AN ABSTRACT OF THE THESIS OF

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Title: Finite Geometries Associated with Dihedral Groups of Regular Polygons

Abstract approved: Thomas Bonner

(A succinct summary of the thesis not to exceed 300 words in length.)

The phrase "finite geometry associated with the dihedral group of a regular polygon" is introduced and defined. Two such geometries are developed in detail: the finite geometries associated with the dihedral groups of the square and the regular hexagon. Models are developed for each of these geometries. A method for developing such a geometry associated with the dihedral group of any even-sided polygon is outlined, as well as a way to develop a model for the geometry. The significance of two subgroups of the dihedral group in the development of both the geometry and the model is emphasized. As stated before, the generalized methods are only applicable to finite geometries associated with dihedral groups of order 4n.

FINITE GEOMETRIES ASSOCIATED WITH DIHEDRAL GROUPS OF REGULAR POLYGONS

A Thesis

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the Department of Mathematics

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by

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Approved for the Graduate Council

Preface

The major purpose of this thesis is to define and illustrate what is meant by "a finite geometry associated with the dihedral group of a regular polygon." In Chapters One and Two, two such finite affine geometries are developed, based on the dihedral groups of the square and regular hexagon, respectively. In Chapter Three, patterns discerned in the first two chapters are refined to make them applicable to the development of a finite geometry based on the dihedral group of any regular even-sided polygon. Chapter Four focuses on the special role played by two subgroups of the dihedral group in the development of the associated geometry.

This thesis is original rather than profound, imaginative rather than rigorous, but it is logical throughout.

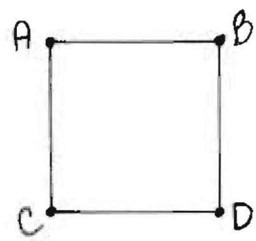
The author wishes to express her sincere appreciation to Dr. Thomas Bonner for his interest and assistance throughout the writing of this thesis.

CONTENTS

Chapter One

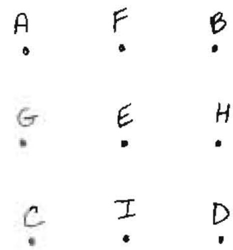
The Finite Geometry Associated With the Dihedral Group of the Square

When considering a square, four essential points are obvious--the vertices. These four points, naturally, are to be included in the development of a related finite geometry. (See figure 1.)



(figure 1)

In order to utilize the transformations of the dihedral group related to the square, one must include the center as a point about which the square can be rotated. One must also consider the lines of symmetry about which the square can be reflected. These lines of symmetry indicate the position of four more essential points, namely, the points of intersection of these lines and the sides of the square. Therefore, these nine natural points (i.e. the center, the vertices and the midpoints of the sides) should be included in any geometry based on the dihedral group of the square. These points are indicated in figure 2. As these are the points suggested by the transformations on the square, one should begin the associated geometry with just them. However, keep in mind, that more points can be added, should that become necessary.



(figure 2)

It is of interest at this point to review the properties of an affine geometry. The basic axioms of such a geometry are:

Axiom 1: There is at least one line.

Axiom 2: There are at least two points on every line.

Axiom 3: Not all points are on the same line.

Axiom 4: There is exactly one line on any two distinct points.

Axiom 5: Given a line and a point not on that line, there is exactly one line on the given point and not on any point of the given line.

The problem, then, is to find a geometry with a minimum number of points satisfying these five requirements and, of course, including the nine natural points determined by the transformations of the dihedral group of the square, D_4.

As the term "point" has been used in referring to the natural points associated with the square, in order to avoid confusion, this author shall refer to the "points" of the finite geometry as Points. The "lines" of the finite geometry are not the natural lines of the square, but rather are to be understood as groups of Points. These groupings will become fixed as the geometry is developed. The "lines" of this geometry will be referred to as Lines. The relation "on" is to be interpreted in such a way that a Line is "on" each Point and each member of that group of Points is "on" the associated Line.

One should require more of the geometry if it is to be considered associated with a dihedral group--namely, that the image of every Point, under the transformations of the dihedral group, be itself a Point and, similarly, that the image of every Line be itself a Line. These requirements necessitate replacing Axiom 2 with Axiom 2': There are the same number of points on every line.

In observing figure 2, it is evident that there are three Points on each of the natural lines. Can the Lines be fixed so that there are also three Points on each Line? If so, some of these Lines are obvious. (Remember, Axiom 4 requires that each pair of Points determine exactly one Line, so no pair of Points may be included on two different Lines.)

Consider Point E. It lies on Lines GEH, FEI, AED and CEB. These four Lines are such that Point E is linked with each of the other eight Points exactly once.

Now consider a Point such as A. If one considers the Lines suggested by the natural lines through this Point, Point A is found on Lines AFB, AGC and AED. These Lines associate Point A with six of the other eight Points. Consider the inclusion of another Line, namely AIH. Now Point A has been grouped with each of the other eight Points exactly once.

Therefore, under the transformation ρ, i.e. a rotation of ninety degrees about the center, Point E, one finds another set of Lines--the images of those Lines on Point A--this time the Lines are those on Point C. These Lines are CGA, CID, CEB and CHF.

Applying the transformation again, one finds the Lines on Point D: DIC, DHB, DEA and DFG, respectively. Likewise, a third application of ρ leads to the Lines on Point B. These are BHD, BFA, BEC and BGI, respectively.

As some Lines have appeared more than once, consider the list of all the Lines determined thus far:

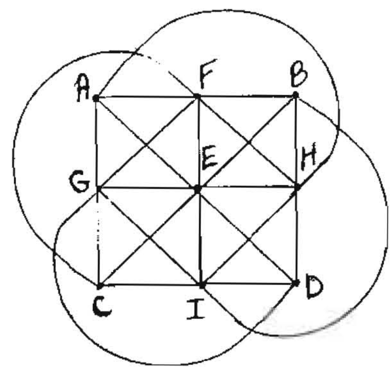
ABF	AHI	BGI	DFG
ACG	BCE	CDI	EFI
ADE	BDH	CFH	EGH

These are the Lines on Points A, B, C, D and E. The task now is to look at the Lines on Points F, G, H, and I. Point F is already found on Lines ABF, CFH, DFG and EFI. Clearly, it has been associated with each of the other eight Points in exactly one way. Through repeated applications of transformation ρ, it can easily be verified that this is also true for Points G, I and H, respectively.

The finite geometry thus developed consists of nine Points and twelve Lines. There are three Points on each Line and four Lines on each Point. Does this geometry satisfy the affine axioms? Clearly it does satisfy Axioms 1, 2, 3 and 4. It only remains for Axiom 5 to be illustrated in this geometry.

As the reader recalls, Axiom 5 requires that "Given a line and a point not on that line, there is exactly one line on the given point and not on any point of the given line." In looking at figure 2, it is obvious that some groups of Lines do satisfy this axiom--namely the Lines associated with the natural horizontal and vertical lines. The Lines ABF, EGH and CDI are one such group of "parallel" Lines, i.e. Lines that have no Point in common. Another group of parallel Lines would be ACG, EFI and BDH. This leaves the six non-horizontal, non-vertical Lines to be considered in light of Axiom 5. These six Lines also fall into two groups of parallel Lines: BCE, AHI and DFG: AED, CFH and BGI. So Axiom 5 is upheld by the arrangement of Lines and Points. Therefore, this geometry is indeed a finite affine geometry.

It would be desirable to be able to illustrate not only the Points of the geometry, but also the Lines. Figure 3 shows one way to do this. The Points are indicated by the dots and the Lines are shown by the curves connecting the associated Points.



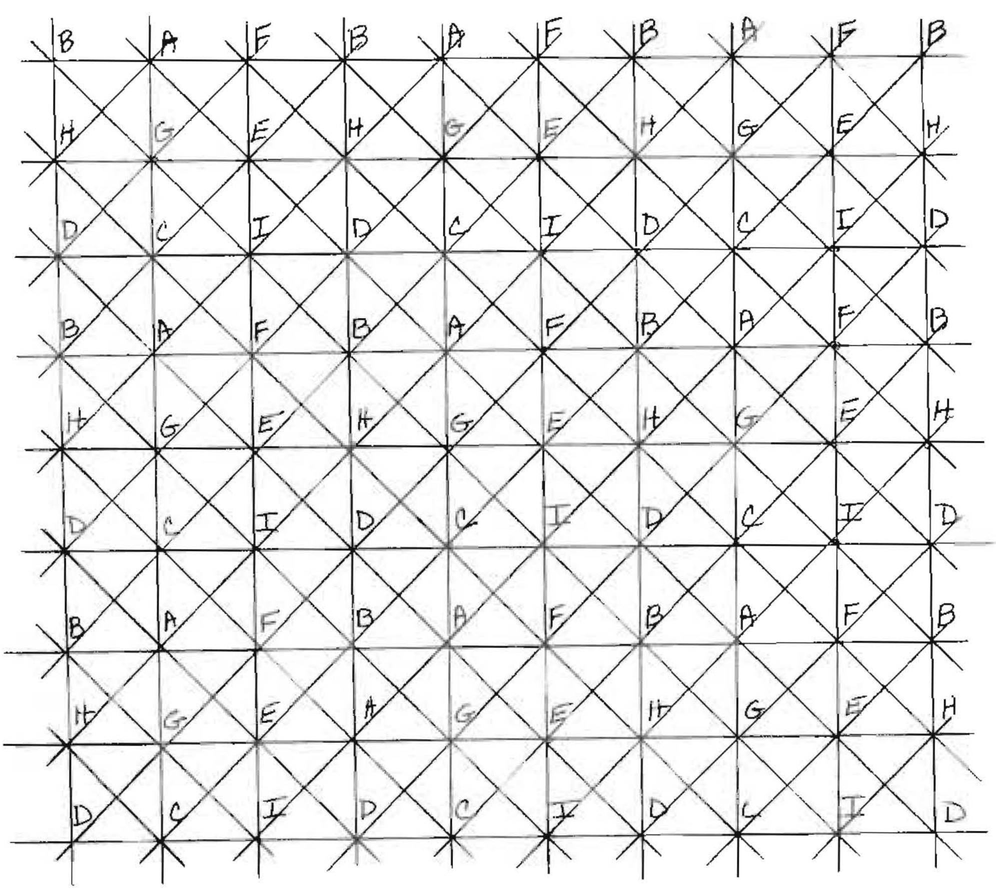
(figure 3)

In view of the fact that aesthetic senses are sometimes upset by the use of curves to represent Lines, it might be preferred to illustrate the geometry in a different manner. One solution to this problem is to "straighten out" the offending Lines through a "repositioning" of some of the Points. This can be done quite simply by imposing a lattice structure on the Points and utilizing the concept of congruence modulo three in establishing coordinates on the lattice structure. Since the distance from the center of the square to the square itself is the same horizontally as vertically, a lattice of 1 X 1 rectangles serves this purpose well. Select a Point, say E, to serve as the origin, (0,0), on the lattice. The lattice is to be arranged so that the other Points can be assigned coordinates in the conventional manner, e.g. so that H is assigned (1,0). However, since it is the arithmetic of modulo three, only the integers 0, 1 and 2 are used in the assignment of coordinates.

Consequently, the coordinates assigned to the Points are:

A	(2,1)	D	(1,2)	G	(2,0)
B	(1,1)	E	(0,0)	H	(1,0)
C	(2,2)	F	(0,1)	I	(0,2)

Please note that there are many lattice points having the same modulo three coordinates. Each of these points is to be designated by the associated Point name. For example $(4,0) \equiv (1,0) \equiv H$. Thus, in this manner, the entire plane of lattice points may be used to represent the nine Points of the geometry. A portion of this lattice is found in figure 4. Points have been designated at the lattice points and the Lines indicated by the lines.



(figure 4)

It is of interest to note that the Points found on the natural lines of the lattice are grouped in the same way as on the Lines of the finite geometry. This is important as it suggests a way to introduce algebraic representation of Lines, as well as Points. Since Lines are groups of Points and the Points have already been assigned modulo three coordinates, one can devise modulo three statements to represent the Lines. Suppose one designates a Line to be the Points in the form (x,y), where x, y are elements of Z_3, such that a certain modulo three statement is true. This would allow the assignment of these statements to the twelve Lines:

EFI:	$x \equiv 0 \pmod 3$	ADE:	$x + y \equiv 0 \pmod 3$
BDH:	$x \equiv 1 \pmod 3$	CFH:	$x + y \equiv 1 \pmod 3$
ACG:	$x \equiv 2 \pmod 3$	BGI:	$x + y \equiv 2 \pmod 3$
EGH:	$y \equiv 0 \pmod 3$	BCE:	$2x + y \equiv 0 \pmod 3$
ABF:	$y \equiv 1 \pmod 3$	DFG:	$2x + y \equiv 1 \pmod 3$
CDI:	$y \equiv 2 \pmod 3$	AHI:	$2x + y \equiv 2 \pmod 3$

Since all other statements of the form $ax + by \equiv c \pmod 3$, where a, b, c are elements of Z_3, are equivalent to one of those listed above, this system of designation has been exhausted. Consequently, one is able to use the lattice structure to devise an algebraic model for the nine Points and twelve Lines. Note that the same Lines are parallel under this system as were parallel originally.

Another method of illustration one might pursue for this geometry involves arrays which indicate the Points on Lines. In looking back on the development of the twelve Lines in the geometry, it is evident that under the transformation ρ , the horizontal Lines are transformed into the vertical Lines and vice versa; likewise, each set of parallel non-horizontal, non-vertical Lines are transformed into the other set of such Lines. It is possible to establish an array of the nine Points such that the rows of the array are the horizontal Lines, and the columns of this array are one set of parallel non-horizontal, non-vertical Lines. Under the transformation ρ , this array becomes a second array which illustrates in its rows and columns the other six Lines of the geometry. Two such arrays are:

$$\begin{matrix} A & F & B \\ E & H & G \\ D & C & I \end{matrix} \quad \text{and} \quad \begin{matrix} C & G & A \\ E & F & I \\ B & D & H \end{matrix}$$

In this representation of the geometry, it is important to remember that the letters of the arrays indicate the Points, but are not "graphs" of the Points. Also, remember that Lines are represented only by the rows and columns of the arrays.

Although the concept of distance is not a requirement for a geometry, if it is to be introduced, then it is obvious that distance cannot be the same on all Lines. This is known since not all Lines can be mapped into each other under the elements of D_4.

No matter which model is used for the geometry of the dihedral group of the square, the Lines fall into two categories: vertical/horizontal versus non-vertical, non-horizontal, or rows versus columns. This suggests that distance must be defined differently on each type of Line. The units that are suggested by the natural geometry of the square are units of 1 along the horizontal and vertical Lines and units of $\sqrt{2}$ along the other Lines.

Therefore, in summary, the finite geometry that has been developed consists of nine Points and twelve Lines. The axioms which define this geometry are:

1. There are exactly nine Points.
2. Every two Points determine exactly one Line.
3. There are exactly three Points on each Line.

The twelve Lines of this geometry can be grouped into four sets of parallel Lines.

This finite geometry can be illustrated by a variety of models. The most simple, perhaps, is on a square where Points are the specified points related to the square and Lines are designated by specified curves on the model. Another geometric model is based on a square lattice imposed on the square. In this model, the points of the extended lattice are identified, by way of repetition, to represent the Points and the natural lines on the lattice points illustrate the twelve Lines.

A third model of the geometry is an algebraic representation of the Points and Lines. Points are designated by modulo three ordered pairs and the Lines are assigned characteristic modulo three statements in such a way that the ordered pairs that satisfy a particular statement are the pairs assigned the Points on the associated Line.

The fourth model for this geometry consists of two 3 x 3 arrays in which letters represent the Points and the rows and columns of the two arrays represent the Lines.

The characteristic of this geometry that is most peculiarly its own is that it is the finite geometry associated with the dihedral group of the square. This implies that only Points essential to the applications of the transformations of D_4 are included in the geometry and only Lines which transform into other Lines under the eight elements of D_4 are included as Lines in the geometry. Thus, the very "substance" of the geometry, its Points and Lines, are determined by the natural points of the square and certain associated points determined by the transformations of the dihedral group.

The author is introducing the phrase "finite geometry associated with the dihedral group of a regular polygon" to mean that geometry with the minimum number of points satisfying:

1. The points intrinsic to the polygon and to the group of transformations of the polygon onto itself are included in the points of the geometry.

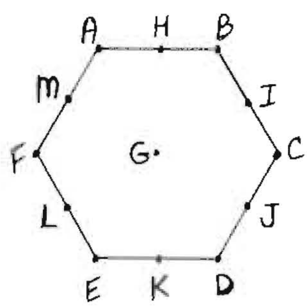
2. The lines of symmetry of the polygon are included as lines of the geometry.
3. The image of any point (or line) of the geometry is itself a point (or line) of the geometry.
4. The geometry must satisfy the five axioms of an affine geometry.

Chapter Two

The Finite Geometry Associated With the Dihedral Group of the Regular Hexagon

The object of Chapter Two is to create a finite geometry associated with the dihedral group of the regular hexagon. In this group, ρ is a sixty degree rotation about the center of the hexagon. Reflections about the six lines of symmetry are other transformations of D_6. One must establish a set of Points and a set of Lines. Lines shall be fixed sets of Points. The relation "on" shall have the same interpretation as in the geometry of nine Points.

Consider the Points required by the dihedral group of the hexagon, as they must be included in the geometry. Naturally, one must include the six vertices of the hexagon, the center, and the points where the lines of symmetry meet the sides of the hexagon. This indicates a minimum of thirteen Points. (See figure 5.)



(figure 5)

On this diagram, many natural lines are associated with three Points. Obvious Lines would be ABH, ADG, AFM, BCI, BEG, CDJ, CFG, DEK, GHK, GIL and GJM. However, there are Point pairs not on any of these Lines. Consider the pair AI. There must be a third Point on that Line. This third Point cannot already be associated with either A or I; and the images of the Line on AI under the elements of D_6 must also be Lines of the geometry.

Points already associated with Point A on other Lines are B, D, F, G, H and M. Points B, C, G and L are associated with Point I on other Lines. This leaves Points E, J or K to be possible on Line AI.

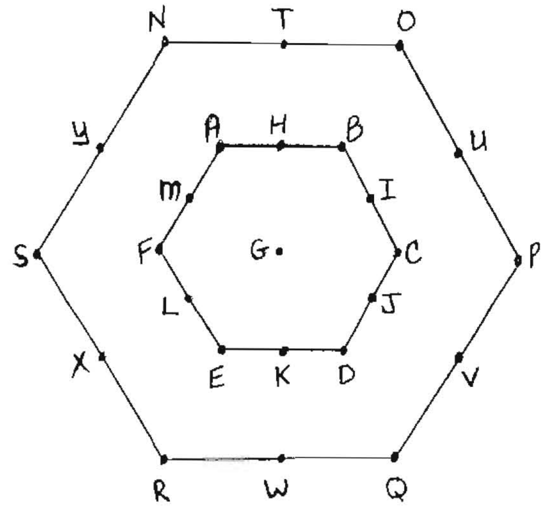
Suppose the Line is AEI. Under the transformation ρ, Line AEI becomes Line FDH. Under transformation ρ, Line FDH becomes Line ECM. Under i, the transformation of reflection about the vertical line of symmetry, Line FDH becomes Line CEH. This puts both Line ECM and Line CEH on the points C and E. Since there cannot be two different Lines on the same pair of Points, E cannot be the third Point on the Line on AI.

Suppose, then, that the desired Line is AIJ. Under ρ, Line AIJ maps into Line FHI. Line FHI, under ρ, becomes Line EMH. Under the transformation i, Line EMH becomes Line DIH. This puts both Line FHI and Line DIH on the pair of Points HI. This is not allowed in the geometry. Therefore, the desired Line cannot be AIJ.

The third possibility, then, is the Line AIK. However, under ρ, AIK becomes Line FHJ. Under ρ, Line FHJ becomes Line EMI. Line EMI becomes Line DIM, under the transformation ι. As there cannot be two Lines on the Points I and M, this third, and last, possibility for the other Point on the Line on AI must also be discarded.

Thus the thirteen Points indicated on figure 5 are not sufficient to determine a Line on Points A and I. Therefore, at least one more Point must be added to the geometry. If the fourteenth Point is on a line of symmetry, under the elements of D_6, five additional Points are determined. Will these nineteen Points suffice? No, for there are now five Points on three Lines, therefore there must be five Points on each Line. This forces the addition of two more Points on each of the other three lines of symmetry, thus, bringing the minimum number of Points to twenty-five. (If the fourteenth Point is not on a line of symmetry, the elements of the dihedral group determine an additional eleven Points, bringing the minimum number of Points to twenty-five, also.) To show that an affine geometry can be formed by this set, consider the twenty-five Points as arranged on two concentric hexagons, such that the radii are in ratio 1 : 2. (See figure 6.)

As the greatest number of Points on any natural line in figure 6 is five, one should strive for a geometry with five Points on each Line. There are a number of such Lines



(figure 6)

suggested by the natural lines in figure 6: YAHBU, SFGCP, XEKDV, THGKW, YMGJV, XLGIU, XFMAT, YFLEW, WDJCU, TBICV, REGBO and NAGDQ. However, there also are "fragments" of lines, such as NTO and MI. What are the other Points on these Lines?

Consider first the Line on NTO. Since N is on this Line, no other Point of Line NAGDQ can be on NTO. Likewise, since T is on the Line, no Point other than T of the Lines THGKW, XFMAT and TBICV can be on the Line NTO; and no Point other than O of the Line REGBO can be on Line NTO. Thus the two other Points on NTO must come from the set J, L, P, S, U and Y. Through a process of eliminating the Points which lead to contradictions, one can determine the other two Points on Line NTO.

Since the Points of fragment OUP are compatible with those of NTO, consider the possibility of Line NTOUP. Under the transformation ρ, NTOUP determines Line SYNTO. Since it is not possible to have two Lines on the Points NTO, Points U and P cannot be the other Points on the Line in question. Similarly, Points S and Y must be eliminated, as Line SYNTO would transform into Line RXSYN, under the transformation ρ, thereby creating a second Line on Points SYN. Therefore, either the other two Points on Line NTO must be L and J, or it is impossible to define a Line on NTO.

Line NTOLJ, under the elements of D_6, determine Lines SYNKI, RXSJH, QWRIM, PVQHL and OUPMK. These Lines satisfy the requirements of this finite geometry. This brings the number of Lines to eighteen.

However, there are still Point pairs which are not on any of the eighteen Lines, for example, Points A and I. No Point already associated with either Point can be on the Line AI. Lines on Point A are YAHBU, XFMAT and NAGDQ; Lines on Point I are TBICV, XLGIU and QWRIM. Therefore the remaining three Points on Line AI must come from the set E, J, K, O, P and S. By eliminating from this set the Points which lead to contradictions, one can determine the three Points that are on Line AI.

Suppose Point O is on Line AI. Two more Points must still be found from the set given above. Obviously, Points E and P cannot be on Line AIO as these Points are already on Lines with Point O. This means that the remaining Points must come from the set J, K and S. Consider a Line on AIOJ, under ρ, this Line has the image FHNI. Under the transformation ρ, FHNI becomes EMSH, while, under transformation i, FHNI becomes CHOM. This indicates the presence of six Points on the same Line, namely EMSHCO. As this cannot be allowed in the geometry, it is clear that J cannot be on the Line AIO. Suppose, then, the Line is AIOKS. Under ρ^2, that is, ρ followed by ρ, this Line has the image EMSIQ, while under $\rho^2 i$, that is, ρ^2 followed by i, the image of AIOKS is DIPMR. This puts two distinct Lines on Points I and M. Therefore, the Line AIOKS must be eliminated. As there are no Points compatible with AIO, Point O must be dropped from the set of Points that might be on AI. This set, then, is reduced to Points E, J, K, P and S.

Next, consider the possibility of Point S being on the same Line as AI. Two more Points from the set of compatible Points must still be found for the Line AIS. As Points P, K and J are already on Lines with S, they cannot be considered. This leaves only Point E to be included on Line AIS. This Line then would not have sufficient Points to be retained in the geometry. Consequently, Point S must not be on the same Line as AI. The set of possible Points on Line AI now contains only E, J, K and P.

Suppose Point K is on Line AI. Line AIK could not be on Point E since there is already a Line on the pair EK. The only possible Line to be considered, then, would be AIKJP. Under ρ , this Line transforms to FHJIO. This puts two Lines on Points J and I. Therefore, this Line must be eliminated and Point K cannot be on Line AI.

The only Points left, then, to be on AI are E, J and P. Is it possible to have a Line AIEJP? In other words, are the images of this Line, under the transformations of D_6, also Lines? The images of AIEJP, under the elements of the dihedral group of the regular hexagon, are FHDIO, EMCHN, DLBMS, CKALR and BJFKQ. These six Lines do not include any Point pairs already on other Lines. This brings the number of Lines to twenty-four.

Now Point A has been associated with all the Points on the inner hexagon. However, there are still some Points on the outer hexagon that are not yet on a Line with Point A. One such Point is Point O. What Points could be on the same Line as AO? They cannot be any of the Points on the Lines on A: YAHBU, XFMAT, NAGDQ, AIEJP and CKALR. Other Points that cannot be on the Line are those already on Lines with O: REGBO and FHDIO. The only Points that remain to be considered are Points S, V and W. Is there a Line AOSVW? The images of this Line, under the elements of D_6, are FNRUV, ESQTU, DRPYT, CQOXY and BPNWX. Under all twelve transformations of D_6, the image of this Line is compatible with the Lines already determined. This brings the total number of Lines to thirty.

Consider this list of Lines to see if they satisfy the requirements of the finite geometry. The thirty Lines are:

ABHUY	BCITV	CEHMN	EFLWY	HJRSX
ACKLR	BDLMS	CFGPS	EQSTU	HLPQV
ADGNQ	BEGOR	COQXY	FNRUV	IKNSY
AEIJP	BFJKQ	DEKVX	GHKTW	IMQRW
AFMTX	BNPWX	DFHIO	GILUX	JLNOT
AOSVW	CDJUW	DPRTY	GJMVY	KMOPU

It is obvious from looking at the first column of Lines that the Lines on Point A satisfy these requirements. Under the transformations of D_6, it is guaranteed that the Lines on Points B, C, D, E and F also satisfy the requirements of a finite geometry.

Next, consider the Lines on Point G. These are ADGNQ, BEGOR, CFGPS, GHKTW, GILUX and GJMVY. These six Lines also satisfy the requirements of the geometry. As Point G is its own image under each transformation of the dihedral group of the regular hexagon, it is in a set by itself.

Now, look at the Lines on Point H: ABHUY, CEHMN, DFHIO, GHKTW, HJRSX and HLPQV. These six Lines satisfy the geometry's requirements. Likewise, so do the Lines on Points I, J, K, L and M, as they are the images of the Lines on H, under the various elements of D_6.

Point N and the Lines on it are the next to be considered. On N, one finds Lines ADGNQ, BNPWX, CEHMN, FNRUV, IKNSY and JLNOT. These Lines satisfy the requirements of the finite geometry, as do their images under the transformations of D_6, namely, the Lines on Points O, P, Q, R and S.

The Lines on Point T are AFMTX, BCITV, DPRTY, EQSTU, GHKTW and JLNOT. These Lines and their images under the transformations of D_6, that is, the Lines on Points U, V, W, X and Y, all satisfy the requirements placed upon the Lines of a finite geometry.

Therefore, the geometry thus determined by the dihedral group of the regular hexagon has twenty-five Points and thirty Lines, with five Points on each Line and six Lines on each Point. This geometry does satisfy the requirements of a finite geometry as formulated by the author on pages 12 and 13.

In addition, this geometry also satisfies the axioms of an affine geometry. If one considers the set of "horizontal" Lines of this geometry, i.e. NTOLJ, YAHBU, SFGCP, XEKDV and RWQMI, it is obvious that these five Lines constitute a set of parallel Lines. Likewise, the set of "vertical" Lines, THGKW, FNRUV, CQOXY, AIEJP and DLBMS, are a second set of parallel Lines. Under the transformations ρ and ρ^2, each of these sets of Lines are transformed into two other sets of parallel Lines. These six groups of Lines satisfy the last axiom for an affine geometry. Therefore, the finite geometry developed is a finite affine geometry.

Now consider some possible models for this finite geometry. To illustrate all thirty Lines on a diagram such as figure 6 would be rather tedious and probably not very helpful. Therefore, consider the other three models used to illustrate the Lines of the nine-Point geometry of Chapter One.

Look at the array method first. Consider the horizontal and vertical Lines. These two sets, under the transformations ρ and ρ^2, determine the other twenty Lines. Suppose one constructed an array of the twenty-five Points such that its columns indicate the Points on the various vertical Lines and the Rows of the array indicate the sets of Points which are the horizontal Lines. This array, under the transformations ρ and ρ^2, would lead to two other arrays in which the rows and columns would be the other twenty Lines.

Here is such an array:

A B H U Y
E D K V X
I M W R Q
J L T N O
P S G F C

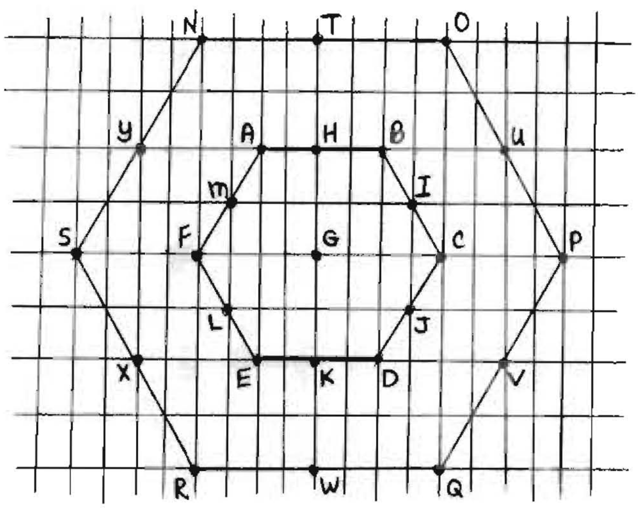
Under ρ and ρ^2, this array determines the following two arrays:

F A M T X		E F L Y W
D C J U W		C B I T V
H L V Q P	and	M K U P O
I K Y S N		H J X R S
O R G E B		N Q G D A

The rows and columns of these arrays do in fact group the Points found on each of the thirty Lines in the geometry.

For those who prefer a "picture" rather than arrays, one can construct a lattice structure to impose on the hexagons and then assign coordinates under the arithmetic of modulo five.

Recall that for a hexagon, the ratio of apothem to radius is $\sqrt{3}$ to 2. Consequently, one would have to use a lattice of $\sqrt{3}$ X 1 rectangles. Let the origin of the lattice, (0,0), coincide with the center of the hexagons, Point G. Let Point C coincide with (4,0) on the lattice and Point H coincide with (0,2) on the lattice. These provisions will guarantee that the Points of the geometry coincide with points of the lattice. (See figure 7.)

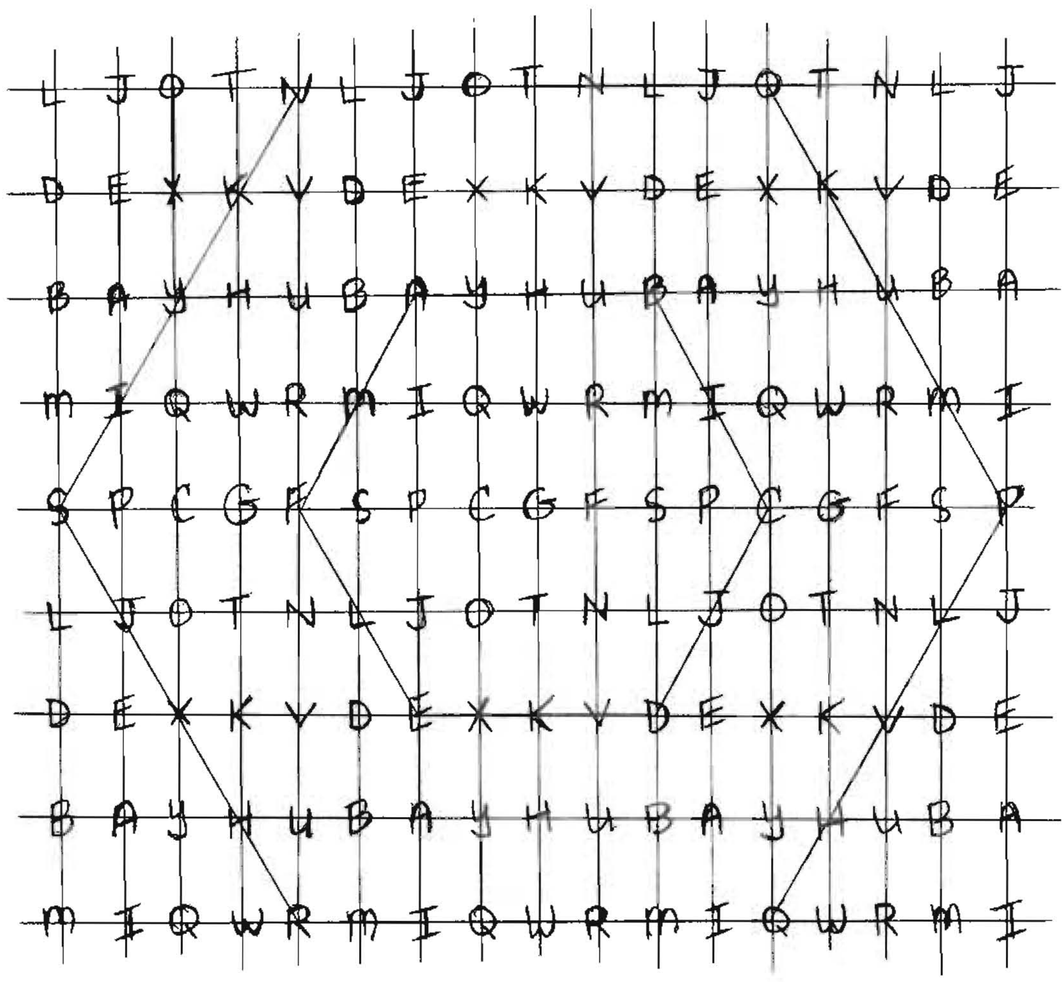


(figure 7)

Under this lattice, each Point is assigned a unique pair of modulo five coordinates:

A	(3,2)	F	(1,0)	K	(0,3)	P	(3,0)	U	(1,2)
B	(2,2)	G	(0,0)	L	(2,4)	Q	(4,1)	V	(1,3)
C	(4,0)	H	(0,2)	M	(2,1)	R	(1,1)	W	(0,1)
D	(2,3)	I	(3,1)	N	(1,4)	S	(2,0)	X	(4,3)
E	(3,3)	J	(3,4)	O	(4,4)	T	(0,4)	Y	(4,2)

As illustrated in figure 8, there are many points on the lattice with identical modulo five coordinates. If each point of the lattice is marked with its corresponding Point, an interesting phenomena is observed: the Points found on the natural lines of the lattice are in the same groupings as on the thirty Lines.



(figure 8)

This relationship between the natural lines and the Lines of the geometry enables one to develop an algebraic representation for this geometry, based on the lattice model.

Let the modulo five coordinates already assigned to the Points by the lattice structure be the algebraic designation of each of the Points. The Lines shall be the group of Points whose modulo five coordinates satisfy a specified modulo five statement. These statements would be as follows:

GHKTW:	$x \equiv 0 \pmod 5$	CFGPR:	$y \equiv 0 \pmod 5$
FNRUV:	$x \equiv 1 \pmod 5$	IMQRW:	$y \equiv 1 \pmod 5$
BDLMS:	$x \equiv 2 \pmod 5$	ABHUY:	$y \equiv 2 \pmod 5$
AEIJP:	$x \equiv 3 \pmod 5$	DEKVX:	$y \equiv 3 \pmod 5$
COQXY:	$x \equiv 4 \pmod 5$	JLNOT:	$y \equiv 4 \pmod 5$
ADGNQ:	$x + y \equiv 0 \pmod 5$	GJMVY:	$2x + y \equiv 0 \pmod 5$
EFLWY:	$x + y \equiv 1 \pmod 5$	BNPWX:	$2x + y \equiv 1 \pmod 5$
HJRSX:	$x + y \equiv 2 \pmod 5$	DFHIO:	$2x + y \equiv 2 \pmod 5$
KMOPU:	$x + y \equiv 3 \pmod 5$	ACKLR:	$2x + y \equiv 3 \pmod 5$
BCITV:	$x + y \equiv 4 \pmod 5$	EQSTU:	$2x + y \equiv 4 \pmod 5$
GILUX:	$3x + y \equiv 0 \pmod 5$	BEGOR:	$4x + y \equiv 0 \pmod 5$
AOSVW:	$3x + y \equiv 1 \pmod 5$	CDJUW:	$4x + y \equiv 1 \pmod 5$
CEHMN:	$3x + y \equiv 2 \pmod 5$	HLPQV:	$4x + y \equiv 2 \pmod 5$
BFJKQ:	$3x + y \equiv 3 \pmod 5$	IKNSY:	$4x + y \equiv 3 \pmod 5$
DPRTY:	$3x + y \equiv 4 \pmod 5$	AFMTX:	$4x + y \equiv 4 \pmod 5$

Since every other congruence statement in the form $ax + by \equiv c \pmod 5$, where a, b, c are elements of Z_5, is equivalent to one of these listed, this modulo five system has been exhausted by the algebraic representations assigned the thirty Lines.

Therefore, one can find several ways to represent this geometry of thirty Lines and twenty-five Points. In each model, there are five Points on each Line and six Lines on each Point. There also are six groups of five parallel Lines in each model.

To take a brief look at the concept of distance in this geometry, one must first realize that once again distance between Points cannot be measured the same way in all directions. The Lines group themselves into two categories: those which can be transformed, under the elements of D_6, into horizontal Lines and those which can be transformed, under the elements of D_6, into vertical Lines. In the array model of this geometry, these groups would be row Lines and column Lines. For the algebraic model, the groups are Lines such that $4x + y \equiv k \pmod 5$, $x + y \equiv k \pmod 5$ or $y \equiv k \pmod 5$, where k is an element of Z_5, and Lines such that $x \equiv k \pmod 5$, $2x + y \equiv k \pmod 5$ or $3x + y \equiv k \pmod 5$, where k is an element of Z_5.

Distance could be determined in the same way along Lines belonging to the same category designated above. The units of measure suggested by the natural geometry of the hexagon would be a unit of 1 for the horizontal Lines (and the Lines associated with them) and a unit of $\sqrt{3}$ along the other Lines.

In summary, through the application of the transformations of the dihedral group of the regular hexagon on two concentric regular hexagons with radii in ratio 1 : 2, there can be developed a finite geometry of twenty-five Points and thirty Lines. Within this geometry, there are five Points on each Line and six Lines on each Point. This geometry satisfies the axioms of a finite affine geometry.

The axioms of this geometry developed in Chapter Two are:

1. There are exactly twenty-five Points.
2. Every two Points determine exactly one Line.
3. There are exactly five Points on each Line.

This geometry can be illustrated on a lattice of $\sqrt{3}$ X 1 rectangles on which the Points are assigned to the lattice points by way of repetition and the Lines are represented by the natural lines on the lattice points.

Another model for this geometry is the use of algebraic representations suggested by the lattice model. The modulo five coordinates assigned the Points in the development of the lattice structure represent the Points while modulo five congruence statements represent the Lines.

The array model for this geometry differs somewhat from that of the nine-Point geometry of Chapter One. The twenty-five Point geometry requires the construction of three five-by-five arrays of letters representing the Points. The rows and columns of these three arrays represent the Lines of the geometry.

Once again, that which is most characteristic of this twenty-five Point, thirty Line geometry is that it is the finite geometry associated with the dihedral group of the regular hexagon.

Chapter Three

Generalizations to Geometries Associated With Dihedral Groups of Regular Polygons with 2n Sides

For a regular polygon with 2n sides, the associated geometry is based on the dihedral group of order 4n. This group is defined by the following relations: $\rho^{2n} = \iota^2 = (\rho\iota)^2 = I$, where ρ is a rotation of 180/n degrees about the center of the polygon and ι is a reflection about the vertical line of symmetry. The elements of this group are in the form $\rho^m\iota^n$, where m = 0, 1, 2,..., 2n-1 and n = 0,1.

For a regular polygon of 2n sides, the associated geometry must have the 2n vertices, the 2n midpoints and the center as Points. So, the minimum number of Points in the geometry is 4n + 1. If these 4n + 1 Points are not sufficient to satisfy the requirements of the geometry, it is necessary to add more Points. However, since the image of a Point must itself be a Point, under all elements of the dihedral group, and every Line must have the same number of Points, another 2n-gon must be added to the diagram. This polygon should be concentric with the first and its radius should be twice as large as that of the first polygon. With the addition of this polygon, there have been added 4n Points to the geometry--2n vertices and 2n midpoints. Continuing this process, it is clear that the number of Points to be included in the geometry is in the form $k(4n) + 1$, where k is the number of polygons. It has been shown that for $n = 2$, $k = 1$; for $n = 3$, $k = 2$.

The number of Points found on the natural lines is clearly dependent on the number of concentric polygons involved. For, while there are k polygons, the maximum number of Points on a natural line is $2k + 1$. As the geometries associated with dihedral groups are to involve only essential Points, the number of Points should be kept to a minimum. However, each such geometry should have a unique number of Points on a Line. This suggests that the number of concentric regular $2n$-gons be different for each type of polygon, namely $n-1$ polygons of $2n$ sides. This implies the imposition of $n-2$ polygons about the original $2n$-gon, such that all the polygons are concentric and the radius of each additional polygon is double that of the previous polygon.

The Points on these polygons are so situated that there are, at most, $2n-1$ Points on each natural line. This suggests that Lines be fixed groups of $2n-1$ Points. This suggestion is compatible with the three Points on a Line in the geometry associated with D_4 and the five Points on a Line in the geometry associated with D_6.

Therefore, let it be conjectured that:

1. $n-1$ concentric regular $2n$-gons are to be used in developing the finite geometry associated with the dihedral group of order $4n$. These concentric polygons are to have radii in ratio $1 : 2 : 4 : \ldots : 2^{n-1}$.

and 2. The number of Points on each Line in such a geometry is $2n-1$.

On the basis of these two conjectures, it is possible to prove several statements about the geometry associated with D_{2n}.

Statement 1: The number of Points in the geometry is $(2n-1)^2$.

On each of the n-1 concentric regular polygons, there are 2n vertices and 2n midpoints, each of which is a Point. These 4n(n-1) points and the center are all the Points of the geometry. Since $4n(n-1) + 1 = 4n^2-4n + 1 = (2n-1)^2$, there are $(2n-1)^2$ Points in the geometry.

Statement 2: The number of Lines in the geometry is $2n(2n-1)$.

The $(2n-1)^2$ Points can be arranged in $(2n-1)^2\left[(2n-1)^2-1\right]$ ordered pairs of different Points. As order is not of consequence in this geometry, there are $(4n^2-4n + 1)(4n^2-4n)/2 = 2n(n-1)(2n-1)^2$ Point pairs to be considered. As every Line consists of 2n-1 Points, there are $\binom{2n-1}{2} = (2n-1)(2n-2)/2 = (n-1)(2n-1)$ pairs of Points on each Line. Therefore, there are $\left[2n(n-1)(2n-1)^2\right] \div \left[(n-1)(2n-1)\right] = 2n(2n-1)$ unique Lines in the geometry.

Statement 3: The number of Lines on a Point is 2n.

Select one of the $(2n-1)^2$ Points of the geometry. The remaining $4n^2-4n$ Points must each be associated with this Point on exactly one Line. There are $2n-2$ other Points on each of these Lines. These $2n-2$ Points are unique for each Line. Therefore, there must be $(4n^2-4n)/(2n-2) = 2n$ Lines on this Point. As the Point was selected arbitrarily, there must be $2n$ Lines on each of the $(2n-1)^2$ Points in the geometry.

The patterns found in the development of the finite geometries associated with the dihedral groups of the square and the regular hexagon suggest several ways to arrive at a model for the geometries associated with the dihedral group of a regular $2n$-gon.

As there are $(2n-1)^2$ Points, these Points can easily be accommodated in a square array $(2n-1) \times (2n-1)$. The $2n(2n-1)$ Lines of the geometry should fall into $2n$ groups of parallel Lines with $2n-1$ Lines in each group. This allows for each Point to be on one Line in each set of parallel Lines. The $2n$ groups of parallel Lines should fall into two categories: those that transform into "horizontal" Lines under the elements of D_{2n} (where a "horizontal" Line is one whose Points fall on natural horizontal lines), and those Lines which do not transform, under the elements of the dihedral group, into horizontal Lines. There should be n groups in each category.

In the square array, assemble one set of horizontal Lines as the rows of the array and place one set of non-horizontal Lines as the columns of the array. This array, under the transformation ρ, becomes a second array in which the rows are a second set of horizontal Lines and the columns are a second set of non-horizontal Lines. Under repeated applications of the transformation ρ, there can be developed a total of n arrays which illustrate the 2n sets of parallel Lines and thereby indicate the Points on each of the 2n(2n-1) Lines. (It is precisely because of this evolution of $\underline{n}$ arrays that this discussion has been limited to finite geometries associated with dihedral groups of regular $\underline{2n}$-gons. Should an odd-sided polygon be the basis of the dihedral group and finite geometry, this type of model is not applicable.)

As illustrated in Chapter Two, as the geometry increases in size, the determination of the Points on the individual Lines is more tedious. But, if the natural lines are used as guides, this is not impossible. The construction of the model for the larger geometries is aided by consistent reference to the concentric polygon configuration on which the geometry is based.

It is of interest to note that if a Line of 2n new Points is added to one of these geometries, a new type of geometry is formed--one in which two Lines always determine a Point. The incorporation of the new Line and new Points is achieved by adding the first new Point to each Line of one set of parallel Lines, then adding the second new Point to the Lines of a second set of parallel Lines, and so on, until each new Point has been added to the Lines of one set of parallels. Thus, each of the $2n(2n-1)$ Lines have received another Point. This brings the total number of Points on each Line to $2n$, including the Line on all the new Points. The number of Lines in this new geometry is $2n(2n-1) + 1 = 4n^2-2n + 1$. The number of Points in the new geometry is $(2n-1)^2 + 2n = 4n^2-2n + 1$. Note that the number of Lines in this geometry equals the number of Points. Thus, it is possible to create a projective geometry by adapting a finite geometry based on the dihedral group of an even-sided polygon.

Therefore, in summary, if the number of concentric regular polygons of $2n$ sides is $n-1$ and each polygon has a radius twice that of the one smaller than itself, it is possible to develop an affine geometry associated with the dihedral group of order $4n$, in which there are $(2n-1)^2$ Points and $2n(2n-1)$ Lines, with $2n-1$ Points on each Line and $2n$ Lines on each Point. There will be $2n$ sets of parallel Lines, with $2n-1$ Lines in each set. The geometry thus developed will satisfy these axioms:

1. There are exactly $(2n-1)^2$ Points.
2. On every two Points, there is exactly one Line.
3. There are exactly $2n-1$ Points on each Line.

A model of n square arrays can be designed for each of these finite affine geometries.

In addition, these geometries can be changed into projective geometries in which two Lines always determine a Point, by the incorporation into the geometry of one Line of 2n Points.

Chapter Four

Relationships Between the Geometry
Associated with D_{2n} and the Subgroups of D_{2n}

As stated in Chapter Three, the dihedral group of a regular 2n-gon is a group of order 4n, with defining relations $\rho^{2n} = i^2 = (\rho i)^2 = I$. The subgroups of this group must be of orders that divide 4n. The subgroups that are of special importance in the development of the related geometry are a normal subgroup of order n, $SG_n = \{I, \rho^2, \rho^4, \ldots, \rho^{2n-2}\}$, and one of order 2n, $SG_{2n} = \{I, \rho, \rho^2, \ldots, \rho^{2n-1}\}$. These two subgroups are essential to the evolution of the geometry.

For example, the dihedral group of the regular hexagon is of order twelve. The elements of the group can be generated by the transformations ρ, a rotation of sixty degrees about the center of the hexagon, and i, a reflection about the vertical line of symmetry of the hexagon. The defining relations of D_6 are $\rho^6 = i^2 = (\rho i)^2 = I$. All elements of D_6 can be written in the form $\rho^m i^n$, where m = 0, 1, ..., 5 and n = 0, 1. Proper subgroups of this dihedral group can be of order 2, 3, 4 or 6. D_6 has thirteen proper subgroups--

two subgroups of order 6: $\{I, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ and $\{I, \rho^2, \rho^4, i, \rho^2 i, \rho^4 i\}$,

three subgroups of order 4: $\{I, \rho^3, i, \rho^3 i\}$, $\{I, \rho^3, \rho^2 i, \rho^5 i\}$ and $\{I, \rho^3, \rho i, \rho^4 i\}$, one subgroup of order 3: $\{I, \rho^2, \rho^4\}$ and

seven subgroups of order 2: $\{I, \rho^3\}$, $\{I, i\}$, $\{I, \rho i\}$, $\{I, \rho^2 i\}$, $\{I, \rho^3 i\}$, $\{I, \rho^4 i\}$ and $\{I, \rho^5 i\}$.

Two of these subgroups, both of them normal subgroups, are of special import in the development of the associated geometry for D_6. These subgroups are $SG_3 = \{I, \rho^2, \rho^4\}$ and $SG_6 = \{I, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$. The role of these two subgroups in the development of the twenty-five Point geometry is illustrated in the paragraphs that follow.

The minimum number of Points in a geometry associated with the dihedral group of a regular 2n-gon is 4n + 1. The Points upon which this is based are the center, one vertex of the 2n-gon, one midpoint of a side of the polygon and their images under the elements of SG_{2n}. The center is its own image under all the elements of SG_{2n}, the vertex has 2n images, including itself, under the transformations of SG_{2n}, as does the midpoint. In the twenty-five Point geometry, the set of images of A, H and G under the subgroup $\{I, \rho, \rho^2, \ldots, \rho^5\}$ is A, B, C,..., M. This is the set of Points on the inner hexagon, thirteen Points in all.

The total number of Points in a geometry associated with D_{2n} is $(2n-1)^2 = 4n(n-1) + 1$. These Points are the set of images of a vertex and a midpoint from each of the concentric polygons and the center, under the elements of SG_{2n}. Once again, the center is its own image under all the transformations. Each of the other 2(n-1) Points has 2n distinct images, including itself. Therefore, the total number of Points in the geometry is 4n(n-1) + 1. In the twenty-five Point geometry, the entire set of Points can be determined by finding the images of the Points A, G, H, N and T, under the transformations of the subgroup $\{I, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

All the Lines of the geometry based on D_{2n} can be found by looking at the images of several lines under the transformations of SG_{2n}. The Lines on the lines of symmetry have n images, including themselves, since order is not of consequence in the geometry. All other Lines of the geometry have 2n distinct images under the elements of SG_{2n}. Therefore, it is necessary to consider the images of 2n Lines, under the transformations of SG_{2n}, to develop the entire set of Lines for the geometry. For example, in the twenty-five Point geometry, Line SFGCP has only three images under the elements of $SG_6 = \{I, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$, namely SFGCP, REGBO and QDGAN, and Line THGKW has only three images under the elements of the same subgroup, namely, THGKW, YMGJV and XLGIU. Every other Line has six images under these elements. In this way, all thirty Lines of the geometry can be determined from six Lines, for example, SFGCP, THGKW, NTOLJ, TAMFX, AIPJE and NFRUV, under the elements of SG_6.

The Lines of the geometry based on D_{2n} have 2n sets of parallel Lines. These parallel sets fall into two categories: those that can be transformed, under elements of D_6, into horizontal Lines and those that cannot. If one set of each type of parallel Lines is determined, the other 2n-2 sets of parallel Lines can be determined under the elements of SG_{2n} or the elements of $SG_n = \{I, \rho^2, \rho^4, \ldots, \rho^{2n-2}\}$. The latter subgroup is preferable as it does not generate any duplications. If the larger subgroup is used, the elements $\rho^n, \rho^{n+1}, \ldots, \rho^{2n-1}$ generate duplicates of previous images. This may be avoided if the subgroup SG_n is used. This illustrates why n arrays are sufficient to identify the entire geometry. While it was suggested in the previous chapters to develop the n arrays through consecutive applications of the transformation ρ, the arrays could be generated, in different order, through repeated application of ρ^2, the generator of SG_n.

Therefore, the subgroups of the dihedral group of a regular polygon with 2n sides are essential to the development of the associated geometry. The use of the elements of the subgroups generated by ρ and ρ^2 simplify the development of the sets of Lines and Points, as well as the determination of the sets of parallel Lines.

Chapter Five

Conclusion

This thesis has defined the term "geometry associated with the dihedral group of a regular polygon." It is that geometry with the minimum number of points necessary to satisfy the following conditions:

1. The points intrinsic to the polygon and to the group of transformations of the polygon onto itself are included in the points of the geometry.
2. The lines of symmetry of the polygon are included as lines of the geometry.
3. The image of any point (or line) of the geometry is itself a point (or line) of the geometry.
4. The geometry must satisfy the five axioms of an affine geometry.

Two such geometries, one associated with D_4 and the other with D_6, have been developed in detail. It has been indicated how to develop the associated geometry for the dihedral group of any even-sided regular polygon. The case of the odd-sided regular polygon has been left for future investigation.

BIBLIOGRAPHY

1. Cundy, H. Martyn. "25 Point Geometry," Mathematical Gazette, XXXVI, (September 1952), 158-166.

2. Davis, Elwyn H. Introductory Modern Algebra. Columbus, Ohio: Charles E. Merrill Publishing Company, 1974.

3, Eves, Howard. A Survey of Geometry (Revised Edition). Boston: Allyn and Bacon, Inc., 1972.

4. Heidlage, Martha. "A Coordinate Approach to the 25-Point Miniature Geometry," The Mathematics Teacher, LVIII, (February 1965), 109-113.

5. Heidlage, Martha. "A Study of Finite Geometry," The Pentagon, XXIII, (Fall 1963), 18-27.

6. Lederman, Walter. Introduction to the Theory of Finite Groups. New York: Interscience Publishers, Inc., 1953.

7. Lieber, Lillian R. and Lieber, Hugh Gray. The Education of T.C. Mits. New York: W. W. Norton & Co., Inc., 1944.

8. Tibbetts, Gene. "Twenty-Five Point Geometry," Unpublished Thesis, Emporia State University, 1977.