ABSTRACT

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Title Introduction to Lie Algebras.

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The purpose of this thesis is to introduce the basic ideas of Lie algebras to the reader with some basic knowledge of abstract and elementary linear algebra.

In this study, Lie algebras are considered from a purely algebraic point of view, without reference to Lie groups and differential geometry. Such a view point has the advantage of going immediately into the discussion of Lie algebras without first establishing the topological machineries for the sake of defining Lie groups from which Lie algebras are introduced.

In Chapter I we summarize for the reader's convenience rather quickly some of the basic concepts of linear algebra with which he is assumed to be familiar. In Chapter II we introduce the language of algebras in a form designed for material developed in the later chapters.

Chapters III and IV were devoted to the study of Lie algebras and the Lie algebra of derivations. Some definitions, basic properties, and several examples are given. In Chapter II we also study the Lie algebra of antisymmetric operators, Ideals and homomorphisms. In Chapter III we introduce a Lie algebra structure on $Der_F(A)$ and study the link between the group of automorphisms of A and the Lie algebra of derivations $Der_F(A)$.

INTRODUCTION TO LIE ALGEBRAS

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CHAPTER ONE

INTRODUCTION

I.I FUNDAMENTAL CONCEPTS OF VECTOR SPACES

The object of this introductory chapter is to provide a short account of the foundations of Linear Algebra that we need in later chapters. Various results are given without proofs and others are given with only sketchy arguments.

DEFINITIONS AND EXAMPLES

DEFINITION 1: Let F be a field. A vector space over F is a set V whose elements are called vectors, together with two operations. The first operation is called vector addition, it assigns to each pair of vectors $v,w \in V$ a vector, denoted by $v+w \in V$. The second operation called scalar multiplication, assigns to each scalar $\alpha \in F$ and vector $v \in V$ a vector denoted by $\alpha v \in V$, such that the following conditions are satisfied:

1. $(u+v) + w = u + (v+w)$, for all $u,v,w \in V$.
   (i.e. addition is associative).
2. There is an element of V, denoted by 0 and is called zero vector, such that
   $0 + u = u + 0 = u$, for all $u \in V$.
3. For each vector $u \in V$, there exists an element, denoted by $-u \in V$ such that
   $u + (-u) = 0$
4. $u + v = v + u$ for all $u,v \in V$.
   (i.e. addition is commutative)
5. $\alpha(u+v) = \alpha u + \alpha v$ for all $u,v \in V$ and $\alpha \in F$.
   (i.e. scalar multiplication is distributive with respect to vector addition).
6. $(\alpha+\beta)u = \alpha u + \beta u$ for all $u \in V$, and $\alpha,\beta \in F$.
   (i.e. scalar multiplication is distributive with respect to scalar addition).
7. $(\alpha\beta)u = \alpha(\beta u)$ for all $u \in V$, and $\alpha,\beta \in F$.
8. $1u = u$ for all $u \in V$.
   (1 here is the multiplicative identity of F).

<u>REMARK</u> 1: In the sequal F will be either the field of real numbers or the field of complex numbers.

<u>REMARK</u> 2: Conditions 1-4 in the definition of vector space is equivalent to say with respect to vector addition, V is an abelian group.

<u>REMARK</u> 3: When no confusion is to be feared a vector space over F will be simply called a vector space.

__EXAMPLE__ 1: Let F be any field. $V = \{(\alpha_1, \ldots, \alpha_n): \alpha_i \in F, 1 <= i <= n\}$. Define a vector addition in V by:

$(\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n) = (\alpha_1+\beta_1, \ldots, \alpha_n+\beta_n)$.

Define a scalar multiplication by:

$\gamma(\alpha_1, \ldots, \alpha_n) = (\gamma\alpha_1, \ldots, \gamma\alpha_n)$ for all $\gamma \in F$.

Then with respect to these operations, V becomes a vector space over F, we denote this vector space by F^n. In particular R^n is a vector space over R and C^n is a vector space over C.

__EXAMPLE__ 2: Let F be any field. Let $Mat_{mxn}(F)$ be the set of all mxn matrices with entries in F. Let $A = (a_{ij})$ and $B = (b_{ij})$ be elements in $Mat_{mxn}(F)$. We define their sum A+B to be the matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then one can immediately verify that $Mat_{mxn}(F)$ is an abelian group under this addition. Let $\alpha \in F$. We define αA, the scalar multiple of A by α to be the matrix $C = (c_{ij})$, where $c_{ij} = \alpha a_{ij}$. Then it can be verified that $Mat_{mxn}(F)$ is a vector space over F.

DEFINITION 2: Let V be a vector space over F. A subset W of V is called a subspace of V if W is itself a vector space over F under the same operations of vector addition and scalar multiplication of V.

THEOREM 1: A nonempty subset W of a vector space V is a subspace of V if and only if for all $w_1, w_2 \in W, \alpha \in F$, we have $w_1 + w_2 \in W$ and $\alpha w_1 \in W$.

I.II LINEAR INDEPENDENCE AND BASES

DEFINITION 1: If $v_1, v_2, \ldots, v_n$ is a set of vectors in a vector space V over F, an expression of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n ,$$

where $\alpha_i \in F$ is called a Linear Combination of the vectors $v_1, v_2, \ldots, v_n$.

THEOREM 1: Let S be any subset (finite or infinite) of a vector space V, then the set L(S) of all linear combinations of vectors from S is a subspace of V.

REMARK 1: The subspace L(S) of all linear combinations of vectors from S is called the subspace spanned or generated by the set S.

REMARK 2: $S \subset L(S)$

REMARK 3: L(S) is the smallest subspace of V that contains S.

DEFINITION 2: Let V be a vector space over F. A subset S of vectors in V is said to be linearly dependent if there are distinct vectors $v_1, \ldots, v_n$ in S and scalar $\alpha_1, \ldots, \alpha_n$ not all zero, such that $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$. A set S of vectors in V is called linearly independent if S is not linearly dependent.

DEFINITION 3: Let V be a vector space. A basis for V is a subset S (finite or infinite) of V, that is linearly independent and that spans V, that is V = L(S).

A vector space V over F is said to be finite dimensional if it has a finite basis.

In spite of the fact that there is no unique choice of basis, for a vector space, there is something common to all of these choices. It is a property that is intrinsic to the space itself.

THEOREM 2: The number of elements in any basis of a finite-dimensional vector space V is the same as any other basis.

DEFINITION 4: The dimension of a finite-dimensional vector space V is the number of elements in a basis of V.

I.III LINEAR TRANSFORMATIONS

DEFINITION 1: Let V and W be vector spaces over the same field F. A mapping T : V ---> W is called a linear transformation if :

1. $T(v_1+v_2) = T(v_1) + T(v_2)$, for all $v_1, v_2 \in V$
2. $T(\alpha v_1) = \alpha T(v_1)$, for all $v_1 \in V$ and $\alpha \in F$.

REMARK 1: A linear transformation T : V ---> W is also called an F-homomorphism or F-linear mapping.

REMARK 2: If T is one-to-one and onto linear transformation, then it is called an isomorphism.

DEFINITION 2: If T : V ---> W is a linear transformation, then the kernel of T, denoted by ker(T) is defined by:

$$\ker(T) = \{v \in V \mid T(v) = 0\}$$

THEOREM 1: Let T : V ---> W be a linear transformation, then,

1. ker(T) is a subspace of V
2. T(V) is a subspace of W

Let V and W be vector spaces over the same field F. Let $Hom_F(V,W)$ be the set of all linear transformation of V into W. We shall now proceed to introduce operations in $Hom_F(V,W)$ in such a way that make $Hom_F(V,W)$ a vector space over F.

For T_1, $T_2 \in Hom_F(V,W)$, we define their sum $T_1 + T_2$ by:

$(T_1+T_2)(v) = T_1(v) + T_2(v)$ for all $v \in V$.

For $\alpha \in F$ and $T \in Hom_F(V,W)$, we define a map $\alpha T : V \longrightarrow W$

by $(\alpha T)(v) = \alpha(T(v))$ for all $v \in V$.

Then it is easy to verify that $T_1+T_2 \in Hom_F(V,W)$ and $\alpha T \in Hom_F(V,W)$. Also it can be checked that these operations makes $Hom_F(V,W)$ a vector space over F. Thus we have:

__THEOREM__ 2: $Hom_F(V,W)$ is a vector space. Moreover if V and W are finite-dimensional vector spaces of dimensions m and n respectively then $Hom_F(V,W)$ is finite-dimensional with dimension mn.

If in particular W = V, we denote $Hom_F(V,V)$ by $End_F(V)$ and in this case an element $T \in End_F(V)$ is called an endomorphism of V.

I.IV THE MATRIX OF A LINEAR TRANSFORMATION

Suppose now that V and W are finite-dimensional vector spaces over the same field F, and let dim V = m, and dim W = n. Let $T : V \longrightarrow W$ be a linear transformation. Let $B = \{v_1, \ldots, v_m\}$ and $B' = \{w_1, \ldots, w_n\}$ be basis for V and W respectively. For each $i = 1, 2, \ldots, m$

$$T(v_i) = \sum_{j=1}^{n} \alpha_{ij} w_j, \quad \alpha_{ij} \in F,$$

then the nxm matrix

$$[A]_B^{B'} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}^t$$

of the elements α_{ij} of F is called the matrix associated with the linear transformation T with respect to the basis B for V and basis B' for W. Of course the matrix $[A]_B^{B'}$ depends on our choice of bases for V and W, in the sense that any change in the bases B and B' would result in a different matrix for T. If these bases are held fixed then each T determines a unique matrix and conversely to each mxn matrix (α_{ij}) over F corresponds a unique linear transformation T determined by:

$$T(v_i) = \sum_{j=1}^{n} \alpha_{ij} W_j$$

we summarize formally:

<u>THEOREM</u> 1: Let V and W be two finite dimensional vector spaces over the same field F, and let dim V = m and dim W = n. Let B and B' be bases for V and W respect-

ively. Then for each linear transformation $T : V \longrightarrow W$ there is an nxm matrix A with entries in F such that

$$[Tv]_{B'} = A[v]_B \text{, for all } v \in V$$

where $[v]_B$ and $[Tv]_{B'}$ are the coordinate matrices of v and Tv relative to the bases B and B' respectively.

Furthermore, $T \mapsto A$ is an isomorphism between the space of all linear transformations from V into W, Hom(V,W) and the space of all nxm matrices over the field F.

In particular we shall be interested in the representation by matrices of endomorphisms, that is of linear transformations of a vector space V into itself. Let $B = \{ v_1, \ldots, v_n \}$ be a basis for V, and T an endomorphism of V, and let $A = (\alpha_{ij})$ be the matrix of T relative to the basis B. If a change of basis is made in V from B to a new basis B', what is the matrix of T relative to this new basis? The following theorem gives the answer.

<u>THEOREM</u> 2: Let V be an n-dimensional vector space over the field F, and let $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ be bases for V. If A is the matrix of T relative to B', then there exists a nonsingular matrix P with columns $P_j = [v'_j]_B$, such that $A' = P^{-1}AP$,

where $[v'_j]_B$ is the coordinate matrix of v'_j relative to the basis B, and A' is the matrix of T relative to B'.

I.V TRACE AND TRANSPOSE OF A MATRIX

Our aim in this short section is to develop the concepts of trace and transpose of a matrix and describe some of their properties that we need in later chapters.

DEFINITION 1: Let A be an nxn matrix over the field F. The trace of A is the sum of the elements of the main diagonal of A. We denote the trace of A by tr A; if $A = (\alpha_{ij})$, then $\operatorname{tr} A = \sum_{i=1}^{n} \alpha_{ii}$

The fundamental properties of the trace are contained in the following theorem:

THEOREM 1: For any nxn matrices A and B over the field F and $\lambda \in F$, we have

1. $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$
2. $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A$
3. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

REMARK 1: Properties 1 and 2 assert that the trace is a linear transformation of the vector space of nxn matrices over F to the one-dimensional vector space F.

REMARK 2: If A is invertible, property 3 implies that $\operatorname{tr}(ABA^{-1}) = \operatorname{tr} B$ for any nxn matrix B.

DEFINITION 2: Let $A = (\alpha_{ij})$ be an mxn matrix over F. The nxm matrix $B = (\beta_{ij})$ such that $\beta_{ij} = \alpha_{ji}$ for each i,j is called the transpose of A. We denote the transpose of A by A^t. In other words, the transpose of A is the matrix obtained by interchanging the rows and columns of A.

THEOREM 2: Let A and B be mxn matrices and let C be an nxn matrix. Then

1. $(A^t)^t = A$
2. $(A+B)^t = A^t + B^t$
3. $(\alpha A)^t = \alpha A^t$ for any $\alpha \in F$
4. $(AC)^t = C^t A^t$

DEFINITION 3: A matrix A is said to be a symmetric matrix if $A^t = A$, and A is called a skew symmetric matrix if $A^t = -A$.

REMARK 3: The concept of trace, and transpose of a matrix can be extended in the obvious way to any linear transformation.

I.VI SERIES OF MATRICES

This section is concerned with the notion of series of matrices, particularly in chapter 4, we need the concept of the exponential function e^A where A is a square matrix over the field R of real numbers.

DEFINITION 1: Let $A = (a_{ij})$ be an mxn matrix over R or C, the norm of A, is denoted by $||A||$ and defined as

$$||A|| = \max_{\substack{1<=i<=m \\ 1<=j<=n}} |a_{ij}|$$

The norm has the following properties:

1. $||A|| >= 0$ for any matrix A, also $||A|| = 0$ if and only if $A = 0$.

2. $||\alpha A|| = |\alpha| \, ||A||$ for any matrix A and any scalar α.

3. $||A+B|| <= ||A|| + ||B||$ for any A and B.

REMARK 1: These properties of the norm implies that the vector space of mxn matrices over R or C is a normed vector space with respect to $||.||$.

DEFINITION 2: Let A_1, A_2, ... to be an infinite sequence of mxn matrices over R. This sequence is said to converge if there exists an mxn matrix A over R such that:

$$\lim_{n\to\infty} ||A_n - A|| = 0.$$

A is called the limit of the sequence and we write

$$\lim_{n \to \infty} A_n = A$$

In order to define convergence of an infinite series of mxn matrices

$$\sum_{n=1}^{\infty} A_n$$

first we construct the sequence $S_1, S_2, \ldots$ of partial sums, $S_1 = A_1$, $S_2 = A_1 + A_2$, ... , $S_k = A_1 + \ldots + A_k$, ... we say that the series converges to the mxn matrix S if the sequence of partial sums converges to S, i.e. if $\lim_{k \to \infty} S_k = S$. In this case we write $\sum_{k=1}^{\infty} A_k = S$.

Using the notion of the norm of a matrix, we can formulate the following test for convergence:

<u>THEOREM</u> 1: Let $A_1, A_2, \ldots$ be mxn matrices. If the series of numbers

$$\sum_{n=1}^{\infty} ||A_n||$$

converges, then the series of matrices $\sum_{n=1}^{\infty} A_n$ converges.

We now define the exponential of a square matrix: Let A be an nxn matrix. First observe that

$$||A^2|| \le n||A||.||A|| = n||A||^2$$

$$||A^3|| \le n||A||.||A^2|| \le n^2||A||^3$$

$$\vdots$$

$$||A^k|| \le n||A||.||A^{k-1}|| \le n^{k-1}||A||^K$$

since the series of numbers

$$\sum_{k=0}^{\infty} \frac{n^{k-1}}{k!} ||A||^k$$

converges (as can be shown by the ratio test), the series of matrices

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges, by the previous theorem, note that A^0 denotes the identity matrix I. Now we define e^A to be the sum, i.e.

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \quad \text{for any square matrix A.}$$

Now we are going to consider a very important function, the exponential matrix function e^{At}, where A is square matrix over R and t is a real variable. This is defined by the formula

$$e^{At} = I + tA + \frac{1}{2!}t^2A^2 + \dots + \frac{1}{k!}t^kA^k + \dots$$

THEOREM 2: For any matrix A and any real number t, e^{At} is nonsingular.

To prove this theorem one can look at its eigenvalues; if λ is an eigenvalue of A then $e^{\lambda t}$ is an eigenvalue of e^{At}, and since $e^{\lambda t} \neq 0$ for any real number t, then e^{At} is nonsingular.

CHAPTER TWO

ALGEBRAS

In this Chapter, we introduce the language of algebras in a form designed for material developed in the later chapters.

We begin with some basic definitions, examples, and basic properties of algebras.

<u>DEFINITION</u> 1: Let F be a field. By F-algebra (or an algebra over F) we mean a vector space V over F together with a binary operation in V, called multiplication in V, the image of an element $(x,y) \in V X V$ under the multiplication is denoted by xy and is called the product of x and y in V. And this satisfies the following conditions:

1. $(x + y)z = xz + yz$
2. $x(y + z) = xy + xz$
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all x, y, and $z \in V$ and $\alpha \in F$.

REMARK 1: In the sequal F will be the field of real numbers R or the field of complex numbers C.

REMARK 2: Conditions 1 - 3 in this definition are equivalent to say the multiplication of the algebra V is F-bilinear mapping of VXV into V.

REMARK 3: When no confusion is to be feared an F-algebra will be simply called an algebra.

REMARK 4: It should be noted that we do not require that multiplication in V to be associative nor it should have a unit element.

DEINITION 2: 1. An F-algebra V is called an associative algebra if multiplication in V is associative, i.e. when

$$(xy)z = x(yz)$$

for all x, y, and $z \in V$

2. When multiplication in V admits an identity element, i.e. there is an element $e \in V$ such that

$$ve = ev = v \quad \text{for all } v \in V$$

V is called an algebra with unity or a unital algebra. Clearly if V has a unit element then it is unique.

3. An F-algebra V is called commutative algebra if the multiplication in V is commutative, i.e. when

$$xy = yx$$

for all $x, y \in V$

<u>EXAMPLE</u> 1: Every vector space V can be considered as an (associative and commutative) algebra by defining multiplication on V by xy = 0 for every x, y $\in$ V.

<u>EXAMPLE</u> 2: The set $Mat_{nxn}(F)$ of all nxn matrices, with entries from a field F, is an associative algebra over the ground field F. The vector space structure on $Mat_{nxn}(F)$ is defined by the ordinary matrix addition and scalar multiplication of a matrix by a scalar. The algebra multiplication is defined by the ordinary matrix multiplication. Note that this algebra is not commutative, it is called the Matrix algebra over F. (Proofs can be found in linear algebra books).

<u>EXAMPLE</u> 3: Let R^3 be the three-dimensional real Euclidean space. The cross product of vectors makes R^3 into a non-associative and non-commutative algebra over R. (Proof can be found in calculus or linear algebra algebra books).

<u>EXAMPLE</u> 4: Let V be an algebra over F. We can define two algebra structures on V by defining new multiplication on V by

$$x * y = xy + yx$$

and the bracket multiplication

$$[x,y] = xy - yx$$

with the same underlying vector space structure as the algebra V. These multiplication are not in general associative, the first multiplication * is always commutative. The bracket multiplication will play an important role in our study of Lie algebras. In particular by defining the bracket multiplication on the matrix algebra $Mat_{nxn}(F)$, we obtain a new algebra, this is called the General Linear Algebra of degree n over F, we denote this algebra by gl(n,F). Now we show that the bracket multiplication defines a multiplication in V.

1. Show $[x+y,z] = [x,z] + [y,z]$

$$\begin{aligned}[x+y,z] &= (x+y)z - z(x+y)\\ &= xz + yz - zx - zy\\ &= (xz-zx) + (yz-zy)\\ &= [x,z] + [y,z]\end{aligned}$$

2. Similarly we can show $[x,y+z] = [x,y] + [x,z]$

3. The third condition is to show $\alpha[x,y]=[\alpha x,y]=[x,\alpha y]$

lets show $\alpha[x,y] = [\alpha x,y]$

$$\begin{aligned}\alpha[x,y] &= \alpha(xy-yx)\\ &= \alpha(xy) - \alpha(yx)\\ &= (\alpha x)y - y(\alpha x)\\ &= [\alpha x,y]\end{aligned}$$

similarly we can show $\alpha[x,y] = [x,\alpha y]$

In this study we shall be interested mainly in the case of algebras over fields which are finite-dimensional as vector spaces. For such an algebra we have a basis $e_1, e_2, \ldots, e_n$ and we can write $e_i e_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k$ where γ's are in F. The n^3 elements γ_{ij}^{k} are called the constants of multiplication (or structural constants) of the algebra (relative to the chosen basis). They give the values of every product $e_i e_j$ for $i, j = 1, 2, \ldots, n$. Moreover, by extending this linearly, these products determine every product in V. For, if x and y are any elements of V,

and $x = \sum_{i=1}^{n} \alpha_i e_i$, $y = \sum_{j=1}^{n} \beta_j e_j$ where $\alpha_i, \beta_j \in F$,

$$
\begin{aligned}
\text{then} \quad xy &= \left(\sum_i \alpha_i e_i\right)\left(\sum_j \beta_j e_j\right) \\
&= \sum_{i,j} (\alpha_i e_i)(\beta_j e_j) \\
&= \sum_{i,j} \alpha_i (e_i(\beta_j e_j)) \\
&= \sum_{i,j} \alpha_i \beta_j (e_i e_j)
\end{aligned}
$$

and this is determined by the $e_i e_j$.

Thus any finite-dimensional vector space V can be given the structure of an algebra over a field F by first selecting a basis $e_1, e_2, \dots, e_n$ in V. Then for every pair (i,j) we define in any way we please $e_i e_j$ as an element in V, say $e_i e_j = v_{ij}$ and extending this linearly to a product in V, that is if

$$x = \sum_{i=1}^{n} \alpha_i e_i \quad \text{and} \quad y = \sum_{j=1}^{n} \beta_j e_j$$

we define $xy = \sum_{i,j=1}^{n} \alpha_i \beta_j (e_i e_j)$

$$= \sum_{i,j=1}^{n} \alpha_i \beta_j (v_{ij})$$

One can check immediately that this multiplication is bilinear in the sense that conditions (1-3) in definition 1, are valid. Letting $e_i e_j = v_{ij} = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k$, we obtain elements (γ_{ij}^{k}) of F which completely determine the product xy, that is to say the choice of $e_i e_j$ is equivalent to the choice of the elements γ_{ij}^{k} in F. Thus, the set of algebras with underlying vector space over F can be identified with the algebra F^{n^3}.

<u>DEFINITION</u> 3: Let V be an F-algebra. By an F-subalgebra of V, we mean a vector subspace W of V which is itself an F-algebra relative to the multiplication on V.

<u>DEFINITION</u> 4: A subset W of an algebra V is called a left ideal (respectively right ideal) of V when W is a subalgebra of V and for each $w \in W$, $v \in V$ we have $wv \in W$ (respectively $vw \in W$). If W is both a left and right ideal of V, then W is called a two-sided ideal (or ideal, for short) of V.

If W is an ideal of an algebra V, then the quotient space $V/W = \{v + W : v \in V\}$ is an algebra with respect to the following multiplication

$$(v_1 + W)(v_2 + W) = v_1 v_2 + W$$

It can be easily verified that this definition of the product on V/W is well-defined and is bilinear. V/W, with this algebra structure, is called the quotient algebra of the algebra V by the ideal W.

<u>EXAMPLES OF SUB-ALGEBRA</u>

We shall consider now some important sub-algebras of the general linear algebra of degree n, gl(n,F).

<u>EXAMPLE</u> 1: The subalgebra $sl(n,F) = \{A \in gl(n,F) : tr(A) = 0\}$ where $tr(A) = \sum_{i=1}^{n} a_{ii}$. This algebra is called the special linear algebra of degree n.

In order to show that $sl(n,F) = \{A \in gl(n,F) \mid tr(A) = 0\}$ is a sub-algebra of gl(n,F), we have to show sl(n,F) is closed under addition, closed under scalar multiplication, and closed under the bracket multiplcation.

1. Show it is closed under addition

let $A, B \in sl(n,F)$,

we need to show that $A+B \in sl(n,F)$

we know $tr(A) = 0$ and $tr(B) = 0$,

$$tr(A+B) = tr(A) + tr(B)$$

$$= 0 + 0 = 0$$

therefore $A+B \in sl(n,F)$

2. Show it is closed under the scalar multiplication

let $A \in sl(n,F)$, and $\alpha \in F$.

we need to show that $\alpha A \in sl(n,F)$

$$tr(\alpha A) = \alpha tr(A)$$

$$= \alpha(0)$$

$$= 0$$

3. Show it is closed under the bracket multiplication

let $A, B \in sl(n,F)$

we need to show that $[A,B] \in sl(n,F)$

$$tr[A,B] = tr(AB-BA)$$

$$= tr(AB) - tr(BA)$$

$$= tr(AB) - tr(AB)$$

$$= 0$$

therefore $[A,B] \in sl(n,F)$

<u>EXAMPLE</u> 2: The sub-algebra of skew-symmetric matrices.

$so(n,F) = \{A \in gl(n,F) \mid A^t = -A\}$, where A^t is the transpose of A.

As in example one we need to show so(n,F) is closed under addition, closed under scalar multiplication, and closed under the bracket multiplication.

1. Show it is closed under addition

   let $A, B \in so(n,F)$

   we need to show $(A+B)^t \in so(n,F)$

   we know $A^t = -A$, and $B^t = -B$

   $$(A+B)^t = A^t + B^t = (-A) + (-B) = -(A+B)$$

2. Show it is closed under scalar multiplication

   let $A \in so(n,F)$, and $\alpha \in F$

   need to show $\alpha A \in so(n,F)$

   $$(\alpha A)^t = \alpha(A)^t = \alpha(-A) = -(\alpha A)$$

3. Show it is closed under the bracket multiplication

   let $A, B \in so(n,F)$

   need to show $[A,B] \in so(n,F)$.

$$[A,B]^t = (AB-BA)^t$$

$$= (AB)^t - (BA)^t$$

$$= B^t A^t - A^t B^t$$

$$= (-B)(-A) - (-A)(-B)$$

$$= BA - AB$$

$$= -(AB-BA)$$

$$= -[A,B]$$

<u>EXAMPLE</u> 3: For n=2m, the symplectic sub-algebra sp(n,F), formed by the matrices $A \in gl(n,F)$ such that $A^t J + JA = 0$ where J has the form:

$$J = \left|\begin{array}{c|c} 0 & I_m \\ \hline -I_m & 0 \end{array}\right|$$

where I_m is the identity matrix of order m, and 0 is the zero matrix of order m.

To show that sp(n,F) is a sub-algebra, we need to show it is closed under addition, closed under scalar multiplication, and closed under the bracket mulitplication.

1. Show it is closed under addition

   let A, B $\in$ sp(n,F)

   we need to show that A+B $\in$ sp(n,F)

   we have $A^t J + JA = 0$, and $B^t J + JB = 0$

let $C = A + B$ then

$$C^t J + JC = (A+B)^t J + J(A+B)$$

$$= (A^t + B^t)J + JA + JB$$

$$= A^t J + B^t J + JA + JB$$

$$= (A^t J+JA) + (B^t J+JB) = 0$$

2. Show it is closed under scalar multiplication

let $A \in sp(n,F)$, and $\alpha \in F$.

we need to show that $\alpha A \in sp(n,F)$

$$A^t J + JA = 0$$

$$(\alpha A)^t J + J(\alpha A) = \alpha(A^t J) + \alpha(JA)$$

$$= \alpha(A^t J+JA)$$

$$= \alpha(0)$$

$$= 0$$

3. Show it is closed under the bracket multiplication

Let $A,B \in sp(n,F)$,

we need to show that $[A,B]^t J + J[A,B] = 0$

$$[A,B]^t J + J[A,B] = (AB-BA)^t J + J(AB-BA)$$

$$= (AB)^t J - (BA)^t J + J(AB) - J(BA)$$

$$= (B^t A^t)J - (A^t B^t)J + J(AB) - J(BA)$$

$$= B^t A^t J - A^t B^t J + J(AB) - J(BA)$$

$$+ B^t JA - B^t JA + A^t JB - A^t JB$$

$$= (B^t A^t J + B^t JA) - (A^t B^t J + A^t JB)$$

$$+ (JAB + A^t JB) - (JBA + B^t JA)$$

$$= B^t (A^t J + JA) - A^t (B^t J + JB)$$

$$+ (JA + A^t J)B - (JB + B^t J)A$$

$$= B^t (0) - A^t (0) + (0)B - (0)A$$

$$= 0$$

<u>**EXAMPLE**</u> **4:** The sub-algebra of upper triangular matrices $ut(n,F) = \{A \in gl(n,F) : a_{ij} = 0 \text{ for } i>j\}$.

1. Show it is closed under addition

   let $A = [a_{ij}]$, and let $B = [b_{ij}]$ be in $ut(n,F)$,

   let $C = A + B$

   we need to show that $A + B \in ut(n,F)$

   we know that $c_{ij} = a_{ij} + b_{ij}$

   for $i>j$, we have $a_{ij} = 0$, and $b_{ij} = 0$

   therefore $c_{ij} = 0$

   therefore it is closed under addition.

2. Show it is closed under the scalar mulitplication

let $A \in ut(n,F), \alpha \in F$.

we need to show that $\alpha A \in ut(n,F)$

let $C = \alpha A$, then

$$c_{ij} = \alpha a_{ij}$$

for $i>j$, we have $a_{ij} = 0$, and hence

$$c_{ij} = \alpha(a_{ij})$$

$$= \alpha(0)$$

$$= 0$$

therefore it is closed under scalar multiplication.

3. Show it is closed under the bracket mutliplication

let $A = [a_{ij}]$, and $B = [b_{ij}]$ be in $ut(n,F)$

let $AB = [c_{ij}]$, first we are going to show

$AB \in ut(n,F)$.

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Now, $a_{ir} = 0$ for $i>r$ and $b_{rj} = 0$ for $r>j$, hence for

$i>r>j$ we have $a_{ir} b_{rj} = 0$. Thus if $i>j$, then

$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = 0$. Hence $AB \in ut(n,F)$.

Similarly it can be shown that $BA \in ut(n,F)$. Thus

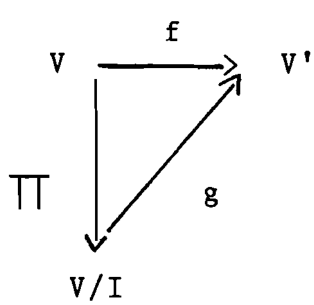
$[A,B] \in ut(n,F)$, therefore it is closed under the

bracket multiplication.

<u>DEFINITION</u> 1: Let V and V' be algebras over the same field F. By an algebra homomorphism of V into V' we mean a mapping f: V ---> V' which is F-linear and has the property that $f(v_1 v_2) = f(v_1)f(v_2)$ for every $v_1, v_2 \in V$. If f is also one-to-one and onto, then it is called an isomorphism.

<u>THEOREM</u> 1: Let V and V' be two F-algebras and f : V ---> V' an algebra homomorphism. The image f(V) is a sub-algebra of V' and the kernel of f, $\ker f = f^{-1}(0)$, (the inverse image of 0) is an ideal in V. The fundamental homomorphism theorems of group and ring theory have their counterparts for algebras we cite:

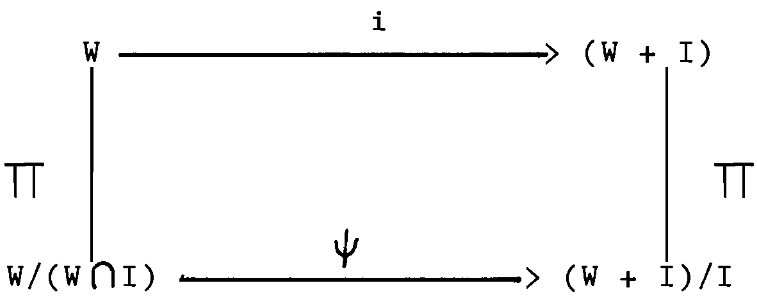
<u>THEOREM</u> 2: If f: V ---> V' is an algebra homomorphism then V/ker f is isomorphic to Im(f). If I is any ideal of V included in ker f, there exists a unique homomorphism g : V/I ---> V' making the following diagram commute.



Where the mapping π : V ---> V/I is the natural homomorphism v|---> v + I.

THEOREM 3: If I and J are ideals of an algebra V such that $I \subset J$, then J/I is an ideal of V/I and (V/I)/(J/I) is isomorphic to V/J.

THEOREM 4: If W is a sub-algebra of an algebra V and if I is an ideal in V, then W + I is a sub-algebra of V, $W \cap I$ is an ideal in W, and there is a unique isomorphism $\psi: W/(W \cap I) \longrightarrow (W + I)/I$ such that the following diagram commutes.



Where the mapping i of W into W + I is the inclusion mapping. (The proofs of the above four theorems can be found in). [1]

EXAMPLE 5: Consider the sub-algebra sl(n,F) of gl(n,F), it can be easily shown that sl(n,F) is in fact an ideal in gl(n,F).

For let $A \in sl(n,F)$, and let $B \in gl(n,F)$

we need to show that $[A,B] \in sl(n,F)$

$[A,B] = AB - BA$

$$\begin{aligned} tr[A,B] &= tr(AB-BA) \\ &= tr(AB) - tr(BA) \\ &= tr(AB) - tr(AB) = 0 \end{aligned}$$

therefore sl(n,F) is an ideal in gl(n,F). Now we can consider the quotient algebra gl(n,F)/sl(n,F). Also F is an F-algebra with respect to the bracket multiplication. Consider the map tr: gl(n,F) ---> F given by A|---> tr(A). Now, we show the map tr is an algebra homomorphism.

1. tr is linear, means

   a) tr(A+B) = tr(A) + tr(B)

   b) $tr(\alpha A) = \alpha tr(A)$

2. tr([A,B]) = [tr(A),tr(B)]

The first condition of linearity follows from properties of the trace.

For condition (2) we have tr([A,B]) = tr(AB-BA) = 0

and [tr(A),tr(B)] = tr(A)tr(B) - tr(B)tr(A) = 0.

3. Moreover tr is onto, for let $\alpha \in F$ be any scalar, consider the matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} \alpha & \text{for } i = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

then $tr\ A = \alpha$.

Also we have $ker(tr) = \{A \in gl(n,F) \mid tr(A) = 0\} = sl(n,F)$, hence by the Fundamental Homomorphism Theorem 2 we have:

$$gl(n,F)/sl(n,F) \cong F.$$

<u>EXAMPLE</u> 6: (Algebra of Endomorphisms). Let V be any vector space over a field F. The set of all F-linear transformations of V into itself denoted by $End_F(V)$ is a vector space over the ground field F. The vector space structure in $End_F(V)$ is defined by ordinary addition of linear transformation and the scalar multiplication of a linear transformation by a scalar. Recall that if $T, T' \in End_F(V)$ and $\alpha \in F$, then $T + T'$ and αT are defined by:

$$(T+T')(v) = T(v) + T'(V)$$

$$(\alpha T)(v) = \alpha(T(v))$$

for every $v \in V$.

If we define a multiplication on $End_F(V)$ by

$$(ToT')(v) = T(T'(v))$$

then it can be shown that this multiplication is bilinear and associative. This algebra is called the algebra of endomorphisms of V.

It is well known that if V is finite-dimensional vector space of dimension n, then $End_F(V)$ is finite-dimensional of dimension n^2 over F. If $e_1, e_2, \ldots, e_n$ is a basis for V, then the linear transformations E_{ij} such that:

$$E_{ij}(e_r) = \begin{cases} e_j & \text{if } r = i \\ 0 & \text{if } r \neq i \end{cases} \qquad 1 <= i, j <= n$$

form a basis for $End_F(V)$ over F. If $T \in End_F(V)$ then we can write $T(e_i) = \sum_{j=1}^{n} \alpha_{ij} e_j$, i = 1, 2, ... , n and the matrix $A = [\alpha_{ij}]^t$ is the matrix of T relative to the basis (e_i), 1<=i<=n. The correspondence $T \mapsto A$ is an algebra isomorphism of the endomorphism algebra $End_F(V)$ onto the matrix algebra $Mat_{nxn}(F)$ of nxn matrices with entries in F. Thus we have:

<u>THEOREM</u> 4: The Matrix Algebra $Mat_{nxn}(F)$ is isomorphic to the algebra of endomorphisms $End_F(V)$, where n = dim V.

(Note that the isomorphism depends upon a choice of basis for V.

Let us consider for any algebra V over F, the algebra of endomorphism $End_F(V)$ of the vector space V. For any $v \in V$ define a map $T_v : V \to V$ by $T_v(x)=vx$ for all $x \in V$. T_v is called the left multiplication by v. Then it can be shown that T_v is an endomorphism of V and hence an element of $End_F(V)$. Also if the algebra V is associative then $T_v(xy) = T_v(x)\, T_v(y)$ for every $x,y \in V$, and in this case, the mapping $\psi: v \mapsto T_v$ is an algebra homomorphism of V into the algebra $End_F(V)$ of endomorphism of the vector space V. Now if V has an identity element 1, then the mapping $\psi: v \mapsto T_v$ is an isomorphism of V into $End_F(V)$. Hence V is isomorphic to an algebra of endomorphism, on the other hand if V does not have an identity, we can adjoin one in a simple way to get an algebra $\overline{V}$ with an identity such that $\dim \overline{V} = 1 + \dim V$ since $\overline{V}$ is isomorphic to an algebra of endomorphism, the same is true for V. Thus we have:

<u>THEOREM</u> 5: If V is a finite-dimensional associative algebra then V is isomorphic to an algebra of endomorphism of a finite-dimensional vector space.

<u>DEFINITION</u> 6: A homomorphism of an algebra A over F into an algebra $End_F(V)$ of endomorphisms of a vector space V over F is called a representation of A. The particular representation $\psi: v \mapsto T_v$ in the above discussion is called the regular represntation of V.

CHAPTER THREE

LIE ALGEBRAS

III.I BASIC DEFINITIONS AND EXAMPLES:

Some basic concepts and definitions of Lie algebras are discussed in this chapter from an algebraic viewpoint.

DEFINITION 1: Let F be a field. A Lie algebra over F is a (non associative) F-algebra, L whose multiplication, denoted by [x,y] for x and y in L, and satisfies in addition the following conditions:

1. [x,x] = 0, for all x in L
2. [[x,y],z] + [[y,z],x] + [[z,x],y] = 0

for all x,y, and z in L.

Condition 2 is called the Jacobi identity. The product [x,y] is often called the Lie bracket or the commutator of x and y.

<u>PROPOSITION</u> 1: In any Lie algebra L we have $[x,y]=-[y,x]$ for all $x,y \in L$. Conversely, if $[x,y] = -[y,x]$ for all $x,y \in L$, then $[x,x] = 0$ provided that the characteristic of F is not 2.

Proof: From 1 and the bilinearity of multiplication we have

$$\begin{aligned} 0 &= [x+y,x+y] \\ &= [x,x] + [x,y] + [y,x] + [y,y] \\ &= [x,y] + [y,x] \end{aligned}$$

thus $[x,y] = -[y,x]$

conversely, if $[x,y] = -[y,x]$ then $[x,x] = -[x,x]$, hence $2[x,x] = 0$, but since the characteristic of F is not 2, then $[x,x] = 0$.

Note that the above proposition implies that the multiplication in any Lie algebra is anticommutative.

<u>EXAMPLE</u> 1: Any vector space L over F can be considered as a Lie algebra over F by defining $[x,y] = 0$, for all $x,y \in L$. Those are the abelian or commutative Lie algebras.

<u>EXAMPLE</u> 2: Let gl(n,F) be the vector space of nxn matrices with entries from field F. Define a Lie product by $[A,B] = AB - BA$, $A,B \in gl(n,F)$, where AB is the ordinary matrix multiplication. We have shown in Chapter II that gl(n,F) with respect to the bracket multiplication is an algebra. Now we are going to prove that the Lie product [A,B] satisfies conditions 1 and 2 in defintion (1) of Lie algebra.

Proof: 1. Show that [A,A] = 0

the proof of this condition is very easy,

$[A,A] = AA - AA = 0$

2. The second condition to be proved is:

$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$

$$
\begin{aligned}
[AB-BA,C] + [BC-CB,A] + [CA-AC,B] &= (AB-BA)C - C(AB-BA) \\
&\quad + (BC-CB)A - A(BC-CB) \\
&\quad + (CA-AC)B - B(CA-AC) \\
&= ABC - BAC - CAB + CBA \\
&\quad + BCA - CBA - ABC + ACB \\
&\quad + CAB - ACB - BCA + BAC \\
&= (ABC-ABC) - (BAC-BAC) \\
&\quad - (CAB-CAB) - (BCA-BCA) \\
&\quad - (ACB-ACB) - (CBA-CBA) \\
&= 0
\end{aligned}
$$

The Lie algebra obtained in this way is called the general linear Lie algebra of degree n over F. For convenience it will be called the linear Lie algebra of degree n over F.

<u>EXAMPLE</u> 3: Example 2 above can be generalized. Let A be any associative algebra over a field F. We can always make A into a Lie algebra by defining a Lie multiplication $[x,y] = xy - yx$ for all $x, y \in A$. One verifies at once that this definition of $[x,y]$ gives A the structure of F-algebra. Clearly $[x,x] = 0$ for any $x \in A$, and the Jacobi identity follows from the associative law in A,

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = (xy-yx)z - z(xy-yx) + (yz-zy)x - x(yz-zy) + (zx-xz)y - y(zx-xz) = 0$$

Thus the product $[x,y]$ satisfies all the conditions of the product in a Lie algebra. The Lie algebra obtained in this way is called the Lie algebra of the associative algebra A, and we shall denote this Lie algebra by A_L.

<u>REMARK</u> 1: The fact that every associative algebra can be turned into a Lie algebra by means of the bracket operation is very important in many aspects of the theory of Lie algebras since it establishes a direct connection between an associative algebra and a Lie algebra. In fact, every Lie algebra is isomorphic to a subalgebra of a Lie algebra A_L, where A is an associative algebra [2].

<u>EXAMPLE</u> 4: Let V be a finite-dimensional vector space over a field F. Since the set of all F-linear endomorphisms of V, $End_F(V)$ forms an associative algebra, we can make $End_F(V)$ a Lie algebra by the bracket operation, we write $(End_F(V))_L$ for $End_F(V)$ viewed as Lie algebra and call it the Lie algebra of endomorphisms of V. When no confusion is to be feared we will denoted $(End_F(V))_L$ simply by $End_F(V)$. Any subalgebra B of a Lie algebra $End_F(V)$ is called a Lie algebra of linear transformation.

In view of the previous remark every Lie algebra is isomorphic to a Lie algebra of linear transformation. In particular the general linear Lie algebra gl(n,F) is isomorphic to the Lie algebra of linear transformation $End_F(V)$, when n = dim V.

III.II STRUCTURE CONSTANTS

Let L be a Lie algebra over a field F and let $\{e_1, \ldots, e_n\}$ be a basis for the vector space L. Expanding the elements $[e_i, e_j]$ of L as a linear combinations of the basis $e_1, \ldots, e_n$ we obtain

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k$$

where the scalars $\gamma_{ij}^{k} \in F$ are called the structure constants of the Lie algebra L with respect to the basis $\{e_1, e_2, \ldots, e_n\}$. Moreover, we have seen in Chapter II, these products determine every product in L. The following theorem characterize Lie algebras in terms of structure constants and basis elements.

THEOREM 1: Let L be a (nonassociative) algebra over a field F with basis $\{e_1, e_2, \ldots, e_n\}$ and let γ_{ij}^{k} be the structure constants of L relative to the basis. For L to be a Lie algebra, it is necessary and sufficient that the basis elements satisfies the following conditions:

a) $[e_i, e_i] = 0$

b) $[e_i, e_j] = - [e_j, e_i]$

c) $[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0$

for all i, j, k = 1, 2, ... , n.

These conditions are equivalent to say that the constants γ_{ij}^{k} satisfies:

a') $\gamma_{ii}^{k} = 0$

b') $\gamma_{ij}^{k} = - \gamma_{ji}^{k}$

c') $\sum_{r} (\gamma_{ij}^{r} \gamma_{rk}^{s} + \gamma_{jk}^{r} \gamma_{ri}^{s} + \gamma_{ki}^{r} \gamma_{rj}^{s}) = 0$

for all i, j, k, and s = 1, 2, ... , n.

Proof: Clearly if L is a Lie algebra then the conditions (a)-(c) are satisfied. Conversely, assume that L is an F-algebra such that (a)-(c) are satisfied. For let $x = \sum_{i=1}^{n} \alpha_i e_i$, $y = \sum_{i=1}^{n} \beta_i e_i$, and $z = \sum_{i=1}^{n} \gamma_i e_i$

1. First we need to show: $[x,x] = 0$

$$[x,x] = [\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \alpha_j e_j]$$

$$= \sum_{i=1}^{n} \alpha_i [e_i, \sum_{j=1}^{n} \alpha_j e_j]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j [e_i, e_j]$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j=i}} \alpha_i \alpha_j [e_i, e_j] + \sum_{i=1}^{n} \sum_{j>i} \alpha_i \alpha_j [e_i, e_j] + \sum_{j=1}^{n} \sum_{i>j} \alpha_i \alpha_j [e_i, e_j]$$

$$= \sum_{i=1}^{n} \alpha_i^2 [e_i, e_i] + \sum_{i=1}^{n} \sum_{j>i} \alpha_i \alpha_j [e_i, e_j] + \sum_{j=1}^{n} \sum_{i>j} \alpha_i \alpha_j [e_i, e_j]$$

$$= \sum_{i=1}^{n} \alpha_i^2 [e_i, e_i] + \sum_{i=1}^{n} \sum_{j>i} \alpha_i \alpha_j [e_i, e_j] - \sum_{j=1}^{n} \sum_{i>j} \alpha_i \alpha_j [e_i, e_j]$$

(by condition b)

$$= \sum_{i=1}^{n} \alpha_i^2 [e_i, e_i] + 0$$

$$= \sum_{i=1}^{n} \alpha_i^2 [e_i, e_i]$$

$= 0$ (by condition a)

2. The second condition to be satisfied is

$[x,y] = -[y,x]$

$$[x,y] = [\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j [e_i, e_j]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j (-[e_j, e_i])$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j [e_j, e_i]$$

$$= -[\sum_{j=1}^{n} \beta_j e_j, \sum_{i=1}^{n} \alpha_i e_i]$$

$$= -[y,x]$$

3. The third condition to be satisfied is

$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

$[[x,y],z] + [[y,z],x] + [[z,x],y]$

$$= [[\sum_{i=1}^{n}\alpha_i e_i, \sum_{j=1}^{n}\beta_j e_j], \sum_{k=1}^{n}\gamma_k e_k] + [[\sum_{j}^{n}\beta_j e_j, \sum_{k}\gamma_k e_k], \sum_{i}\alpha_i e_i]$$

$$+ [[\sum_{k}\gamma_k e_k, \sum_{i}\alpha_i e_i], \sum_{j}\beta_j e_j]$$

$$= [\sum_{i}\sum_{j}\alpha_i\beta_j[e_i,e_j], \sum_{k}\gamma_k e_k] + [\sum_{j}\sum_{k}\beta_j\gamma_k[e_j,e_k], \sum_{i}\alpha_i e_i]$$

$$= \sum_{i}\sum_{j}\sum_{k}\alpha_i\beta_j\gamma_k[[e_i,e_j],e_k] + \sum_{j}\sum_{k}\sum_{i}\beta_j\gamma_k\alpha_i[[e_j,e_k],e_i]$$

$$+ \sum_{k}\sum_{i}\sum_{j}\gamma_k\alpha_i\beta_j[[e_k,e_i],e_j]$$

Now since addition is commutative and the $\alpha_i\beta_j\gamma_k = \beta_j\gamma_k\alpha_i = \gamma_k\alpha_i\beta_j$ for all $i,j,k = 1,2,...,n$, then the last expression can be written as:

$$= \sum_{i}\sum_{j}\sum_{k}\alpha_i\beta_j\gamma_k\{[[e_i,e_j],e_k] + [[e_j,e_k],e_i] + [[e_k,e_i],e_j]\}$$

Condition (c) implies each term in the last summation is zero, and this complete the prove of Jacobi's identity.

As an application to Theorem 1, we have:

<u>**EXAMPLE**</u> 1: Let V be a 2-dimensional vector space over a field F. Pick a basis $\{e_1,e_2\}$ for V, by defining a multiplication table for the base elements by:

$[e_1,e_2] = e_1$, $[e_2,e_1] = -e_1$, and $[e_1,e_1] = [e_2,e_2] = 0$

and extending this linearly to a product in V. We are going to show the multiplication satisfies the conditions of theorem 1, and hence this multiplication turns V into a two-dimensional Lie algebra.

Conditions 1 & 2 follows directly from the definition.

For condition 3 we have,

$$[[e_1,e_2],e_1] + [[e_2,e_1],e_1] + [[e_1,e_1],e_2]$$

$$= [e_1,e_1] + [-e_1,e_1] + [[e_1,e_1],e_2]$$

$$= 0 + 0 + [0,e_2]$$

$$= 0$$

therefore the multiplication satisfies the conditions of theorem 1.

<u>REMARK</u> 1: There are couple of simplifying remarks. First, we note that if $[e_i,e_i] = 0$ and $[e_i,e_j]=-[e_j,e_i]$, then the validity of

$$[[e_i,e_j],e_k] + [[e_j,e_k],e_i] + [[e_k,e_i],e_j] = 0$$

for a particular triple i, j, k implies

$$[[e_j,e_i],e_k] + [[e_i,e_k],e_j] + [[e_k,e_j],e_i] = 0$$

Since cyclic permutation of i, j, k are clearly allowed, it follows that the Jacobi identity for

$[[e_{\sigma(i)},e_{\sigma(j)}],e_{\sigma(k)}]$ is valid, where σ is a permutation of i, j, k. Next let i = j. Then the Jacobi identity becomes:

$$[[e_i,e_i],e_k] + [[e_i,e_k],e_i] + [[e_k,e_i],e_i]$$

$$= 0 + [[e_i, e_k], e_i] - [[e_i, e_k], e_i]$$

$$= 0$$

Hence $[e_i, e_i] = 0$ and $[e_i, e_j] = -[e_j, e_i]$ implies that the Jacobi identities are satisfied for e_i, e_i, e_k. In particular, the Jacobi identity for a Lie algebra L, with dim L <= 2 is a consequence of $[x,x] = 0$, and if dim L = 3, the only identity we have to check is:

$$[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = 0.$$

<u>**EXAMPLE**</u> **2:** Let R^3 be the 3-dimensional real Euclidean space. We can make R^3 into a Lie algebra by defining a Lie multiplication $[a,b] = aXb$ for all $a,b \in R^3$, where aXb is the usual cross product of a and b.

That the Lie multiplication satisfies $[a,a] = 0$, for all a in R^3 is evident, and by taking advantage of the above remark it suffices to prove the Jacobi identity holds for the orthonormal standard basis of R^3.

Let $e_1 = \langle 1,0,0\rangle$, $e_2 = \langle 0,1,0\rangle$, and $e_3 = \langle 0,0,1\rangle$, then $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, and $[e_3, e_1] = e_2$. Hence we have $[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2]$

$$= [e_3, e_3] + [e_1, e_1] + [e_2, e_2]$$

$= 0$, and the Jacobi identity holds.

Before proceeding with more examples of Lie algebras we need to study some interesting facts about the Lie algebra structure of R^3. First we are going to give a geometric interpretation of the Jacobi identity.

A three-dimensional analog of the triangle is the trihedron, i.e. the figure formed by the noncoplanar vectors a,b,c. These vectors correspond to the vertices of the triangle, and the faces of the trihedron correspond to the sides of the triangle. The faces of the trihedron are planes for which we may substitute their normal vectors, i.e. the vectors, bXc, cXa, and aXb. Using the same correspondence between planes and vectors, we see that the vectors, aX(bXc), bX(cXa), and aX(bXc) correspond to the altitudes of the trihedron, i.e. the planes containing an edge and perpendicular to the opposite face.

If the sum of three vectors is the zero vector, the three vectors must be coplanar. The normal vectors of three planes having a point in common are coplanar if and only if, the planes also have a line in common. Hence the geometric interpretation of the Jacobi identity is: The altitudes of the trihedron are three planes having a line in common. This is a generalization of the familiar theorem from plane geometry asserting that the altitudes of the triangle are three lines having a point (the orthocenter) in common.

It is customary to talk about the "Paradox" of linear algebra: while every vector space can be converted to an inner product space by endowing it with a dot product, regardless of its dimension, only the three-dimensional vector space can be converted into a Lie algebra by the introduction of the cross product. This implies the cross product seems to lack a higher-dimensional generalization. This Paradox is the result of a vicious formulation of the problem, as we shall see in the next section.

III.III THE LIE ALGEBRA OF ANTISYMMETRIC OPERATORS

First let us consider the special case of the three dimensional vector space R^3. Let a be a fixed vector, the map $f_a : R^3 \longrightarrow R^3$ given by $f_a(v) = aXv$ for all $v \in R^3$ is linear. Let $A(R^3) = \{f_a : a \in R^3\}$ be the set of all such linear operators on R^3. For $f_a, g_b \in A(R^3)$ define the sum $f_a + g_b$ and scalar multiplication f_a as usual, i.e. $(f_a + g_b)(v) = f_a(v) + g_b(v) = (axv) + (bxv) =$ and $(\alpha f_a)(v) = \alpha(f_a(v)) = \alpha(axv)$. Then it can be easily shown that $A(R^3)$ is a vector subspace of $End_R(R^3)$. By introducing the Lie bracket multiplication we have:

<u>THEOREM</u> 1: $A(R^3)$ is a Lie algebra with respect to the multiplication $[f_a, g_b] = f_a o g_b - g_b o f_a$. Moreover there is a natural Lie algebra isomorphism between R^3 with the cross product and $A(R^3)$.

Proof: It is routine to show that $A(R^3)$ is a Lie algebra. To show there is a Lie algebra isomorphism between R^3 and $A(R^3)$,

define $\psi: R^3 \dashrightarrow A(R^3)$.

by $\psi: a \mapsto f_a$, where $f_a(v) = aXv$ for all $v \in R^3$

We claim ψ is a Lie algebra isomorphism.

1. To show ψ is linear

First we need to show that $\psi(a+b) = \psi(a) + \psi(b)$

$$\begin{aligned}\psi(a+b)(v) &= f_{a+b}(v)\\ &= (a+b)Xv\\ &= (aXv) + (bXv)\\ &= f_a(v) + f_b(v)\\ &= \psi(a)(v) + \psi(b)(v)\end{aligned}$$

Second we must show

$\psi(\lambda a) = \lambda\psi(a)$, where $\lambda \in R$

$$\begin{aligned}\psi(\lambda a)(v) &= f_{\lambda a}(v) = \lambda(aXv) = \lambda f_a(v)\\ &= \lambda(\psi(a)(v))\end{aligned}$$

2. To show that ψ is one-to-one we need the following

REMARK : If $a \in R^3$ is a fixed vector such that $aXv = 0$ for all $v \in R^3$, then $a = 0$.

Proof:

let $a = a_1 i + a_2 j + a_3 k$ and $v = 1i+0j+0k$ be vectors in R^3 then the cross product aXv is:

$$aXv = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ 1 & 0 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}$$

$$= 0i - a_3 j - a_2 k \Longrightarrow (0, -a_3, -a_2) = (0, 0, 0)$$

therefore $a_3 = a_2 = 0$,

similarly we find $a_1 = 0$ when $v = (0, 0, 1)$

hence $a_1 = a_2 = a_3 = 0$, and $a = 0$.

Now, $\ker \psi = \{a \in R^3 \mid \psi(a) = 0\}$

$= \{a \in R^3 \mid f_a = 0\}$

$= \{a \in R^3 \mid f_a(v) = 0, \text{ for all } v \in R^3\}$

$= \{a \in R^3 \mid aXv = 0, \text{ for all } v \in R^3\}$

$= \{0\}$,

Hence ψ is one-to-one.

3. It is clear that ψ is onto.

4. Finally we show ψ preserves product, this means we need to show that $\psi([a,b]) = [\psi(a),\psi(b)]$

$$\psi([a,b])(v) = \psi(aXb)(v)$$

$$= f_{aXb}(v) = (aXb)Xv = aX(bXv) - bX(aXv)$$

$$= f_a(bXv) - f_b(aXv) = f_a(f_b)(v) - f_b(f_a)(v)$$

$$= (f_a of - f_b of_a)(v)$$

$$= [\psi(a),\psi(b)](v)$$

therefore ψ preserves product.

Let $f_a \in A(R^3)$; The matrix of f_a relative to the standard basis in R^3 is

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

where $a = \langle a_1, a_2, a_3 \rangle$.

Note that this matrix is antisymmetric, in this case we say the linear operator f_a is antisymmetric operator.

In general we define antisymmetric operators on the n-Euclidean space R^n as:

<u>**DEFINITION**</u> 1: Let $V = R^n$ be the n-dimensional Euclidean space. A linear operator $f : V \text{---}> V$ is called antisymmetric if $f^t = -f$, where f^t is the transpose of f

Let $f : V \text{---}> V$ be antisymmetric. Select a basis B for V, and let $[M]_B$ be the matrix associated with f relative to B. Then we have $[M]_B$ is antisymmetric matrix. This notation is independent of the choice of the basis, i.e. if $[M]_{B'}$ is the matrix of f relative to another basis B', then it is known that there exists invertible matrix N such that $[M]_{B'} = N^{-1}[M]_B N$, and hence

$$\left([M]_{B'}\right)^t = \left(N^{-1}[M]_B N\right)^t = N^t [M]_B^t \left(N^{-1}\right)^t$$
$$= N^t\left(-[M]_B\right)\left(N^{-1}\right)^t$$
$$= -\left(N^{-1}[M]_B N\right)^t$$
$$= -[M]_{B'}$$

i.e. $[M]_{B'}$ is antisymmetric if $[M]_B$ is antisymmetric. The entries of the matrix M must satisfy $m_{ij} = -m_{ji}$, and in particular, the diagonal entries of M must be 0.

Now we want to show that antisymmetric operators of R^3 are precisely the cross product by a fixed vector n.

<u>THEOREM</u> 2: Let f: $R^3 \longrightarrow R^3$ be antisymmetric operator. Then there exists a unique vector n such that f(v) = nXv for all v in R^3. The converse of this also true.

Proof: Every antisymmetric operator of R^3 has a matrix of the form:

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

and hence f(v) = nXv, for all v with

$$n = -a_{23} e_1 + a_{13} e_2 - a_{12} e_3 ,$$

where e_1, e_2, and e_3 is the standard orthonormal basis for R^3. Conversely we have seen the matrix of the linear operator f(v) = nXv is antisymmetric.

This theorem brings out the importance of the antisymmetric operators; they are distinct to generalize the cross product to higher dimensions.

In the sequal let $V = R^n$.

Let $AO(V) = \{f \in End(V) \mid f^t = -f\}$. Under the usual addition and scalar multiplication AO(V) is a vector subspace of End(V). The product gof of two antisymmetric operators f and g in general fails to be antisymmetric but by introducing a bracket multiplication we have :

<u>THEOREM</u> 3: AO(V) is a Lie subalgebra of $(End(V))_L$.

Proof: To show that AO(V) is closed under the bracket multiplication, [f,g] = fog - gof, let $f,g \in AO(V)$, then $f^t = -f$ and $g^t = -g$

$$[f,g]^t = (fog - gof)^t = (fog)^t - (gof)^t$$
$$= (g^t o f^t) - (f^t o g^t)$$
$$= (-g)o(-f) - (-f)o(-g)$$
$$= (gof) - (fog)$$
$$= -[f,g]$$

this implies $[f,g] \in AO(V)$.

Now we are going to find the dimension of AO(V) by costructing a basis for it as follows:

Let $B = \{e_1, e_2, \ldots, e_n\}$ be an arbitrary basis of V.

Consider the dual basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ for the dual space V^* of V. Recall that the dual basis satisfy the following condition:

$$e_i^*(e_j) = \delta_{ij} \quad \text{(Kronecker's delta)}$$

Now we define $s_{ij}(v) = e_j^*(v)e_i$, $\quad (i,j = 1, 2, \ldots, n)$,

for each $v \in V$.

It is easy to check that $s_{ij} \in End(V)$, and satisfy the following conditions:

1. $s_{ij}^2(v) = s_{ij}(v)\,\delta_{ij}$

2. $[s_{ij}, s_{rs}] = \begin{cases} s_{ij} & \text{iff } j = r \text{ and } i = s \\ 0 & \text{otherwise} \end{cases}$

3. $\{s_{ij} \mid i,j = 1, \ldots, n\}$ form a basis for End(V)

In fact this proves that $\dim(End(V)) = n^2$.

If the basis $\{ e_1, e_2, \ldots, e_n \}$ is orthonormal, it coincides with its dual, i.e. $e_i^* = e_i$, $i = 1, 2, \ldots, n$ in this case $s_{ij}^t = s_{ji}$ for $i, j = 1, 2, \ldots, n$.

Define $t_{ij} = (-1)^{i+j}(s_{ij} - s_{ij}^t)$, $i,j = 1, 2, \ldots ,n$.

One can show the set $\{t_{ij} \mid 1 <= i < j <= n \}$ satisfies the following conditions:

1. The set has $\frac{n(n-1)}{2}$ elements and each element is antisymmetric operator.
2. $[t_{ij} , t_{rs}] = t_{is} . \delta_{jr}$
3. The set is linearly independent in AO(V)
4. The set spans AO(V).

Now let us pull things together, we have shown that if dim V = n, then the Lie algebra of antisymmetric operators AO(V) has dimension $\frac{n(n-1)}{2}$. Therefore if $n \neq 0$, then

$$\frac{n(n-1)}{2} = n \iff n = 3$$

Since there is a natural Lie algebra isomorphism between R^3 and $AO(R^3)$, we have the following theorem:

<u>THEOREM</u> 4: There is a natural Lie algebra isomorphism between R^n and the Lie algebra of antisymmetric operators $AO(R^n)$ if and only if $n = 3$.

One final remark about the Lie algebra structure of R^3, it is compatible with the usual inner product on R^3. By this we mean that

$$|aXb|^2 = |a|^2 |b|^2 - (a.b)^2$$

i.e. the length of aXb equal to the area of the parallelogram spanned by a and b.

In fact $n = 3$ is the only case in which it is possible to convert R^3 into a noncommutative Lie algebra over R so that the Lie product is compatible with the inner product on R^3. This is rather a deep result and its proof is beyond the scope of this thesis.

III.IV IDEALS AND HOMOMORPHISMS

In this section we study analogues, for Lie algebras, of some of the concepts we encountered in algebras, concerning quotient algebras and algebra homomorphisms.

DEFINITION 1: Let L be a Lie algebra. By a sub-Lie algebra of L we mean a subalgebra L' of L which is itself a Lie algebra relative to the multiplication on L.

DEFINITION 2: A sub-Lie algebra B of a Lie algebra L is called an ideal of L if $[b,a] \in B$, for every $b \in B$ and $a \in L$.

REMARK 1: Since $[b,a] = -[a,b]$, the condition in the definition could just as well be written $[a,b] \in B$. Thus in the case of Lie algebras "left ideal" coincides with " right ideal ".

Ideals play the role in Lie algebra theory which are played by normal subgroups in group theory, and by two sided ideals in ring theory; They arise as kernels of homomorphisms.

DEFINITION 3: Let L be a Lie algebra. The center of L is defined by $Z(L) = \{x \in L : [x,y] = 0 \text{ for all } y \in L\}$.

THEOREM 1: $Z(L)$ is an ideal of L.

Proof: To show $Z(L)$ is an ideal we need to show

1. $Z(L)$ is a vector subspace of L.
2. for all $z \in Z(L)$, and for all $x \in L$, then $[z,x] \in Z(L)$.

1. To show $Z(L)$ is a vector subspace of L, let $x,x' \in Z(L)$ then $[x,y] = [x',y] = 0$ for all $y \in L$.

   The bilinearity of multiplication implies:

   $[x+x',y] = [x,y] + [x',y] = 0$ for all $y \in L$.

   $[\alpha x,y] = \; [x,y] = \; (0) = 0$ for all $\alpha \in F$ and $y \in L$.

   Hence $Z(L)$ is a vector subspace of L.

2. Let $z \in Z(L)$ and $x \in L$, we need to show $[z,x] \in Z(L)$.

   For any $y \in L$, Jacobi's identity implies:

   $$\begin{aligned}[][[z,x],y] &= -[[x,y],z] - [[y,z],x] \\ &= -[0,z] - [0,x] \\ &= 0\end{aligned}$$

   Hence $[z,x] \in Z(L)$. This complete the proof.

THEOREM 2: If A and B are two ideals of a Lie algebra L, then $A + B = \{a + b \mid a \in A, b \in B\}$ and

$$[A, B] = \{\sum_{i=1}^{n} [a_i, b_i] \mid a_i \in A, b_i \in B\} \text{ are}$$

ideals of L.

Proof: Let $a \in A$, $b \in B$, and $x \in L$.

If $a \in A$ means $[a,x] \in A$, and if $b \in B$ means $[b,x] \in B$,

$[a + b,x] = [a,x] + [b,x]$

but $[a,x] + [b,x] \in A + B$, therefore $A + B$ is an ideal.

To prove [A,B] is an ideal of L we follow the same steps of the proof of the first part of this theorem. Let $a \in A$, and $b \in B$. Then $[a,x] \in A$ and $[b,x] \in B$ for every x L. Then we have the following

$[[a,b],x] = [a',x] \in A$ where $[a,b] = a'$,

$[[a,b],x] = [b',x] \in B$ where $[a,b] = b'$,

therefore [A,B] is an ideal of L

By using this theorem we can define the ideal $L^2 = [L,L]$ which is called the derived algebra or commutator algebra of L.

EXAMPLE 1: Let L = gl(n,F), the general linear Lie algebra. The center of L is the set of all nxn scalar matrices, i.e. $Z(gl(n,F)) = \{dI_n : d \in F$ and I_n is the identity matrix of order $n\}$.

Clearly the set of all nxn scalar matrices is contained in the center, since

$$[dI_n, A] = (dI_n)A - A(dI_n) = dA - Ad = 0$$

To see the reverse inclusion, let $A = (a_{ij})$ be an element in the center of L and consider the matrix $E_{pq} = (\delta_{ip} . \delta_{qj})_{i,j}$ for any p and q with $1 <= p,q <= n$, where as usual δ_{ij} is the Kronecker's delta. Now, $A \in Z(L)$ implies $[A, E_{pq}] = 0$, which implies $AE_{pq} = E_{pq}A$, which in turn implies $a_{ip}\delta_{qj} = \delta_{ip}a_{qj}$ for all i, j with $1 <= i,j <= n$. Given i and j, and choose $p = q = j$ to obtain $a_{ij} = \delta_{ij}a_{jj}$, which implies $a_{ij} = 0$ if i = j and also $a_{ii} = a_{jj}$ for any i and j. Thus A is a scalar matrix. Note that L is abelian if and only if L = 0, and in this case Z(L) = L.

<u>**EXAMPLE**</u> **2:** In this example we are going to show that if $L = gl(n,f)$, then $L^2 = [L,L] = sl(n,F)$. Let $X \in L^2$, then $X = \sum_{i=1}^{m} [A_i, B_i]$, where $A_i, B_i \in gl(n,F)$ for all $1 <= i <= m$. Since $Tr([A_i,B_i]) = Tr(A_iB_i - B_iA_i) = 0$ for each $i = 1, 2, \ldots, m$, then $L^2 \subseteq sl(n,F)$.

In order to show the reverse inclusion we will make use of the matrices E_{pq} introduced in the previous example.

First note that every element of sl(n,F) can be written in the form:

$$diag(\alpha_1, \alpha_2, \ldots, \alpha_n) + \sum_{i=j} a_{ij}E_{ij}, \text{ where } \sum_{i=1}^{n} \alpha_i = 0.$$

$$diag(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n-1} \alpha_i (E_{ii} - E_{nn}).$$

Since $[E_{ik}, E_{kj}] = E_{ij}$ for $i \neq j$ it follows that $\sum_{i \neq j} a_{ij}E_{ij}$ belongs to [L,L]; and since $[E_{in}, E_{ni}] = E_{ii} - E_{nn}$

for all i, it follows that diag $(\alpha_1, \ldots, \alpha_n)$ belongs to $[L,L]$. Hence $sl(n,F) \subseteq [L,L]$, and thus $sl(n,F) = [L,L]$.

DEFINITION 4: Let L be a Lie algebra. If L has no ideals except itself and {0}, and if moreover $L^2 = [L,L] \neq \{0\}$, then L is called simple.

The condition $L^2 \neq 0$, which is equivalent to saying L is non abelian, is imposed in order to avoid giving over prominence to the one-dimensional Lie algebra.

EXAMPLE 3: The three-dimensional Lie algebra R^3 with multiplication defined by the cross product is a simple Lie algebra. R^3 has no proper subalgebras other than the one-dimensional subalgebras, which are clearly not ideals. To see that R^3 has no two-dimensional Lie subalgebras, assume the contrary, i.e. assume S is a two dimensional Lie subalgebra of R^3. Then S contain two Linearly independent vectors e_1 and e_2, then it would follow that $a = [e_1, e_2]$ would have to be distinct from 0 and perpendicular to the plane S, which is impossible since $a = [e_1, e_2] \in S$.

The construction of a quotient Lie algebra L/B, where B is an ideal of L is formally the same as the construction of a quotient algebra: as a vector space L/B is just the quotient space, while its Lie multiplication is defined by $[x+B,y+B] = [x,y] + B$. This multiplication is well-defined, since if $x+B = x'+B$ and $y+B = y'+B$, then we have $x' = x+b_1 \mid (b_1 \in B)$, and $y'= y+b_2 \mid (b_2 \in B)$, whence

$$[x',y'] = [x+b_1,y+b_2] = [x,y] + ([b_1,y]+[x,b_2]+[b_1,b_2])$$

and therefore $[x',y']+B = [x,y]+B$, since the terms in the parenthesis are all in B.

<u>**DEFINITION**</u> 5: Let L and L' be Lie algebras over a field F. A Linear transformation $\psi: L \longrightarrow L'$ is called a Lie algebra homomorphism if $\psi([x,y]) = [\psi(x),\psi(y)]$, for all $x,y \in L$.

If ψ is also one-to-one, then it is called an isomorphism.

<u>**THEOREM**</u> 3: Let L and L' be Lie algebras over F. And $\psi : L \longrightarrow L'$ a Lie algebra homomorphism, then the image of ψ, $\psi(L)$ is a sub-Lie algebra of L', and the kernel of ψ, $\ker(\psi) = \{x \in L : \psi(x)=0\}$ is an ideal.

Proof: 1. We need to show that $\psi(L)$ is closed under the bracket multiplication. That is let $a'_1, a'_2 \in \psi(L)$, then we need to show $[a'_1,a'_2] \in \psi(L)$

we know $[a'_1,a'_2] = [\psi(a_1), \psi(a_2)]$

$$= \psi[a_1,a_2]$$

therefore $[a'_1,a'_2] \in \psi(L)$

2. First we show, $\psi(L) = \mathrm{Im}(\psi)$ is a verctor subspace.

$\psi(L) = \{ \psi(a) \mid a \in L\} = \{ a' \in L' \mid a' = \psi(a)$ for some $a \in L\}$.

1. Show $\psi(L)$ is closed under addition.

   Let $a'_1, a'_2 \in \psi(L)$, we need to show $a'_1 + a'_2 \in \psi(L)$.

   $a'_1 = \psi(a_1)$ and $a'_2 = \psi(a_2)$ for some $a_1, a_2 \in L$

   then $a'_1 + a'_2 = \psi(a_1) + \psi(a_2)$

   $= \psi(a_1 + a_2)$

   therefore $a'_1 + a'_2 \in \psi(L)$

2. Show $\psi(L)$ is closed under scalar multiplication.

   Let $a'_1 \in \psi(L)$ and $\alpha \in F$. What we need to show

   $\alpha a'_1 \in \psi(L)$

   $\alpha a'_1 = \psi(\alpha a_1)$

   $= \alpha\psi(a_1)$

   but $\psi(a_1) \in \psi(L)$

   therefore $\alpha a'_1 \in \psi(L)$

3. We need to show that $\ker(\psi) = \{x \in L \mid \psi(x) = 0\}$ is an ideal of L.

   First we need to show $\ker(\psi)$ is a vector subspace of L.

   1. Show $\ker(\psi)$ is closed under addition.

      Let $a, b \in \ker(\psi)$ we need to show $a + b \in \mathrm{Ker}(\psi)$

$\psi(a + b) = \psi(a) + \psi(b)$

$= 0$

therefore $a + b \in \ker(\psi)$.

2. Show $\ker(\psi)$ closed under scalar multiplication.

Let $a \in \ker(\psi)$, $\alpha \in F$. We need to show

$\alpha a \in \ker(\psi)$

$\psi(\alpha a) = \alpha\psi(a)$

$= 0$

therefore $\alpha a \in \ker(\psi)$

3. If $a \in \ker(\psi)$ and $b \in L$, we need to show that

$[a,b] \in \ker(\psi)$

$\psi([a,b]) = [\psi(a), \psi(b)]$

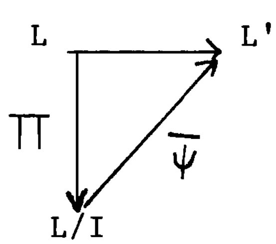
$= [0,b']$

$= 0$

therefore $[a,b] \in \ker(\psi)$.

The standard homomorphism theorems of algebras have their counterparts for Lie algebras we cite:

<u>THEOREM</u> 4: If ψ: L ---> L' is a homomorphism of Lie algebras, then $L/\ker(\psi)$ is isomorphic to $\mathrm{Im}(\psi)$. If I is any ideal of L included in $\ker(\psi)$, there exists a unique homomorphism $\bar{\psi}$: L/I ---> L' making the following diagram commute:



Here Π : L ---> L/I is the natural homomorphism x |---> x + I.

<u>THEOREM</u> 5: If I and J are ideals of L such that $I \subset J$ then J/I is an ideal of L/I and (L/I)/(J/I) is isomorphic to L/J.

<u>THEOREM</u> 6: Let B be a sub-algebra and I an ideal of a Lie algebra L; then B+I is a subalgebra of L, $B \cap I$ is an ideal of B and $B/(B \cap I)$ is isomorphic to (B+I)/I as Lie algebras.

Here we shall recall the sub-Lie algebras of the general Linear Lie algebra gl(n,F),

1. The special Linear sub-Lie algebra sl(n,F) = $\{ A \in gl(n,F) : trace(A)=0 \}$. In fact sl(n,F) is an ideal of gl(n,F).
2. The sub-Lie algebra of skew-symmetric matrices
$$so(n,F) = \{ A \in gl(n,F) : A^t = -A \}$$
3. Let n=2m, the symplectic sub-Lie algebra sp(n,F), which by definition consists of all matrices $A \in gl(n,F)$ such that $A^t J + JA = 0$, for some matrix J, which has the form:
$$J = \begin{vmatrix} 0 & I_m \\ -I_m & 0 \end{vmatrix}$$
where I is the identity matrix of order m, and 0 is the zero matrix of order m.
4. The sub-Lie algebra of upper triangular matrices, $ut(n,F) = \{A \in gl(n,F) : a_{ij} = 0 \text{ for } i>j\}$.

CHAPTER FOUR

LIE ALGEBRA OF DERIVATIONS

IV.I DERIVATION ALGEBRA

Some Lie algebras of linear transformations arise most naturally as derivations of algebras. In this section we will study the Lie algebra of derivations.

DEFINITION 1: Let A be any algebra over F (not necessarily associative). A derivation D in A is a linear mapping D : A ---> A satisfying:

$$D(xy) = D(x).y + x.D(y)$$

for all x, y $\in$ A.

EXAMPLE 1: Let A be the R-algebra of functions of R into R which have derivatives of all orders. Let D be the differential operator. Then the mapping D: A ---> A given by D(f) = f' (the derivative of f) is a derivation A.

We denote the set of all derivation of an F-algebra A by $Der_F(A)$. Since derivation mappings are F-endomorphisms of A we can define the sum of two derivations D_1 and D_2 by: $(D_1+D_2)(x) = D_1(x) + D_2(x)$ for every $x \in A$ and the multiplication of derivation D by a scalar α by:

$$(\alpha D)(x) = \alpha(D(x)) \quad \text{for every } x \in A.$$

With respect to these operations we have the following:

<u>THEOREM</u> 1: $Der_F(A)$ is a vector subspace of $End_F(A)$.

Proof: We need to show if $D_1, D_2 \in Der_F(A)$ and $\alpha_1, \alpha_2 \in F$ then $\alpha_1 D_1 + \alpha_2 D_2 \in Der_F(A)$. That is we must show

$$(\alpha_1 D_1+\alpha_2 D_2)(xy) = (\alpha_1 D_1+\alpha_2 D_2)(x).y + x(\alpha_1 D_1+\alpha_2 D_2)(y)$$

$$\begin{aligned}(\alpha_1 D_1+\alpha_2 D_2)(xy) &= (\alpha_1 D_1)(xy) + (\alpha_2 D_2)(xy)\\ &= \alpha_1 D_1(xy) + \alpha_2 D_2(xy)\\ &= \alpha_1(D_1(x).y + xD_1(y)) + \alpha_2(D_2(x).y + xD_2(y))\\ &= \alpha_1 D_1(x).y + \alpha_1 xD_1(y) + \alpha_2 D_2(x).y + \alpha_2 xD_2(y)\\ &= (\alpha_1 D_1(x) + \alpha_2 D_2(x)).y + x(\alpha_1 D_1(y) + \alpha_2 D_2(y))\\ &= (\alpha_1 D_1 + \alpha_2 D_2)(x).y + x(\alpha_1 D_1 + \alpha_2 D_2)(y)\end{aligned}$$

which completes the proof.

On $Der_F(A)$ it is possible to define an algebraic composition of two derivation D_1 and D_2 by $D_1 \circ D_2$, the composition of D_1 and D_2 in the ordinary sense. Then for every $x,y \in A$, we have

$$(D_1 o D_2)(xy) = D_1(D_2(xy)) = D_1(D_2(x)y + xD_2(y))$$
$$= (D_1(D_2(x))(y) + D_2(x)D_1(y)$$
$$+ (D_1(x))(D_2(y)) + x(D_1(D_2(y)))$$
$$= ((D_1 o D_2)(x))(y) + (D_2(x))(D_1(y))$$
$$+ (D_1(x))(D_2(y)) + x((D_1 o D_2)(y))$$

which, in general, is not equal to

$((D_1 o D_2)(x))(y) + x((D_1 o D_2)(y))$ because the sum

$(D_2(x))\ (D_1(y)) + (D_1(x))\ (D_2(y)) \neq 0$, generally.

Hence $Der_F(A)$ is not closed under this operation. However, we shall see that $Der_F(A)$ can be made into a Lie algebra if we define the bracket multiplicaion by:

$$[D_1, D_2] = D_1 o D_2 - D_2 o D_1 .$$

To see that $Der_F(A)$ is a Lie algebra with respect to the bracket operation we first show the following property:

LEMMA 1: For every D_1, $D_2 \in Der_F(A)$, $[D_1,D_2]$ $Der_F(A)$.

i.e. the bracket multiplication is closed in $Der_F(A)$.

Proof: For any x and y in $Der_F(A)$ we have

$$
\begin{aligned}
[D_1,D_2](x.y) &= (D_1 o D_2 - D_2 o D_1)(x.y) \\
&= (D_1 o D_2)(x.y) - (D_2 o D_1)(x.y) \\
&= (D_1(D_2))(x.y) - (D_2(D_1))(x.y) \\
&= D_1((D_2 x).y + x.(D_2 y)) \\
&\quad - D_2((D_1 x).y - x.(D_1 y)) \\
&= ((D_1 o D_2)x).y + x.((D_1 o D_2)y \\
&\quad - ((D_2 o D_1)x).y + x.((D_2 o D_1)y) \\
&= ([D_1,D_2]x).y + x.([D_1,D_2]y)
\end{aligned}
$$

Thus $[D_1,D_2] \in Der_F(A)$.

Hence it follows that $Der_F(A)$ is a subalgebra of the algebra $End_F(A)$ of endomorphism of the vector space A.

Finally we state the following:

THEOREM 2: $Der_F(A)$ is a Lie algebra with respect to the bracket multiplication.

We shall call $Der_F(A)$ the Lie algebra of derivations in A or simply the derivation algebra of A.

Next we are going to study the link between the Lie algebra of derivations $Der_R(A)$ and the group of automorphisms of A, where A is finite-dimensional algebra over the field R of real numbers. First, we state several useful properties of derivation mappings.

<u>THEOREM</u> 3: (Leibniz Rule) Let A be an algebra. For any $D \in Der_F(A)$ and $x,y \in A$, we have:

$$D^n(xy) = \sum_{j=0}^{n} \binom{n}{j} D^j(x).D^{n-j}(y),$$

where D^o is the identity map on A, $D^{i+1} = DoD^i$ for all i, and $\binom{n}{j}$ is the binomial coefficient,

Proof: Using mathematical induction

for n=1,

$$D^1(xy) = \binom{1}{0} xD(y) + \binom{1}{1} D(x).y$$

$$= x(D(y)) + y(D(x))$$

next we assume that $D^n(xy)$ is true for n=m, i.e.

$$D^m(xy) = \sum_{j=0}^{m} \binom{m}{j} D^j(x).D^{m-j}(y)$$

Now we are going to show the formula is true for n=m+1,

$$D^{m+1}(xy) = D(D^m(xy)) = \sum_{j=0}^{m} \binom{m}{j} D(D^j(x).D^{m-j}(y))$$

but $D(D^j(x).D^{m-j}(y)) = D(D^j(x)).D^{m-j}(y)+D^j(x).D(D^{m-j}(y))$

$$= D^{j+1}(x).D^{m-j}(y) + D^j(x).D^{m-j+1}(y)$$

$$D^{m+1}(xy) = \sum_{j=0}^{m} \binom{m}{j} D^{j+1}(x).D^{m-j}(y)+ \sum_{j=0}^{m} \binom{m}{j} D^j(x).D^{m-j+1}(y)$$

By changing the subscripts j to i-1 for the first sum above, and j to i for the second sum above, we obtain

$$D^{m+1}(xy)= \sum_{i=1}^{m+1} \binom{m}{i-1} D^i(x).D^{m-i+1}(y)+ \sum_{i=0}^{m} \binom{m}{i} D^i(x).D^{m-i+1}(y)$$

$$= \sum_{i=1}^{m+1} \left[\binom{m}{i-1}+\binom{m}{i}\right] D^i(x).D^{m-i+1}(y) +\binom{m}{0} D^0(x).D^{m+1}(y)$$

since $\binom{m}{0} = \binom{m+1}{0}$ and $\binom{m}{i-1} + \binom{m}{i} = \binom{m+1}{i}$, the above expression can be written into:

$$D^{m+1}(xy)= \sum_{i=1}^{m+1} \binom{m+1}{i} D^i(x).D^{m-i+1}(y) +\binom{m+1}{0} D^0(x).D^{m+1}(y)$$

$$= \sum_{i=0}^{m+1} \binom{m+1}{i} D^i(x).D^{m-i+1}(y)$$

now change i to j and m+1 to n we get the following:

$$D^n(xy) = \sum_{j=0}^{n} \binom{n}{j} D^j(x).D^{n-j}(y)$$

which completes the proof.

<u>THEOREM</u> 4: Let A be a commutative and associative algebra with identity element 1. For any $D \in Der_F(A)$ we have:

1) $D(\alpha.1) = 0$ for all $\alpha \in F$

2) $D(x^n) = nx^{n-1} D(x)$, for any $x \in A$, and $n >= 0$ where $x^o = 1$.

Proof: (1) To show $D(\alpha.1) = 0$, first we need to show $D(1) = 0$,

$$
\begin{aligned}
D(1) &= D(1.1) \\
&= 1.D(1) + D(1).1, \text{ but 1 is identity element} \\
&= D(1) + D(1)
\end{aligned}
$$

$$D(1) - D(1) = D(1) + D(1) - D(1)$$

$$0 = D(1)$$

but
$$
\begin{aligned}
D(\alpha.1) &= \alpha D(1) \\
&= \alpha.0 \\
&= 0
\end{aligned}
$$

(2) The proof is by using mathematical induction on n. For $n = 0$, it is always true.

suppose it is true for $n = k$. That is assume

$D(x^k) = kx^{k-1} D(x)$ is true

for $n = k+1$

$$
\begin{aligned}
D(x^{k+1}) = D(x.x^k) &= D(x).x^k + x.D(x^k) \\
&= D(x)x^k + x.kx^{k-1} D(x) \\
&= D(x).x^k + kx^k D(x) \\
&= (1+k).x^k D(x) = (k+1)x^k D(x)
\end{aligned}
$$

Thus $D(x^n) = nx^{n-1} D(x)$, for all $n>=0$.

We now assume F = R, the field of real numbers. Since char R = 0, we can divide both sides of

$D^n(xy) = \sum_{j=0}^{n} \binom{n}{j} D^j(x) D^{n-j}(y)$ by n! and obtain

$$\frac{1}{n!} D^n(xy) = \sum_{j=0}^{n} \left(\frac{1}{j!} D^j(x)\right)\left(\frac{1}{(n-j)!} D^{n-j}(y)\right)$$

Therefore, we can write down formally the series:

$$\sum_{n=0}^{\infty} \frac{D^n}{n!} = I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \ldots$$

where I is the identity map on A.

We want to show that in the case of A is a finite-dimensional algebra over the field R of reals the series converges for every derivation mapping D. To see this let dim A = m, and select a basis B for A, then for every $D \in Der_R(A)$, there is an mxm matrix with entries in R associated with D relative to B. Denote this matrix by $[M]_D$. Then the series above has the form:

$$\sum_{n=0}^{\infty} \frac{M^n}{n!} = I + M + \frac{M^2}{2!} + \ldots$$

where M stands for $[M]_D$ and I is the mxm identity matrix.

We have seen in chapter I that this series converges for any square matrix M. Moreover if N is the matrix of D relative to another basis B', then $\sum_{n=0}^{\infty} \frac{N^n}{n!}$ converges to the same limit.

Hence we shall write $\sum_{n=0}^{\infty} \frac{D^n}{n!} = \exp D.$

<u>THEOREM</u> 5: Let A be a finite-dimensional algebra over R. Then for every $D \in \mathrm{Der}_R(A)$, exp D is an algebra automorphism of A.

Proof: Clearly exp D is linear. For every $x,y \in A$, we have

$$(\exp D)(x)(\exp D)(y) = \left(\sum_{k=0}^{\infty}\left(\frac{D^k(x)}{k!}\right)\right)\left(\sum_{m=0}^{\infty}\left(\frac{D^m(y)}{m!}\right)\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} D^k(x) D^{n-k}(y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} D^k(x).D^{n-k}(y)$$

By Leibniz Rule

$$= \sum_{n=0}^{\infty} \frac{1}{n!} D^n(xy)$$

$$= (\exp D)(xy)$$

Therefore exp D is an algebra homomorphism for every $D \in \mathrm{Der}_R(A)$. Now, we need to show that exp D is a one-to-one map. If $\lambda_1, \ldots, \lambda_m$ is the set of all distinct eigenvalues of D then $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_m}$ is the set of all distinct eigenvalues of exp D with the same multiplicities. Hence the exponential of any matrix of D is non-singular. Therefore exp D is one-to-one.

COROLLARY : The Lie algebra of derivations $Der_R(A)$ is isomorphic to the Lie algebra of automorphisms of A, Aut(A).

IV.II INNER DERIVATIONS OF ASSOCIATIVE AND LIE ALGEBRAS

Let A be an associative algebra over a field F. If a is any element of A, then a determines two mappings $a_L : x \mapsto ax$ and $a_R : x \mapsto xa$ of A into A. These are called the left multiplication and the right multiplication by a. The bilinearity conditions for the algebra multiplication in A, implies that a_L and a_R are linear mappings. Let $D_a = a_R - a_L$. Hence, D_a is a linear mapping of A into A. We also have

$$\begin{aligned} D_a(xy) &= xya - axy \\ &= xya - axy + xay - xay \\ &= (xa - ax)y + x(ya - ay) \\ &= D_a(x)y + xD_a(y) \end{aligned}$$

hence D_a is a derivation in A. We call D_a the inner derivation determined by a.

THEOREM 1: If A is an associative algebra then the inner derivations in A is an ideal of $Der_F(A)$.

Proof: Let $D_a \in Inn(A)$ and $D \in Der_F(A)$

where $\text{Inn}(A) = \{D_a \mid a \in A\}$

$$= \{a_R - a_L \mid a \in A\}$$

Show that $[D_a, D] \in \text{Inn}(A)$. That is we must show

$[D_a, D] = b_R - b_L$ for some $b \in A$.

$$[D_a, D](x) = (D_a o D - D o D_a)(x)$$

$$= ((a_R - a_L) o D - (D o (a_R - a_L)))(x)$$

$$= ((a_R - a_L) o D)(x) - (D o (a_R - a_L))(x)$$

$$= a_R(D)(x) - a_L(D)(x) - (D(a_R(x)) + D(a_L(x))$$

$$= D(x)a - aD(x) - D(xa) + D(ax)$$

$$= D(x)a - aD(x) - D(x)a - xD(a) + D(a)x + aD(x)$$

$$= D(a)x - xD(a)$$

$$= x(-D(a)) - (-D(a)x)$$

$$= (-D(a))_R - (-D(a))_L$$

$= b_R - b_L$ by letting $b = -D(a)$

therefore $[D_a, D]$ is an inner derivation, hence the inner derivations in A is an ideal in $\text{Der}_F(A)$.

Next let L be a Lie algebra, with the algebra multiplication in L denoted by $[x,y]$ for all $x,y \in L$. Now we are going to study the concept of inner derivations in L. We first introduce the very useful concept of " adjoint mappings ".

<u>**DEFINITION**</u> **1:** Let L be a Lie algebra and a an element of L. The linear mapping $x \mapsto [a,x]$ of L into L is called the adjoint mapping of a and is denoted by ad a.

<u>**THEOREM**</u> **2:** If L is a Lie algebra, then ad a is a derivation in L, for each $a \in L$.

Proof: Evidently ad a is an endomorphism of L. Moreover

$$(\text{ad } a)[x,y] = [a,[x,y]]$$
$$= -[y,[a,x]] - [x,[y,a]] \text{ by the Jacobi Identity}$$

Since multiplication in a Lie algebra is anticommutative we have

$$(\text{ad } a)[x,y] = [[a,x],y] + [x,[a,y]]$$
$$= [(\text{ad } a)(x),y] + [x,(\text{ad } a)(y)]$$

Thus ad a is a derivation in L.

<u>**DEFINITION**</u> **2:** The mapping ad a is also called the inner derivation determined by $a \in L$.

Let L be a Lie algebra, let $\text{Adj}\,(L) = \{\text{ad } a : a \in L\}$ denote the set of all adjoint mappings in L (i.e the set of all inner derivations in L).

<u>**THEOREM**</u> **3:** If L is a Lie algebra over a field F, then Adj (L) is an ideal in $\text{Der}_F\,(L)$.

Proof: We first show that Adj(L) is a vector subspace of $\text{Der}_F\,(L)$, by showing

$$\text{ad } a + \text{ad } b = \text{ad}(a+b) \quad \text{and}$$
$$\alpha\, \text{ad } a = \text{ad}(\alpha a), \text{ for all } a,b \in L \text{ and } \alpha \in F.$$

This follows immediately from the identities below:

$$\text{ad }(a+b)(x) = [a+b,x] = [a,x] + [b,x]$$
$$= (\text{ad } a)(x) + (\text{ad } b)(x)$$

$$\text{and } (\text{ad } \alpha a)(x) = [\alpha a,x] = \alpha[a,x]$$
$$= \alpha(\text{ad } a)(x)$$

Next we show that $[D,\text{ad } a] \in \text{Adj}(L)$ for any $a \in L$ and $D \in \text{Der}_F(L)$, thus establishing the fact that Adj (L) is closed under multiplication of elements from $\text{Der}_F(L)$. Consider:

$$[D,\text{ad } a](x) = (D\text{o}\text{ad } a - \text{ad } a \text{ o } D)(x)$$
$$= D\text{o}[a,x] - [a,D(x)]$$
$$= [D(a),x] + [a,D(x)] - [a,D(x)]$$
$$= [D(a),x]$$
$$= (\text{ad } D(a))(x)$$

Therefore Adj(L) is an ideal in $\text{Der}_F(L)$.

Now since Adj(L) is an ideal in $\text{Der}_F(L)$ we can construct the quotient Lie algebra of $\text{Der}_F(L)$ by Adj(L). We call it the Lie algebra of outer derivations on L and we denoted by

$$\text{Out}(L) = \text{Der}_F(L)/\text{Adj}(L)$$

<u>THEOREM</u> 4: Out(L) is an ideal of L.

The proof follows immediately from the following lemma.

<u>LEMMA</u> 1: If L is a Lie algebra and J an ideal of L then L/J is an ideal of L.

Proof: Let a+J be an element in L/J where $a \in L$, then for any $a' \in L$, $[(a+J),a'] = [a,a'] + J \in L/J$.
Hence L/J is an ideal of L.

<u>THEOREM</u> 5: Let L be a Lie algebra.
Then ad : L ---> $End_F(L)$ is a Lie algebra homomorphism.

Proof: We have seen that ad (a+b) = ad (a) + ad (b) and ad $(\alpha a) = \alpha$ ad (a) for all $a,b \in L$ and $\alpha \in F$, hence ad is a vector space homomorphism of L into $End_F(L)$.
It remains to show ad preserves multiplication.
Let $a,b \in L$, then we must show
ad [a,b] = (ad a o ad b) - (ad b o ad a)
That is we must show for any $x \in L$,
ad [a,b](x) = (ad a o ad b)(x) - (ad b o ad a)(x)
ad [a,b](x) = [[a,b],x]
= - [[b,x],a] - [[x,a],b] by Jacobi's identity
= [a,[b,x]] - [b,[a,x]]
= (ad a)([b,x]) - (ad b)([a,x])
= (ad a)(ad b(x)) - (ad b)(ad a(x))
= (ad a o ad b)(x) - (ad b o ad a)(x).
Therefore ad is a Lie algebra homomorphism.
The kernel of this homomorphism,

$$\ker(ad) = \{ x \in L : [x,y] = 0 \text{ for all } y \in L \}$$

is an ideal of L. This ideal is called the center of L.

<u>DEFINITION</u> 3: Let L be a Lie algebra over F.
A subspace B of L is called a characteristic ideal of L if B is stable under every derivation of L.

i.e. $D(b) \in B$ for every $D \in Der_F(L)$ and $b \in B$.

THEOREM 6: Let L be a Lie algebra, then the center Z(L) is a characteristic ideal of L.

Proof: Firts recall that Z(L) is an ideal of L (Theorem 1 chapter 3). It remains to show Z(L) is stable under every $D \in Der(L)$.

Let $z \in Z(L)$ and $D \in Der(L)$, we need to show $D(z) \in Z(L)$, this means we must show $[D(z),y] = 0$ for any $y \in L$.

Now since $z \in L$, then $[z,y] = 0$ for all $y \in L$, then $D([z,y]) = 0$, also we have

$$D([z,y]) = [D(z),y] + [z,D(y)]$$
$$= [D(z),y] + 0,$$

thus $[D(z),y] = 0$ and hence $D(z) \in Z(L)$.

Therefore Z(L) is a characteristic ideal of L.

DEFINITION 4: A Lie algebra L is said to be complete if

1. $Z(L) = \{0\}$
2. $Der_F(L) = Adj(L)$.

THEOREM 7: Let L be a Lie algebra and I an ideal of L. If I is complete, there is an ideal J of L such that $L = I \oplus J$.

Proof: Consider the set $J = \{ x \in L : [x,a] = 0,$ for every $a \in I\}$.

Claim: J is an ideal in L.

Evidently J is a subspace of L. Let $b \in J$ and $x \in L$,

then by Jacobi identity $[a,[b,x]] = -[x,[a,b]]-[b,[x,a]]$
$= 0 - [b,a']$, where
$a' = [x,a] \in I$; hence $[a,[b,x]] = 0$ for all $a \in I$, and $[b,x] \in J$. Hence J is an ideal.

Next we will show that $I \cap J = \{0\}$. For let $c \in I \cap J$, then $[c,a] = 0$ for all $a \in L$, hence c is in the center of I, but since I is complete $Z(I) = \{0\}$, and thus $c = 0$ Hence $I \cap J = \{0\}$.

Now let $x \in L$, since I is an ideal of L, we define a derivation ad x which maps I into itself and hence by restricting ad x to I induces a derivation D in I. This is inner and so we have an element $a \in I$ such that $D(y) = [y,x] = [y,a]$ for all $y \in I$. Then $b = x - a \in J$ and $x = b + a$, thus $L = I + J$ and since $I \cap J = \{0\}$ then we have $L = I \oplus J$.

EXAMPLE 1: Let us consider the two-dimenisional Lie-algebras with basis $\{e_1,e_2\}$, where $[e_1,e_2] = e_1$, $[e_2,e_1] = -e_1$ and all other products of base elements are 0. The derived algebra $L^2 = [L,L] = \{\alpha e_1 : \alpha \in F\} = Fe_1$.

If D is a derivation in L then $D(e_1) = \alpha e_1$, for some $\alpha \in F$. Also ad (αe_2) has the property

$(\text{ad } (\alpha e_2))(e_1) = [\alpha e_2,e_1] = \alpha[e_2,e_1] = -\alpha e_1$. Hence

if $E = D + \text{ad } (\alpha e_2)$ then E is a derivation in L and

$$E(e_1) = D(e_1) + ad\,(\alpha e_2)(e_1) = \alpha e_1 + [\alpha e_2, e_1]$$

$$= \alpha e_1 - \alpha e_1 = 0.$$

Then $[e_1, e_2] = e_1$ implies $E(e_1) = E([e_1, e_2]) ===> 0$

$$= [E(e_1), e_2] + [e_1, E([e_2)] = 0 + [e_1, E(e_2)]$$

$$= [e_1, E(e_2)]$$

which implies $E(e_2) = \gamma e_1$ for some $\gamma \in F$.

Now consider $ad\,(\gamma e_1)(e_1) = [\gamma e_1, e_1] = 0$, and

$ad\,(\gamma e_1)(e_2) = [\gamma e_1, e_2] = \gamma e_1$.

Hence $E = ad\,(\gamma e_1)$ is an inner derivation and

$D = E - ad\,(\alpha e_2)$ is also inner derivation thus we have

shown that every derivation in L is inner.

Now let us find the center of L,

$$Z(L) = \{ x \in L : [x, y] = 0 \text{ for all } y \in L \}$$

$$= \{\alpha_1 e_1 + \alpha_2 e_2 \quad L : [\alpha_1 e_1 + \alpha_2 e_2 + \beta_1 e_1 + \beta_2 e_2] = 0\}$$

$$= \{\alpha_1 e_1 + \alpha_2 e_2 \quad L : \alpha_1 \beta_2 [e_1, e_2] + \alpha_2 \beta_1 [e_2, e_1] = 0\}$$

$$= \{\alpha_1 e_1 + \alpha_2 e_2 \quad L : \alpha_1 \beta_2 e_1 - \alpha_2 \beta_2 e_1 = 0 \}$$

$$= \{\alpha_1 e_1 + \alpha_2 e_2 \quad L : (\alpha_1 \beta_2 - \alpha_2 \beta_2) e_1 = 0$$

$$= \{0\} \text{ since } \{ e_1, e_2 \} \text{ is a basis.}$$

Thus L is a complete Lie algebra.

CHAPTER FIVE

SUMMARY AND CONCLUSION

The subject of Lie algebras has much to recommend it as a subject for study immediately following courses on general abstract algebra and linear algebra, both because of the beauty of its results and its structure, and because of its many contacts with other branches of mathematics.

In this thesis I have tried not to make the treatment too abstract and have consistently followed the point of view of treating the theory as a branch of liear algebra. No attempt has been made to indicate the historical development of the subject. I just want to point out that the theory of Lie algebras is an outgrowth of the Lie theory of continuous groups.

The purpose of this thesis is to introduce the basic ideas of Lie algebras to the reader with some basic knowledge of abstract and elementary linear algebra.

In this study, Lie algebras are considered from a purely algebraic point of view, without reference to Lie groups and differential geometry. Such a view point has the advantage of going immediately into the discussion of Lie algebras without first establishing the topological machineries for the sake of defining Lie groups from which Lie algebras are introduced.

In Chapter I we summarize for the reader's convenience rather quickly some of the basic concepts of linear algebra with which he is assumed to be familiar. In Chapter II we introduce the language of algebras in a form designed for material developed in the later chapters.

Chapters III and IV were devoted to the study of Lie algebras and the Lie algebra of derivations. Some definitions, basic properties, and several examples are given. In Chapter II we also study the Lie algebra of antisymmetric operators, Ideals and homomorphisms. In Chapter III we introduce a Lie algebra structure on $Der_F(A)$ and study the link between the group of automorphisms of A and the Lie algebra of derivations $Der_F(A)$.

Some of the materials introduced in this thesis consists mainly of materials of fairly recent origin, including some material on the general structure of Lie algebras.

Finally, through out this thesis I made a lot of efforts to only introduce the very basic concepts of this very extensive topic, and study the relationship between different concepts. Some important theorems are proved in details, and most of the examples are worked out completely.

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