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A Product of the Intuitive Mind

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Mathematical induction is prevalent in many varied areas of mathematics, and has several very necessary and useful applications. Yet induction, as a deductive method, is often not fully comprehended by the person applying it.

This thesis presents mathematical induction in both a theoretical and applicational light. The reason for this format is to encourage a more indepth and comprehensive understanding of mathematical induction and the theory behind it.

A further purpose of this thesis, is to expose the reader to a variety of applications and variations in the principle of induction. This is done by presenting induction from the first principle to the transfinite case, and fully documenting each with the appropriate theory and examples.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

A Product of the Intuitive Mind

A Thesis in Mathematics

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To Doctor Marion Emerson, for instilling in me the inspiration and drive to strive for higher knowledge;

To my family and friends, who helped me through difficult moments;

And to the commemoration of Doctor Donald Bruyr, a great teacher, and a great man.

Preface

The sole purpose of this work is to help the reader gain a precise, clear cut, indepth understanding of the principle of mathematical induction. To reach this end it is necessary that you, the reader, have (1) a fundamental knowledge of set notation and theory and (2) a working knowledge of functions and sequences. The first shall be assumed, since fundamental set theory is basic to mathematics and should be common knowledge to one interested in a concept of the magnitude of induction. The second, a working knowledge of functions and sequences, shall be afforded an introduction and brief review.

Therefore, the body of this work will consist of three chapters. Chapter One includes the necessary introductory material and a brief discussion on the meaning of induction and its importance. Chapter Two will present finite mathematical induction, implementated with demonstrations of its uses and applications through examples. Chapter Three will discuss transfinite induction through a similar format.

As you proceed through this work, realize that the beauty of mathematical induction is that it is not derived from experience, but rather it is an inherent, intuitive, almost instinctive property of the mind.

CONTENTS

Chapter One

A Beginning

To comprehend mathematical induction, one must possess a fundamental knowledge of set theory, functions and sequences, since these topics are essential to the discussion of the concept of mathematical induction. In fact, these topics are employed in the very definition of the first principle of mathematical induction.

Therefore, to insure a solid basis for and full understanding of this work, the following introductory material is briefly presented.

SECTION 1: Sets

A fundamental knowledge of set notation and theory is basic to mathematics, and is assumed to be common knowledge to one interested in a concept of the magnitude of induction. Thus there shall be no formal presentation here. The student interested in reacquainting himself with the set concepts may find reference to Flora Dinkines' Elementary Theory of Sets helpful, or any of several other works on basic set theory. (3)

SECTION 2: Functions and Sequences

The discussion of functions is dependent on the ideas of (1) ordered pairs and (2) relations, which are directly involved in its definition. Therefore the discussion shall begin with the definition of ordered pairs, follow through an operation on ordered pairs known as the Cartesian Product, and culminate with a discussion on relations, functions, and sequences.

First consider the concept of ordered pairs of numbers.

DEFINITION: An ordered pair is denoted by (x,y) where x is called the first coordinate and y is called the second coordinate. (13, p.3)

DEFINITION: The ordered pair (x,y) equals the ordered pair (u,v) if and only if x=u and y=v. (13, p.3)

Therefore, an ordered pair is a set of two numbers in which order is important, for example: $(4,5) \neq (5,4)$.

Secondly, consider an operation which yields ordered pairs when performed on two sets.

DEFINITION: The Cartesian Product of two sets A and B (symbolized A x B) is the set of all ordered pairs having the first coordinate from set A and the second coordinate from set B. (8, p.70)

The Cartesian Product of A and B where A = (1,2,3) and B = (a,b) is the set of ordered pairs:

$$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

Now, with the knowledge of the above definitions, the following definition of relation should be more easily comprehended.

DEFINITION: A relation from a set A to a set B is any set of ordered pairs in A x B. The set of first coordinates in a relation is the domain. The set of second coordinates in a relation is the range. (8, p.71)

A relation, then, is a rule which relates elements of the domain with elements of the range. The relation less than on the two sets $\{0,1,2\}$ and $\{1,2\}$ yields the set of ordered pairs: $\{(0,1), (0,2), (1,2)\}$ since $0<1$, $0<2$, and $1<2$.

By computing A x B where $A=\{0,1,2\}$ (domain) and $B=\{1,2\}$ (range), one obtains:

$$A \times B = \{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\}.$$

It is seen that the relation less than yields a set of ordered pairs contained in the larger set A x B, thereby satisfying the definition of a relation.

With relation defined, the final step of a series of definitions that leads to the important concept of a function, is complete.

DEFINITION: A function F is a relation in which no two ordered pairs have the same first coordinate and different second coordinates. (3, p.83)

The set of ordered pairs $\{(0,1), (1,3), (2,3)\}$ is a function; however, $\{(0,1), (0,2), (1,3)\}$ is not, since 0 has two different second coordinates, 1 and 2. In more formal terms, a function F is a rule that associates with each element x in the domain a unique element y in the range. This unique element y is often denoted by F(x), read "F of x". F(x) or y is called the image of x under F.

The function or rule $y=x^2$, with a domain $D=\{x \mid x \in \text{Reals } R\}$, has a range consisting of a set of non-negative real numbers, and $F(0)=0^2=0=y$, so (0,0) is an ordered pair of the function, as is (1/2,1/4), (-1/2,+1/4), (3,9), and (-3,9).

The definition of a sequence in terms of a function is supplied below:

DEFINITION: A sequence is a function whose domain is the set of positive integers N. (13, p.5)

The value of the sequence S at n, where n is an element of the positive integers N, is denoted S_n, and is called the n^{th} term of the sequence. So the ordered pairs of this special function would be of the form (n, S_n), where n is some element of the domain of positive integers and S_n is the value of n under the given rule S. However, since the domain of a sequence is always the set of positive integers, it is often written $\{S_n\}$ rather than $\{(n,S_n)\}$.

The sequence $\{\frac{1}{n}\}$ can be denoted as $\{(n,\frac{1}{n})\}$, where n is an element of N, and the terms of the sequence are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.... The fifth term of this sequence would be the value of this sequence when n=5, that is: $S_n = 1/5$.

In finishing, this presentation was meant as a brief review of fundamental terms involved in the discussion and understanding of mathmetical induction. It was not intended to be a detailed or comprehensive study of the above terms. If more detailed discussion is necessary for fuller comprehension of the above subjects, then it may be helpful to refer to Fundamental College Algebra by Mervin L. Keedy. (8)

SECTION 3: The Mystic Undertaking of Mathematical Induction

The principle of mathematical induction provides one of the most powerful methods of proof available to the mathematician. Its applications are widespread, touching areas of math from algebra and geometry to trigonometry. Its application in the definitions of certain mathematical concepts allows a level of clarity and precision which would be otherwise unattainable. Mathematical induction is, in fact, invaluable, for it supplies a process of proving which eliminates the necessity for verification for all positive integers. Yet induction fought a long embittered battle for recognition as an acceptable method in "higher" science.

In 1739, David Hume's A Treatise of Human Nature questioned the world as to the validity of induction as a plausible means of inference for the higher, pure sciences. His skeptical problem about the future, often called the problem of induction, stated in short that: "Our expectations are formed by customs and habit, but lack justification." (4, p.176) Closely related is the skeptical problem about generalizations: "Can any number of observed instances, short of a complete survey, ever make it reasonable to believe a generalization?" (4, p.176) Hume therefore stated the doubt that any occurrence could be "fully" proven until every case had been checked. This method, however, was stifling and preposterous, for in dealing with the infinite as mathematics often does, it is impossible to verify every case.

Through years of study, and with the help of great mathematicians such as Blaise Pascal, (1623-1662), Pierre de Fermat, (1601-1665), and Jacob Bernoulli, (1654-1705), the beginning of mathematical induction had already begun to form when Hume completed his famous Treatise. But its acceptance into the higher sciences had to wait until the 19th century. It was during this time that Guiseppo Peano, (1858-1932), stated his famous fifth axiom. Peano's Axioms were meant as an axiomatic basis for the natural numbers, and Peano's fifth axiom is essentially the first principle of mathematical induction.

Peano's axiomatic approach (paired with and compiled on others' achievements) took mathematical induction from an inductive process to a method of deductive proof. The inductive process involves mere conjectures, or incomplete induction, from which properties of the positive integers may be discovered, but not assumed true without proof. However, the accepted form of mathematical induction is a rigorous method of deductive proof that eliminates the necessity of verification for all positive integers. Mathematical induction literally takes the step from inductive assumption to deductive fact.

Jules Henri Poincare, (1854-1912), a famous French mathematican, put it best when he said, ".... at once necessary to mathematics and irreducible to logic. Mathematical induction is not derived from experiences, rather it is an inherent, intuitive, almost instinctive property of the mind. What we have done, we can do again." (7, p.5)

Chapter Two

Finite Induction

With the building blocks supplied, and an insight into the history and development of induction, the first principle of mathematical induction is at hand.

The first principle of mathematical induction can be proven by application of the well-ordering axiom for the positive integers, or it may be stated as an axiom. Guiseppo Peano chose to state the first principle as his fifth axiom; this approach shall be examined in Section 3 of this chapter. Immediately ensuing is a proof of the first principle of mathematical induction using the well-ordering axiom.

SECTION 1: The Well-ordering Axiom

The concept of order in mathematics is defined as follows:

DEFINITION: A set X is ordered if for every x and y in X, either $x \leq y$ or $y \leq x$. (9, p.54)

Thus the natural numbers are ordered, because for any two elements of the set chosen, one is either less than the other, or they are equal.

A further property of the set of natural numbers is that it is well ordered.

WELL-ORDERING PRINCIPLE: An ordered set is well-ordered if and only if every non-empty subset has a smallest element. (1, p.213)

Therefore there exists an element L for every non-empty subset of any well-ordered set such that: $L \leq x$ for all x an element of the well-ordered set. In the set of natural numbers, L would equal 1, since $1 \leq x$ no matter what other element of the set is chosen for x.

Examples of sets that are not well ordered (i.e. contain no least element in the set) are:

$T = \{x | x$ is an element of rationals, $0<x<1\}$

$S = \{x | x$ is an element of negative integers, $x<-1\}$

The following proof is presented as an illustration of the Well-ordering axiom, and a demonstration of using it to prove a statement about the natural numbers.

STATEMENT: For every natural number n, 2 is a factor of $n^2 + n$

PROOF: Let P(n) denote the open sentence "2 is a factor of $n^2 + n$". Let S be the set of all natural numbers x such that P(x) is false. Therefore if x is an element of S then 2 is not a factor of $x^2 + x$. The approach to this proof lies in the fact that if the set S is empty, then the statement must be true for all natural numbers n. Therefore use a proof by contradiction, and assume that S is not empty.

If S is not empty then by the Well-ordering Axiom there exist a smallest element t in S. Hence, t is the smallest integer such that 2 is not a factor of $t^2 + t$. Since $P(1) = 1^2 + 1 = 2$ has 2 as a factor, then $t>1$. Furthermore, $P(t-1) = (t-1)^2 + (t-1)$ has 2 as a factor, since $t - 1<t$ and t is the smallest integer such that 2 is not a factor of $P(t) = t^2 + t$. Since 2t has 2 as a factor (it is obvious 2 divides 2t), the sum of $((t-1)^2 + (t-1)) + 2t$ must have a factor of 2 (this holds since $2x + 2y = 2(x+y)$ is true for all natural numbers). But $(t-1)^2 + (t-1) + 2t = t^2 - 2t + 1 + t-1 + 2t = t^2 + t$. Therefore, $t^2 + t$ has 2 as a factor, this contradicts the assumption that 2 is not a factor of $t^2 + t$.

Thus, the conclusion is that there exists no smallest integer t such that 2 is not a factor of $P(t) = t^2 + t$; the set S is empty, and 2 is a factor of $P(n) = n^2 + n$ for every positive integer n. (13, p.9)

SECTION 2: The First Principle of Mathematical Induction as a Theorem

In situations where mathematical induction is applicable, there exists a one-to-one correspondence between an infinite set of statements, $\{S_n\}$, and the natural numbers n.

For example S_n is the statement:

$$S_n: \ 1 + 3 + 5 + \cdot \cdot \cdot + (2n-1) = n^2$$

Then for each natural number n there exists a statement S_n, as follows:

$$S_1: \ 1 = 1^2$$

$$S_2: \ 1 + 3 = 2^2$$

$$S_3: \ 1 + 3 + 5 = 3^2$$

$$S_4: \ 1 + 3 + 5 + 7 = 4^2, \text{ etc.}$$

Concurrently, since a sequence is a function, then sequence and function notation may be interchanged, therefore the above may be restated equivalently as:

$$P(n): \ 1 + 3 + 5 + \cdot \cdot \cdot + (2n - 1) = n^2$$

$$P(1): \ 1 = 1^2$$

$$P(2): \ 1 + 3 = 2^2$$

$$P(3): \ 1 + 3 + 5 = 3^2$$

$$P(4): \ 1 + 3 + 5 + 7 = 4^2, \text{ etc.}$$

This is simply concurrent symbolism and should not confuse the reader.

The statements S_n (or $P(n)$) can not be proven for all natural numbers n by validating a few cases, nor even by validating a vast number of instances, for this does not imply that every case holds true. In fact, this would be incomplete induction. Consequentially, the need for a method of proof arises which does not necessitate the validation of every case. The method known as mathematical induction satisfies this need.

Mathematical induction is used to prove a statement of the type: For every natural number n, S_n is true. Even more specifically, to prove that all of the following are true:

S_1, S_2, S_3, S_4, ..., S_n, ...

The following principle of mathematical induction can be used for such a proof, and is presented along with a proof of the principle by way of the well-ordering axiom.

<u>THEOREM</u>: Let P(n) be a function over the natural numbers, and assume the following:

(a) P(1) is a true statement

(b) For any natural number $k \geq 1$, if P(k) is true then P(k + 1) is true.

Conclusion: P(n) is true for every positive integer.

Let $S = \{x \mid x$ is a positive integer and $P(x)$ is a false statement$\}$. Assume S is not empty, then by the well-ordering axiom there exists an integer t in S which is the smallest; that is, t is the least positive integer such that P(t) is false. By hypothesis (a) P(1) is true, thus $t \neq 1$. Furthermore P(t - 1) is true since $t - 1 < t$ and t is the smallest integer such that P(t) is false.

By hypothesis (b), since P(t-1) is true, then P((t-1) + 1) is true; that is P(t) is true. This contradicts the statement that P(t) is false. Thus, the set S must be empty, and therefore P(n) is true for every positive integer. (13, p.10)

From the above theorem, the first principle of finite induction may be more clearly stated:

PRINCIPLE OF MATHEMATICAL INDUCTION: Let S_n (or P (n)) be a statement for all natural numbers n, and show:

(a) Basis Step: S_1 is true (or: P(1) is true)

(b) Induction Step: For all natural numbers k, if S_k, then S_{k+1} (or: If P(k) then P(k+1))

Conclusion: S_n is true for all natural numbers n (or: P(n) is true for all natural numbers n).

According to the theorem then, to prove S_n is true, one must show (a) S_1 is true and (b) assuming S_k, show S_{k+1} is true.

To see this more clearly, suppose both parts of the proof are completed, resulting in the following endless sequence of statements:

S_1	Basis Step
$S_1 \rightarrow S_2$	These can be calculated up to any value
$S_2 \rightarrow S_3$	of n an element of the natural numbers, however,
$S_3 \rightarrow S_4$	cannot be assumed for values greater than n.
⋮	
$S_k \rightarrow S_{k+1}$	Induction Step
⋮	This step allows the conclusion, using the axiom or principle of induction, that S_n is true for the remaining cases greater than the n calculated (i.e.: true for all natural numbers).

An illustration of the first principle of mathematical induction follows:

STATEMENT: For every natural number n,

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2$$

PROOF:

S_n: $1 + 3 + 5 + \cdot \cdot \cdot \cdot + (2n - 1) = n^2$

S_1: $1 = 1^2$

S_k: $1 + 3 + 5 + 7 + \cdot \cdot \cdot \cdot + (2k - 1) = k^2$

S_{k+1}: $1 + 3 + 5 + 7 + \cdot \cdot \cdot \cdot + (2(k+1)-1) = (k+1)^2$

(a) Basis Step: S_1: $(2(1)-1) = 2 - 1$

$= 1$

$= 1^2$

therefore true for S_1.

(b) Induction Step:

Assume S_k: $1 + 3 + 5 + \cdot \cdot \cdot \cdot + (2k-1) = k^2$

Deduce S_{k+1}: $1 + 3 + 5 + \cdot \cdot \cdot \cdot + (2k-1) + (2(k+1)-1) = (k+1)^2$

(1) $1 + 3 + 5 + \cdot \cdot \cdot \cdot + (2k-1) = k^2$

(2) $1 + 3 + 5 + \cdot \cdot \cdot \cdot + (2k-1) + (2(k+1)-1) = k^2 + (2(k+1)-1)$

(3) $= k^2 + 2k + 2 - 1$

(4) $= k^2 + 2k + 1$

(5) $= (k+1)^2$

therefore by adding $(2(k+1)-1)$ to both sides of S_k and simplifying the right hand side, one deduces or arrives at S_{k+1}.

Since S_n is true for S_1 and for S_{k+1} when S_k assumed, then by mathematical induction S_n is true for <u>all</u> natural numbers n.

SECTION 3: The First Principle of Mathematical Induction as an Axiom

As is stated in Section 1 of this chapter, the first principle of mathematical induction may be proven by way of the well-ordering axiom or may be stated as an axiom. The difference between these two choices is pedagogical, since it can be proven that the well-ordering axiom and the first principle of mathematical induction are equivalent statements; that is, they imply one another.

It is in the study of the natural numbers that the principle of mathematical induction is applied as an axiom. Guiseppo Peano constructed an axiomatic structure for the natural numbers, laying down five axioms, which were to become known as Peano's Axioms. The fifth of these axioms, of which a simplified form is supplied below, is essentially the first principle of mathematical induction.

Axiom I: 1 is an element of N where N is the set of natural numbers.

Axiom II: To each x an element of N there corresponds an unique element x' an element of N called the successor of x. (x' = x + 1)

Axiom III: For each x an element of N there exists an x' ≠ 1, that is, 1 is not the successor of any number.

Axiom IV: If x,y are elements of N such that x' = y', then x=y.

Axiom V: Let S be a set of elements of N. Then S=N provided the following conditions are satisfied:

(1) 1 is an element of S

(2) If x is an element of S, then x' is an element of S. (9, p.46)

These assertions came to be "considered as the fountainhead of all mathematical knowledge. From them it is possible to define integers, rational numbers, real numbers and complex numbers, and to derive their usual arithmetic and analytic properties." (5, p.47-48) In this work, the applications of Peano's fifth axiom are of immediate importance. The proof of the case below differs little from that presented when mathematical induction was proven as a theorem.

<u>STATEMENT</u>: For every natural number n,

S_n: $1 + 2 + 3 + \cdots + n = n(n+1)/2$

S_1: $1 = 1(2)/2$

S_k: $1 + 2 + 3 + \cdots + k(k+1)/2$

S_{k+1}: $1 + 2 + 3 + \cdots + (k+1) \cdot (k+1+1)/2$

<u>PROOF</u>: Define S by $S = \{k \mid k$ is an element of N and $1+2+\cdots+k=k(k+1)/2\}$

(a) Basis Step: S_1: $1 = 2/2$

$= 1(2)/2$

$= 1(1+1)/2$

therefore true for S_1.

(b) Induction Step:

Assume S_k: $1+2+\cdots+k=k(k+1)/2$

Deduce: S_{k+1}: $1+2+\cdots+k+(k+1)=(k+1)(k+1+1)/2$

$=(k+1)(k+2)/2$

(1) Add k+1 to both sides of S_k:

$1+2+3+\cdots+k+(k+1)=k(k+1)/2+(k+1)$

(2) Factor out k+1 and find a common denominator for the second term:

$=(k+1)((k/2)+1))$

$=(k+1)((k/2)+(2/2))$

$=(k+1)(k+2)/2$

therefore by assuming S_k, S_{k+1} can be deduced.

Therefore S=N by Peano's fifth axiom. Since S is the set of all natural numbers for which the statement is true, and since it has been proven S=N, then the proposition is true for all natural numbers.

SECTION 4: Danger

Before proceeding with this discussion of mathematical induction, a warning should be given. Mathematical induction is a beautiful and powerful tool of mathematics. It provides methods of proof, means for verification of formulas, and a clear cut precise way of defining certain mathematical concepts, which is discussed in Section 7. Yet mathematical induction is very dangerous, for it is easy to misuse.

It must always be remembered that mathematical induction is a two part method. The order in which these two parts are verified is not significant, but the verification of each separately, is the crux of an inductive proof.

One must always verify that the proposition, or formula, under scrutiny is true not only for the least integral value of n for which it holds, but also for any value n=k+1, assuming n=k is true. (This idea is expounded upon in the coming section.) Then and only then will the proposition be true for all natural numbers n.

If, for example, one performs the basis step of an inductive proof, and finds it true for any number of natural number values, but neglects to perform the inductive step, then that proposition still remains unverified. The best one can offer from this is a conjecture as to the nature of the proposition. Examples in which the basis steps can be shown, but the inductive steps are impossible to demonstrate are:

(1) $2+4+6+\cdots+2n = n^3-5n^2+12n-6$ which is true for 1,2, and 3, but fails for n=4.

(2) $n^2 = n$

(3) x^2+x+41 is a prime number.

Similarily, if one demonstrates the inductive step, but fails to perform the basis step, then the proof is incomplete. The best one can offer from this situation, is a case of incomplete induction. Examples of these types are:

(1) $1+2+3+\cdots+n=1/2n(n+1)+1$

(2) $x+5=x+7$

which holds for the general case or inductive step, yet no value of n makes the proposition true——try a few!

In conclusion, one of the two steps being true, does not constitute a proof for all natural numbers n. Both cases must be shown for an inductive proof; otherwise, one is just conjecturing or performing incomplete induction.

SECTION 5: Inductive Variations

With the discussion of the first principle of induction completed, the text will now advance into a discussion of variations of this basic principle. Three such variations will be discussed, and reasons for their existence will be supplied.

The first principle of induction is repeated here for ease of reference in comparison to the given variations.

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be an open sentence about natural numbers and assume the following:

(a) P(1) is a true statement

(b) For any integer $k\geq 1$, if P(k) is true then P(k+1) is true.

Conclusion: P(n) is true for every natural number. (13, p.10)

Variation I:

The first variation comes about when one is confronted with an open sentence P(n) such that P(1) is false, P(2) is false, and P(t) is false, if t is an integer less than some integer a; however, P(n) is true for every integer $n \geqslant a$.

Thereby, the basis step does not have to begin with the natural number 1, but may begin with any natural number, even more so, any integer, positive or negative—with the given stipulation that any chosen integer k is greater than or equal to the given least integral value for which the proposition holds (a).

And so the first variation is obtained by simply substituting in an integral value for which the proposition holds, some a, in place of 1.

First Variation:

Let P(t) be an open sentence about integers, and assume the following:

(a) For some integer a, P(a) is a true statement.

(b) For every integer $k \geqslant a$, if P(k) is true, then P(k+1) is true.

Conclusion: P(t) is true for every integer t such that; $t \geqslant a$.

Consider the mapping from M to A as follows:

$M \leftrightarrow A$

$1 \leftrightarrow a$

$2 \leftrightarrow a+1$

$3 \leftrightarrow a+2$

.
.
.

$Q(m) = P(a)$

PROOF: Let Q(m) be the open sentence P(a+(m-1)). If m is any positive integer, then $a+(m-1) \geqslant a$. Therefore, if one can prove Q(m) is true for every positive integer, then P(t) is true for every integer $t \geqslant a$. So apply the first principle of induction to show that Q(m) is true for every positive integer. By the definition of Q(m), Q(1) is the statement P(a+(1-1))=P(a).

By hypothesis (a) P(a) is true, and it follows that Q(1) is true.

Assume that Q(k) is true for some integer $k \geq 1$. Again by definition of Q(m), this means P(a+(k-1)) is true.

By hypothesis (b) P(a+(k+1)-1) is true; that is P(a+k)=Q(1+k) is true.

Therefore, by the first principle of induction one concludes that Q(m) is true for every positive integer and P(t) is true for every positive integer $t \geq a$ (a is not assumed positive). (13, p.14)

An application of the first variation follows:

STATEMENT: Let P(n) be the following open sentence: $2^n < n!$

Now P(n) is false for n=1,2,or 3, however P(n) is true for all integers greater than or equal to 4.

PROOF: Show P(t) is true for every integer greater than or equal to a, where a=4.

(a) Basis Step: Q(1) where Q(1)=P(a) and a=4.

(1) $2^4 < 4!$	(1) Prove
(2) $16 < 4!$	(2) Definition of exponents
(3) $16 < 4\cdot 3\cdot 2\cdot 1$	(3) Definition of factorial (!)
(4) $16 < 24$; true	(4) Closure for multiplication

therefore true for Q(1) thus by definition true for P(a) or P(4).

(b) Inductive Step:

Assume $2^k < k!$

Deduce $2^k+1 < (k+1)!$

(1) $2^k < k!$	(1) Hypothesis

(2) $2\cdot 2^k < 2k!$	(2) Multiplication
(3) $2\cdot 2^k = 2^{k+1}$	(3) Exponential Laws
(4) $2^{k+1} < 2k!$	(4) Substitution into Step 2
(5) $2 < k+1$, for any k	(5) Hypothesis: $k \geq a$ and $a=4$ therefore $2 < 4+1$, is true.
(6) $2k! < k!(k+1)$	(6) Multiplication and Step 5
(7) $k!\cdot(k+1) = (k+1)!$	(7) Definition of factorial (!)
(8) $2k! < (k+1)!$	(8) Substitution into Step 6
(9) $k! < 2k!$	(9) Order Properties
(10) $2^{k+1} < (k+1)!$	(10) Order Properties

therefore, P(t) is true for every integer $t \geq 4$ by the first variation.

A second example of this variation, which may be considered by the reader, is: In every polygon of n sides the sum of the interior angles is $(n-2)\cdot 180^\circ$

Variation II: The Second Principle of Mathematical Induction.

The second principle of mathematical induction is another method of proof which draws the same conclusion as that of the first principle: a proposition P(n) is true for every positive integer (natural number). However, one begins with a slightly different assumption, although P(1) must still be held true, now consider that for any positive integer, k, if the proposition is true for every positive integer less than k, then it is true for k.

The formal statement and proof of this theorem follow:

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be an open sentence about the positive integers and assume the following:

(a) P(1) is a true statement

(b) For any positive integer k, if P(y) is true for every positive integer $y < k$, then P(k) is true.

Conclusion: P(n) is true for every natural number. (13, p.15)

PROOF: Let $S = \{x | x$ is a positive integer and P(x) is false$\}$. If S is not empty, let t be the lease integer in S; (Since P(1) is true, $t > 1$) that is, t is the smallest positive integer such that P(t) is false. Then, P(y) is true for every positive integer $y < t$, and by hypothesis (b) P(t) is true. This contradiction proves that P(n) is true for every positive integer. (13, p.14)

The second principle of induction can be applied to prove the following statement concerning a Fibonacci Sequence. A Fibonacci Sequence being a sequence in which the nth term equals $a_{(n-1)} + a_{(n-2)}$, where a_1 is the first term and a_2 is the second term of the sequence; therefore, if $a_1=1$, $a_2=2$, then $a_n=a_n-1+a_n-2$ for all $n>2$.

STATEMENT: $a_n < (7/4)^n$ for every positive integer n.

PROOF:

(a) Basis Step: n=1

(1) $a_1=1$	(1) Given
(2) $1 < (7/4)^1$, true	(2) Substitution

then for n=2.

(1) $a_2=2$	(1) Given
(2) $2 < (7/4)^2$	(2) Substitution
(3) $2 < 49/4$, true	(3) Exponential Laws

therefore true for P(1) and P(2).

(b) Induction Step:

Assume for an integer $k > 2$ that P(y) is true for all $y < k$.

Deduce P(k) is true; show $a_k < (7/4)^k$

(1) P(k-1) and P(k-2), true by assumption

(2) therefore $a_{k-1} < (7/4)^{k-1}$ and $a_{k-2} < (7/4)^{k-2}$

(3) $a_k = a_{k-1} + a_{k-2} < (7/4)^{k-1} + (7/4)^{k-2} = (7/4)^{k-2} \cdot ((7/4)+1)$

(4) $a_k < (7/4)^{k-2} \cdot (11/4) < (7/4)^{k-2}(7/4)^2 = (7/4)^k$

(5) therefore $a_k < (7/4)^k$

therefore by the second principle of induction, P(n) is true for every positive integer.

Variation III:

This final variation is not often used, but because of its "wide open" nature, it gives the reader an idea of the variety of induction methods that are available.

In this variation one is again operating with the basis step of the inductive method. Recall from the first variation the discussion of substituting for one an integral value a, for which the proposition holds, and applying this as the basis step—the first domino with which one starts the chain reaction of induction. Now in place of this basis element substitute a new element—an a_k where $\{a_k\}$ is any unbounded sequence of positive integers with the property that $a_k < a_{k+1}$, (in essence, then, any unbounded increasing sequence of positive integers such as the natural numbers $\{1,2,3,4,\ldots n,\ldots\}$ or the sequence $S_n = 2^n$). This inductive method takes the following form:

Variation III: Let P(n) be an open sentence about the positive integers and assume the following:

(a) $P(a_k)$ is true for every positive integer k.

(b) For any positive integer u, if P(u) is true, then P(u-1) is true.

Conclusion: P(n) is true for every positive integer.

To gain a better understanding of this variation it will be discussed through the following special case. The special case chosen is the one mentioned above, $S_n=2^n$. Therefore substitute 2^k in place of a_k and formulate a proof for that case.

Variation III: (2^k): Let P(n) be an open sentence about the positive integers and assume the following:

(a) $P(2^k)$ is true for every positive integer k

(b) For any positive integer u, if P(u) is true, then P(u-1) is true.

Conclusion: P(n) is true for every positive integer.

PROOF: Let $S=\{x|x$ is an integer and $P(x)$ is false$\}$. If S is non-empty, there exists a smallest integer t such that P(t) is false. By hypothesis (a), P(n) is true for every integral power of 2. Hence, there exists some positive integer v such that $2^v>t$. Let d be the difference 2^v-t; that is, $d=2^v-t$.

Now, if P(t+1) is true, by hypothesis (b) one would have that P(t) is true. This is a contradiction. If P(t+1) is false, by a similar argument one could conclude that P(t+2), P(t+3), P(t+4), ... , P(t+d) is false. However, since $t+d=2^v$, this would imply that $P(2^v)$ is false, a contradiction of hypothesis (a). Thus, P(n) is true for every positive integer. (2, p.15)

This case of variation III is used to prove the Jensen's Inequality: A function f defined on a closed interval $[a,b]$ is called convex if for each pair of numbers y and z in $[a,b]$ we have

$$f((y+z)/2)\leq (f(y)+f(z))/2.$$

This proof, due to its length and lack of use to this work except as an example, is omitted. The interested reader may find the proof in its entirety in Bevan Youse's Mathematical Induction. (13, p.15)

SECTION 6: Multiple Induction

One more type of finite mathematical induction is of enough consequence to be discussed separately, this is the idea of performing induction on more than one element at the same time. Thus this new type of induction is really not new at all, but is the performance of the types of induction previously discussed, on multiple elements, simultaneously. This technique is known as n-induction.

The simplest form of n-induction is double induction, the performance of induction on two elements at the same time.

Double Induction: D=N x N = $\{(x,y) \mid$ x is an element of N and y is an element of N$\}$; that is, D is the set of all ordered pairs of natural numbers. If S is a subset of D such that the following are true:

(a) (1,1) is an element of S;

(b) If (h,k) is an element of S, then (h+1,k) is an element of S;

(c) If (h,k) is an element of S, then (h,k+1) is an element of S;

Then S=D

PROOF: Let S'= $\{(x,y) \mid$ x and y are elements of N and (x,y) is not in S$\}$. As in previous proofs, attempt to show S' is empty through a contradiction, and thereby proof that S=D. $(x,y) \neq (1,1)$ since (1,1) is an element of S by the hypothesis (a), therefore consider the cases where (x,y) is unequal to (1,1), and x and y are the least elements in their respective positions.

CASE I: x=1, y is unequal to 1. If $y \neq 1$ then, since y is a natural number by hypothesis, $y > 1$ or $y-1 > 0$. Since y is chosen as the least element for the second position of (x,y) where (x,y) is an element of S', then (1,y-1) is an element of S. This, however, implies that (1,(y-1)+1) is an element of S or (1,y) is an element of S. Therefore no such element y exists when x=1, such that (1,y) is an element of S'.

CASE II: $x \neq 1$, $y=1$ follows the same form as Case I, simply apply the argument performed on y in the first case, to the variable x in the second case.

CASE III: $x \neq 1$ and $y \neq 1$ then both x and y are greater than one, so $x-1 > 0$ and $y-1 > 0$. Select (x,y) such that x has miminal value of all $(x,y) \in S'$, then (x-1,y) is an element of S. However by hypothesis (b) ((x-1)+1,y)= (x,y) is an element of S. This contradicts the hypothesis that (x,y) is an element of S'. Selecting (x,y) such that y has miminal value of all (x,y) belonging to S' can be treated similarly. Therefore no $x > 1$ and $y > 1$ exists such that (x,y) is an element of S'.

Therefore all possible cases lead to contradictions, and S' must be empty. This then provides that S=D where D= N x N.

This proof of Double Induction is seen to parallel the previous proofs of induction; indeed, it is identical except for the checking of an extra case. This identical proof, extended to include one more case or element, would provide the proof for Triple Induction (the induction of three elements performed simultaneously: (x,y,z) is an element of S). Triple Induction will now be formally stated so the reader can more fully comprehend the above discussion of performance for one more case.

Triple Induction: Let T=N x N x N = $\{(x,y,z) \mid$ x is an element of N, y is an element of N, and z is an element of N$\}$. If S is a subset of T such that the following are true:

(a) (1,1,1) is an element of S;

(b) If (h,k,ℓ) is an element of S, then $(h+1, k,\ell)$ is an element of S;

(c) If (h,k,ℓ) is an element of S, then $(h,k+1,\ell)$ is an element of S;

(d) If (h,k,ℓ) is an element of S, then $(h,k,\ell+1)$ is an element of S.

Hypothesis (a), (b), and (c) can be drawn verbatim from the proof of Double Induction. The fourth hypothesis (d) is the extra case that must be proven. This is readily done by applying the same method to the third element of the ordered triple, as was applied to the first two elements (i.e.: simply show that there is no element z such that (x,y,z) is an element of S').

By continuing this identical argument n times, one should be able to prove that induction can be performed on any finite number of elements simultaneously. This proof is the proof of n-induction.

<u>n-induction</u>: Let $Z_n = N_1 \times N_2 \times N_3 \times \cdots \times N_n = \{(a_1, a_2, a_3, \ldots a_n) \mid a_i$ is an element of N; $i=1,2,3,\ldots\}$ where the subscripts indicate the number and position of each object in the n-tuple. If Y_n is a subset of Z_n then $Y_n = Z_n$ if:

(1) $(1_1, 1_2, \ldots, 1_n)$ is an element of Y_n

(2) If $(b_1, b_2, \ldots, b_n)$ is an element of Y_n then $(b_1+1, b_2, \ldots, b_n)$ is an element of Y_n

(3) If $(b_1, b_2, \ldots, b_n)$ is an element of Y_n then $(b_1, b_2+1, b_3, \ldots, b_n)$ is an element of Y_n

.
.
.

(n+1) If $(b_1, b_2, \ldots, b_n)$ is an element of Y_n then $(b_1, b_2, \ldots, b_n+1)$ is an element of Y_n.

PROOF: Let $K = \{k \mid Y_k = Z_k$, k is an element of N, and conditions 1 through $(k+1)$ are given$\}$.

CASE I: Show 1 is an element of K.

If $Y_1 \subset Z_1$, then $Y_1 = Z_1 = N$ if conditions (1) and (2) with $k=1$ are satisfied. This follows directly from the first principle of mathematical induction; therefore, 1 is an element of K.

CASE II: If k is an element of K, then $k+1$ is an element of K.

Let X be the set of ordered $(k+1)$-tuples of positive integers not in Y_{k+1}. Show X is empty. The assumption that $k=K$ allows no elements $b_1, b_2, \ldots,$ or b_k such that $(b_1, b_2, \ldots, b_k, b_{k+1})$ is an element of X. Choose b'_{k+1} as the least element in the $k+1$ position of the elements of X. By hypothesis (1) $(1_1, 1_2, \ldots, 1_{k+1})$ is an element of Y_{k+1}, then $(b_1, b_2, \ldots, b'_{k+1}) \neq (1_1, 1_2, \ldots, 1_{k+1})$ and $b'_{k+1} \neq 1$, $b'_{k+1} > 1$ or $b'_{k+1} - 1 > 0$. Since b'_{k+1} was chosen to be the least element in the $k+1$ position of the elements of X, $(b_1, b_2, \ldots, b'_{k+1} - 1)$ is an element of Y_{k+1}. But by $((k+1)+1)$ of the hypothesis, if $(b_1, b_2, \ldots, b'_{k+1} - 1)$ is an element of Y_{k+1}, then $(b_1, b_2, \ldots, (b'_{k+1} - 1) + 1)$ is an element of Y_{k+1} or $(b_1, b_2, \ldots, b'_{k+1})$ is an element of Y_{k+1}. Therefore, X must be empty and $Y_{k+1} = Z_{k+1}$ if $Y_k = Z_k$. Then if k is an element of K then $k+1$ is an element of K. Therefore $K=N$ and $Y_n = Z_n$ for all n is an element of N, with the conditions (1) to $(n+1)$ given.

So with this proof, the discussion of methods of finite mathematical induction is completed. The next section will present an important application of finite mathematical induction.

SECTION 7: Application

The applications of mathematical induction stretch beyond the means of its use as a proof or as a method of verifying formulas. Among its other applications is its use in Inductive or Recursive Definitions.

Mathematical definitions must be both precise and rigorous, this unique combination can be difficult to accomplish when discussing certain mathematical concepts. In fact some concepts, such as the polygon, are very difficult to define.

When a concept involving the positive integers is defined for one and is also defined for the integer k+1 when it is defined for the integer k, then it is defined for every positive integer. It is clear that in this case one is applying the inductive principle, and it is these cases that result in the use of inductive or recursive definitions.

A well known inductive definition is the definition of exponential notation, (a^k) where k is an element of the positive integers.

DEFINITION: For any real number a, define

(1) $a^1=a$

(2) $a^{k+1}=(a^k)\cdot a$ where k is a positive integer.

Therefore $a^5=(a^4)\cdot a$

and similarly $= (a^3)\cdot a\cdot a$

$= (a^2)\cdot a\cdot a\cdot a$

$= (a^1)\cdot a\cdot a\cdot a\cdot a$

Thereby any exponential power can be recursively broken down to any desired level, or inversely; increased to any exponential power.

With this final topic, the discussion of finite mathematical induction is completed. The text now advances forward into the transfinite realm!

Chapter Three

Transfinite Induction

According to Webster the adjective transfinite has the following definition:

TRANSFINITE: 1. going beyond or surpassing any finite number; 2a. being a power of a mathematical aggregate whose cardinal number is not finite; 2b. being either an index by purely algebraic means. i.e.: ordinal numbers. (11, p.2,427)

First, it is given in this definition that transfinite means beyond the finite numbers, or simply infinite. Therefore transfinite induction deals with mathematical induction on infinite numbers or sets. However, the definition also brings about two new terms in its description of transfinite: (1) Cardinal Numbers and (2) Ordinal Numbers.

In order to obtain a complete and knowledgeable understanding of what transfinite induction is, it is necessary to first introduce cardinal and ordinal numbers. Thus, this chapter on transfinite induction is divided into three main sections: (1) Cardinal Numbers, (2) Ordinal Numbers, (3) Transfinite Induction.

SECTION 1: Cardinal Numbers

Cardinal numbers are a means of dividing sets into classes, assigning two sets to the same class if and only if they are equivalent. (10, p.132)

For example, if given two sets, each with five elements, then these two sets would be assigned to the same class since they are equivalent.

This class would be the class of all sets containing five elements, and is given the cardinal number 5. Some examples of sets with a cardinal number of 5 are:

$S=\{1,2,3,4,5\}$

$T=\{6,12,18,24,30\}$

$V=\{m,n,o,p,q\}$

Similarly, the cardinal number 1 corresponds to all sets containing one element, the cardinal number 10 corresponds to all sets containing ten elements, and the cardinal number 1,000 corresponds to all sets containing one-thousand elements.

Cantor formalized the concept of cardinal numbers as a (1-1) - correspondence between the elements of sets:

<u>DEFINITION OF CARDINAL NUMBERS</u>: If A and B are two sets such that there exists a (1-1)- correspondence between the elements of A and the elements of B, then we shall say that A and B have the same cardinal number. (12, p.84)

EXAMPLE:

$A=\{1,2,3,4,5,6\}$

$B=\{10,20,30,40,50,60\}$

There exists an obvious (1-1)- correspondence between A and B, therefore they have the same cardinal numbers. That cardinal number is the number of elements in the set; thus in this case A and B have the cardinal number 6.

It is obvious that as long as a set has a finite number of elements, then its cardinal number simply corresponds to the number of elements in the set. But what is the cardinal number of an infinite set?

Cantor, to assign cardinal numbers to infinite sets, had first to decide a means of classifying infinite sets. To achieve this purpose of classification, he analyzed one of the "simplest" infinite sets, the set N of natural numbers.

Upon analysis of the set N, Cantor found it to possess a distinctive property. The set of natural numbers, though infinite, is still denumerable. By denumerable, is meant, that the natural numbers have a "natural" order by which any element may be counted (the process is achieved by simply adding one to the previous element until the desired element is attained).

Cantor further discovered that some infinite sets, when arranged in a specified order by some predetermined index or rule, can be set up in a (1-1) - correspondence with the natural numbers N. Such sets, through this (1-1)- correspondence, are also denumerable or countable. On the other hand, some infinite sets could not be put in a (1-1)- correspondence with the natural numbers N, these sets are said to be non-denumerable or uncountable.

Cantor, therefore, used denumerability and non-denumerability to classify infinite sets. To the infinite sets which are denumerable or countable, Cantor assigned the cardinal number $\aleph_0$ ("aleph-null"). To the infinite sets which are non-denumerable or uncountable, Cantor assigned the cardinal number c.

As previously stated, to show an infinite set has a cardinal number $\aleph_0$, it is necessary to provide an index or rule by which the set may be specifically ordered, such that a (1-1)- correspondence between the ordered set and the set of natural numbers exists.

This can be done with the set of rational numbers R as follows:

(1) Definition of R: R= $\{p/q$ an element of R$|$p is an integer, q is an element of N and $(p,q)=1\}$.

(2) Index of p/q is $|p| + q$ (thus the index of 2/3 is $|2| + 3=5$).

(3) Order the elements of a set of a given index in pairs of absolute values, the relative order between pairs is determined by numerical magnitude (thus the order of the set of index 5 is: $\{-1/4,1/4;-2/3,2/3;-3/2,3/2;-4/1,4/1\}$.

(4) Arrange finite sets of indexes according to size of their indexes.

Therefore the final ordering of the elements of R is:

R is an element of $\{0,-1,1;-1/2,1/2;-2,2;-1/3,1/3;-3,3;\ldots\}$

Every element of R has a unique position and a definite index. Thus to obtain a (1-1)- correspondence between R and N, let the nth number in R, starting from the left, be denoted by r_n, and make the pairing (n,r_n), n is an element of N, r_n is an element of R. (12, p.80)

As an example of a set which has the cardinal number c, take the set of all reals $\mathcal{R}$. The set of reals is non-denumerable since there does not exist a (1-1)- correspondence between the natural numbers N and the reals $\mathcal{R}$.

In conclusion, cardinal numbers tell literally <u>how many</u> elements are in a set. The finite cardinals are 0 and the natural numbers 1,2,3,4,...,n... The infinite cardinals, termed transfinite cardinals, are numbers of the type $\aleph_0$ and c. There are other transfinite cardinals; in fact, infinitely many. However, for this text, the cardinal numbers have been sufficiently explored.

SECTION 2: Ordinal Numbers

To define ordinal numbers it is necessary to understand the concepts of (1) simply ordered, (2) order types, and (3) well-ordered sets. Therefore each of these concepts will be defined and discussed previous to the introduction of ordinal numbers.

Simply Ordered: A Set C (having exactly n elements, n is an element of N) is simply ordered relative to $<$ (precedes) if:

(1) If x,y are elements of C and if $x \neq y$, then $x<y$ or $y<x$.

(2) If x,y are elements of C and $x<y$, then $x \neq y$.

(3) If x,y,z are elements of C and $x<y$ and $y<z$, then $x<z$. (6,p.10)

Order Type: An order type is a property common to two sets that have a (1-1)- correspondence that preserves the order relations.

A set must be simply ordered to have an order type, but what exactly is an order type? A few examples might best exemplify the meaning of order type.

EXAMPLE # 1:

Let S be a set with cardinal number 2, then $S=\{a,b\}$ (since S must have 2 elements). These 2 elements can be ordered in two ways (either $a<b$ or $b<a$), but either result is still an ordered pair, therefore, any set with two elements has order type two.

EXAMPLE # 2:

Let the set S have cardinal number 3, then S has three elements that can be ordered in 3! or $1\cdot2\cdot3=6$ ways. But still, each of the six orderings would be an ordered triple, therefore any set with three elements has order type three.

In general then, any set with a finite cardinal number n can be ordered in n! ways. Each of the orderings is an ordered n-tuple, therefore the order type must be n.

The next question is naturally, does the previous general statement hold also for infinite sets (or sets with transfinite cardinal numbers $\aleph_0$ or c)? The answer is no. The reason can be easily demonstrated.

Consider the set N of natural numbers. N can be assigned its natural order: $n<n+1$ for all n an element of N. Call this order type ω. Another simple ordering of N is: (1) if a,b are elements of N, and a is odd, b is even, then $a<b$; (2) if a and b are both odd, then $a<b$ denotes the natural order; (3) if a and b are both even, then $a<b$ means $b<a$ in the natural order. The order just defined is: 1,3,5,...,2n+1,...;..., 2n,...,6,4,2. Call this order type y. Then there exists at least two order types on the set of natural numbers.

The conclusion can be drawn that for simply ordered sets of n elements, there exists only one order type n. But for infinite sets there may exist many order types.

Well-ordering Sets: A simply ordered set is well ordered if it has a first element for every non-empty subset of itself. (14, p.159)

Every finite simply ordered set is well ordered. The set of natural numbers N, when ordered in "natural order" ω, is well ordered; however, the set N under order type y is not well ordered, since the subset of even numbers has no first element. Thus, an infinite set may or may not be well ordered, depending on the order type of the simple order.

Finally, with simply ordered, order types, and well ordered all defined, the definition of ordinal numbers is at hand.

Ordinal Numbers: The order types of well-ordered sets. (12, p.121)

From the above discussion it is known that all finite sets are well ordered and have a unique ordering type. In fact, the ordering type for a finite set is equivalent to the cardinal number n of that set. Then it is true for all finite sets that the cardinal and ordinal numbers are equivalent. Thus, the set S= $\{1,3,5,7\}$ has both a finite cardinal and finite ordinal number of 4.

Infinite sets, however, may be so ordered as to belong to more than one well-ordering type, thus transfinite cardinals and transfinite ordinals are not equivalent. This is why for the set N, the cardinal number is $\aleph_0$, while the ordinal number for the "natural order" of N is ω. (Other existing well-ordering types for the set N are symbolized differently.) Literally, an ordinal number specifies which one of the well-ordering types over a given set.

Generating "new" well-ordering types, from established "older" well-ordering types, such as the finite ordinals or ω, can be accomplished by a method of addition defined below:

ADDITION OF ORDER TYPES: If A and B are order types, then A+B is the order type determined by A$\cup$B, so that the elements maintain the original order for A and B, unless an element belongs to both sets, then the order type of A supersedes the order type of B. (14, p.163)

Since this method of addition is generating "new" well-ordering types, it is, per se, generating "new" transfinite ordinal numbers. Intuitively, these new ordinal numbers can be visualized by first regarding the elements of A in the order in which they occur, and then follow these elements with the elements of B in the order they occur. For example, if A is the odd natural numbers $\{1,3,5\cdots\}$ and B is the even natural numbers $\{2,4,6\cdots\}$ then A+B is $\{1,3,5,\cdots;2,4,6,\cdots\}$. The order type of this set is $\omega+\omega$, since both A and B have order type ω.

The operation is not commutative, however; for if the ordinal number 1 is represented by $\{0\}$ and combined with ω, then $1+\omega$ is $\{0,1,2,3\cdots\}$. This order type is obviously still ω, therefore $1+\omega=\omega$. But $\omega+1$ would be represented by $\{1,2,3\cdots,0\}$, which is a new order type. Continuing this argument for n an element of the natural numbers, infinitely many transfinite ordinals of the form ω+n can be generated. In fact, as with transfinite cardinals, there are infinitely many transfinite ordinal numbers.

The following table summarizes some of the information provided in the sections over cardinal and ordinal numbers.

I. Equivalence of Finite Cardinal and Ordinal Numbers.

Cardinal/Ordinal Number		Cardinal Representation	Ordinal Representation
1	First	$\{a\}$	$\{a\}$
2	Second	$\{a,b\}$	$\{(a,b)\}$
3	Third	$\{a,b,c\}$	$\{(a,b,c)\}$
.	.	.	.
.	.	.	.
.	.	.	.
For any n Naturals		$\{1,2,3,\cdots n\}$	$\{(1,2,3,\cdots n)\}$

II. Cardinals ordered according to magnitude.

$1<2<3<\cdots<n<\cdots\ \aleph_0$ (1-1 correspondence with Naturals)

$2^{\aleph_0} = c$ (1-1 correspondence with the Reals)

$2^{c} = f$ (set of all subsets of the Reals)

In fact, for any cardinal number α, 2^{α} is a new cardinal number such that $\alpha < 2^{\alpha}$.

This can be corrected by revising the induction so that: (2') for any n, if every element less than n has a property, then n also has that property. This revision yields transfinite induction.

Transfinite Induction Principle: A simple ordered set W is said to satisfy the transfinite induction principle provided that:

(1) W has a first element

(2) If $W_1 \subset W$ such that (a) W_1 contains the first element of W, and (b) if W_1 contains a section W/w (denotes the set $\{x \mid x < w\}$) then it contains w; then $W_1 = W$. (12, p.116)

Statement number (2) is the actual transfinite induction principle. It is used to show that all elements of a well-ordered set W possess a given property.

It should be noted that finite mathematical induction is a special type of transfinite induction. More specifically: transfinite mathematical induction, performed on a set W which has an order type of ω becomes identical with the finite or ordinary principle of mathematical induction. To more fully comprehend this relationship between finite and transfinite induction, compare the methods of performing both types.

In proving that the elements of a denumerable set S have a certain property P, one applies the finite form of mathematical induction:

Since S is denumerable then its elements may be ordered in a form of a type ω sequence:

$x_1, x_2, x_3, \ldots x_n, \ldots$

(1) Prove x_1 has property P (Basis step)

(2) Prove if for any x_n an element of S, the property P holds, then the property P holds for x_{n+1} also. (Induction step)

Then by finite mathematical induction all x an element of S have the property P.

Now suppose that W is any set, be it denumerable or non-denumerable. In order to prove that all its elements have a certain property P, apply the transfinite induction principle:

Even though W may be uncountable, it might be well-ordered. If W is well-ordered then:

(1) Prove the first element, w_1 of W has property P. (Basis step)

(2) Prove if for any w, an element of W, all the elements of W/w (i.e. $\{x \mid x<w\}$) have the property P, then the property P holds for w also. (Induction step)

Then by transfinite mathematical induction all w an element of W have the property P.

Therefore it is evident, from the above discussion, that finite mathematical induction is a special case of transfinite mathematical induction. Thus several special cases of proof by transfinite induction have been previously demonstrated, that is: all previous proves involving finite mathematical induction were a form of transfinitely mathematical inductive proof. However, in the interest of further clarification of transfinite mathematical induction, and its application in proving theorems on sets W with ordering other than ω, the following proof is presented.

THEOREM: Let W and W' be well-ordered sets. Then either W and W' are of the same order type, or one is of the same order type as a section of the other.

PROOF:

CASE I:

If W is empty, the theorem holds trivially. (This is not the case involving induction, and is of little interest, therefore it is not further discussed.)

CASE II:

W is not empty. Then since W is non-empty and well-ordered, W has a first element w_1.

(1) Either all elements of W' are in order-preserving (1-1)-correspondence with elements of the section W/w, or not.

(2) If they are not, then pair w_1 with the first element w'_1 of W'.

(3) Therefore, if w is an element of W and each element of the section W/w is already paired with some element of W', then either all elements of W' are already in order-preserving (1-1)-correspondence with elements of W/w or not.

(4) If they are not, pair w with the first element of W' not already paired with elements of W.

(5) Then by the transfinite induction principle, either all elements of W' are paired in this manner with elements W, or conversely. (12, p.120)

SECTION 4: Definitions by Transfinite Induction

Definition by transfinite induction is performed in much the same format as that of definition by finite mathematical induction.

Begin by considering a well-ordered set W with a first element w_1. Then if some mathematical entity E is defined for:

(1) $E(w_1)$ and then;

(2) for each w is an element W, E(w) is defined in terms of the section W/w or its elements.

Then from the transfinite induction principle it can be concluded that E(w) is defined for every w is an element of W.

This type of definition allows a precise and clear cut method of defining mathematical concepts involving sets of order type other than ω.

Chapter Four

Conclusion

The original concept of this thesis was to explore, define, and clarify finite mathematical induction.

To achieve this end a ground work was laid in sets, functions, and sequences. From this ground work, the first principle of mathematical induction was introduced as an axiom. Then, with the aid of the Well-ordering Axiom for the positive integers, this first principle was itself proven.

After the first principle was explored through examples, the text advanced to variations of finite mathematical induction such as: The second principle of mathematical induction and multiple induction. These variations were then similarily explored. In conclusion of finite mathematical induction, there was a discussion of their application in definitions of mathematical terms.

The author was then urged---"strongly"---by Doctor Marion Emerson to include in this thesis transfinite induction. Thus, the appropriate introduction in cardinal and ordinal numbers was presented, so that transfinite induction could be included.

This thesis, thereby, is a fairly complete study of mathematical induction. In its pages I hope you found the answer to most of your questions, but not all of them. Because, for me, the interest in mathematical induction was much like the intuitive idea behind it: Every time I answered a question, there was still one more.

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