi-measure of singletons As in m_q^*, one possibility is the value of zero; perhaps a different quasi-measure gives singletons a value of $\varepsilon > 0$; possibly another quasi-measure assigns an infinite value to each singleton. In any case, translation invariance forces the quasi-measure to assign the same particular value to all singletons. That is, under translation invariance,

$$m_q\{x_1\} = m_q[\{x_1\}+(x_2-x_1)] = m_q\{x_2\} \text{ for } x_1, x_2 \in \mathbf{Q}.$$

Furthermore, a quasi-measure defined for all sets of rationals [property (i)] is certainly defined on (0,1), which may be enumerated as $\{r_i\}_{i=1}^{\infty}$. Thus, following the same argument of (*), the quasi-measure of (0,1) is either zero or infinite if countable additivity is assumed.

$$m_q(0,1) + m_q[\cup\{r_i\}] = m_q\{r_1\} + m_q\{r_2\} + \cdots = \begin{cases} 0 & \text{if } m_q\{r_1\} = 0. \\ \infty & \text{if } m_q\{r_1\} > 0. \end{cases}$$

Any quasi-measure that satisfies (i) and (iv) must fail at either (ii) or (iii).

The above comments can be generalized to any countably infinite set, so we have the following theorem:

Theorem 3.5: If $S \subsetneq \mathbf{R}$ and if m: $\mathbb{P}(S) \rightarrow \mathbf{R}^*$ is countably additive and translation invariant, and if $T \subsetneq S$ is countably infinite then $mT = 0$ or $mT = \infty$.

Thus, if we retain countable additivity on a translation invariant quasi-measure defined on all subsets of **Q** we must give up the property that the measure of an interval is its length. We may now say that it is impossible to obtain a quasi-measure which satisfies all four of our ideals at once. Even stronger, we won't find an m_q which satisfies the first three ideals at one time.

Proposition 3.6: If m_q: $\mathbb{P}(\mathbf{Q}) \rightarrow \mathbf{R}^*$ has the property that for any interval I, $m_q I = \mathbb{L}(I)$, then m_q is not countably additive.

Proof: We'll prove it by contradiction. Assume m_q satisfies the hypothesis and is countably additive. Since $m_q(a,b) = m_q[a,b) = b - a$, it must be that $m_q\{a\} = 0$. Or, in general, $m_q\{x\} = 0$ for all $x \in \mathbf{Q}$. Let $\{r_i\}_{i=1}^{\infty}$ enumerate [a,b]. Then, if we assume countable additivity, $m_q[a,b] = m_q\{r_1\} + m_q\{r_2\} + \ldots = 0$. Hence $m_q[a,b] \neq \mathbb{L}[a,b]$. Therefore m_q is not countably additive.

In conclusion, properties (ii) and (iii) are incompatible for any quasi-measure defined on $\mathbb{P}(\mathbf{Q})$.

An alternative approach is to weaken one of the ideal properties and examine which of the others are maintained or possibly regained. Our initial approach will be to weaken property (iii) of countable additivity to finite additivity.

Definition 3.7: A quasi-measure m_q is *finitely additive* if, given a finite sequence $\{S_1, S_2, \ldots, S_N\}$ of disjoint sets then,

$$m_q\left(\bigcup_{n=1}^{N} S_n\right) = \sum_{n=1}^{N} m_q(S_n)$$

Again the interval (0,1) on $\mathbf{Q}$ provides a counterexample for m_q^*. Let S_1 be the set of all rationals in (0,1) with even denominators and S_2 the set with odd denominators (where all fractions are reduced). Together S_1 and S_2 comprise (0,1) and since, they are disjoint, finite additivity would require that

$$m_q^*(S_1 \cup S_2) = m_q^* S_1 + m_q^* S_2.$$

But every open interval in (0,1) contains an element of S_1, as is with S_2 [S_1 and S_2 are then said to be *dense* in (0,1)]. Thus, every finite cover of S_1 is without gaps in (0,1). In other words, $m_q^* S_1 = 1$. Similarly, $m_q^* S_2 = 1$. The conclusion is

$$m_q^*(0,1) = m_q^*(S_1 \cup S_2) \neq m_q^* S_1 + m_q^* S_2 = 2.$$

Another approach is to weaken property (iii) to *countable subadditivity*. This property says that, given any sequence of sets $\{S_n\}$, then

$$m_q(\cup S_n) \leq \Sigma\, m_q S_n$$

Given the definition of m_q^*, the quasi-measure of the set of integers, **Z**, must be infinite since at least one open interval in any finite cover of **Z** must be unbounded. On the other hand, the infinite sum of the quasi-measure of singletons on **Z** is zero since each one is zero. With $\{x_i\}_{i=1}^{\infty} = \mathbf{Z}$, we see that m_q^* is not countably subadditive because

$$m_q^*[\cup \{x_i\}] = m_q^*\mathbf{Z} = \infty \text{ while } \Sigma m_q^*\{x_i\} = 0.$$

Still another approach is to weaken property (iii) to finite subadditivity.

Proposition 3.8: Given sets $S_1, \ldots, S_N$, then $m_q^*\left(\bigcup_{n=1}^{N} S_n\right) \leq \sum_{n=1}^{N} m_q^*(S_n)$

Proof: If $m_q^* S_n = \infty$ for some $n = 1, 2, \ldots N$ it follows immediately. Suppose $m_q^* S_n < \infty$ for each n and let $\varepsilon_n > 0$. For each set S_n there is a finite collection of open intervals $\{I_{n,i}\}_i$ such that

$$S_n \subset \bigcup_i (I_{n,i}) \quad \text{and} \quad \Sigma\, \ell(I_{n,i}) < m_q^* S_n + \varepsilon_n\,.$$

Now the collection $\{I_{n,i}\}_{n,i}$ is also finite, being the union of a finite collection, and it covers $\cup\, S_n$. Thus, by the definition of m_q^*,

$$m_q^*(\cup S_n) \leq \sum_n \sum_i \ell(I_{n,i}) \leq \sum_n (m_q^* S_n + \varepsilon_n) = \sum_n m_q^* S_n + \sum_n \varepsilon_n.$$

Since ε_n is arbitrary for each n, we have

$$m_q^*(\bigcup S_n) \leq \sum m_q^* S_n.$$

■

We thus have a finitely subadditive, translation invariant quasi-measure defined on all sets that has the property that the measure of any interval is the interval's length. The compromises made on ideal property (iii) allowed the other 3 ideals to stand. As we will see next, however, these compromises on (iii) are not the only approach.

§4 Quasi-measurable Sets

The elimination of sets like S_1 and S_2 defined earlier could gain back some of what was compromised. Our second approach, will be to weaken the first ideal (i), that the quasi-measure be defined for all sets, and again check for different additivities. Note that S_2 can also be defined as the set of all rationals in (0,1) that aren't in S_1; S_2 is then called the *complement* of S_1 [in (0,1)], denoted by $\widetilde{S_1}$. A good way, though not exactly intuitive, to eliminate the S_1/S_2 situation is to use an idea of Caratheodory's which guarantees that the sum of the parts is equal to the whole. We can reduce the family of sets for which m_q^* is defined to those sets meeting the following definition:

Definition 4.1: The set S is said to be *quasi-measurable* if, given any set A, $m_q^*A = m_q^*(A \cap S) + m_q^*(A \cap \widetilde{S})$.

Since $S_2 = \widetilde{S_1} \cap (0,1)$ we see that $m_q^*(0,1) \neq m_q^*[(0,1) \cap S_1)] + m_q^*[(0,1) \cap \widetilde{S_1}]$. Thus, S_1 is an example of a nonquasi-measurable set. Let's now re-examine m_q^* in view of a domain restricted to quasi-measurable sets on **Q**.

There are several facts to notice given this definition. First, to prove a set S quasi-measurable, it is sufficient to show that for any set A, $m_q^*A \geq m_q^*(A \cap S) + m_q^*(A \cap \widetilde{S})$ since the opposite inequality is true by the finite subadditivity of m_q^*. Second, the definition is symmetric; if S is quasi-measurable, $\widetilde{S}$ is too. Third, both **Q** and ∅, the empty set, are quasi-measurable. Fourth, all sets of quasi-measure 0 are quasi-measurable, as we see from the following:

Lemma 4.2: If $m_q^*S = 0$, then S is quasi-measurable.

Proof: Since $(A \cap S) \subset S$, then by proposition 1, $m_q^*(A \cap S) \leq m_q^*S = 0$.
So $m_q^*(A \cap S) = 0$.
Also $A \supset A \cap \widetilde{S}$, thus $m_q^*A \geq m_q^*(A \cap \widetilde{S})$.
Therefore $m_q^*A \geq m_q^*(A \cap S) + m_q^*(A \cap \widetilde{S})$. ■

Furthermore, the union of two quasi-measurable sets is also quasi-measurable.

Lemma 4.3: If S_1 and S_2 are quasi-measurable, so is $S_1 \cup S_2$.

Proof: Let A be any set. The quasi-measurability of S_2 implies that

$$m_q^*(A \cap \widetilde{S_1}) = m_q^*(A \cap \widetilde{S_1} \cap S_2) + m_q^*(A \cap \widetilde{S_1} \cap \widetilde{S_2})$$

Also, since $A \cap (S_1 \cup S_2) = (A \cap S_1) \cup (A \cap S_2 \cap \widetilde{S_1})$, we have

$$m_q^*[A \cap (S_1 \cup S_2)] \leq m_q^*(A \cap S_1) + m_q^*(A \cap S_2 \cap \widetilde{S_1}).$$

Thus,

$$m_q^*[A \cap (S_1 \cup S_2)] + m_q^*(A \cap \widetilde{S_1} \cap \widetilde{S_2})$$

$$\leq m_q^*(A \cap S_1) + m_q^*(A \cap S_2 \cap \widetilde{S_1}) + m_q^*(A \cap \widetilde{S_1} \cap \widetilde{S_2})$$

$$= m_q^*(A \cap S_1) + m_q^*(A \cap \widetilde{S_1}) = m_q^* A$$

since S_1 is quasi-measurable. Because $\sim(S_1 \cup S_2) = \widetilde{S_1} \cap \widetilde{S_2}$, $S_1 \cup S_2$ is proven quasi-measurable. ■

As stated earlier we have that $\widetilde{S}$ is in our domain whenever S is (closure under complements). By lemma 13, we have that $A \cup B$ is in our domain whenever A and B are (closure under unions). A collection of subsets possessing these two properties is call an *algebra* of sets. Note that closure under both complements and unions implies closure under intersection since $A \cap B = \sim(\widetilde{A} \cup \widetilde{B})$. Consequently we have the following corollary:

Corollary 4.4: The family $\mathbb{M}_q$ of quasi-measurable sets is an algebra of sets.

It's encouraging that our compromise of the first ideal led to an algebra; we at least have finite unions, intersections and complements to work with. What's more is the gain made in additivity.

Lemma 4.5: Let A be any set and $S_1, S_2, \ldots S_N$ a finite sequence of disjoint quasi-measurable sets. Then

$$m_q^*(A \cap \left[\bigcup_{i=1}^{N} S_i\right]) = \sum_{i=1}^{N} m_q^*(A \cap S_i)$$

Proof: It's definitely true for N = 1. Assume it's true for N – 1 of the S_i sets. Since the sets are disjoint,

$$A \cap \left[\bigcup_{i=1}^{N} S_i\right] \cap S_N = A \cap S_N \qquad \text{and}$$

$$A \cap \left[\bigcup_{i=1}^{N} S_i\right] \cap \widetilde{S_N} = A \cap \left[\bigcup_{i=1}^{N-1} S_i\right]$$

thus, by the quasi-measurability of S_N,

$$m_q^*(A \cap \left[\bigcup_{i=1}^{N} S_i\right]) = m_q^*(A \cap S_N) + m_q^*(A \cap \left[\bigcup_{i=1}^{N-1} S_i\right])$$

$$= m_q^*(A \cap S_N) + \sum_{i=1}^{N-1} m_q^*(A \cap S_N)$$

$$= \sum_{i=1}^{N} m_q^*(A \cap S_i). \qquad \blacksquare$$

Now, letting A in the above lemma be **Q**, we have:

Corollary 4.6: If $\{S_1, S_2, \ldots S_N\}$ is a finite sequence of disjoint quasi-measurable sets,

$$m_q^*\left[\bigcup_{i=1}^{N} S_i\right] = \sum_{i=1}^{N} m_q^* S_i.$$

And indeed, a gain from finite subadditivity to finite additivity is made in the domain restriction of $\mathbb{M}_q$.

It is possible to compromise the first ideal in a different way and retain completely the last three ideals. However, the domain of such a quasi-measure is not optimal in that it is not defined on singletons. For example, let $\mathbb{I}_i$ for i = 1,2,3,... denote disjoint unions of intervals. Then the set function $\mathbb{L}$ applied to this domain of disjoint unions of intervals, achieves the following:

ii. $\mathbb{L}(\mathbb{I}) = \mathbb{L}(\mathbb{I})$ for any interval $\mathbb{I}$

iii. $\mathbb{L}\left(\bigcup_{i=1}^{\infty} \mathbb{I}_i\right) = \sum_{i=1}^{\infty} \mathbb{L}(\mathbb{I}_i)$

iv. $\mathbb{L}(\mathbb{I}+y) = \mathbb{L}(\mathbb{I})$.

But, if $\mathbb{L}(\{r\})$ is defined for every $r \in \mathbb{I}$ then, letting $\{r_i\}_{i=1}^{\infty}$ enumerate $\mathbb{I}$, $\mathbb{L}(\mathbb{I}) = \mathbb{L}(\cup\{r_i\}) = 0$ or ∞ by Theorem 2.5. The contradiction of condition (ii) shows that we can not extend $\mathbb{L}$ to be defined on singletons if (ii), (iii), and (iv) are to be satisfied. And in general we have:

Proposition 4.7 A translation invariant, countably additive quasi-measure with the property that the quasi-measure of an interval is its length is not defined on singletons.

Put simply, a quasi-measure satisfying the last two of our ideal properties is not defined on intervals and singletons simultaneously. The restricted domain of $\mathbb{M}_q$ for m_q^* is preferred.

This restricted domain needs a closer examination. We've seen that it includes the empty set, singletons, and other sets of quasi-measure zero. The rationals, **Q**, are included and the next lemma helps show that the integers, **Z**, are quasi-measurable.

Lemma 4.8: For some set A, if $m_q^*A < \infty$, there exists two integers M_1 and M_2 such that $M_1 < x < M_2 \ \forall \ x \in A$.(That is, $A \subset [M_1,M_2]$.)

Proof: Assume for every integer M_1, there is an $x \in A$ such that $x < M_1$. Then any cover of A must extend to $-\infty$, and since we are considering finite covers, at least one of the open intervals is unbounded. That means $m_q^*A = \infty$, a contradiction. Then there is an M_1 such that $x \geq M_1 \; \forall \; x \; \varepsilon \; A$. Similarly, there is an M_2 with $x \leq M_2 \; \forall \; x \; \varepsilon \; A$. ■

This sets up our next proposition.

Proposition 4.9: The integers are quasi-measurable.

Proof: For some set A consider the statement

$$m_q^*A \geq m_q^*(A \cap \mathbf{Z}) + m_q^*(A \cap \widetilde{\mathbf{Z}}) \qquad (1)$$

If $m_q^*A = \infty$, then (1) is definitely true.
If $m_q^*A < \infty$, let M_1 and M_2 be as in Lemma 18, then

$$A \cap \mathbf{Z} \subseteq \{x \mid M_1 \leq x \leq M_2, x \in \mathbf{Z}\}$$

so $\quad m_q^*(A \cap \mathbf{Z}) \leq m_q^*\{x \mid M_1 \leq x \leq M_2; x \; \varepsilon \; \mathbf{Z}\} = 0 \qquad (2)$

and $\quad m_q^*A \geq m_q^*(A \cap \widetilde{\mathbf{Z}}) \qquad$ since $A \supset (A \cap \widetilde{\mathbf{Z}})$.

thus again (1) is true. Hence the integers are quasi-measurable. ■

The intersection of the integers with any finite interval on **Q** is a finite set and, therefore, of quasi-measure zero. This idea, the basis of Proposition 4.9 is generalized a bit more in Proposition 3.9.

Proposition 4.10: If S is a set such that $m_q^*\{[-M,M] \cap S\} = 0 \; \forall \; M$, then S is quasi-measurable.

Proof: Again we are to verify line (1) of the proof of Proposition 4.9. If $m_q^*(A) = \infty$, then we're done. Otherwise $A \subset [-M_1,M_1]$ for some integer M_1 and $m_q^*([-M_1,M_1] \cap S) = 0$. Then $m_q^*(A \cap S) = 0$ by monotonicity. For the same reason $m_q^*A \geq m_q^*(A \cap \widetilde{S})$. Thus, line (1) is valid and S is quasi-measurable. ■

So far we have found quasi-measurable sets whose quasi-measure is either zero or infinity. The quasi-measurability of intervals, if proven, would offer sets of any rational quasi-measure.

Proposition 4.11: The interval (a,∞) is quasi-measurable.

Proof: Let A be any set. We need to show

$$m_q^*A \geq m_q^*\{A \cap (a,\infty)\} + m_q^*\{A \cap (-\infty,a]\}.$$

For convenience let $A_1 = A \cap (a,\infty)$ and $A_2 = A \cap (-\infty,a]$.
If $m_q^*A = \infty$ then we're done. Suppose $m_q^*A < \infty$ and $\varepsilon > 0$. There exists a finite collection of open intervals $\{I_n\}$ such that

$$A \subset \bigcup_{i=1}^{n} I_i \quad \text{and} \quad \sum_{i=1}^{n} \ell(I_i) \leq m_q^*A + \varepsilon$$

Let $I_n' = I_n \cap (a,\infty)$ and $I_n'' = I_n \cap (-\infty,a]$.

These are intervals (or empty), not necessarily open, and

$$\ell(I_n) = \ell(I_n') + \ell(I_n'') = m_q^*(I_n') + m_q^*(I_n'').$$

Since $A_1 \subset \bigcup I_n'$, then $m_q^*A_1 \leq m_q^*(\bigcup I_n') \leq \sum m_q^*(I_n')$.

Likewise $A_2 \subset \bigcup I_n''$, then $m_q^*A_2 \leq m_q^*(\bigcup I_n'') \leq \sum m_q^*(I_n'')$.

Thus, $m_q^*A_1 + m_q^*A_2 \leq \sum(m_q^*I_n' + m_q^*I_n'') = \sum \ell(I_n) < m_q^*A + \varepsilon$.

Since ε is arbitrary, $m_q^*A_1 + m_q^*A_2 \leq m_q^*A$, and the interval (a,∞) is shown quasi-measurable. ■

Because $\mathbb{M}_q$ is an algebra of sets, the quasi-measurability of (a,∞) implies that its complement, $(-\infty,a]$, is also quasi-measurable. Hence the finite union $(-\infty, a] \cup (a,\infty) = (-\infty,\infty)$ is as well. Since singletons are quasi-measurable, we have that $\{a\} \cup (a,\infty) = [a,\infty)$ is quasi-measurable as is its complement, $(-\infty,a)$. Thus, for $c < a$, we see the quasi-measurability of $(c,\infty) \cap (-\infty,a) = [c,a)$. For $a < b$, $(a,\infty) \cap (-\infty,b] = (a,b]$ and $[\{a\} \cup (a,\infty)] \cap (-\infty,b] = [a,b]$ and $(a,\infty) \cap (-\infty,b) = (a,b)$ are all quasi-measurable. Every interval is quasi-measurable.

Let's examine the third property, translation invariance. By Proposition 3.4, it was shown that $m_q^*(S+y) = m_q^*S$ for any set S. The next proposition assures us that S in $\mathbb{M}_q$ implies S+y is also an element of $\mathbb{M}_q$. First, let's establish a helpful lemma.

Lemma 4.12: Given sets A and B and $y \in \mathbf{Q}$,

1. $(A \cap B) + y = (A+y) \cap (B+y)$
2. $\widetilde{A}+y = \widetilde{A+y}$

Proof: 1. Let $x \in (A+y) \cap (B+y)$. Then $x \in A+y$ and $x \in (B+y)$. So $x = x' + y$ for some $x' \in A$ and $x = x'' + y$ for some $x'' \in B$. It must be that $x' = x''$, so $x' \in A \cap B$ and $x = x' + y \in (A \cap B)+y$.

Let $x \in (A \cap B) + y$, then $x = x' + y$ where $x' \in A \cap B$. So $x' \in A$ and $x' \in B$, then $x' + y \in A + y$ and $x' + y \in B+y$. Hence, $x = x' + y \in (A+y) \cap (B+y)$.

2. Let $x \in \widetilde{A}+y$, then $x = x' + y$ for some $x' \in \widetilde{A}$. So, $x' \notin A$ then $x = x' + y \notin A+y$. Hence $x \in \widetilde{A+y}$.

Let $x \in \widetilde{A+y}$, then $x \notin A+y$ and so there does not exist an $x' \in A$ such that $x = x' + y$. Hence $x - y \notin A$. In other words, $x - y \in \widetilde{A}$. Thus $x = (x - y) + y \in \widetilde{A}+y$. ■

Now the proposition follows:

Proposition 4.13: If S is a quasi-measurable set, then each translate S+y of S is also quasi-measurable.

Proof: By the quasi-measurability of S we know, given any set A,

$$m_q^* A = m_q^*(A \cap S) + m_q^*(A \cap \widetilde{S}) \qquad (3)$$

Since m_q^* is translation invariant, we know

$$m_q^* A = m_q^*(A+y),$$
$$m_q^*(A \cap S) = m_q^*[(A \cap S)+y]$$
and $$m_q^*(A \cap \widetilde{S}) = m_q^*[(A \cap \widetilde{S})+y].$$

Thus, applying this along with Lemma 3.11 to line (3), we have

$$\begin{aligned} m_q^*(A+y) &= m_q^*[(A \cap S)+y] + m_q^*[(A \cap \widetilde{S})+y] \\ &= m_q^*[(A+y) \cap (S+y)] + m_q^*[(A+y) \cap (\widetilde{S}+y)] \\ &= m_q^*[(A+y) \cap (S+y)] + m_q^*[(A+y) \cap (\widetilde{S+y})]. \end{aligned}$$

Then by making the substitution $A = A' - y$, where A' is any set, we have

$$m_q^* A' = m_q^*[A' \cap (S+y)] + m_q^*[A' \cap (\widetilde{S+y})]$$

Hence S+y is another quasi-measurable set. ■

§5 Summary on m_q^*

Let's gather what we have seen so far in the rationals. We began with a search to find a quasi-measure m_q satisfying four most desirable properties:

i. m_qS is defined for all sets S
ii. $m_qI = \ell(I)$ for any interval I
iii. m_q is countably additive
iv. m_q is translation invariant.

The search for one possessing all four ideals proved futile, as did the chance of finding one satisfying the first three. While it is possible to construct a quasi-measure with properties (ii) - (iv), for example length restricted to the domain of disjoint unions of intervals (denoted by $\mathbb{I}$), the domain is less than desirable. We've seen that the counting measure n_q (Definition 2) possesses (i), (iii) and (iv) but lacks (ii). That, for us, is too much to lose. Recall from the beginning of section 1 that we are trying to generalize from the idea of length of an interval. So naturally, we want the quasi-measure of an interval to be the length of the interval, even if this demands compromises on other properties. The quasi-measure m_q^* satisfies (ii) as well as (i) and (iv) but is limited to finite subadditivity. However, a restriction in property (i) to the algebra of quasi-measurable sets $\mathbb{M}_q$ advances finite subadditivity to finite additivity. These finding are summarized in the following table where T indicates that the quasi-measure has that particular property.

IMPOSSIBLE		COUNTING MEASURE	DEFINED ON AN INTERVAL
i. T	i. T	i. T	i. not on singletons
ii. T	ii. T	ii.	ii. T
iii. T	iii. T	iii. T	iii. T
iv. T	iv.	iv. T	iv. T

m_q^*	m_q^* on $\mathbb{M}_q$
i. T	i. an algebra
ii. T	ii. T
iii. finite subadd.	iii. finite add.
iv. T	iv. T

It seems that m_q^* most favorably reaches toward our original ideals. As we have seen, all finite sets are of m_q^*-measure zero. The m_q^*-measure of the two infinite sets, **Z** and **Q** are both infinite. And since $m_q^*(I) = \ell(I)$ for any interval I, we have quasi-measurable sets of every quasi-measure between 0 and ∞. Furthermore, there exist infinite sets of quasi-measure zero

Example: The m_q^*-measure of the infinite sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is zero.

Proof: Let $\varepsilon > 0$. Choose N such that $\frac{1}{N} < \varepsilon$.

$$m_q^*\left[\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right] = m_q^*\left(\left[\left\{\frac{1}{n}\right\}_{n=1}^{N}\right] \cup \left[\left\{\frac{1}{n}\right\}_{n=N+1}^{\infty}\right]\right)$$

$$\leq m_q^*\left[\left\{\frac{1}{n}\right\}_{n=1}^{N}\right] + m_q^*\left[\left\{\frac{1}{n}\right\}_{n=N+1}^{\infty}\right] \text{ by finite subadditivity}$$

$$= 0 + m_q^*\left[\left\{\frac{1}{n}\right\}_{n=N+1}^{\infty}\right] \text{ since } \left\{\frac{1}{n}\right\}_{n=1}^{N} \text{ is finite and}$$

$$< \varepsilon \text{ since } \left\{\frac{1}{n}\right\}_{n=N+1}^{\infty} \subset (0,\tfrac{1}{N}) \text{ and } m_q^*(0,\tfrac{1}{N}) = \frac{1}{N} < \varepsilon.$$

With ε arbitrary, this means that $m_q^*\left[\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right] = 0.$ ■

Thus, m_q^* offers a complete range of quasi-measure. Not only does m_q^*, which is defined on *all* sets, offer a complete range of quasi-measure, but the *quasi-measurable* sets offer a complete range. Perhaps we were lucky with our construction, perhaps selective. Let's re-examine m_q^*.

Recall the definition of m_q^*, the infinum of sums of interval lengths taken over all finite covers. Since this definition came without introduction or explanation, there may be questions over its development rather than that of another defined function. One question may concern the use of open interval covers. Their use in the definition of a quasi-measure is a logical extension of the set function, length, whose domain is the set of all intervals. This, of course, was our initial intention -- to extend the notion of length on intervals to a set function defined on a more complicated domain. Another question may be over the use of finite covers rather than infinite. That is, we could have used the following definition of a quasi-measure:

$$m_q^{\infty} S = \inf \left\{ \Sigma\, \ell(I_i) \ / \ S \subseteq \bigcup_{i=1}^{n} I_i \right\} \qquad (3)$$

where we allow the cover to be a countably infinite collection of intervals. This however would be a rather dull choice considering all sets would be of quasi-measure zero.

Lemma 5.1: If S is countable then $m_q^{\infty} S = 0$.

Proof: Let S be a countable set, $\{s_i\}_{i=1}^{\infty}$ an enumeration of its elements, and $\varepsilon > 0$. Define a sequence of intervals as such:

$$I_1 = (s_1 - \frac{\varepsilon}{2^2}, s_1 + \frac{\varepsilon}{2^2}) \ ; \ \ell(I_1) = \frac{\varepsilon}{2}$$

$$I_2 = (s_2 - \frac{\varepsilon}{2^3}, s_2 + \frac{\varepsilon}{2^3}) \ ; \ \ell(I_2) = \frac{\varepsilon}{2^2}$$

.

.

.

$$I_j = (s_j - \frac{\varepsilon}{2^{j+1}}, s_j + \frac{\varepsilon}{2^{j+1}}) \ ; \ \ell(I_j) = \frac{\varepsilon}{2^j}.$$

.

.

.

Now $S \subset \bigcup I_j$ and $m_q^{\infty}(\bigcup I_j) \leq \sum \ell(I_j) = \sum \frac{\varepsilon}{2^j} = \varepsilon$

Since ε is arbitrary, $m_q^{\infty}(\bigcup I_j) = 0$. And it follows that $m_q^{\infty} S \leq m_q^{\infty}(\bigcup I_j) = 0$,

thus $m_q^{\infty} S = 0$. ■

Since we are working in the rationals, all subsets are countable and thus for every set we have a quasi-measure of zero for m_q^{∞}. This is a vast difference: the range of m_q^* is 0 to ∞ on countable sets while the range of m_q^{∞} is 0 on countable sets. It is a difference we will examine more closely in the reals. While in the rationals we have played a game of give and take of ideals. When all of what we wanted was impossible we made compromises, as little as possible to hold as much as possible. We discovered some impossibilities, some limitations, some favorable properties, and perhaps further study on a different quasi-measure would bring improvements over what we have we found in m_q^*. This is basically what measure theory is. A look at Royden's game of give and take in the reals will shed more light on the heart of measure theory.

§6 Measure Theory in the Reals

Royden's introduction of measure theory in *Real Analysis* parallels what we have done in the rationals. He states that the idea is to construct a set function like length but which goes beyond the domain of just intervals on **R**, and instead, "assigns to each set E in some collection $\mathbb{M}$ of sets of real numbers a nonnegative extended real number mE called the "measure of E" ([Ro], pg. 52]. Ideally m would have the following properties:

i. $\mathbb{M} = \mathbb{P}(\mathbf{R})$

ii. $mI = \mathbb{L}(I)$, for any interval I

iii. m is countably additive

iv. m is translation invariant.

Let's follow Royden's game of give and take. First, we may note that the counting measure n_q defined on **Q** continues to have properties (i), (iii), and (iv) when redefined on **R**. Call it n. Thus n is an example of a measure holding 3 of the 4 ideal properties. However, we wish (ii) to hold. Next consider for each set E of real numbers the countable collections $\{I_n\}$ of open intervals which cover E [$E \subset \cup I_n$]. For each such collection consider the sum of the lengths of the intervals in the collection. We define the *outer measure* m*E of E to be the following:

$$m^*E = \inf_{E \subset \cup I_n} \sum \mathbb{L}(I_n)$$

That is, the outer measure is the infinum of all considered sums.

Now if we would pursue this definition like that of our outer quasi-measure, as Royden shows, we would find (1) m* is defined on all sets, (2) the m* of an interval is the length of the interval, (3) m* is countably subadditive, and (4) m* is translation invariant. It's interesting to compare the differences between the proofs of outer quasi-measure properties in **Q** and those proofs of outer measure properties in **R**. Of course, this is a reflection of the difference between **Q** and **R**. Lemma 6 points out such a difference. It follows from this lemma that m*E = 0 whenever E is countable. Then m***Q** = 0. That means the entire number system of the Pythagorean school has a measure of zero. Now to them that would have been unutterable: with all rationals removed from the real number line, the line would remain unchanged from the point view of measure theory. That's quite a demonstration of the difference between the reals and the rationals. At any rate, m* again has 3 of the 4 ideal properties; it lacks countable additivity.

If our domain is of sets which satisfy the same definition of quasi-measurability given for the rationals (Def. 4), then m* becomes countably additive. In **R** we say that such sets are measurable. The family of measurable sets, besides bringing countable additivity to m*, is a σ-algebra. That is, the family is closed under countable unions (verses finite unions for an algebra) and under complementation. Hence, the domain is favorable and properties (ii)-(iv) hold for m*. It nearly meets the original ideals. If all sets were measurable [if $\mathbb{M} = \mathbb{P}(\mathbf{R})$], then m* would meet them. The construction of m* is credited to Henri Lebesgue. In general, the construction of functions which reach toward the 4 ideals, as Ciesielski states in his article, has been in question since the beginning of the nineteenth century ([Ci], pg. 54). It was examined by several well known mathematicians, but Lebesgue's solution is now considered to be the best answer, though the question is not completely solved even today.

As mentioned above, if $\mathbb{M} = \mathbb{P}(\mathbf{R})$ then the search for our ideal function is complete. Unfortunately, this is not the case. In 1905, Giuseppe Vitali constructed a subset of **R** that is not in $\mathbb{M}$. This proof of an existing nonmeasurable set, shown by Royden, is fairly easy to follow and from it, it implies that no measure can simultaneously satisfies all 4 ideals. Vitali's proof, we may note, is dependent on the Axiom of Chioice, which Royden states as follows:

Axiom of Choice (AC): Let $\mathbb{C}$ be any collection of nonempty sets. Then there is a function F defined in $\mathbb{C}$ which assigns to each set $A \in \mathbb{C}$ an element F(A) in A.

Early in the twentieth century AC was not commonly accepted. Hence Lebesgue questioned Vitali's construction. Today AC is generally accepted; Vitali's proof stands firm. But even more, Robert Solovay showed in 1964 that we can not prove $\mathbb{M} \neq \mathbb{P}(\mathbf{R})$ without AC. Briefly stated, if AC is not accepted, he proved that there is a "mathematical world" where all subsets of **R** are Lebesgue measurable, i.e. $\mathbb{M} = \mathbb{P}(\mathbf{R})$. This world, though it denies power to AC, holds true with a related axiom. It has a disadvantage however in that Solovay's proof uses and must use an additional controversial axiom in set theory ([Ci], pg. 55).

Let's go back to accepting the Axiom of Choice. Since $\mathbb{M} \neq \mathbb{P}(\mathbf{R})$, an appropriate step is to see if we can improve, or extend, Lebesgue measure. Can we find a measure u such that for a σ-algebra of sets, properties (ii), (iii), and (iv) hold and $\mathbb{M} \subset \mathbb{M}^*$? Such functions u: $\mathbb{M}^* \rightarrow [0,\infty)$ are called extensions of Lebesgue measure. Any function defined on a σ-algebra and satisfying (ii)-(iv) is called an *invariant measure.*

Members of the Polish Mathematical School proved several results concerning extensions. For one, there does exist an extension of Lebesgue measure and it is an invariant measure. For another, there is no final extension that outdoes all others as far as domain size (there is no maximal invariant measure). Thus if we restrict our view to invariant measures, then Lebesgue measure is not the richest. However, the same defect exists for any other invariant measure. Thus, by only comparing the sizes of domains of invariant measures there is no best answer. If we carefully restrict our view without the Axiom of Choice then all sets are Lebesgue measurable ([Ci], pg. 56). So, this fact along with the natural ease of the m* construction, makes Lebesgue's measure a great unique candidate.

Of course, like in the rationals, there are alternatives besides just compromising property [(i)]. One idea is to drop translation invariance and construct a countably additive [(iii)] measure defined on all sets [(i)] where the measure of an interval is the interval's length [(ii)]. This construction is possible but besides losing translation invariance we must additionally assume a very strong axiom that mathematicians usually don't accept [the continuum hypothesis]. This is a high price to pay. So another idea is to weaken countable additivity to finite additivity. Does there exist a finitely additive translation invariant measure defined on all sets? In 1923, Stefan Banach proved that on a plane ($n = 2$) and on a line ($n = 1$) such a measure exists. But due to Banach and Alfred Tarski in 1928, we know there is no such measure for $n \geq 3$ ([Ci], pg. 57). Before continuing, let's summarize the give and take of measures in **R** with the following:

IMPOSSIBLE	NOT KNOWN	COUNTING MEASURE
i. T	i. T	i. T
ii. T	ii. T	ii.
iii. T	iii. T	iii. T
iv. T	iv.	iv. T

m*	m* on $\mathbb{M}$
i. T	i. σ-algebra
ii. T	ii. T
iii. Subadditivity	iii. T
iv. T	iv. T

<u>FOR R^1 AND R^2</u>

i. T

ii. T

iii. finite addivity

iv. T

Banach and Tarski's surprising finding that a finitely additive translation invariant measure does not exist for $n \geq 3$ leads to a more startling conclusion known as the Banach-Tarski Paradox. It is often stated in expressive form: "a pea may be taken apart into finitely many pieces that may be rearranged using rotations and translation to form a ball the size of the sun" ([Wa], pg. 3). This result is probably as discomforting to us now as the unutterable length of $\sqrt{2}$ was to Pythagorean school. It is also one of the strongest arguments against the use of the Axiom of Choice since the construction depends on it. Here is where much future study awaits and where this paper ends. We began with what appeared to the Pathagoreans as a paradox but instead was a need for more information. We'll end with this new paradox that awaits more information. Enlarging the number system from **Q** to **R** solved the first paradox. Perhaps, an enlargement in our notion of volume will solve the second. Together, it's a continued study in the heart of measure theory.

§ Bibliography

[Ci] K. Ciesielski, How Good is Lebesque Measure?, *Math. Intel.*, 11(1989), 54-58.

[Ku] J. Kupka, Measure Theory: The Heart of the Matter, *Math. Intel.*, 8(1986), 47-56.

[Ro] H.L. Royden, *Real Analysis* (2 ed.), New York, Macmillan, 1968.

[Wa] Stan Wagon, *The Banach-Tarski Paradox*, Cambrigde Univ. Press, 1985.

[Wi] Alan J. Wier, *Lebesgue Integration and Measure*, Cambridge Univ. Press, 1973.