VANISHING TRIPLE PRODUCTS

A Thesis

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Title: VANISHING TRIPLE PRODUCTS.

Abstract approved: Stefanos Gialamas

In studying Knot Theory, one fundamental problem is to determine whether two links are equivalent. Many polynomials are defined axiomatically or algebraically which answered partially the question of determining the equivalence of two links. Using the Linking Number, one can classify 2-component links into two classes: those that have Linking Number zero and those that do not. Using Triple Products one can classify links with 3-components into two classes: those that have all Triple Products zero and those that have at least one non-zero Triple Product. Determining Vanishing Triple Products using the definition is beyond the scope of this thesis since it requires an intensive study of Cohomology Group Theory and Lie Algebras. In this thesis, an algorithm developed by Dr. Stefanos Gialamas is used, in order to detect vanishing Triple Products in the complement of a link with 3-components. The algorithm requires a presentation of the fundamental group of the link (Wirtinger Presentation) and techniques from the Commutator Calculus and the Fox Derivatives.

The algorithm is applied to closed braids, which are links, and answers the question: which closed braids have all Triple Products vanished?

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DEDICATIONS

This thesis is dedicated to my father, Hj. Talib Mohd. Amin, my mother Hjj. Yang Chik Hj. Ahmad, my grandmother Hjj. Maimunnah, my brothers: Roslan and his wife, and Rosdi, and to my sisters: Rosnah, Rozita, and Zanariah and her husband Mohd. Salleh Mohd. Nor. This thesis is also dedicated to my late grandfather, Hj. Ahmad Embok who will always be in my memory. To all of the above, my deepest appreciation for your help, encouragement, support, patience, and trust during the course of my study in the United States.

VANISHING TRIPLE PRODUCTS

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CHAPTER 0

INTRODUCTION

A. Historical backgroud.

Triple Products on a complement of a link were introduced by Hiroshi Uehara and W. S. Massey in their paper entitled "The Jacobi Identity for Whitehead Products" as higher order cohomology operation. In 1967, W. S. Massey used homology theory to define Triple Products in his paper "Higher Order Linking Number" presented at the Conference of Algebraic Topology, University of Illinois at Chicago, Chicago 1968. David Kraines extended and analyzed the theory of Triple Products in his paper entitled "Loop Operations" which was also presented at the same conference.

A different approach into the study of k-fold products using Hodge Theory was introduced by John Morgan and Alan Durfee. Topologists such as Clint McCrory, Larry Lambe, and Richard Hain also contributed to the study of k-fold products by using Intersection Theory.

An algebraic approach to determine vanishing Triple Products was developed by Stefanos Gialamas in his Ph. D. desertation and later, k-folds Products. Moreover, this approach was applied to closed braids.

B. The Triple Products.

Let L be a link with more than two components. Let $H^1(\mathbf{R}^3-L;\mathbf{Z})$ be the first cohomology group of the complement of the link. Let $f,g,h \in H^1(\mathbf{R}^3-L;\mathbf{Z})$ and choose $\bar{f}$, $\bar{g}$, and $\bar{h}$

one-cocycles such that $[\bar{f}]=f$, $[\bar{g}]=g$, and $[\bar{h}]=h$. Choose one-cochains θ and φ such that $\bar{f}.\bar{g}=d\theta$ and $\bar{g}.\bar{h}=d\varphi$.

The two-cycles $c=\bar{f}\varphi-\theta\bar{h}$ represents the element

$$\langle f,g,h\rangle \in H^1(\mathbb{R}^3-L;\mathbb{Z})$$

which we call the Triple Product of f, g, and h.

C. Purpose of this thesis.

As stated in the abstract, the purpose of this thesis is not to determine Vanishing Triple Products by using the definition but to give the algorithm determining vanishing Triple Products developed by Stefanos Gialamas and apply it to closed braids and answer the question: Given a closed braid, determine if all Triple Products vanish?

We are only concerned with closed braids, since all links are combinatorially equivalent to some closed braids. If a closed braid has some non-vanishing Triple Product then the associated link cannot be pulled apart, as in the case of the Borromean Rings.

In order to use the algorithm, we need some backgroud on Commutator Calculus and Associate Algebra. Chapter 1 of this thesis is written for this purpose. In Chapter 2, we introduce the notions of links and braids, and we give the algorithm to find the Fundamental Group of the complement of a link, and its associated closed braid. The algorithm to determine vanishing Triple Products is given by Theorem 3.2. We also present some problems and their solutions concerning Vanishing Triple Products.

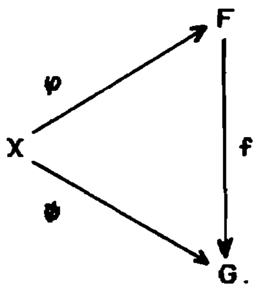
CHAPTER 1

COMMUTATOR CALCULUS and ALGEBRA

A. Free Groups.

DEFINITION 1.1.

Let X be an arbitrary nonempty set. A free group on X is a group F together with a map $\varphi : X \longrightarrow F$ such that for any map $\phi : X \longrightarrow G$ where G is any group, there exists a unique homomorphism $f : F \longrightarrow G$ such that the following diagram commutes :



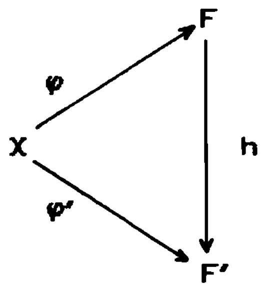
Remark 1.1.

This definition only characterizes a free group. We are yet to show the uniqueness and the existence of such a group. We are going to denote the free group F on a set X with respect to the function $\varphi : X \longrightarrow F$ by the pair (F, φ).

The following theorem gives another characteristic of a free group.

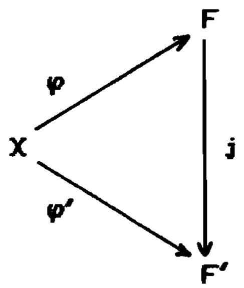
THEOREM 1.1. (Uniqueness theorem)

Let (F, φ) and (F', φ') be free groups on the same set X. Then there exists a unique isomorphism $h : F \longrightarrow F'$ such that the following diagram is commutative:

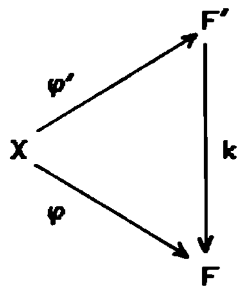


Proof.

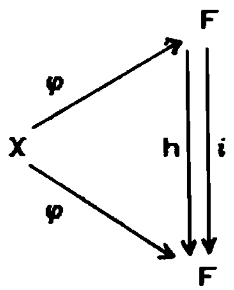
Since (F,φ) is a free group, then it follows from the definition that there exists a unique homomorphism $j:F \longrightarrow F'$ such that $j\circ\varphi=\varphi'$.



Similarly, there exists a unique homomorphism $k:F' \longrightarrow F$ such that $k\circ\varphi'=\varphi$.



Let $h=k\circ j$. Consider the following diagram:



Here, i denotes the identity mapping. Moreover,

$h\circ\varphi=k\circ j\circ\varphi=k\circ\varphi'=\varphi$.

$i\circ\varphi=\varphi$.

Hence, it follows from the uniqueness in the Definition 1.1. that $k\circ j=h=i$. But, since i is a monomorphism, j is one-to-one. Similarly, it can be shown that $j\circ k=i$ which implies that j is also onto. Therefore, j is an isomorphism.□

Let X be a nonempty (finite or infinite) set of symbols x_i, $i\in I$. We think of X as an alphabet and the x_i as letters in the alphabet. We shall denote these symbols also by x_i^1 and we construct another set X^{-1} that is disjoint from X such that $|X|=|X^{-1}|$ and denote the elements of X^{-1} by x_i^{-1}, $i\in I$ (for example take $X^{-1}=\{(x,1);x\in X\}$ and identify $(x,1)$ by x^{-1}).

DEFINITION 1.2.

A word w in X is a finite sequence of symbols from $X\cup X^{-1}$, written for convenience in the form

$$w=x_{\alpha_1}^{\epsilon_1}x_{\alpha_2}^{\epsilon_2}\ldots x_{\alpha_n}^{\epsilon_n}$$

where $x_{\alpha_i}\in X$, $\epsilon_i=\pm1$, and $n\geq 0\in I$. In case n=0 the sequence is empty and w is called the empty word which will be denoted by $\emptyset$. Two words are said to be equal if and only if they have the same symbols in corresponding positions. w is said to be reduced if it contains no pair of consecutive symbols of the form $x_{\alpha_i}x_{\alpha_i}^{-1}$ or $x_{\alpha_i}^{-1}x_{\alpha_i}$.

Let F(X) be the set of all reduced words on X. Let multiplication be the binary operation on the elements of F(X) where it is defined to be as follows:

If w_1 and w_2 are two reduced words where

$$w_1 = x_{\alpha_1}^{\varepsilon_1} x_{\alpha_2}^{\varepsilon_2} \cdots x_{\alpha_n}^{\varepsilon_n} \qquad (\varepsilon_i = \pm 1)$$

$$w_2 = x_{\beta_1}^{\delta_1} x_{\beta_2}^{\delta_2} \cdots x_{\beta_k}^{\delta_k} \qquad (\delta_i = \pm 1)$$

then, the product of w_1 and w_2, denoted by $w_1 w_2$, can be found by writing w_2 immediately following w_1, i.e.

$$w_1 w_2 = x_{\alpha_1}^{\varepsilon_1} x_{\alpha_2}^{\varepsilon_2} \cdots x_{\alpha_n}^{\varepsilon_n} x_{\beta_1}^{\delta_1} x_{\beta_2}^{\delta_2} \cdots x_{\beta_k}^{\delta_k}$$

But, the word on the right may not be reduced if $x_{\alpha_n}^{\varepsilon_n} = x_{\beta_1}^{-\delta_1}$. Therefore, we redefine the product of w_1 and w_2 by juxtaposition and (if necessary) carry out certain cancellations, that is to delete successive pairs of symbols with opposite exponent standing next to one another. Clearly, it can happen that in performing these cancellations we delete all the symbols of one of the factors w_1, w_2, or both.

The identity element for the multiplication of reduced words so defined is the empty word.

The inverse of w_1 is

$$w_1^{-1} = x_{\alpha_n}^{-\varepsilon_n} x_{\alpha_{n-1}}^{-\varepsilon_{n-1}} \cdots x_{\alpha_1}^{-\varepsilon_1}.$$

The proof of the associative law of the multiplication is a little laborious and will be omitted. Hence, the following lemma is proved:

LEMMA 1.1.

F(X) is a group with respect to the operation defined above.

The following theorem will show that the group F(X) is the free group on X.

THEOREM 1.2. (Existence theorem)

Let X be a non-empty set and F(X) be the group of all reduced words on X. Let $\varphi:X \longrightarrow F(X)$ be a map defined by $\varphi(x)=x'\in F$ for any $x\in X$. Then (F,φ) is the free group on the set X.

Proof.

Let G be any group and $\psi:X \longrightarrow G$ be any function. Define $f: F(X) \longrightarrow G$ as follows:

Let $w\in F(X)$. If w is the empty word $\emptyset$, we define $f(w)=1_G$, otherwise if w is in the form

$$w=x_{\alpha_1}^{\varepsilon_1}x_{\alpha_2}^{\varepsilon_2}\ldots x_{\alpha_n}^{\varepsilon_n}$$

we define

$$f(w)=[\psi(x_{\alpha_1})]^{\varepsilon_1}\ldots[\psi(x_{\alpha_n})]^{\varepsilon_n}.$$

Let w_1 and w_2 be reduced words as defined previously. Then

$$f(w_1w_2)=f(x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x_{\beta_1}^{\delta_1}\ldots x_{\beta_k}^{\delta_k})$$

$$=[\psi(x_{\alpha_1})]^{\varepsilon_1}\ldots[\psi(x_{\alpha_n})]^{\varepsilon_n}[\psi(x_{\beta_1})]^{\delta_1}\ldots[\psi(x_{\beta_k})]^{\delta_k}$$

$$=f(w_1)\circ f(w_2).$$

Therefore, f is a homomorphism. Moreover, $f\circ\varphi=\psi$.

To prove the uniqueness of f, let $g:F \longrightarrow G$ be an arbitrary homomorphism such that $g\circ\varphi=\psi$. Then, for any $w=x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}\in F(X)$ we have

$$g(w)=g(x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n})$$

$$=[g(x'_{\alpha_1})]^{\varepsilon_1}\ldots[g(x'_{\alpha_n})]^{\varepsilon_n}$$

$$=[g(\varphi(x_{\alpha_1}))]^{\varepsilon_1}\ldots[g(\varphi(x_{\alpha_n}))]^{\varepsilon_n}$$

$$=[\psi(x_{\alpha_1})]^{\varepsilon_1}\ldots[\psi(x_{\alpha_n})]^{\varepsilon_n}$$

$$=[f(x_{\alpha_1})]^{\varepsilon_1}\dots[f(x_{\alpha_n})]^{\varepsilon_n}$$

$$=f(w)$$

Hence $g(w)=f(w)$ which implies that $g\equiv f$.□

DEFINITION 1.3.

The group $F(X)$ is called the free group on the set X.

As we can see, the free group $F(X)$ does not depend on the individual properties of the elements of X. The rank of $F(X)$ is define to be the cardinal number of the set X. Let us shift our attention back to the map $\varphi:X\longrightarrow F(X)$ defined by $\varphi(x)=x'$. Since φ is one-to-one, we may identify x with its image $\varphi(x)$ in $F(X)$. Having done so, we can think of X as a subset of $F(X)$ since each element of $F(X)$ can be written as a product of elements of X. Thus, X constitutes a generating set for $F(X)$. The group $F(X)$ sometimes is referred to as the free group generated by X.

EXAMPLE 1.1.

Let $X=\{x_1,\dots,x_5\}$. Let w_1 and w_2 be two words of the elements of $X\cup X^{-1}$ such that $w_1=x_1x_2^{-1}x_3x_4x_4^{-1}x_3^{-1}x_5x_1$ and $w_2=x_1^{-1}x_5^{-1}x_3x_4^{-1}x_1$. The word w_2 is in the reduced form but w_1 is not. $w_1=x_1x_2^{-1}x_5x_1$ is the reduced form of w_1.

$$w_1w_2=x_1x_2^{-1}x_5x^1x_1^{-1}x_5^{-1}x_3x_4^{-1}x_1$$

$$= x_1x_2^{-1}x_3x_4^{-1}x_1.$$

$$w_1^{-1} = x_1^{-1}x_5^{-1}x_2x_1^{-1} \quad \text{and}$$

$$w_2^{-1} = x_1^{-1}x_4x_3^{-1}x_5x_1.$$

B. Group Presentation.

DEFINITION 1.4.

Let G be a group generated by a subset X of G. By a relation among elements of X, we mean a finite product $u_1u_2...u_n$ of elements of X or their inverses where $u_1u_2...u_n=1_G$.

THEOREM 1.3.(Nielsen-Schreier)

Any subgroup of a free group is free.

Remark 1.2.

The proof of the above theorem can be found in [9] page 95.

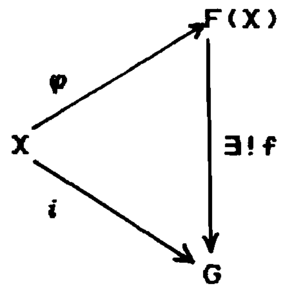
THEOREM 1.4.

Any group is a homomorphic image of a free group.

Proof.

Let G be any group. Let X be a set of generators of G (we can take X=G). By the existence theorem, there exists a free group F(X) with X as its set of generators.

Consider the following diagram:



f is a unique homomorphism such that

$$f\circ\varphi=i \Rightarrow (f\circ\varphi)(x)=f(\varphi(x))=i(x)$$

Moreover, f is onto. For any g∈G, g can be written as a finite product of elements of X, i.e.

$$g=x_{\alpha_1}^{\epsilon_1}x_{\alpha_2}^{\epsilon_2}...x_{\alpha_n}^{\epsilon_n} \quad \text{where } \epsilon_i=\pm 1.$$

Since $\varphi(x_{\alpha_1}^{\epsilon_1}x_{\alpha_2}^{\epsilon_2}\ldots x_{\alpha_n}^{\epsilon_n})\in F(X)$, $f(\varphi(x_{\alpha_1}^{\epsilon_1}x_{\alpha_2}^{\epsilon_2}\ldots x_{\alpha_n}^{\epsilon_n}))=g$. Hence $f(F(X))=G$.□

By the Fundamental Theorem of group homomorphism, we have,

COROLLARY 1.1.

Every group is isomorphic to a factor group of a free group.

Consider the following diagram:

$$\begin{array}{ccc} f:F(X) & \longrightarrow & G \\ \downarrow & \nearrow \simeq & \\ F(X)/\text{Ker } f & & \end{array}$$

Let R be a set of generators of the free group F(X). Since F(X) is completely determined by X and $N(R)\triangleleft F(X)$ is completely determined by R, then the group $G\simeq F(X)/N(R)$ can be completely described by specifying a set X, whose elements are called the generators of G and the set R, whose elements are called the defining relations of G. We denote this by $G=\{X|R\}$ where G is generated by the set $X\subset G$. This is the presentation of the group G.

From the above discussion, we can see that given a set of generators X and a set of relations R among the elements of X, we can find the group that is presented by $\{X|R\}$.

To explain the terminology let $r=u_1u_2\ldots u_n=1$ be a relation among the generators of G, where $u_i\in X\cup X^{-1}$. Let F(X) be the free group on the set X and let $i:X\longrightarrow F(X)$ be an inclusion identity mapping. Next, we will extend i to X^{-1} by setting $i(x^{-1})=(i(x))^{-1}$ $\forall x\in X$. So, i is define as

$$i(r)=i(u_1)i(u_2)\ldots i(u_n)$$

which is a unique element of F(X).

Let the homomorphism $f:F(X) \longrightarrow G$ be onto which satisfies

$$f(i(x))=x \text{ and}$$

$$f(i(x^{-1}))=x^{-1} \quad \forall x \in X.$$

Then,

$$f(i(r))=u_1u_2...u_n=r=1.$$

Hence, $i(r) \in \text{Ker } f$.

Conversely, let $\gamma \in \text{Ker } f$ and let

$$\gamma = i(u_1)i(u_2)...i(u_n).$$

Since $\gamma \in \text{Ker } f$, we have $f(\gamma)=\emptyset$. But

$$f(\gamma)=u_1u_2...u_n=1 \text{ in } G.$$

Thus, the "reduced" relations among the elements of X and the elements of the free group F(X) which lie in Ker f are in one-to-one correspondence.

EXAMPLE 1.2.

Let $G=\{1,a,a^2\}$ be a group where

$\bullet$	1	a	a^2
1	1	a	a^2
a	a	a^2	1
a^2	a^2	1	a

Then, G has a presentation $\{a|a^3=1\}$.

Remark 1.3.

Presentation of a group is not unique.

C. <u>Commutator Subgroup and Group Ring</u>.

DEFINITION 1.5.

The commutator of two elements x and y in a group G is an expression of the form $[x,y]=xyx^{-1}y^{-1}$. If X and Y are subsets of G then [X,Y] is the subgroup generated by all elements [x,y] where $x \in X$ and $y \in Y$.

DEFINITION 1.6.

The lower central series of a group G is the sequence of $G^{(n)}$ $(n \geq 1)$ defined inductively by

$$G^{(1)}=G,$$

$$G^{(n)}=[G^{(n-1)},G]$$

where $[G^{(n-1)},G]$ denotes the n-th commutator subgroup generated by all commutators $[x,y]=xyx^{-1}y^{-1}$ with $x \in G^{(n-1)}$ and $y \in G$.

Remark 1.4.

$G^{(1)} \triangleright G^{(2)} \triangleright \ldots \triangleright G^{(n-1)} \triangleright G^{(n)} \triangleright \ldots$. Moreover, $G^{(n-1)}/G^{(n)}$ is an abelian group.

DEFINITION 1.7.

A ring is a nonempty set R together with two binary operations (usually denoted as addition (+) and multiplication) such that:

(i) (R,+) is an abelian group.

(ii) $(ab)c=a(bc)$ for all $a,b,c \in R$.

(iii) $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$.

If in addition:

(iv) $ab=ba$ for all $a,b \in R$,

then R is said to be a commutative ring. If R contains an element 1_R such that

(v) $1_R a=a 1_R=a$ for all $a \in R$,

then R is said to be a ring with identity.

DEFINITION 1.8.

Let R be a ring with identity 1_R and G a multiplicative group. We define the group ring RG to be the set of all formal sums

$$RG=\left\{ \sum_{g\in G} r_g.g \mid r_g\in R \text{ and } r_g=0 \text{ except for finitely many } g\in G \right\}.$$

where the addition in RG is defined by:

$$\left(\sum_{g\in G} r_g.g \right) + \left(\sum_{g\in G} r'_g.g \right) = \sum_{g\in G}(r_g + r'_g).g \quad .$$

and multiplication in RG is define by:

$$\left(\sum_{g\in G} r_g.g \right) . \left(\sum_{g\in G} r'_g.g \right) = \sum_{g\in G} \left(\sum_{g_1g_2=g} r_{g_1}.r'_{g_2} \right).g \quad .$$

Remark 1.5.

RG with these two operations can be shown to form a ring with identity $1_R.1_G$ denoted by 1_{RG}.

EXAMPLE 1.3.

Let $\mathbb{Z}$ be the ring of integers and G be any group. Then the group ring $\mathbb{Z}G$ is define as follows:

$$\mathbb{Z}G=\left\{ \sum_{i} n_ig_i \mid n_i\in\mathbb{Z},\ g_i\in G \right\}$$

where the summation is a finite sum.

Remark 1.6.

The map $i:\mathbb{Z} \longrightarrow \mathbb{Z}G$ defined by $i(n)=n.1_G$ is a ring monomorphism. Thus under the identification $n\equiv n.1_G$, $\mathbb{Z}$ becomes a subring of the group ring $\mathbb{Z}G$.

Remark 1.7.

The map $j:G \longrightarrow \mathbb{Z}G$ given by $j(g)=1_{\mathbb{Z}}.g$ is a group monomorphism and under the identification $g\equiv 1_{\mathbb{Z}}.g$, G becomes a subgroup of $(\mathbb{Z}G,+)$.

D. <u>Commutator</u> <u>calculus</u>.

Remark 1.8.

Throughout this section, $\mathbf{F}$ will denotes the free group on a non-empty set.

DEFINITION 1.9.

Let G be any group and $\varepsilon:\mathbb{Z}G \longrightarrow \mathbb{Z}$ be defined by

$$\varepsilon\Big(\sum_{g\in G} n_g \cdot g\Big)=\sum_{g\in G} n_g.$$

It is called the augmentation map.

DEFINITION 1.10.

A map $d:\mathbb{Z}G \longrightarrow \mathbb{Z}G$ is called a derivation if

1. $d(\mu + \nu) = d(\mu) + d(\nu)$
2. $d(\mu\nu) = d(\mu)\varepsilon(\nu) + \mu d(\nu)$

where μ, $\nu\in\mathbb{Z}G$.

THEOREM 1.5.

a) $d(n\mu) = nd(\mu)$

b) $d(n) = 0$

c) $d(g^{-1}) = -g^{-1}d(g)$

for any $n\in\mathbb{Z}$, $g\in G$, and $\mu\in\mathbb{Z}G$.

COROLLARY 1.2.

If g, $h\in G$, then $d(gh)=d(g) + gd(h)$.

LEMMA 1.2.

Let $\mathbf{F}$ be the free group generated by $\{a_1,a_2,\ldots,a_n\}$. Then, to each generator a_i of $\mathbf{F}$ there corresponds a unique derivation $d_{a_i}:\mathbb{Z}\mathbf{F} \longrightarrow \mathbb{Z}\mathbf{F}$, called the derivation with respect to a_i, which has the property that

$$d_{a_i}(b)=\begin{cases} 1 \text{ if } a_i=b \\ 0 \text{ if } a_i\neq b \end{cases}.$$

PROOF.

The existence of the derivations d_{a_i} follow from the following formula:

Let $w=w_1a_i^{\epsilon_1}w_2a_i^{\epsilon_2}\ldots w_ka_i^{\epsilon_k}w_{k+1}$ where $\epsilon_i=\pm 1$ and $w_1,\ldots,w_{k+1}$ are words in $\mathbf{F}$ which do not involve a_i, then

$$d_{a_i}(w)=\sum_{i=1}^{k}\epsilon_i w_1a_i^{\epsilon_1}w_2a_i^{\epsilon_2}\ldots w_{i-1}a_i^{\epsilon_{i-1}}w_ia_i^{(\epsilon_i-1)/2}.$$

We can extend the above formula to $\mathbf{ZF}$ by defining:

$$d_{a_i}\Big(\sum_j n_jw_j\Big)=\sum_j n_jd_{a_i}(w_j)$$

The uniqueness of d_{a_i} follows from the observation that the value of any derivation is determined by its values on the generators of $\mathbf{F}$.□

Remark 1.9.

In the sequal we will use the word derivative in place of derivation.

DEFINITION 1.11.

The higher order derivatives are defined inductively by

$$d_{a_1a_2\ldots a_k}(w)=d_{a_1}\big((d_{a_2\ldots a_k})(w)\big).$$

The order of the derivatives is given by the integer k.

DEFINITION 1.12.

The augmented derivatives are defined inductively as

$$\epsilon_{a_1}(w)=\epsilon(d_{a_1}(w))$$

and

$$\epsilon_{a_1\ldots a_k}(w)=\epsilon(d_{a_1\ldots a_k}(w))$$

where $w\in\mathbf{ZF}$, and $\epsilon:\mathbf{ZF}\longrightarrow\mathbf{Z}$ is the augmentation map.

DEFINITION 1.13.

Let w be a word in F. Write w as $\prod_{i=1}^{k} a_i^{\varepsilon_i}$ where $a_i \in X$ (the set of generators of F) and $\varepsilon_i = \pm 1$. An occurrence of the pair x,y occurs when $a_i = x$, $a_j = y$, $i < j$. The signed of the occurrence $\ldots x^{\varepsilon_i} \ldots y^{\varepsilon_j} \ldots$ is defined to be $\varepsilon_i \varepsilon_j$.

LEMMA 1.3.

If $w \in F$ and $a_1, a_2, \ldots, a_k$ satisfy $a_i \neq a_{i+1}$, $i = 1, \ldots, k-1$ then $\varepsilon_{a_1 a_2 \ldots a_k}(w)$ is the total number of signed occurrences of $a_1 a_2 \ldots a_k$ in the word w.

EXAMPLE 1.4.

Let $w = x^{-1} y x^{-1} y x y$. Compute d_x, d_y, d_{xy}, d_{yx}, d_{xyx}, d_{yxy}, ε_x, ε_y, ε_{xy}, and ε_{xyx}.

$$d_x(w) = -x^{-1} - x^{-1} y x^{-1} + x^{-1} y x^{-1} y.$$

$$d_y(w) = x^{-1} + x^{-1} y x^{-1} + x^{-1} y x^{-1} y x.$$

$$d_{xy}(w) = d_x(d_y(w))$$

$$= -x^{-1} - x^{-1} - x^{-1} y x^{-1} - x^{-1} - x^{-1} y x^{-1} + x^{-1} y x^{-1} y.$$

$$d_{yx}(w) = d_y(d_x(w))$$

$$= -x^{-1} + x^{-1} + x^{-1} y x^{-1}$$

$$= x^{-1} y x^{-1}.$$

$$d_{xyx}(w) = d_x(d_{yx}(w))$$

$$= -x^{-1} - x^{-1} y x^{-1}.$$

$$d_{yxy}(w) = d_y(d_{xy}(w))$$

$$= -x^{-1} - x^{-1} + x^{-1} + x^{-1} y x^{-1}$$

$$= -x^{-1} + x^{-1} y x^{-1}.$$

$$\varepsilon_x(w) = -1 - 1 + 1 = -1.$$

$$\varepsilon_y(w) = 1 + 1 + 1 = 3.$$

$$\varepsilon_{xy}(w) = -1 - 1 - 1 - 1 - 1 + 1 = -4.$$

$$\epsilon_{xyx}(w)=1-1-1-1=-2.$$

DEFINITION 1.14.

Let X be a non-empty set. A string J on the elements of X is defined as follows:

$$J=x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_n}$$

where for each $x_{\alpha_i}, x_{\alpha_{i+1}}\in X$, $x_{\alpha_i}\neq x_{\alpha_{i+1}}$. If n=0 we have an empty string, which is also will be denoted by $\emptyset$. The length of J, denoted by $\ell(J)$, is given by n.

LEMMA 1.4.

For any string $J=x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_n}$ and $a,b\in \mathbb{Z}F$ we have

$$\epsilon_J(ab) = \sum \epsilon_I(a)\epsilon_K(b)$$

where the sum is taken over all ordered pairs (I,K) where $I=x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_j}$ and $K=x_{\alpha_{j+1}}x_{\alpha_{j+2}}\ldots x_{\alpha_n}$ such that J=IK (the juxtaposition of strings I and K) including $(J,\emptyset)$ and $(\emptyset,J)$.

COROLLARY 1.3.

If string $I\neq\emptyset$ and $g\in F$, then

$$\epsilon_I(g^{-1})=\sum(-1)^k\epsilon_{I_1}(g)\ldots\epsilon_{I_k}(g)$$

where the sum is taken over all $I_1I_2\ldots I_k=I$ with $I_j\neq\emptyset$, $j=1,2,\ldots,k$.

LEMMA 1.5.

For $A,B\in\mathbb{Z}F$,

$$d_{a_1a_2\ldots a_k}(AB)=\sum_{j=1}^{k} d_{a_1a_2\ldots a_j}(A)\epsilon_{a_{j+1}a_{j+2}\ldots a_k}(B) + A d_{a_1a_2\ldots a_k}(B).$$

LEMMA 1.6.

Let $g_1\in G^{(i)}$ and $g_2\in G^{(j)}$ and let I be a string on generators of the group G.

(i) If $\ell(I)<i$ then $\epsilon_I(g_1)=0$

(ii) If $\ell(I)\le\min\{i,j\}$ then $\epsilon_I(g_1g_2)=\epsilon_I(g_1)+\epsilon_I(g_2)$

(iii) If $\ell(I)=i+j$ and $I=I_1I_2=I'_2I'_1$ where $\ell(I_1)=\ell(I'_1)=i$ and $\ell(I_2)=\ell(I'_2)=j$, then

$$\epsilon_I[g_1,g_2]=\epsilon_{I_1}(g_1)\epsilon_{I_2}(g_2)-\epsilon_{I'_1}(g_1)\epsilon_{I'_2}(g_2).$$

COROLLARY 1.4.

The element $g\in G^{(n)}$ if and only if $\epsilon_I(g)=0$ for all strings I satisfying $0<\ell(I)<n$.

EXAMPLE 1.5.

Let $w=ababa^{-1}b^{-1}ab^{-1}a^{-1}ba^{-1}bab^{-1}a^{-1}b^{-1}$ be an element of F. Show that $w\in F^{(4)}$.

If $\ell(I)=1$, then

$\epsilon_a(w)=1+1-1+1-1-1+1-1=0$

and $\epsilon_b(w)=1+1-1-1+1+1-1-1=0$.

If $\ell(I)=2$, then $\epsilon_{ab}(w)=0$ and $\epsilon_{ba}(w)=0$.

If $\ell(I)=3$, then $\epsilon_{aba}(w)=0$ and $\epsilon_{bab}(w)=0$.

Let I=abab. Then $\epsilon_{abab}(w)=2$.

Since $\ell(abab)=4$ and $\epsilon_{abab}(w)\ne0$, by Corollary 1.4, $w\in F^{(4)}$.

Remark 1.10.

The proof of the theorems, corollaries, and lemmas stated in this section can be found in [4] and [15].

E. <u>Algebras</u>.

DEFINITION 1.15.

A bracket arrangement $\mathfrak{B}^n$ of weight n, is defined recursively as a certain sequence of asterisks (which act as place holders) and brackets (which indicate the order in which commutation is performed) in the following manner:

There is only one bracket arrangement of weight one

$\mathfrak{B}^1=(*)$.

A bracket arrangement $\mathfrak{B}^n$ of weight $n>1$ is obtained by choosing bracket arrangement $\mathfrak{B}^k$ and $\mathfrak{B}^m$ of weight k and m respectively such that $k+m=n$ and setting

$$\mathfrak{B}^n=(\mathfrak{B}^k,\mathfrak{B}^m),$$

that is, juxtaposing the sequences $\mathfrak{B}^k$ and $\mathfrak{B}^m$ and enclosing the resulting sequence in a pair of brackets.

EXAMPLE 1.6.

According to the definition, the only bracket arrangement of weight two is $(*,*)$ and the bracket arrangements of weight three are

$$(*,(*,*)) \text{ and } ((*,*),*).$$

DEFINITION 1.16.

Let G be a group and let $a_1,\ldots,a_n$ be a finite sequence of elements of G and $\mathfrak{B}^n$ be the bracket arrangement of weight n. We define the elements

$$\mathfrak{B}^n(a_1,\ldots,a_n) \text{ of } G$$

recursively as follows:

$$\mathfrak{B}^1(a_1) = a_1,$$

and if $n>1$ and $\mathfrak{B}^n=(\mathfrak{B}^k,\mathfrak{B}^m)$ then

$$\mathfrak{B}^n(a_1,\ldots,a_n)=(\mathfrak{B}^k(a_1,\ldots,a_k),\mathfrak{B}^m(a_{k+1},\ldots,a_n)).$$

We call $\mathfrak{B}^n(a_1,\ldots,a_n)$ a commutator of weight n on the components $a_1,\ldots,a_n$.

DEFINITION 1.17.

Let R be a ring. An R-module is an abelian group A together with a function $R\times A \longrightarrow A$ (the image (r,a) being denoted by ra) such that for all $r,s\in R$ and $a,b\in A$:

(i) $r(a+b)=ra+rb$

(ii) $(r+s)a=ra+sa$

(iii) $r(sa)=(rs)a$.

If R has an identity element 1_R and

(iv) $1_R a=a$ for all $a\in A$,

then A is said to be a unitary R-module.

DEFINITION 1.18.

Let R be a commutative ring with identity. An algebra over R (or an R-algebra) A, is a ring A such that

(i) $(A,+)$ is a unitary R-module.

(ii) $r(ab)=(ra)b=a(rb)$ for all $r\in R$ and $a,b\in A$.

EXAMPLE 1.7.

Let G be an additive abelian group and $\mathbb{Z}$ the ring of integers. Let $\varphi:\mathbb{Z}\times G \longrightarrow G$ be defined as

$$\varphi(n,g)=ng=\sum_{i=1}^{n} g$$

for any $n\in\mathbb{Z}$ and $g\in G$. Then for any $n,m\in\mathbb{Z}$, and $a,b\in G$,

(i) $n(a+b)=\sum_{i=1}^{n}(a+b)=\sum_{i=1}^{n}a + \sum_{i=1}^{n}b= na+nb$.

(ii) $(n+m)a=\sum_{i=1}^{n+m}a=\sum_{i=1}^{n}a + \sum_{i=1}^{m}a=na+nb$.

(iii) $n(ma)=\sum_{i=1}^{n}\left(\sum_{j=1}^{m}a\right)=\sum_{i=1}^{nm}a=(nm)a$.

(iv) $1.a=a$.

Therefore G is a unitary $\mathbb{Z}$-module.

(v) $n(ab)=\sum_{i=1}^{n}(ab)=\left(\sum_{i=1}^{n}a\right)b=(na)b$.

$$n(ab)=\sum_{i=1}^{n}(ab)=a\sum_{i=1}^{n}b=a(nb).$$

Hence, G is a $\mathbb{Z}$-algebra.

THEOREM 1.6.

Let R be a ring with identity. Let R[x] denote the set of all sequences of elements of R as follows:

$$R[x]=\{(a_0,a_1,\ldots)|a_i\in R \text{ and } a_i=0 \text{ except for finitely many}\}.$$

a) R[x] is a ring with addition and multiplication defined by:

$$(a_0,a_1,\ldots)+(b_0,b_1,\ldots)=(a_0+b_0,a_1+b_1,\ldots)$$

and

$$(a_0,a_1,\ldots)(b_0,b_1,\ldots)=(c_0,c_1,\ldots),$$

where

$$c_n=\sum_{i=0}^{n}a_{n-i}b_i.$$

b) If R is commutative then so is R[x].

c) The map $R \longrightarrow R[x]$ given by $r \longrightarrow (r,0,0,\ldots)$ is a monomorphism of rings.

DEFINITION 1.19.

The ring R[x] is called the ring of polynomials over R, and its elements are called polynomials.

We are going to identify R with its isomorphic image in R[x] and write $(r,0,0,\ldots)$ by r. We will now explain the notation R[x] and develop a more familiar notation for polynomials.

THEOREM 1.7.

Let R be a ring and denote by x the element $(0,1_R,0,\ldots)$ of R[x]. Then

a) $x^n=(0,0,\ldots,0,1_R,0,\ldots)$, where 1_R is the (n+1)st coordinate.

b) If $r\in R$, then for each $n\geq 0$, $rx^n=(0,\ldots,0,r,0,\ldots)$, where r is the $(n+1)$st coordinate.

c) For every nonzero polynomial f in $R[x]$ there exists an integer $n\in N$(natural numbers) and elements $a_0,\ldots,a_n\in R$ such that $f=a_0x^0+a_2x^2+\ldots+a_nx^n$. The integer n and the elements a_i are unique in the sense that

$$f=b_0x^0+b_1x^1+\ldots+b_mx^m \quad (b_i\in R)$$

implies $m\geq n$; $a_i=b_i$ for $i=1,2,\ldots,n$; and $b_i=0$ for $n<i\leq m$.

Remark 1.11.

It is convenient to write

$$f=a_0x^0+a_1x^1+a_2x^2+\ldots+a_nx^n$$

as

$$f=a_0+a_1x^1+a_2x^2+\ldots+a_nx^n.$$

We will call x the indeterminate. Let us extend $R[x]$ to more than one indeterminate, namely $R[x_1,\ldots,x_n]$. For simplicity, we will only consider the case where $n<\infty$.

Let $N^n=N\times\ldots\times N$ (n is a positive integer) be the set of all n-tuples of elements of N.

THEOREM 1.8.

Let R be a ring and denote by $R[x_1,\ldots,x_n]$ the set of all functions $f:N^n \longrightarrow R$ such that $f(u)\neq 0$ for at most a finite number of elements u of N^n.

i) $R[x_1,\ldots,x_n]$ is a ring with addition and multiplication defined by

$$(f+g)(u)=f(u)+g(u) \text{ and}$$

$$(fg)(u)=\sum_{v+w=u} f(v)g(w) \qquad \forall v,w\in N^n$$

where $f,g\in R[x_1,\ldots,x_n]$ and $u\in N^n$.

ii) If R is commutative then so is $R[x_1,\ldots,x_n]$.

iii) The map $R \longrightarrow R[x_1,\ldots,x_n]$ given by $r \longrightarrow f_r$, where $f_r(0,\ldots,0)=r$ and $f(u)=0$ for all other $u\in N^n$, is a monomorphism of rings.

Remark 1.12.

We can identify R with its homomorphic image in $R[x_1,\ldots,x_n]$ under the mapping described in Theorem 1.8.(iii).

Let n be a positive integer and for each $i=1,\ldots,n$, let

$$e_i=(0,\ldots,0,1,0,\ldots,0)\in N^n,$$

where 1 is the i-th coordinate of e_i. If $k\in N^n$, let $ke_i=(0,\ldots,0,k,0,\ldots,0)$. Then every element of N^n may be written in the form $k_1e_1+k_2e_2+\ldots+k_ne_n$.

Let us find a more convenient notation for elements of $R[x_1,\ldots,x_n]$.

THEOREM 1.9.

Let R be a ring with identity and n a positive integer. For each $i=1,\ldots,n$ let $x_i\in R[x_1,\ldots,x_n]$ be defined by $x_i(e_i)=1_R$ and $x_i(u)=0$ for $u\neq e_i$.

i) For each integer $k\in N$, $x_i^k(ke_i)=1_R$ and $x_i^k(u)=0$ for $u\neq ke_i$;

ii) For each $(k_1,\ldots,k_n)\in N^n$,

$$x_{\alpha_1}^{k_1}\ldots x_{\alpha_n}^{k_n}(k_1e_1+\ldots+k_ne_n)=1_R \text{ and}$$

$$x_{\alpha_1}^{k_1}\ldots x_{\alpha_n}^{k_n}(u)=0 \text{ for } u\neq k_1e_1+\ldots+k_ne_n;$$

iii) $x_i^tr=rx_i^t$ for all $r\in R$ and all $t\in N$;

iv) For every nonzero polynomial f in $R[x_1,...,x_n]$ there exists a unique nonzero elements $(k_{11},k_{12},...,k_{1n})$, $(k_{21},k_{22},...,k_{2n}),...,(k_{n1},k_{n2},...,k_{nn})$ of N^n and unique elements $a_0,a_1,...,a_n$ of R such that

$$f=a_0x_{\alpha_1}^0x_{\alpha_2}^0...x_{\alpha_n}^0+a_1x_{\alpha_1}^{k_{11}}x_{\alpha_2}^{k_{12}}...x_{\alpha_n}^{k_{1n}}+a_2x_{\alpha_1}^{k_{21}}x_{\alpha_2}^{k_{22}}...x_{\alpha_n}^{k_{2n}}$$

$$+...+a_nx_{\alpha_1}^{k_{n1}}x_{\alpha_2}^{k_{n2}}...x_{\alpha_n}^{k_{nn}}.$$

Remark 1.13.

If R is a ring with identity, then $x_1,...,x_n$ are called indeterminates. The elements $a_0,a_1,...,a_n$ in Theorem 1.9(iv) are called the coefficients of the polynomial f. A polynomial of the form $ax_{\alpha_1}^{k_1}x_{\alpha_2}^{k_2}...x_{\alpha_n}^{k_n}$ is called a monomial. For convenient, we will omit x_i that appear with zero exponent in a monomial, i.e.

$$ax_{\alpha_1}^0x_{\alpha_2}^0...x_{\alpha_n}^0\equiv a.$$

Then f in Theorem 1.9.(iv) can be written as

$$f=a_0+a_1x_{\alpha_1}^{k_{11}}x_{\alpha_2}^{k_{12}}...x_{\alpha_n}^{k_{1n}}+a_2x_{\alpha_1}^{k_{21}}x_{\alpha_2}^{k_{22}}...x_{\alpha_n}^{k_{2n}}$$

$$+...+a_nx_{\alpha_1}^{k_{n1}}x_{\alpha_2}^{k_{n2}}...x_{\alpha_n}^{k_{nn}}.$$

As we observed from Theorem 1.9.(iv), a polynomial is a sum of monomials.

THEOREM 1.10.

Let R be a ring and denote by R[[x]] the set of all sequences of elements of R, $(a_0,a_1,...,a_n,...)$.

i) R[[x]] is a ring with addition and multiplication defined by:

$$(a_0,a_1,...)+(b_0,b_1,...)=(a_0+b_0,a_1+b_1,...) \quad \text{and}$$

$$(a_0,a_1,...)(b_0,b_1,...)=(c_0,c_1,...)$$

where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{k+j=n} a_k b_j.$$

ii) The polynomial ring $R[x]$ is a subring of $R[[x]]$.

iii) If R is commutative, then so is $R[[x]]$.

DEFINITION 1.20.

The ring $R[[x]]$ is called the ring of formal power series over the ring R. If R has an identity, then $x=(0,1_R,0,\ldots)\in R[[x]]$ is called an indeterminate or a variable.

The power series $(a_0,a_1,\ldots)\in R[[x]]$ is denoted by the formal sum $\sum_{i=0}^{\infty} a_i x^i$. The elements a_i are called coefficients and a_0 is called the constant term.

Remark 1.14.

Let $R[[x_1,x_2,\ldots,x_k]]$ be the set of all formal sums

$$\sum_{\lambda\in I_k} a_\lambda x^\lambda,$$

where

$$x^\lambda = x_{\alpha_1}^{n_1} x_{\alpha_2}^{n_2} \ldots x_{\alpha_k}^{n_k},$$

$a_\lambda = a_{n_1} a_{n_2} \ldots a_{n_k}$, and

$$I_k = \{\lambda=(n_1,n_2,\ldots,n_k) \mid n_i \in \mathbb{Z}^+ \cup \{0\}\}.$$

Let us define addition and multiplication, respectively, on the elements of $R[[x_1,x_2,\ldots,x_k]]$ by

$$\sum_{\lambda\in I_k} a_\lambda x^\lambda + \sum_{\lambda\in I_k} b_\lambda x^\lambda = \sum_{\lambda\in I_k} (a_\lambda + b_\lambda) x^\lambda$$

$$\left(\sum_{\lambda\in I_k} a_\lambda x^\lambda\right)\left(\sum_{\lambda\in I_k} b_\lambda x^\lambda\right) = \sum_{\lambda\in I_k} c_\lambda x^\lambda$$

where

$$c_\lambda = \sum_{\gamma+\beta=\lambda} a_\gamma b_\beta .$$

Then $R[[x_1,x_2,...,x_k]]$ is a ring of formal power series in n variables with respect to the operations defined above.

Remark 1.15.

Let $\mathbb{Z}[[x_1,...,x_k]]$ be the ring of formal power series in k indeterminates. Scalar multiplication in $\mathbb{Z}[[x_1,...,x_k]]$ is defined as follows:

$$z\Big(\sum_{\lambda\in I_k} m_\lambda x^\lambda\Big)=\sum_{\lambda\in I_k} (zm_\lambda)x^\lambda$$

Remark 1.16.

The degree of a nonzero monomial

$$ax_{\alpha_1}^{n_1}x_{\alpha_2}^{n_2}...x_{\alpha_k}^{n_k}\in R[[x_1,x_2,...,x_k]]$$

is the nonnegative integer $n_1+n_2+...+n_k$.

Remark 1.17.

Proof of the theorems on ring of polynomials and ring of formal power series can be found in [5].

DEFINITION 1.21.

The algebra $A(\mathbb{Z},n)$ is the associative $\mathbb{Z}$-algebra of formal power series in the non-commuting variables $x_1,...,x_n$. This algebra consists of formal power series in $x_1,...,x_n$ with integer coefficients.

Remark 1.18.

The bracket $[,]_0$ is defined in the associative algebra $A(\mathbb{Z},n)$ and is called the bracket multiplication or bracket product. For two elements u and v in $A(\mathbb{Z},n)$ we define

$$[u,v]_0=uv-vu.$$

LEMMA 1.7.

The set of all elements $g\in A(\mathbf{Z},n)$ with constant term 1 is a group under multiplication. If $g=1+h$, then

$$g^{-1}=1-h+h^2-h^3+\dots+(-1)^k h^k+\dots \quad .$$

THEOREM 1.11.

If $A(\mathbf{Z},n)$ is freely generated by $x_1,\dots,x_n$, then the elements

$$a_\rho=1+x_\rho,\ \rho=1,2,\dots,n$$

of $A(\mathbf{Z},n)$ are generators of a free group $\mathbf{F}(n)$ of rank n. Moreover,

$$a_\rho^{-1}=1-x_\rho+x_\rho^2-\dots+(-1)^k x_\rho^k+\dots \quad .$$

DEFINITION 1.22.

Let W be a word in the free generators a_ρ of $F(n)$. Using the mapping $a_\rho \longrightarrow 1+x_\rho$, W can be expressed as an element of the power series ring $A(\mathbf{Z},n)$ in the form $1+u_k+u_{k+1}+\dots+u_n+\dots$ where u_k is the non-vanishing homogeneous component of the lowest positive degree. The deviation $\delta(W)$ of W is defined by

$$\delta(W)=\begin{cases} 0 & \text{if } W=\emptyset \\ u_k & \text{otherwise} \end{cases}$$

DEFINITION 1.23.

The bracket arrangements in $A(\mathbf{Z},n)$ is defined recursively as follows:

$$\mathfrak{B}_0^1(g_1)=g_1 \qquad \text{and}$$

$$\mathfrak{B}_0^n(g_1\dots g_n)=[\mathfrak{B}_0^k(g_1\dots g_k),\mathfrak{B}_0^m(g_{k+1}\dots g_n)]_0$$

where $g_1,\dots,g_n\in A(\mathbf{Z},n)$ and $k+m=n$.

EXAMPLE 1.8.

Let $g_1,g_2\in A(\mathbf{Z},2)$. Then,

$\mathfrak{B}_0^2(g_1g_2)=([g_1,g_2]_0,[g_2,g_1]_0)$.

LEMMA 1.8.

Let U and V be words (non trivial) in the $a_\rho=1+x_\rho$ and let $\delta(U)=u_j$, $\delta(V)=v_k$.

(i) Then, for all integers k,

$\delta(U^k)=ku_j$.

(ii) If j<k, then

$\delta(UV)=\delta(VU)=u_j$.

(iii) If j=k and $u_j+v_k\neq 0$, then

$\delta(UV)=\delta(VU)=u_j+v_k$.

(iv) If j=k and $u_j+v_k=0$, then

$UV=\emptyset$ or

degree $\delta(UV)$=degree $\delta(VU)=j+1$.

(v) If $u_jv_k-v_ku_j\neq 0$, then

$\delta([U,V])=u_jv_k-v_ku_j$.

(vi) If $u_jv_k-v_ku_j=0$, then

$UV=VU$ or

degree $\delta([U,V])=j+k+1$.

(vii) $\delta(U^{-1}VU)=v_k$. [15]

REMARK 1.19.

Let W be a word in F(n) under the mapping

$F(n) \longrightarrow A(\mathbb{Z},n)$.

It can be shown that $\delta(W)=u_k=\sum_i \lambda_i \mathfrak{B}_0^m(g_1...,g_m)$ where $\lambda_i \in \mathbb{Z}$ and $g_1,...,g_k \in A(\mathbb{Z},n)$. The weight of the bracket arrangement $\mathfrak{B}_0^m(g_1...,g_m)$ may or may not be equal to the degree of the monomial u_k. The degree of the deviation $\delta(W)$ is the weight of the bracket arrangement $\mathfrak{B}_0^m(g_1...,g_m)$, i.e., degree $\delta(W)=m$.

EXAMPLE 1.9.

Suppose $\{x_1,\dots,x_n\}$ is a set of free generators of $A(Z,5)$. Then $a_i=1+x_i$ for $i=1,\dots,5$ are the generators for the free group $F(5)$. Let $U=a_1a_3$ and $V=a_2a_4^{-1}$ be words in the generators of $F(5)$. Find $\delta(U^2)$, $\delta(V^2)$, $\delta(UV)$, $\delta(U^{-1}VU)$, $\delta(V^{-1}UV)$, and $\delta([U,V])$. Moreover, find the degree of each deviation.

1) $a_1a_3 \longrightarrow (1+x_1)(1+x_3)$

$$=1+(x_1+x_3)+x_1x_3.$$

Hence, $\delta(U)=x_1+x_3=u_1$.

2) $a_2^{-1}a_4^{-1} \longrightarrow (1+x_2)(1-x_4+x_4^2-\dots+(-1)^n x_4^n+\dots)$

$$=(1-x_4+x_4^2-\dots+(-1)^n x_4^n+\dots)$$

$$+ (x_2-x_2x_4+x_2x_4^2-\dots+(-1)^n x_2x_4^n+\dots)$$

$$=1+(x_2-x_4)+(x_4^2-x_2x_4)+\dots$$

Hence, $\delta(V)=x_2-x_4=v_1$.

Therefore,

1) $\delta(U^2)=2(x_1+x_3)$ and degree $\delta(U^2)=1$.

2) $\delta(V^2)=2(x_2-x_4)$ and degree $\delta(V^2)=1$.

3) $\delta(UV)=\delta(VU)=x_1+x_3+x_2-x_4$ and degree $\delta(UV)=1$.

4) $\delta(U^{-1}VU)=x_2-x_4$ and degree $\delta(U^{-1}VU)=1$.

5) $\delta(V^{-1}UV^{-1})=x_1+x_3$ and degree $\delta(V^{-1}UV^{-1})=1$.

6) $\delta([U,V])=(x_1+x_3)(x_2-x_4)-(x_2-x_4)(x_1+x_3)$

$$=x_1x_2-x_1x_4+x_3x_2-x_3x_4-x_2x_1-x_2x_3+x_4x_1+x_4x_3$$

$$=[x_1,x_2]_0+[x_4,x_1]_0+[x_3,x_2]_0+[x_4,x_3]_0.$$

Hence, degree $\delta([U,V])=2$

Remark 1.20.

The proof of the theorems, lemmas, and corollaries in this section can be found in [9] unless specified otherwise.

CHAPTER 2

LINKS and BRAIDS

A. Links.

DEFINITION 2.1.

A link is a finite collection of disjoint simple closed curves in 3-dimensional space $\mathbb{R}^3$, the individual simple closed curves being called the components of the link. A link of just one component is a knot.

DEFINITION 2.2.

In Formal Knot Theory, we closely abstract the rope drawings that represent knots.

Figure 2.1

We call the picture on the right as knot diagram. It contains all necessary information for constructing the knot out of rope and it presents a specific form for an embedding of a circle, S^1, in $\mathbb{R}^3$. To see this embedding, we must understand that a broken line indicates where one part of the curve undercrosses the other part.

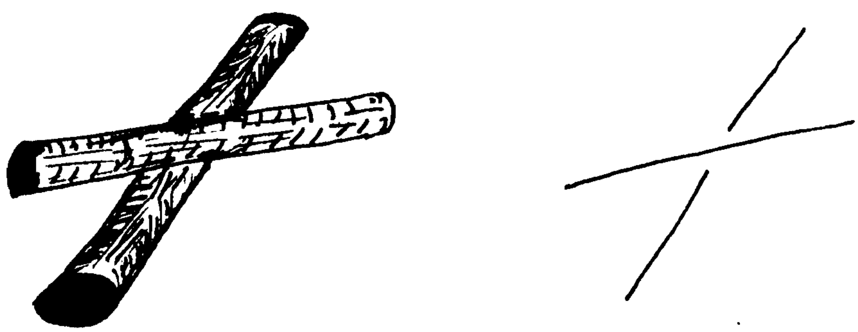


Figure 2.2

DEFINITION 2.3.

Two link diagrams, L and L', are equivalent if there exists a finite sequence of Reidemeister moves (R-moves) that transforms the link diagram L into the link diagram L' or vice versa.

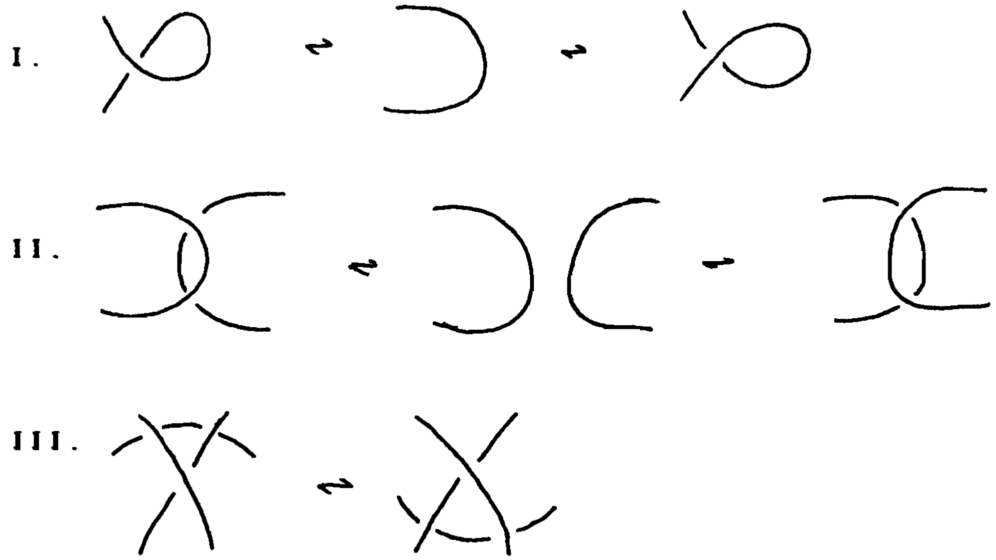


Figure 2.3.

For simplicity, we will use the word link to denotes the link diagram.

EXAMPLE 2.1.

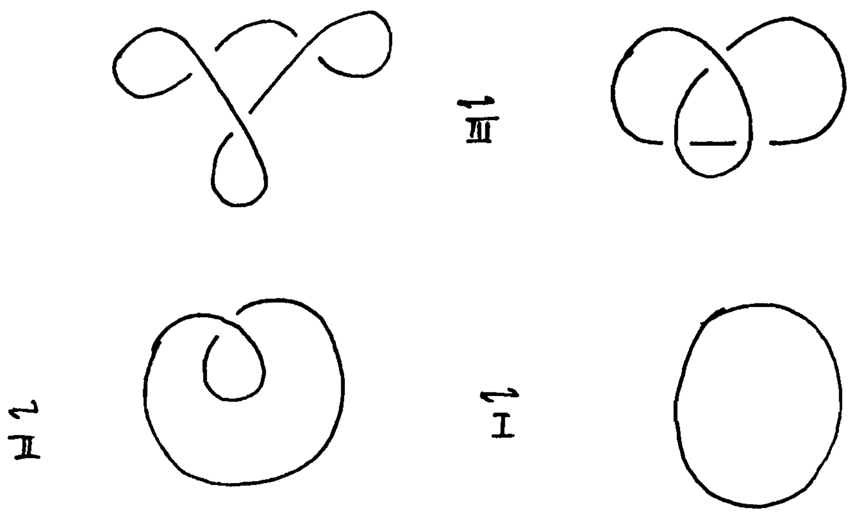


Figure 2.4.

Remark 2.1.

A link is split if it is equivalent to a link with diagram containing two nonempty parts that live in disjoint neighborhoods.

EXAMPLE 2.2.

Thus

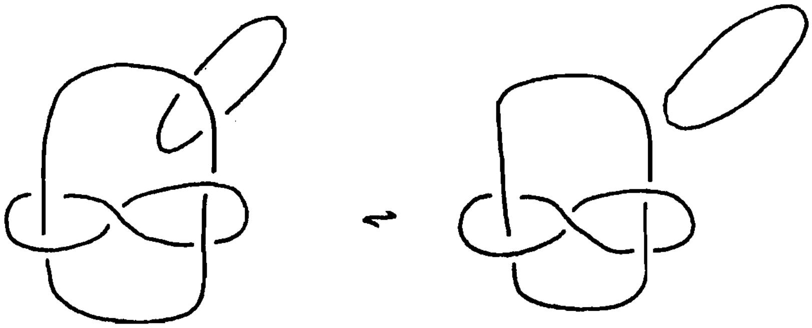


Figure 2.5.

is a split link.

B. Linking Number.

To each crossing in an oriented link, we associate a sign η such that

$\eta=+1$ for

$\eta=-1$ for

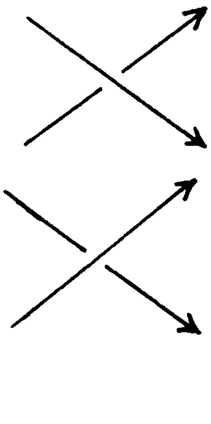


Figure 2.6.

DEFINITION 2.4.

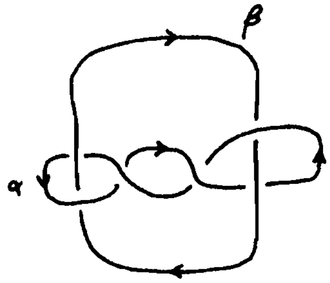
Let $L=\alpha\sqcup\beta$ be a link of two components. Let $\alpha\sqcup\beta$ denote the set of crossings of α with β. Then

$$\ell k(L)=\ell k(\alpha,\beta)=\frac{1}{2}\sum_{p\in\alpha\sqcup\beta}\eta(p).$$

This formula defines the linking number for a given diagram.

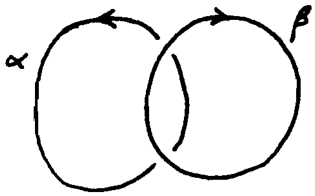
EXAMPLE 2.3.

a)



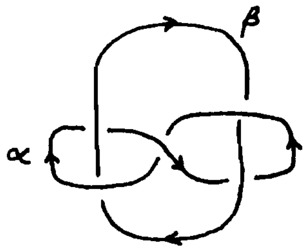
$\ell k(\alpha, \beta) = +2$.

b)



$\ell k(\alpha, \beta) = 0$.

c)



$\ell k(\alpha, \beta) = 0$.

Figure 2.7.

Remark 2.2.

If a 2-component link L splits, then $\ell k(L) = 0$. The converse, however, is false as shown in the examples 2.3(b) and 2.3(c).

C. <u>Fundamental Group</u>.

Associated to a link L in $\mathbb{R}^3$ is the fundamental group of the complementary space, $\mathbb{R}^3 - L$, of the link, and it is denoted by $\pi_1(\mathbb{R}^3 - L)$. For simplicity, we will denote the fundamental group of a link L by $\pi_1(L)$. There exists an algorithm called the Wirtinger presentation for finding $\pi_1(L)$.

<u>The Wirtinger Presentation</u>.

This is a procedure for writing down a presentation of the group of a knot K in $\mathbf{R}^3$, given the diagram of the knot. We labelled each arc in K by $\alpha_1,\ldots,\alpha_n$ such that each α_i is assumed connected to α_{i-1} and α_{i+1} (mod n) by undercrossing arcs exactly as pictured below.

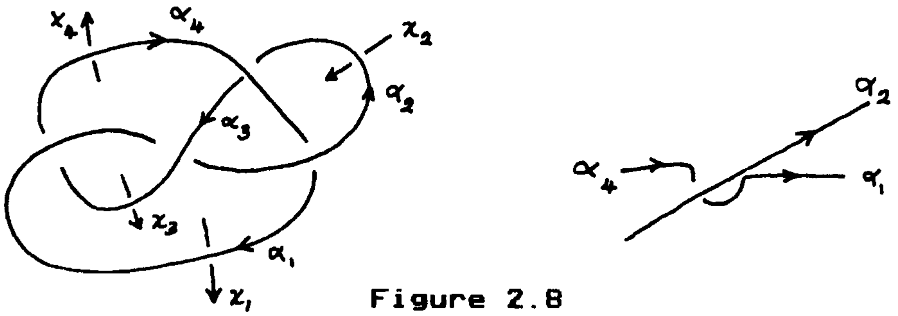


Figure 2.8

We assume for convenience that all α_i's are oriented compatibly with the order of their subscripts. Draw a short arrow labelled x_i passing under each α_i in a right-left direction. This is supposed to represent a loop in $\mathbf{R}^3$–K where a point ★ is taken to be the basepoint (best imagined as the eye of the viewer), and the loop consists of the oriented triangle from ★ to the tail of x_i, along x_i to the head, thence back to ★.

There is a certain relation among the x_i's which must hold. The two possibilities are:

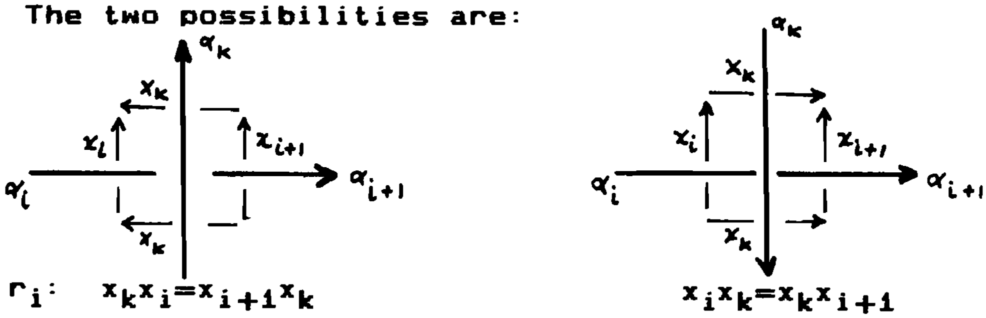


Figure 2.9.

Here α_k is the arc passing over the gap from α_i to α_{i+1} (k=i or i+1 is possible). Let r_i denote whichever of the two equations holds. In all, there are exactly n relations $r_1,\ldots,r_n$ which may be read off this way. These comprise a complete set of relations.

THEOREM 2.1.

The group $\pi_1(K)$ is generated by (homotopy classes) x_i and has presentation

$$\pi_1(L)=\{x_1,\ldots,x_n \mid r_1,\ldots,r_n\}.$$

Moreover, any one of the r_i may be omitted and the above remains true.[12]

EXAMPLE 2.4.

The Figure-Eight Knot

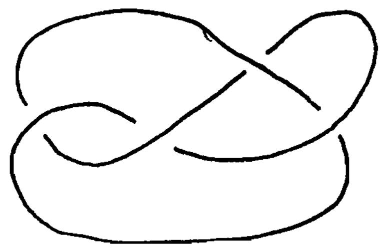


Figure 2.10.

For the Figure-Eight knot, we have a presentation with generators x_1, x_2, x_3, x_4 and relations

(1) $x_1x_3=x_3x_2$,

(2) $x_4x_2=x_3x_4$,

(3) $x_3x_1=x_1x_4$.

We may simplify, using (1) and (3) to eliminate $x_2=x_3^{-1}x_1x_3$ and $x_4=x_1^{-1}x_3x_1$ and substitute into (2) to obtain the equivalent presentation $\pi_1(\text{figure-eight knot})=\{x_1,x_3 \mid r\}$

where

$$r: x_1^{-1}x_3x_1x_3^{-1}x_1x_3=x_3x_1^{-1}x_3x_1.$$

The argument establishing the Wirtinger presentation theorem adapts in an obvious manner to links.

EXAMPLE 2.5.

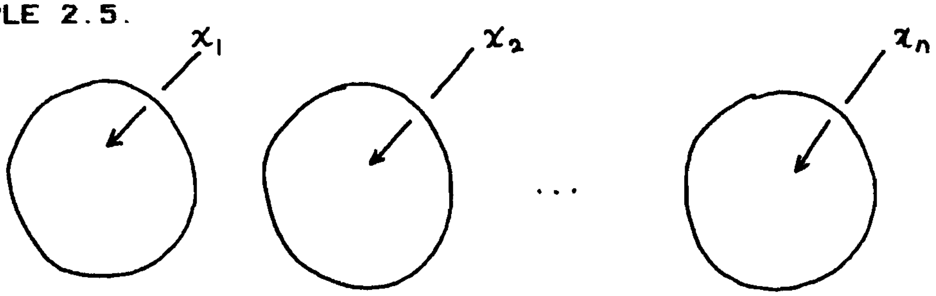


Figure 2.11

The trivial link (disjoint circles in a plane) of n components has group

$$\{x_1,\ldots,x_n \mid x_1=\ldots=x_2=1\} = \text{Free group of rank } n.$$

EXAMPLE 2.6.

The Borromean Ring.

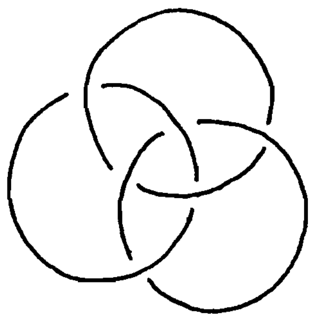


Figure 2.12.

The fundamental group, $\pi_1(L)$, has a presentation

$$\pi_1(L)=\{a_1,a_2,a_3 \mid r_1,r_2\}$$

where

$$r_1=[a_2,[a_1,a_3^{-1}]] \quad \text{and}$$

$$r_2=[a_3,[a_2,a_1^{-1}]].$$

D. <u>Braids</u>.

Let

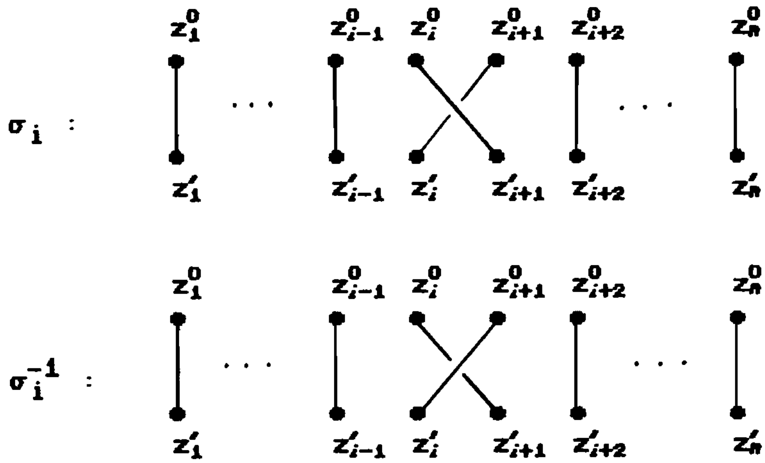


Figure 2.13.

We define the multiplication of σ_i and σ_j, denoted by $\sigma_i\sigma_j$, by

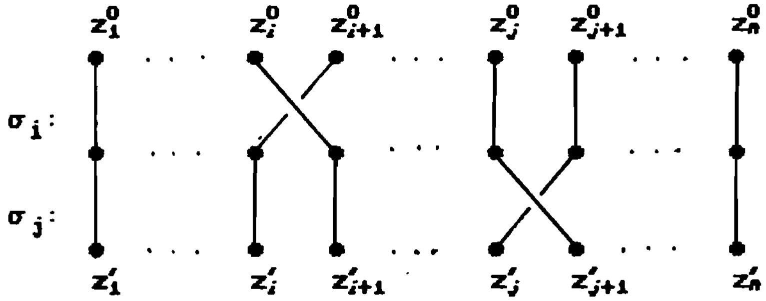


Figure 2.14.

If $j \neq i$ or $i+1$ then we can use the following diagram to define $\sigma_i\sigma_j$

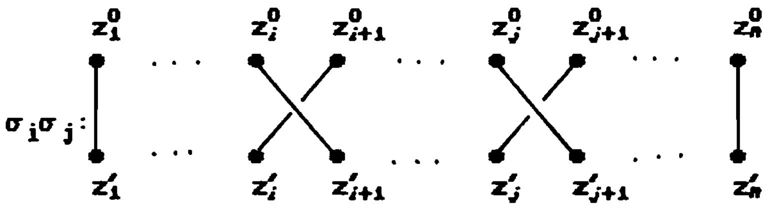


Figure 2.15.

THEOREM 2.2. [Artin,1925]

The group $\pi_1(B_n)$ admits a presentation with generators

$\sigma_1, \ldots, \sigma_{n-1}$ and defining relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i \qquad \text{if } |i-j|\geq 2,\ 1\leq i,\ j\leq n-1$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} \qquad 1\leq i\leq n-2.$$

(Here B_n denotes braid with n strings.)

For simplicity, we will use B_n when we mean $\pi_1(B_n)$.

Remark 2.3.

Define

$$\Delta=(\sigma_1\sigma_2\ldots\sigma_{n-1})(\sigma_1\sigma_2\ldots\sigma_{n-2})\ldots(\sigma_1\sigma_2)(\sigma_1).$$

Then, in Garside's treatment, it is shown that each element $\beta\in B_n$ has a unique normal form:

$$\beta=\Delta^m P$$

where m is an integer, and P is a positive word. m is called the power of β, and P is the tail of β.

EXAMPLE 2.7.

Let $\beta=\sigma_1\sigma_2\sigma_1\sigma_2^2$. Then $\beta\in B_3$ since β is in a normal form where $\Delta=\sigma_1\sigma_2\sigma_1$, m=1, and $P=\sigma_2^2$. The diagram of β is as follows:

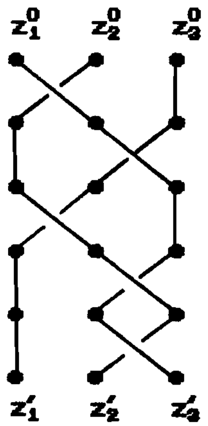


Figure 2.16.

Remark 2.4.

$\Delta^{2k}, k>0\in\mathbb{Z}$ is a pure braid.

EXAMPLE 2.8.

$\beta = (\sigma_1\sigma_2\sigma_1)^2 \in B_3$ is a pure braid.

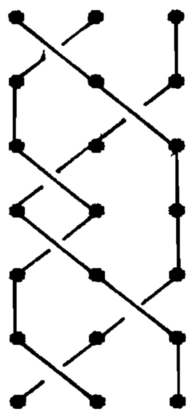


Figure 2.17

<u>Closed braid and Link</u>.

Let L be an oriented link. Choose a point p not on any strand of L (p can be viewed as the point of intersection between a line β that is orthogonal to the plane on which the link $L \in \mathbb{R}^3$ is projected onto). Assign a positive direction of rotation obout p (consider the right hand rule being applied to β). An edge ab of L is said to be positively (resp. negatively) oriented if a radius vector from p to ab rotates in a positive (resp. negative) direction about p in going from a to b along ab.

DEFINITION 2.5.

A link L is said to be a closed braid, denoted by $\hat{C}$ if all of its edges are positively oriented.

Remark 2.5.

The height of a link L, denoted h(L), is the number of negative edges, and is the measure of how far the link is from being a closed braid.

EXAMPLE 2.9.

Trefoil knot types.

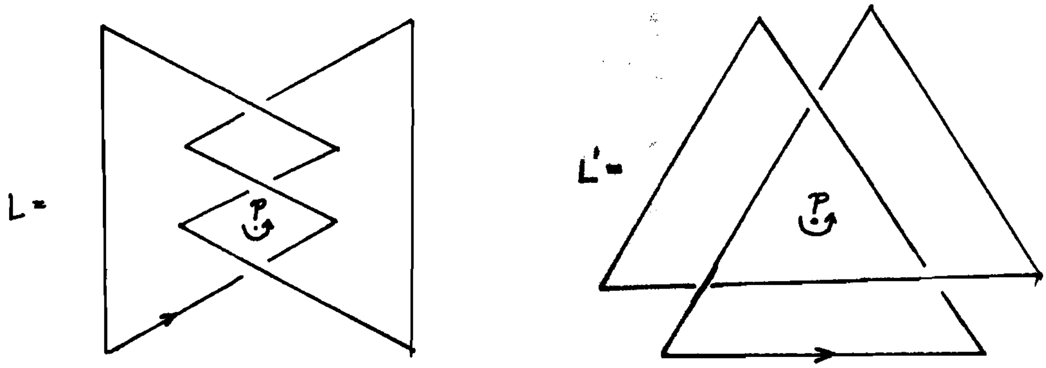


Figure 2.18.

L′ is a closed braid since it has height 0. L has height 4, hence not a closed braid.

Remark 2.6.

An open braid β may be used to construct a closed braid $\hat{\beta}$, by identifying the initial points and end points of each of the braid strings.

EXAMPLE 2.10.

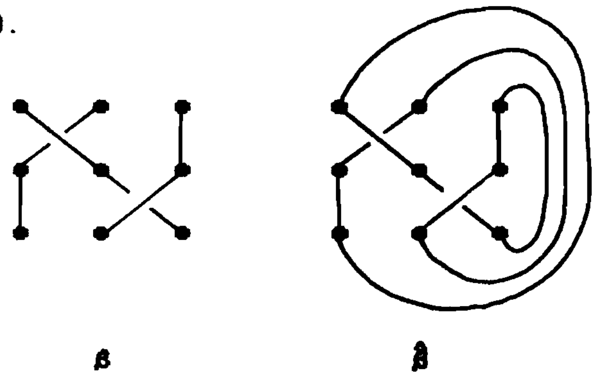


Figure 2.19.

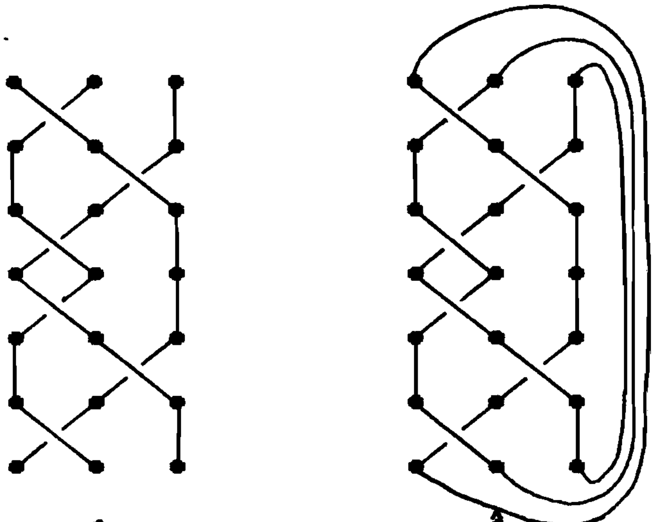
THEOREM 2.3.[Alexander]

Every link is combinatorially equivalent to a closed braid.

Remark 2.7.

For any pure braid $\beta \in B_n$, the closed braid $\hat{\beta}$ associated to β is a link with n components.

EXAMPLE 2.11.



β $\qquad$ $\hat{\beta}$

Figure 2.20.

COROLLARY 2.1.

The braid group B_n has a faithful representation as a group of (right) automorphisms of a free group F_n generated by $a_1,\ldots,a_n$, of rank n. The representation is induced by a mapping ξ from B_n to Aut F_n defined by:

$$(\sigma_i)\xi : a_i \longrightarrow a_i a_{i+1} a_i^{-1}$$
$$a_{i+1} \longrightarrow a_i$$
$$a_j \longrightarrow a_j \quad \text{if } j \neq i, i+1.$$

EXAMPLE 2.12.

Consider B_3. Let $F_3 = \langle a_1, a_2, a_3 \rangle$. Then under the mapping ξ as defined above, B_n has a representation as an Aut F_3 where

$$\sigma_1 : a_1 \to a_1 a_2 a_1^{-1} \qquad a_1^{-1} \to a_1 a_2^{-1} a_1^{-1}$$

$$a_2 \to a_1 \qquad a_2^{-1} \to a_1 a_2^{-1} a_1 a_2 a_1^{-1}$$

$$a_3 \to a_3 \qquad a_3^{-1} \to a_3^{-1}$$

$$\sigma_2 : a_1 \to a_1 \qquad a_1^{-1} \to a_1^{-1}$$

$$a_2 \to a_2a_3a_2^{-1} \qquad a_2^{-1} \to a_1a_2^{-1}a_1^{-1}$$

$$a_3 \to a_2 \qquad a_3^{-1} \to a_2a_3^{-1}a_2a_3a_2^{-1}$$

THEOREM 2.4. [Artin, 1925]

Let $F_n=\langle a_1,\ldots,a_n\rangle$ be a free group of rank n. Let β be an endomorphism of F_n. Then $\beta\in B_n\subset \mathrm{Aut}\, F_n$ if and only if β satisfies the two conditions

$$(a_i)\beta=A_ia_{\mu_i}A_i^{-1} \qquad 1\leq i\leq n$$

$$(a_1a_2\ldots a_n)\beta=a_1a_2\ldots a_n$$

where $(\mu_1,\ldots,\mu_n)$ is a permutation of $(1,\ldots,n)$, and $A_i=A_i(a_1,\ldots,a_n)$ is a word in the generators of F_n.

EXAMPLE 2.13.

Consider $\beta\in B_3$. Let $\beta=\sigma_1\sigma_2$ and $F_3=\langle a_1,a_2,a_3\rangle$. Then

$$\sigma_1\sigma_2:\ a_1 \to \sigma_1(\sigma_2(a_1))=a_1a_2a_1^{-1}$$

$$a_2 \to \sigma_1(\sigma_2(a_2))=\sigma_1(a_2a_3a_2^{-1})$$

$$=a_1a_3a_1a_2^{-1}a_1a_2a_1^{-1}$$

$$a_3 \to \sigma_1(\sigma_2(a_3))=\sigma_1(a_2)=a_1$$

$$a_1a_2a_3 \to \sigma_1(\sigma_2(a_1a_2a_3))=a_1a_2a_3$$

Since $\sigma_1\sigma_2(a_2)$ is not in the form described as in the theorem above, $\beta\in B_n\not\subset \mathrm{Aut}\, F_3$.

THEOREM 2.5. [Artin]

Let $\beta\in B_n$ and suppose that the action on the free group F_n is given by the Theorem 2.4. Let $\hat{\beta}$ be the link determined by the braid β. Then the fundamental group $\pi_1(S^3-\hat{\beta})$ of the complement of $\hat{\beta}$ in S^3 admits the presentation:

generators: $a_1,\ldots,a_n$

defining relations: $a_i=A_i(a_i,\ldots,a_n)a_{\mu_i}A_i^{-1}(a_1,\ldots,a_n)$

where $1 \leq i \leq n-1$ and μ_i is a permutation of $(1,...,n)$. Moreover, every link group admits such a presentation.

EXAMPLE 2.14.

See example 2.6. for the fundamental group of the Borromean Rings.

Remark 2.8.

Proofs of the theorems and the corollaries in this section can be found in [2].

CHAPTER 3

Determining Vanishing Triple Products

DEFINITION 3.1.

Let F be a free group generated by a_i, and let W be a word on a_i ($W\neq\emptyset$). The weight of W, denoted by $\omega(W)$, is the largest integer n such that $W\in F^{(n)}$, but $W \notin F^{(n+1)}$.

THEOREM 3.1.

Let F be a free group generated by $a_1,\ldots,a_k$, and let W be a word on $a_1,\ldots,a_k$ ($W\neq\emptyset$). If $\omega(W)=n$ and $\delta(W)=u_m$, then the degree of $\delta(W)$ is greater or equal to n.[15]

THEOREM 3.2.

Let $\beta\in B_k$ and let $G=\{a_j|r_i\}$ be the fundamental group of $\hat{\beta}$, where $r_i=a_iA_ia_{\mu_i}^{-1}A_i^{-1}$. Then,

(i) If $\mu_i=i$ for some i, $\omega(A_i)=2$, degree $\delta(A_i)$ is not zero, and $\delta(r_i)$ is not zero, then there exists a non-vanishing Triple Product.

(ii) If $i=\mu_i$ for some i, $\omega(A_i)$ is the same for all i and $\omega(A_i)>2$, then all Triple Products vanish.

(iii) For every $W\in F^{(3)}\cap N$ where N is the normal subgroup generated by the generators of G, there are integers k,p, and $(n_{11},n_{12},\ldots,n_{1k})$, $(n_{21},n_{22},\ldots,n_{2k}),\ldots,$ $(n_{p1},n_{p2},\ldots,n_{pk})$ such that

$$W=\left(r_{j_1}^{n_{11}}r_{j_2}^{n_{12}}\cdots r_{j_k}^{n_{1k}}\right)\left(r_{j_1}^{n_{21}}r_{j_2}^{n_{22}}\cdots r_{j_k}^{n_{2k}}\right)\cdots\left(r_{j_1}^{n_{p1}}r_{j_2}^{n_{p2}}\cdots r_{j_k}^{n_{pk}}\right)$$

for $j_1,j_2,\ldots,j_k \in \{1,\ldots,s\}$ where s is the number of generators of G. If $\mu_i\neq i$ for all i, $\omega(A_i)$ is the same for all i, and

$$\sum_{i=1}^{p}\left(n_{i1}+\ldots+n_{ik}\right)\left(\delta(a_i)-\delta(a_{\mu_i})\right)\neq 0,$$

then all Triple Products vanish.[16]

PROBLEM 1.

Let $G=\{a_1,a_2,a_3 | r_1=a_1a_2a_3a_2^{-1}a_3^{-1}a_1^{-1}a_3a_2a_3^{-1}a_2^{-1}\}$ be the fundamental group of a link. Determine if all Triple Products vanish.

r_1 is of the form $a_iA_i(a_1,a_2,a_3)a_{\mu_i}^{-1}A_i^{-1}(a_1,a_2,a_3)$ where $i=1$, $\mu_1=1$, and $A_1(a_1,a_2,a_3)=a_2a_3a_2^{-1}a_3^{-1}$.

1) For $I=a_1$, $I=a_2$, or $I=a_3$ we are going to show that $\epsilon_I(A_1)=0$.

$\epsilon_{a_1}(A_1)=0$ since A_1 does not have any sequence of a_1's.

$\epsilon_{a_2}(A_1)=1-1=0$ by Lemma 1.3.

$\epsilon_{a_3}(A_1)=1-1=0$ by Lemma 1.3.

2) We will show that for at least one of the following strings $I=a_1a_2$, $I=a_1a_3$, or $I=a_2a_3$ will have $\epsilon_I(A_1)\neq 0$.

$\epsilon_{a_1a_2}(A_1)=0$ since A_1 does not have any occurrences of a_1a_2.

$\epsilon_{a_1a_3}(A_1)=0$ since A_1 does not have any occurrences of a_1a_3.

$\epsilon_{a_2a_3}(A_1)=1-1+1=1$ by Lemma 1.3.

Hence, by Corollary 1.4. $A_1\in F^{(2)}$ and $A_1\notin F^{(3)}$. Therefore, by Definition 3.1 we have $\omega(A_1)=2$.

3) Next, we will show that the deviation of r_1, $\delta(r_1)$, and the degree of $\delta(r_1)$ do not vanish.

Let $U=a_1$ and $V=a_2a_3a_2^{-1}a_3^{-1}$. Under the mapping

$F_3 \longrightarrow A(\mathbb{Z},3)$ defined by

$$a_i \to 1+x_i$$

$$a_i^{-1} \to 1-x_i+x_i^2-x_i^3+\ldots+(-1)^n x_i^n+\ldots$$

we have

$$U \to 1+x_1$$

$$V \to 1+(x_2x_3-x_3x_2)$$
$$+(x_3x_2x_2-x_2x_3x_2+x_3x_2^2-x_2x_3^2+x_2^3-x_3^3)+\ldots$$

Hence, by Definition 1.22, we have

$$\delta(U)=x_1$$

$$\delta(V)=x_2x_3-x_3x_2$$

By Lemma 1.8(v),

$$\delta(UVU^{-1}V^{-1}) =x_1(x_2x_3-x_3x_2)-(x_2x_3-x_3x_2)x_1$$
$$=[x_1,[x_2,x_3]_0]_0$$

Hence, the degree of $\delta(A_1)=3$.

Finally, by Theorem 3.2(i) we can conclude that there exists a non-vanishing Triple Product.

PROBLEM 2.

Let $G=\{a_1,a_2,a_3,a_4 | r\}$ be the fundamental group of a link where

$$r=a_1a_2a_3a_4a_3^{-1}a_4^{-1}a_2^{-1}a_4a_3a_4^{-1}a_3^{-1}a_1^{-1}a_3a_4a_3^{-1}a_4^{-1}a_2a_4a_3a_4^{-1}a_3^{-1}a_2^{-1}.$$

Determine if all Triple Products vanish.

r is also of the form $a_iA_i(a_1,a_2,a_3)a_{\mu_i}^{-1}A_i^{-1}(a_1,a_2,a_3)$

where $i=1$, $\mu_i=1$, and $A_1=a_2a_3a_4a_3^{-1}a_4^{-1}a_2^{-1}a_4a_3a_4^{-1}a_3^{-1}$. As in problem 1 we will find the weight of A_1. Since

$$\epsilon_{a_2}(A_1)=\epsilon_{a_3}(A_1)=\epsilon_{a_4}(A_1)=0,$$

$\epsilon_{a_2a_3}(A_1)=\epsilon_{a_2a_4}(A_1)=\epsilon_{a_3a_4}(A_1)=\epsilon_{a_4a_3}(A_1)=0$, and

$\epsilon_{a_2a_3a_4}(A_1)=1$,

we conclude that $A_1 \in \mathcal{F}^{(3)}$ and $A_1 \notin \mathcal{F}^{(4)}$. Hence $\omega(A_1)=3>2$. Therefore by Theorem 3.2(ii), all Triple Products vanish.

PROBLEM 3.

Let L be the Borromean Rings. Then

$$\pi_1(L)=\{a_1,a_2,a_3|r_1,r_2\}$$

where $r_1=[a_2,[a_1,a_3^{-1}]]$ and $r_2=[a_3,[a_2,a_1^{-1}]]$.

$$r_1=a_2a_1a_3^{-1}a_1^{-1}a_3a_2^{-1}a_3^{-1}a_1a_3a_1^{-1},$$

$$r_2=a_3a_2a_1^{-1}a_2^{-1}a_1a_3^{-1}a_1^{-1}a_2a_1a_2^{-1}.$$

Both r_1 and r_2 are of the form as described in Theorem 3.2. where $A_1=a_1a_3^{-1}a_1^{-1}a_3$, $A_2=a_2a_1^{-1}a_2^{-1}a_1$, $\mu_1=1$, and $\mu_2=2$. Then

1) $\epsilon_{a_1}(A_1)=\epsilon_{a_3}(A_1)=0$.

2) $\epsilon_{a_1a_3}(A_1)=-1+1-1=-1$.

3) $\epsilon_{a_1}(A_2)=\epsilon_{a_2}(A_2)=0$.

4) $\epsilon_{a_1a_2}(A_1)=1$.

By Corollary 1.4, $A_1,A_2 \in \mathcal{F}^{(2)}$ and $A_1,A_2 \notin \mathcal{F}^{(3)}$ which imply that $\omega(A_1)=\omega(A_2)=2$.

Computing the deviation for r_1 and r_2 we obtain

$\delta(a_1)=x_1$ $\qquad\qquad$ $\delta(a_3)=x_3$

$\delta(A_1)=x_3x_1-x_1x_3$ $\qquad\qquad$ $\delta(A_2)=x_1x_2-x_2x_1$

Hence,

$$\delta(r_1)=\delta([a_1,A_1])$$

$$=x_1(x_3x_1-x_1x_3)-(x_3x_1-x_1x_3)x_1$$

$$=x_1x_3x_1-x_1x_1x_3-x_3x_1x_1+x_1x_3x_1$$

$$=[x_1,[x_3,x_1]_o]_o$$

Similarly $\delta(r_2)=[x_3,[x_1,x_2]_o]_o$.

Therefore, by Theorem 3.2(i) there exists a non-vanishing Triple Product.

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