AN ABSTRACT OF THE THESIS OF

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The purpose of this thesis is to enlighten a person with a calculus background to the composition and use of the Student's T Distribution.

The thesis is divided into basically two parts. The first part shows the composition of the Student's T Distribution and how the distribution was formulated from two other distributions.

The second part demonstrates how one tests hypothesis from the various tests using the T distribution. There are six different tests which are presented in this thesis.

THE STUDENT'S T DISTRIBUTION

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THE HISTORY OF THE STUDENT'S T DISTRIBUTION

Until the beginning of the twentieth century the normal distribution was estimated with the Z test despite the size of the sample of the population. In small samples this did not always prove satisfactory.

In 1908 William S. Gosset, a consultant at the Guinness Brewery in Dublin, Ireland, needed and created a test for small samples taken from a normal distribution. It was the T test which approximates the normal distribution for small samples by using degrees of freedom to determine the distribution which will be used.

The T statistic consists of one variable with a normal distribution divided by the square root of the quantity of another variable with a chi-squared distribution divided by its degrees of freedom. The T test has different distributions due to the degrees of freedom in the chi-squared distribution.

The distribution is usually called the "Student's T" distribution because as a consultant to the Guinness Brewery Gosset was not allowed to publish under his real name. Gosset instead chose the pseudonym "Student" in the publication Biometrika.

Sir R. A. Fisher verified the distribution in 1923 in the publication Metron.

THE COMPOSITION OF THE STUDENT'S T DISTRIBUTION

THE NORMAL DISTRIBUTION

The normal distribution is of great significance to the Student's T distribution. It is the distribution which is approximated by the Student's T Distribution plus it is the distribution of the variable in the numerator in the Student's T statistic.

The normal distribution is the most important probability distribution yet discovered. There are several reasons for this. It is the limit of the binomial distribution; in the physical sciences many distributions are normally distributed; and the Central Limit Theorem greatly expands the use of the normal distribution.

The Central Limit Theorem is as follows:

If the random variables $X_1, \ldots\ldots, X_n$ form a random sample of size n from a given distribution with mean μ and variance σ^2 $(0<\sigma^2<\infty)$, then for any fixed number x,

$$\lim_{n\to\infty} \Pr\left[\frac{n^{1/2}(\overline{X}_n - \mu)}{\sigma} \leq x\right] = \phi(x) \text{ [1]}$$

As one can clearly see from this theorem any expected [2] value can be estimated by means of the normal distribution. This fact leads to the normal distribution being extensively used.

The normal distribution is defined as follows:

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2 \frac{(X-\mu)^2}{\sigma^2}} .$$

The fact that the normal distribution is really a probability distribution can be shown by proving that the area under the integral is equal to one.

First one sets up the integration

$$I=c\int_{-\infty}^{\infty} e^{-1/2y^2}\ dy.$$

In this integration $c = 1/\sqrt{2\pi}$. Since c is a constant it can be moved to the outside of the integral.

Now one squares the integral for added convenience, except one uses different variables in one equation.

$$I^2 = c^2\left(\int_{-\infty}^{\infty} e^{-1/2y^2}\ dy\right)\left(\int_{-\infty}^{\infty} e^{-1/2z^2}\ dz\right)$$

$$I^2 = c^2\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-1/2(y^2+z^2)}\ dy\ dz\right)$$

Now it becomes convenient to change the coordinate system from Cartesian coordinates to polar coordinates. In doing this $\cos\theta = \frac{z}{r}$ and $\sin\theta = \frac{y}{r}$. In the integral from the preceding page the variables are transformed as such:

$$y^2+z^2=(r\cos\theta)^2+(r\sin\theta)^2=r^2.$$

The integration now becomes

$$c^2\int_0^{2\pi}\int_0^{\infty} e^{-1/2r^2} r\ dr d\theta = -c^2\int_0^{2\pi} e^{-1/2r^2}\Big|_0^{\infty} d\theta = c^2\int_0^{2\pi} 1 d\theta = c^2\theta\Big|_0^{2\pi} = 2\pi c^2$$

$$I^2=c^2 2\pi,\ I=c\sqrt{2\pi}=1$$

This integration obviously proves that the normal distribution is a probability distribution.

At this point the expectation and the variance need to be found. Using the moment generating function [3] this problem is easily solved.

The moment generating function of the normal distribution is

$$M_x(t)=e^{\mu t+1/2\sigma^2 t^2} .$$

In order to find the expected value one finds the first derivative of the moment generating function. This derivative is solved for the expected value to x by taking the limit as t approaches zero.

$$\lim_{t\to 0} \frac{dM_x(t)}{dt^2} = \lim_{t\to 0} e^{\mu t+1/2\sigma^2 t^2} (\mu+\sigma^2 t)=\mu.$$

In order to find the variance one finds the second derivative of the moment generating function, takes its limit as t approaches zero, and subtracts the square of the expected value. Using this information one has

$$\lim_{t\to 0} \frac{d^2M_x(t)}{dt^2} = \lim_{t\to 0} e^{\mu t+1/2\sigma^2} (\mu+\sigma^2 t)^2+\sigma^2 e^{\mu t+1/2\sigma^2 t^2} = \mu^2+\sigma^2.$$

Thus the variance is $\mu^2+\sigma^2-(\mu)^2=\sigma^2$.

The normal distribution has a graph that resembles a bell (see page 8). Because of this it has often been called the "bell shaped curve."

There is a special type of normal distribution. The standard normal distribution is the normal distribution with μ equal to zero and the variance equal to one.

The Z test is associated with the standard normal distribution. This test uses the normal distribution by standardization in order to apply the standard normal distribution.

In order to use the Z test one must have a null hypothesis and an alternative hypothesis along with an alpha level which will decide whether or not to accept the null hypothesis.

The Z test is run in the following fashion:

$$Z=\frac{X-\mu}{\sigma}$$

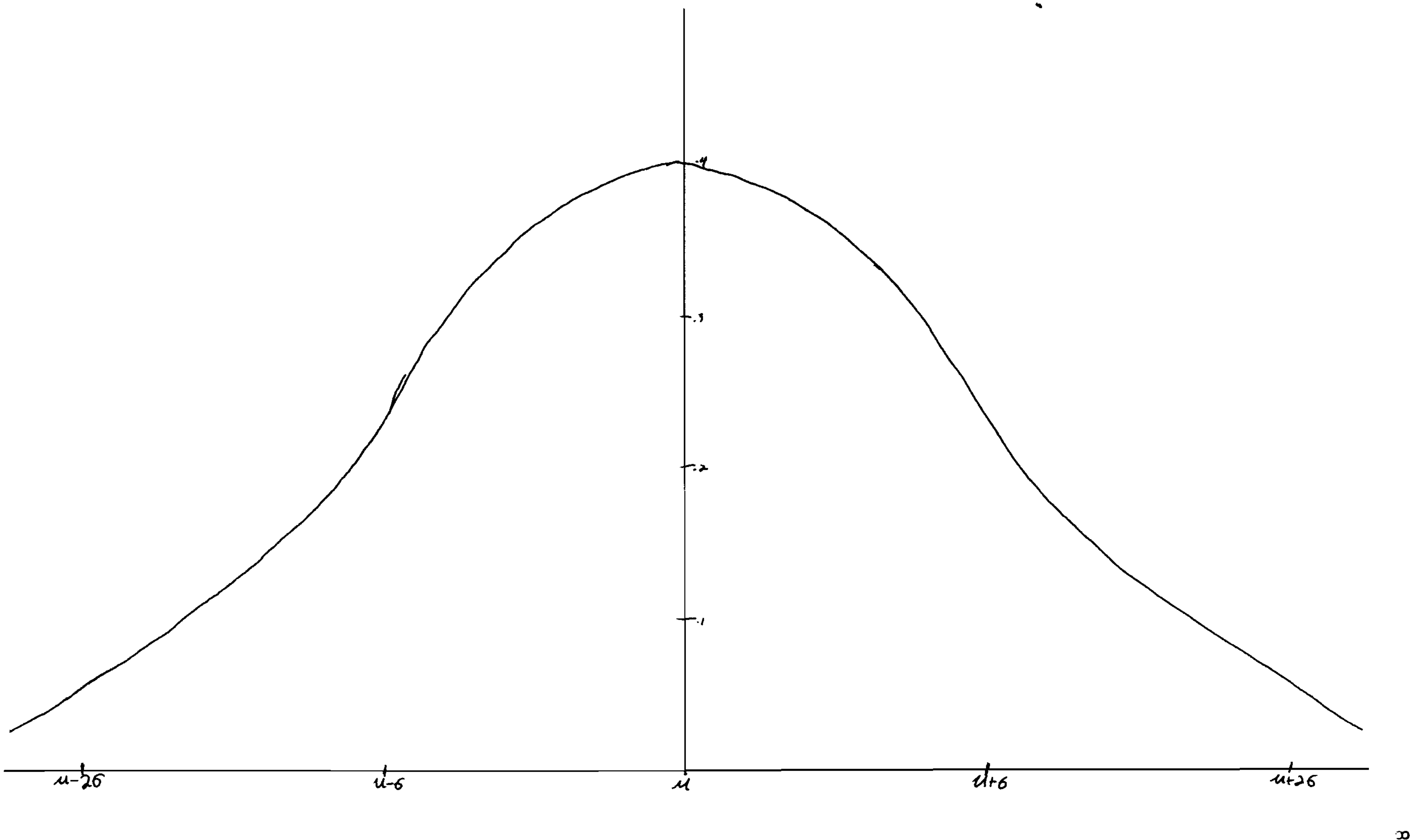
The Z scores allow one to determine the probability of X being less than Z. The probability percentages are determined from a Z table which is based on the standard normal distribution. The null hypothesis is accepted if the probability of X is less than the alpha level, a number between zero and one.

As an example suppose that a patient arrives at Hornet Hospital. In order to determine if the patient has a fever the staff assumes the null hypothesis "The patient does not have a fever." The alternative hypothesis is "The patient does have a fever." An alpha level of .05 is chosen.

Suppose that the mean temperature is 98.6 with a variance of 3. The patient's temperature is taken and it is 100.1. Placing this data in the Z test equation one has

$$Z=\frac{100.1-98.6}{\sqrt{3}} = 0.866.$$

The Z score is compared to a Z table. The table says that 80.79% of the population is to the left of the Z score. This means that 19.21% of the population lies to the right of the Z score. Since 80.79 is less than 100-5 one accepts the null hypothesis, "The patient does not have a fever."



THE NORMAL DISTRIBUTION

THE CHI-SQUARED DISTRIBUTION

The chi-squared distribution is important to the T distribution because it is related to the distribution of the variable in the denominator of the T statistic. The chi-squared distribution is used as the probability for the sample variances of samples from the normal distribution.

The chi-squared distribution is a special gamma distribution. In a chi-squared distribution the parameters of alpha and beta are assigned the values of n/2 and 1/2 respectively. n becomes the degrees of freedom of the distribution.

In order to demonstrate that the gamma distribution is a true probability distribution one needs to look at the gamma function. The gamma function is defined as follows:

$$\Gamma(\alpha)=\int_0^\infty X^{\alpha-1}e^{-X}dX.$$

If $\alpha>1$ the $\Gamma(\alpha)=(\alpha-1)\ \Gamma(\alpha-1)$ which is shown by integrating by parts. Let $u=X^{\alpha-1}$ and $dv=e^{-X}dX$, then $du=(\alpha-1)X^{\alpha-2}dX$ and $v=-e^{-X}$.

$$\Gamma(\alpha)=\quad \Big[uv\Big]_0^\infty-\int_0^\infty vdu=X^{\alpha-1}(-e^{-X})-\int_0^\infty -e^{-X}(\alpha-1)X^{\alpha-2}dX$$

$$=(\alpha-1)\ \Gamma(\alpha-1).$$

To evaluate $\Gamma(n)$, $\Gamma(1)$ is needed. $\Gamma(1)=\int_0^\infty e^{-X}dX=1$, thus $\Gamma(n)=(n-1)\ (n-2)\ldots\ldots\ldots(1)\ \Gamma(1)=(n-1)!$.

For later use one needs to evaluate $\Gamma(n+1/2)$. As shown previously for all $\alpha>1$ one needs to look at $\Gamma 1/2$.

$$\Gamma(1/2)=\int_0^{\infty} X^{-1/2}e^{-X}dX$$

If one lets $X=1/2y^2$, then $dX=ydy$ and

$$\Gamma(1/2)=2^{1/2}\int_0^{\infty} e^{-1/2y^2}dy.$$

As one can clearly see this integral is related to the standard normal distribution. This integral equals the inverse of the constant of the normal distribution divided by two since the integral only covers half the area.

$$1/2(2\pi)^{1/2}=(\frac{\pi}{2})^{1/2}$$

It now follows that

$$2^{1/2}(\frac{\pi}{2})^{1/2}=\pi^{1/2}.$$

As an example of how $\Gamma(n+1/2)$ works one should look at $\Gamma(7/2)$. $\Gamma(7/2)=(5/2)\ (3/2)\ (1/2)\ \Gamma(1/2)=(15/8)\pi^{1/2}$.

Since the gamma function is now defined one can look at the gamma distribution. It is necessary to show that it is a probability distribution.

The gamma distribution is defined as follows:

$$\begin{cases} \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} X^{\alpha-1}e^{-\beta X} & \text{for } X>0 \text{ and } \alpha,\beta>0 \\ 0 & \text{for } X\leq 0 \end{cases}$$

In order to integrate the gamma distribution one takes the constants to the outside of the integration.

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} X^{\alpha-1}e^{-\beta X}dX$$

As was shown previously $\int_0^{\infty} X^{\alpha-1}e^{-\beta X}dX = \frac{\Gamma(\alpha)}{\beta^{\alpha}}$ and thus $\frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\beta^{\alpha}} = 1$ for all possible values of α or β.

Since the chi-squared distribution is only a gamma distribution with certain values for alpha and beta it becomes obvious that the chi-squared distribution is a true probability distribution.

The expected value of the gamma distribution is α/β. The variance of the distribution is α/β^2. This means that in the chi-squared distribution the mean is n and the variance is 2n.

The chi-squared distribution is the distribution of the squares of the differences of X about the mean in a normal distribution. For this reason the chi-squared distribution is the distribution of the variable in the denominator in the T statistic.

There is a theorem that states, "If a random variable X is normally distributed with expectation zero and variance one, then X^2 has a chi-squared distribution with one degree of freedom."

Of course a normal distribution with expectation zero and a variance of one is the standard normal distribution. By using this form of the normal distribution one can later standardize other normal distributions into the standard normal distribution.

One can prove the theorem by working out the probability that $\left[X^2 < U\right]$.

It is $G(U)=(2\pi)^{-1/2}\int_{-\sqrt{u}}^{\sqrt{u}} e^{-1/2z^2} dz=2(2\pi)^{-1/2}\int_0^{\sqrt{u}} e^{-1/2z^2} dz.$

If one changes the variable of integration to $y=z^2$, then one obtains the result of

$$G(U)=(2\pi)^{-1/2}\int_0^u y^{-1/2}e^{-1/2y} dy.$$

One can now proceed to the general case. The moment generating function of the chi-squared distribution is

$$\varphi(t)=\int_0^\infty u^{\lambda-1}e^{-1/2u+tu}du.$$

If one changes the variable to $v=(1/2-t)u$, one has

$$\varphi(t)=2^{\lambda}(1-2t)^{-\lambda}\alpha\int_0^\infty v^{\lambda-1}e^{-v}dv.$$

The first and second moments of the chi-squared distribution can easily be calculated by definition of the moment.

$$\alpha_n=\int_0^\infty u^n dG(U)=\int_0^\infty u^n g(u)du.$$

The expectation and the variance of a random variable y with a chi-squared distribution are

$E(y)=\alpha_1=f$ where f is the degrees of freedom and

$$\sigma^2=E(y^2)-(E(y))^2=\alpha_2-\alpha_1^2=2f.$$

If y and z are independent random variables having chi-squared distributions with f and f' degrees of freedom, then as shown on the previous page the moment generating functions are

$$(1-2t)^{-1/2f} \text{ and } (1-2t)^{-1/2f'}.$$

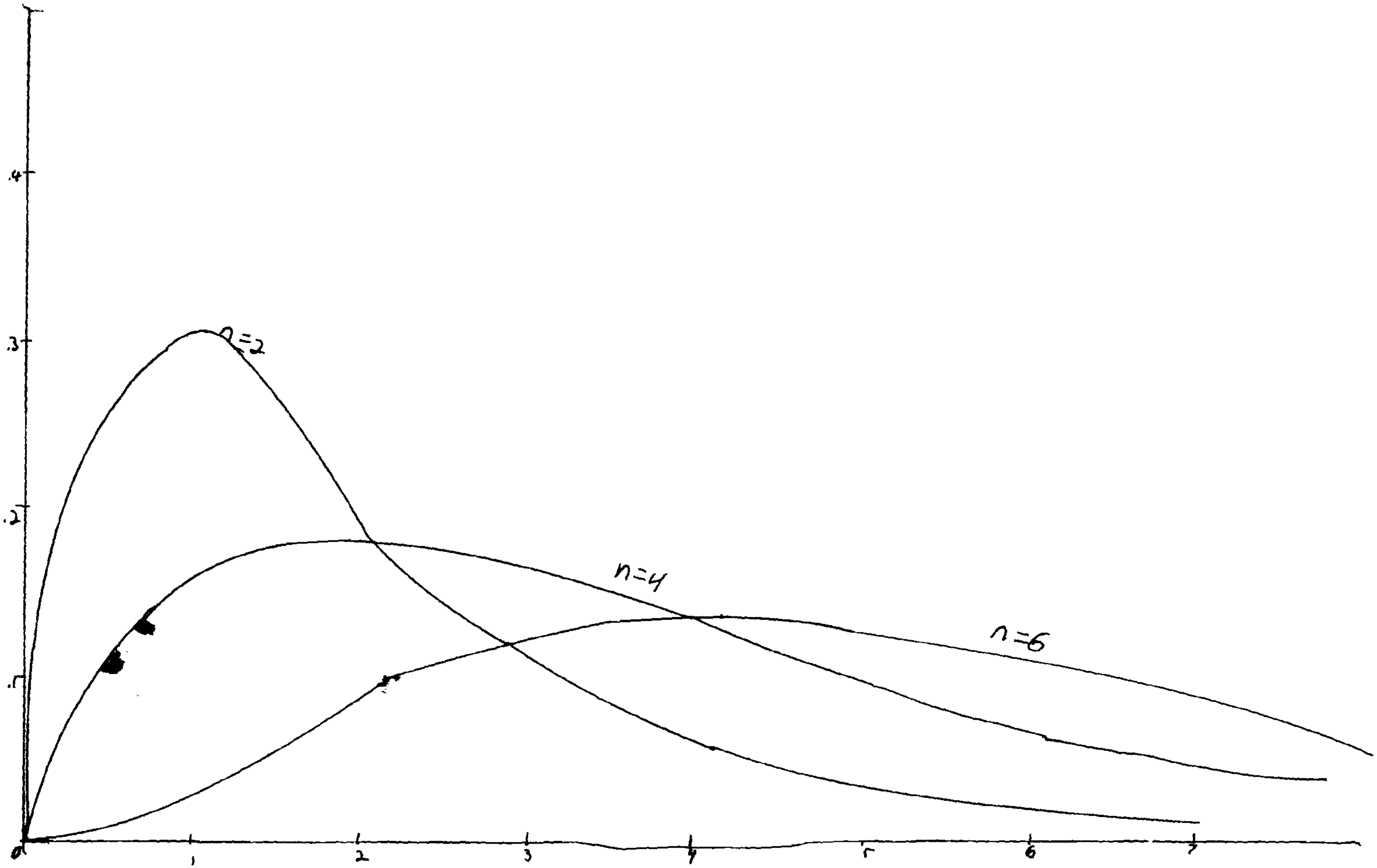
The moment generating function for the sum y+z is the product of the moment generating functions for y and z. Hence it follows that: If two independent random variables have chi-squared distributions with f and f' degrees of freedom, then their sum has a chi-squared distribution with f+f' degrees of freedom.

A general theorem can be drawn from this result. <u>If $X_1, X_2, \ldots, X_n$ are independent normally distributed random variables with expectation zero and variance one, the sum of squares</u>

$$\underline{\chi^2 = X_1^2 + X_2^2 + \ldots + X_n^2}$$

<u>has a chi-squared distribution with n degrees of freedom.</u>

There are graphs of several chi-squared distributions on the following page.



THE CHI-SQUARED DISTRIBUTION

THE STUDENT'S T DISTRIBUTION

The student's T distribution, or called simply the T distribution, is a combination of the normal distribution and the chi-squared distribution and is used when the only data that are available are from a small sample.

The T distribution depends on a degrees of freedom value in order to determine the appropriate probability function. The degrees of freedom is a characteristic inherited from the chi-squared distribution. The T distribution changes from a Cauchy distribution when the degree of freedom is equal to one to the normal distribution as the degrees of freedom approach infinity.

The T distribution is defined as a function with

$$\frac{Y}{\left(\frac{Z}{n}\right)^{1/2}}$$

where Y is normally distributed and Z has a chi-squared distribution with a degrees of freedom of n. By the change of variable technique, this function of Y and Z can be reduced to a distribution function of a single variable.

If $X = \frac{Y}{\left(\frac{Z}{n}\right)^{1/2}}$ and $W=Z$, then Y equals $(1/n^{1/2})XW^{1/2}$ and Z equals W.

One now sets up the Jacobian Transformation [4] in order to find the necessary multiplier in which the function in X and W will be equivalent to the function Y and Z upon integration.

This Jacobian matrix is as follows:

$$\begin{bmatrix} n^{-1/2}W^{1/2} & (2n^{1/2})^{-1}X(W)^{-1/2} \\ 0 & 1 \end{bmatrix}$$

The determinant [5] of this matrix is $n^{-1/2}W^{1/2}$.

The probability density function $f(Y,Z)=Jf(X,W)$. By replacing Z and Y with their equivalent in X and W one has

$$f(X,W)=cW^{9n-1/2}\ e^{-1/2(1+X^2)2W/n}$$

with the constant $c=\left[2^{(n+1)/2}(n\pi)^{1/2}\ \Gamma(n/2)\right]^{-1}$.

In order to turn the function into a single variable function it is now necessary to remove the W by integration.

$$g(X)=\int_0^\infty f(X,W)dW = \frac{\Gamma(\frac{n+1}{2})}{(n\pi)^{1/2}\ \Gamma(\frac{n}{2})}\ (1+\frac{X^2}{n})^{-(n+1)/2}$$

The T distribution is a probability distribution since it is the product of two probability distributions with one variable integrated out.

The expectation of the T distribution is present only when n is greater than one. When the expectation exists it is zero. This can be proved by using the derived function for the T distribution; multiplying it by X and then integrating the function from infinity to negative infinity.

Let $c=\dfrac{\Gamma(\frac{n+1}{2})}{(n\pi)^{1/2}\ \Gamma(\frac{n}{2})}$

$$c\int_{-\infty}^{\infty} X(1+X^2/n)^{-(n+1)/2}dX = c\cdot \frac{(1+X^2/n)^{(-n+1)/2}}{\frac{-n+1}{2}}\Bigg|_{-\infty}^{\infty} = c\cdot 0=0$$

As one can clearly see that when n equals one the integral cannot be defined because division by zero is undefined.

The variance does exist for n greater than two. This can be proved by using a substitute for X.

The integral for finding the variance around the expectation of zero is as follows with c equalling the same as it did in the previous integration.

$$\int_{-\infty}^{\infty} cX^2(1+X^2/n)^{-(n+1)/2}dX = E(X-0)^2$$

The integration is divided in half and the integration only from zero to infinity is considered.

If $y=\dfrac{\frac{X^2}{n}}{1+\frac{X^2}{n}}$, then $\dfrac{X^2}{n}=\dfrac{y}{1-y}$ and $\dfrac{dX}{dy}=\dfrac{n^{1/2}}{2}\left(\dfrac{y}{1-y}\right)^{-1/2}\dfrac{1}{(1-y)^2}\,dy.$

Because of the definition of X^2/n by y the new integral is integrated over the interval of zero to one since as X approaches zero y approaches zero and as X approaches infinity y approaches one.

The new integrand becomes

$$\frac{(ny)}{1-y}\left(1+\frac{y}{1-y}\right)^{-(n+1)/2}\frac{n^{1/2}}{2}\left(\frac{y}{1-y}\right)^{1/2}\left(\frac{1}{(1-y)^2}\right).$$

This can easily be changed to

$$(ny)(1-y)^{-1}(1-y)^{(n+1)/2}\frac{n^{1/2}}{2}(y)^{-1/2}(1-y)^{1/2}(1-y)^{-2}.$$

This newer form of the integrand allows the exponents to be combined more easily.

Now one integrates the complete function by putting the constants outside the integration and integrating only the variables. This gives the following integration:

$$c\frac{n}{2}^{1/2}(n)\int_0^1 (y)^{1/2}(1-y)^{\frac{n-4}{2}-2}\,dy.$$

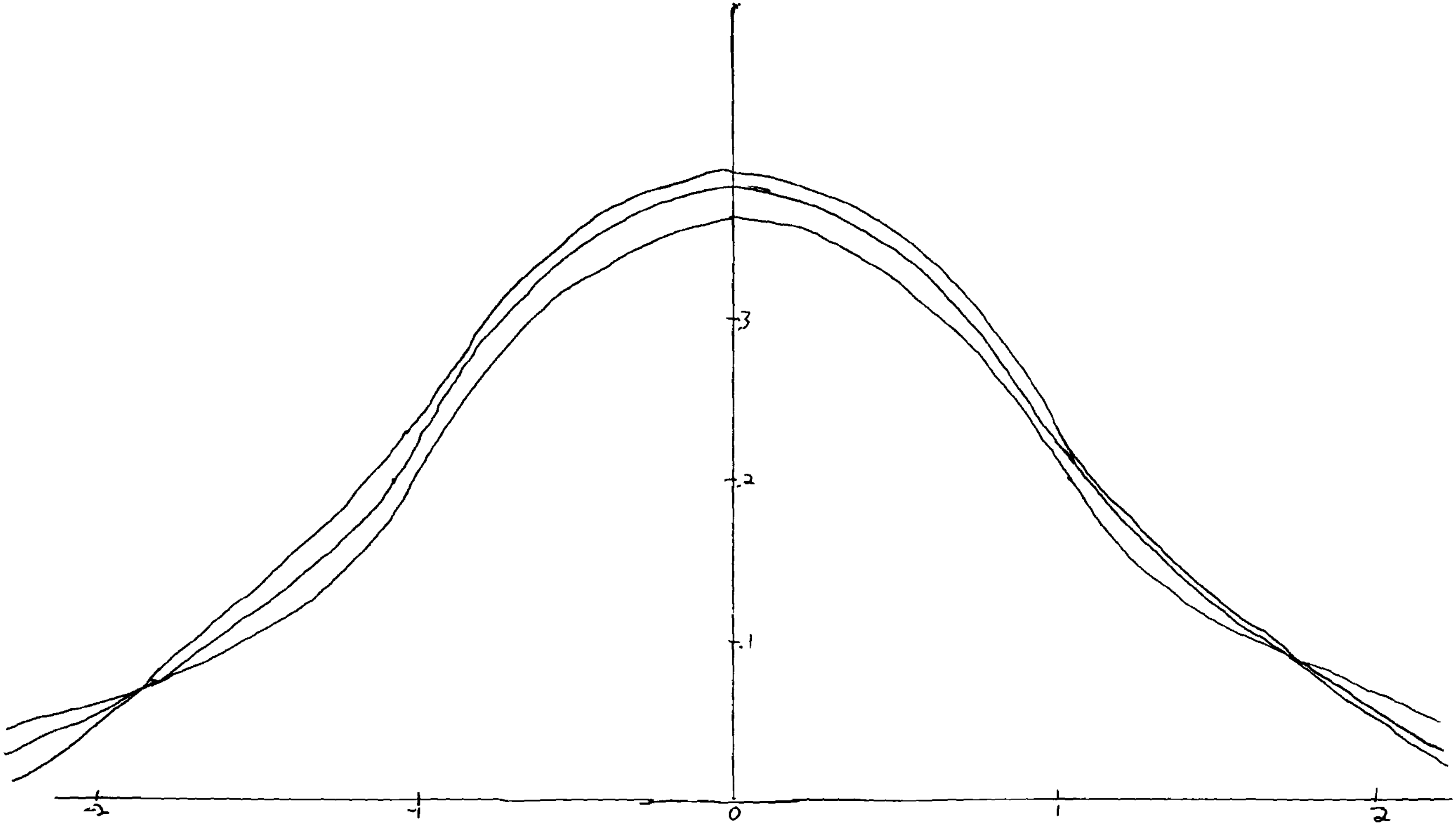
This integration yields the following:

$$(n)\frac{n}{2}^{1/2}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{n+2}{2}}{\Gamma\frac{n+1}{2}}\;\frac{\Gamma\frac{n+2}{2}}{n^{1/2}\pi^{1/2}\,\Gamma\frac{n}{2}}=\frac{1/2\;\Gamma(\frac{n-2}{2})\;n.}{2\Gamma\frac{n}{2}}$$

Since this only covers half the interval it becomes essential to multiply by two. Thus, the function loses the one one half term and it becomes

$$\frac{\Gamma\frac{n-2}{2}\,\frac{n}{2}}{\Gamma\frac{n}{2}}$$

Regardless of whether n is even or odd the ratio of the two gamma functions is $1/((n-2)/2)$. Thus the final quantity left is $\frac{n}{n-2}$. Therefore this must be the variance of the T distribution. If n were less than two the variance would be negative, which is impossible. If n were two the division by zero would not be defined. Therefore the variance does not exist for n less than three.



THE STUDENT'S T DISTRIBUTION

THE F DISTRIBUTION

The F distribution (the F is for Fisher) has some importance to the T distribution in two facts. The F test is used to determine if two samples have the same variance and two, the F distribution with one degree of freedom in its numerator is equal to the square of the T distribution with the same degrees of freedom as the denominator under a zero hypothesis.

The F distribution is used to run a test in order to see if variances are similar or not. It consists of two variables with degrees of freedom for each. The two variables both have a chi-squared distribution. One variable, divided by its degrees of freedom, is divided by the other variable, divided by its degrees of freedom.

The F distribution is composed of two chi-squared distributions. Due to this relationship the F distribution is a probability distribution.

In creating the F distribution $X=(Y/m)/(Z/m)$ where Y and Z are independent and have chi-squared distributions with m and n degrees of freedom respectively.

The equation of the F distribution with Y and Z is as follows:

$$g(Y,Z)=cY^{(m/2)-1}Z^{(n/2)-1}e^{-(Y+Z)/2}$$

with c equal to $\left(2^{(n+m)/2}\,\Gamma\left(\frac{m}{2}\right)\,\Gamma\left(\frac{n}{2}\right)\right)^{-1}$.

The variables are now changed from Y and Z to X and Z. By definition of X, $Y = XmZ/n$. All one needs to do is replace Y with its equivalent in X and Z and then multiply the factor by dY/dX, mZ/n.

The new function becomes

$$g(X,Z)=c(XmZ/n)^{(m/2)-1}Z^{(n/2)-1}e^{-((XmZ/n)+Z)/2}mZ/n.$$

In order to change the function from a two variable function one needs to integrate the function by Z.

$$g(X)=c(m/n)^{(m/2)}X^{(m/2)-1}\int_{-\infty}^{\infty} Z^{((m+n)/2)-1}e^{-((Xm/n)/2)Z}dZ$$

This function equals

$$g(X)=c(m/n)^{(m/2)}X^{(m/2)-1}\,(m+n)/2\quad(((Xm/n)+1)/2)^{-(m+n)/2}.$$

As was stated earlier the F distribution with one degree of freedom in the numerator is equal to the square of the T distribution with the same degrees of freedom as the denominator of the F distribution when using the hypothesis that the mean is zero.

One should compare the definitions of the two distributions. The T distribution is defined as follows:

$$X=\frac{Y}{(\frac{Z}{n})^{1/2}}$$

where Y and Z are independent with n the degrees of freedom.

As one can clearly see that when m of the F statistic equals one and Y equals zero the difference between the terms is the square of the denominator.

Using this comparison in the F test allows one to use the T test instead of the F test in regression analysis with a hypothetical value of zero. This subject will be discussed further in the section on regression analysis.

APPLICATIONS OF THE T TESTS

THE T TEST

When one is testing a hypothesis of a normal distribution the Z test is used with known parameters. However, if the parameters μ and σ^2 are not known then they will need to be estimated. The Z test can be used if one knows the variance, but if the variance needs to be estimated then the T test must be used.

The T test is used to estimate the probability of a hypothetical value which is assumed to have a normal distribution. It resembles the Z test.

When the T test is used a number of things must be completed before the actual computations are started. First one must select a hypothetical value for a mean. This is called the null hypothesis. An alternate hypothesis is selected such that if the null hypothesis is false then the alternate hypothesis is true.

Another value that must be known before the T test is started is the alpha level. Alpha is a number between zero and one. A probability area under the T curve is created such that the probability that the hypothetical value lying in the area is equal to one minus alpha. If the hypothetical value lies in the specified area the null hypothesis is accepted. If the hypothetical value lies outside the specified area the alternate hypothesis is accepted.

When the hypothesis says that the hypothetical value is greater than or equal to the sample mean then the area in which the value must lie in order to be acceptable lies to the right of a line under the T curve which divides the area under the T curve into two sections. The section to the left has the area of alpha. The section to the right has the area of one minus alpha. The sections are reversed when the null hypothesis claims that the hypothetical value is less than or equal to the sample mean.

When the null hypothesis states that the hypothetical value equals the sample mean the acceptable area under the curve is limited on two sides. The acceptable area, the area in the middle, is equal to one minus alpha, however the area of the two unacceptable areas, which are equal to each other, are equal to alpha divided by two.

At this point the mean and the variance need to be estimated. In the normal distribution $\overline{X}_n$ is the unbiased estimator of μ. This is because for some unknown mean θ the $E_\theta(\delta(X_1,X_2,\ldots,X_n))= \theta$ for any possible value of θ.

Now it is necessary to estimate the variance σ_n^2. One considers the estimate S_0^2 for σ^2. $S_0^2= (1/n) \sum_{i=1}^{n} (X_i-\overline{X}_n)^2$. Using the identity we observe

$$\sum_{i=1}^{n} (X_i-\mu)= \sum_{i=1}^{n} (X_i-\overline{X}_n)^2+n(\overline{X}_n-\mu)^2.$$

Since X_i has a mean μ and a variance σ^2, then $E\left[(X_i-\mu)^2\right]=(1/n)n\,\sigma^2= \sigma^2$. Therefore $E\left[(1/n) \sum_{i=1}^{n} (X_i-\mu)^2\right] =$

$$\frac{1}{n} \sum_{i=1}^{n} E\left[(X_i-\mu)^2\right] = \frac{1}{n}\, n\, \sigma^2= \sigma^2.$$

The sample mean $\overline{X}_n$ has a mean μ and a variance σ^2/n. Because of this fact $E\left[(\overline{X}_n-\mu)^2\right] = \sigma^2/n$.

It now follows that since

$$E(S_0^2)= \sigma^2 - \frac{1}{n}\sigma^2= \frac{n-1}{n}\ \sigma^2$$

S_0^2 is not the unbiased estimator. However, S_1^2 is the unbiased estimator since S_1^2 is defined as

$$S_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2.$$

The T test is in the following form:

$$T = \frac{(\overline{X} - \mu_0)}{S_1} \sqrt{n}.$$

When the T score is obtained it can be compared with the table of values which is in the back of this paper. It should be compared with values in the test's degrees of freedom which are n-1.

In order to show how this works one should consider the following example.

Coach Welch is the track coach at Hornet High School. He has been told that the average long jumper can jump seven meters. Coach Welch feels that he has better than average long jumpers on his team. He has decided to check his hypothesis with a T test.

Coach Welch's null hypothesis is "The long jump squad is average or below." The alternate hypothesis is "The long jump squad is better than average."

Coach Welch chose an alpha level of .05. Since this is only a one sided test the only boundary of the test is dependent on the degrees of freedom. Since there are fifteen members of the long jump squad the degrees of freedom will be fourteen. This means that the T score cannot be greater than 1.761.

Coach Welch has all the members jump for their best length. The results are as follows:

7.3, 6.9, 7.6, 6.6, 7.5, 7.6, 8.0, 7.9, 6.7, 8.1, 6.9, 7.1, 7.0, 7.2, 7.9.

The mean is 7.35. The variance is 0.238. This makes the standard deviation 0.488.

Coach Welch puts these numbers into the T test and he has

$$T= \frac{(7.35-7.00)}{0.488} \sqrt{15}.$$

The T value is approximately 2.78. Since 2.78 is greater than 1.761 Coach Welch rejects the null hypothesis and accepts the alternate hypothesis. Therefore Coach Welch has a better than average long jump squad.

THE PAIRED SAMPLE T TEST

When two samples are taken from a single set a modified T test can be used. This test will use a covariance of the two samples.

In this test one would gather data in pairs in this manner:

Identity	Data
1	X_{11}, X_{21}
2	X_{12}, X_{22}
.	.
.	.
.	.
n	X_{1n}, X_{2n}

Since the two samples are assumed to be dependent they have a covariance and a correlation value. The correlation value and the covariance are needed in order to operate this test.

In this test the mean is obtained the same way as in the regular T test. The variance is found in the same manner, also. However, the covariance is found in a specific manner. Let S_{ij} be the estimate of the covariance. S_{ij} is found in the following manner:

$$S_{ij} = \frac{1}{n-1} \sum_{k=1}^{n} (X_{ik}-\overline{X}_{i.}) (X_{jk}-\overline{X}_{j.}).$$

By definition the correlation equals $\sigma_{ij}/ \sigma_i \sigma_j$. By using the maximum likelihood estimate [6] and letting r_{ij} be the estimate of the correlation

$$r_{ij} = \frac{S_{ij}}{S_i S_j} .$$

In the paired sample T test, after one has determined one's hypothesis and one's alpha value, the T score is determined as follows:

$T=\dfrac{(\bar{d}-\delta)\sqrt{n}}{S_d}$ where δ is the hypothetical value, $\bar{d}$ equals $\bar{X}_1-\bar{X}_2$,

and S_d^2 equals $S_1^2+S_2^2-2S_1S_2r_{12}$ with n-1 degrees of freedom.

As an example one should look at another problem of Coach Welch's track team.

Coach Welch has decided to change the high jump style from the flop to the roll. However, Coach Welch would like to know what the difference, if any, there is between the heights achieved using the different methods.

There are seven members of the high jump squad. Coach Welch will use the null hypothesis "There is no difference in heights using the different methods." The alternate hypothesis is "There is a difference using the different methods." Coach Welch chose the alpha level to be .05.

With this information Coach Welch knows that there are two boundaries with six degrees of freedom. The alpha level is divided in half and the critical points are 2.447 and -2.447.

The data are as follows in meters:

Identity	Flop	Roll
1	1.82	1.92
2	2.03	1.88
3	1.74	2.15
4	1.97	1.99
5	1.76	2.00
6	1.69	1.89
7	2.09	1.91

From these data Coach Welch knows the mean value of the flop is 1.871, the mean value for the roll is 1.963, the standard deviation of the flop is 0.157, and the standard deviation of the roll is 0.0948. The covariance is -.0133.

Coach Welch now places the appropriate value in the correct places and he has

$$T = \frac{(-.092-0)\ \sqrt{7}}{0.245} = -0.994.$$

This is clearly not beyond the critical value and therefore there is little probability that there is any difference in heights due to the change in methods.

THE TWO SAMPLE T TEST

When there is a single population and a factor Y which divides the population into subpopulations, the subpopulations can be compared using the two sample T test.

In this test it is not important to have equal numbers of data. There is no covariance nor correlation involved in performing the test. The only fact that needs to be added to the means and the variance is whether the variances can be assumed to be the same. This requires the F test.

After one finds that the variance can be assumed to be equal, the parameters are found in the following ways:

$$\overline{X}=(1/n)\sum_{i=1}^{n} X_i$$

$$S_1^2=(1/n-1)\sum_{i=1}^{n} (X_i-\overline{X})$$

The T score is found in the following manner:

$$T=\frac{(\overline{X}_1-\overline{X}_2)-\delta}{S_p\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^{1/2}}$$

$$S_p^2=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}\ .$$

δ is the hypothetical value with n_1+n_2-2 degrees of freedom.

As an example one can look at the efforts of Coach Welch, again. Coach Welch has two groups from his track team, the shot putters and the discus throwers. Coach Welch wants to see if there is any difference in the amount of weight the different members can bench press.

There are seven shot putters and nine discus throwers. The null hypothesis is "There is no difference in the weight the two groups can press." The alternate hypothesis is "There is a difference between the two groups in the amount of weight that can be pressed by each group." The alpha level is .10.

The data in kilograms for the shot putters are as follows: 113, 145, 126, 172, 119, 168, 150.

The data in kilograms for the discus throwers are as follows: 134, 167, 189, 174, 125, 149, 188, 154, 133.

The mean for the shot putters is 141.9; the standard deviation is 23.38. The mean of the discus throwers is 157.0; the standard deviation is 23.92.

The F test boundaries are 5.14 and .2203. The F test is set up as follows:

$$F= \frac{23.38/6}{23.92/8} = 1.303$$

Since the F statistic is within the critical boundaries Coach Welch can use the two sample T test. The critical values, or boundaries, for the T test are 1.761 and -1.761. The T score is

$$T= \frac{(141.9-157.0)-0}{23.69(1/7+1/9)^{1/2}} = -2.51.$$

The T score is obviously greater than the larger critical value. Therefore Coach Welch concluded that the discus throwers are better at the bench press than the shot putters.

THE WELCH T TEST

If one has the same conditions as needed for the two sample T test, two subpopulations of the same population, except that the variances cannot be assumed to be equal, then the Welch T test is needed to compare the two subpopulations.

The degrees of freedom and the combined standard deviation are determined differently than the way they are in the two sample T test. The T score is figured as follows:

$$T = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}.$$

The degrees of freedom are determined by

$$\frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{S_1^4}{n_1^2(n_1-1)} + \frac{S_2^4}{n_2^2(n_2-1)}}.$$

This does not always result in an integer, but one can interpolate for the right critical values.

One should look at another example from Coach Welch's problems. Coach Welch decided that he wanted to know which group could create the greatest distance from the highest reach standing to the highest reach jumping. He took the data from his high jump squad and his pole vault squad.

Coach Welch's null hypothesis is, "There is no difference between the distance of the two squads in the difference from the standing reach to the jumping reach." The alternate hypothesis is, "There is a difference between the distances of the two squads in the differences from the standing reach to the jumping reach." The alpha level is .10.

There are eight members of the high jump squad. Their data are as follows: .65, .86, .77, .69, .80, .76, .75, .72.

There are nine members of the pole vault squad. Their data are as follows: .92, .42, .97, .45, .50, .76, .41, .41, .96.

From these data Coach Welch knows that the mean of the high jump squad is 0.75 with a standard deviation of 0.065, and that the mean of the pole vault squad is 0.744 with a standard deviation of 0.281.

The degrees of freedom for the F test are 7 and 8. Since the alpha level equals 0.10 the critical values are 5.40 and 0.201. The F test is as follows:

$$F = \frac{0.004225/7}{0.07896/8} = 0.06115.$$

Since the F score is less than 0.201, which is the lower boundary to the acceptable interval, Coach Welch does not believe that the variances are equal. Therefore Coach Welch needs to use the Welch T test.

The degrees of freedom needed for this test is determined as follows:

$$\frac{(.004225/8 + .078961/9)^2}{\frac{.000017851}{64(7)} + \frac{.0062348}{81(8)}} = 8.988.$$

Now Coach Welch will find the T score knowing that in order for the null hypothesis to be accepted it must lie in the range between -1.835 and 1.835.

The test is as follows:

$$T = \frac{.75 - .744 - 0}{\sqrt{\frac{.004225}{8} + \frac{0.078961}{9}}} = .06221.$$

Since this score is within the interval Coach Welch accepts the null hypothesis.

THE T TEST FOR REGRESSION ANALYSIS

A regression function exists between a dependent and an independent variable when there is a correlation of the two variables. When a function is linear in parameters then one has a linear regression model. In other words, the dependent variable is regressed on the independent variable.

In one's work with regression analysis using the T test one will try to obtain the best estimate of the parameters and hypothesis about the parameters.

The strength of the linear relation is measured by the simple correlation. The correlation lies between one and negative one. The greater the absolute value of the correlation the greater the strength of the linear relation. If the correlation is one or negative one then there is a completely linear relationship between the dependent and the independent variable.

In simple linear regression one has a pair of observations from a single population (X_1,Y_1), (X_2,Y_2),.....,(X_n,Y_n). There are two methods by which one can create a regression function. In the first method X is set up at different intervals and Y is then obtained and a least sum of squares line is drawn. In the second method the X is a random variable and thus so is the Y variable.

The correlation value r is equal to

$$r = \frac{\sum_{i=1}^{n} (X_i-\overline{X})\ (Y_i-\overline{Y})}{\left[\sum_{i=1}^{n} (X_i-\overline{X})^2 \ \sum_{i=1}^{n} (Y_i-\overline{Y})^2\right]^{1/2}} .$$

One could check the strength of a regression equation from the absolute value of r.

The theoretical model of a simple regression equation is

$$y_i = \beta_0 + \beta_1 x_i + e_i.$$

This is called the simple linear regression model of Y on X where $E(e_i)=0$ and the variance of e_i equals σ^2, $i=1,2,\ldots,n$.

b_0 and b_1 are the estimators of β_0 and β_1 by minimizing the sum of squares of the deviations.

$$S = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

The estimators are referred to as the least-square estimators.

$$b_0 = \overline{Y} - b_1\overline{X}$$

$$b_1 = \frac{\sum_{i=1}^{n} (X_i - \overline{X}) y_i}{\sum_{i=1}^{n} (x_i - \overline{X})^2}$$

The estimated regression equation is

$$\hat{y} = b_0 + b_1 x.$$

b_1 is called the regression coefficient, and b_0 is called the intercept.

Before one can hypothesize one needs to find S^2. S^2 is the variance around the line of the error.

One finds it by the following:

$$S^2 = \frac{\sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2}{n-2}$$

In order to find T one would compute the following equation:

$$T = \frac{b_1 - \beta_1^{(0)}}{\left[V(b_1)\right]^{1/2}} \quad \text{where } V(b_1) = \frac{S^2}{\sum_{i=1}^{n} (x_i - \bar{X})^2}$$

The T score for testing y is obtained by the equation

$$T = \frac{\hat{y} - y^{(0)}}{P} \quad \text{where } P = S \left[(1/n) + \frac{(x - \bar{X})^2}{\sum_{i=1}^{n} (x_i - \bar{X})^2} \right]^{1/2}.$$

As an example one should look again at the problems of Coach Welch. Coach Welch wishes to recruit taller runners because Coach Welch believes that taller athletes run faster than shorter ones.

Coach Welch has taken data from twenty runners. He arrives at the equation that regresses height in meters to velocity in meters per second. The equation is as follows:

y m/sec = .45+.55x meters.

Coach Welch also found that $\hat{V}(b_1)$ equals .025. Coach Welch states his null hypothesis as follows: "The correlation coefficient, or regression coefficient, is zero." The alternate hypothesis is naturally "The correlation coefficient is not zero." The alpha level is .05. The degrees of freedom are 18, n-2.

$$T = \frac{.55 - 0}{.5} = 1.1.$$

T had to lie within the interval (-2.101,2.101) which it does. Therefore Coach Welch concludes that there is no correlation between an athlete's height and an athlete's velocity.

Multiple regression is a form of regression. The difference from simple regression is that one tries to find a relation between a single dependent variable and two or more independent variables. A regression equation of multiple coefficients would look something like the following:

$$y=\beta_0+\beta_1x_1+\ldots\ldots+\beta_nx_n.$$

The test of hypothesis for each single coefficient is an F test which looks like

$$F=\frac{b_k^2}{\left[se(b_k)\right]^2}$$ where $se(b_k)$ is the standard error of the coefficient. However, as shown in the section about the F distribution, when one uses a hypothesis in which the coefficient is zero, the square root of the F test is really a T test with the denominator's degrees of freedom. In this case it is n-p-1 where p is the total number of coefficients in the regression equation.

If one only wanted to test whether the correlation is zero one could use a T test that is related to the T test just mentioned. Let r equal the correlation. If one had a hypothesis such as $H_0=r_{yh.c}=0$ one could test using

$$T=\frac{r_{yh.c}\sqrt{(n-k-2)}}{\sqrt{1-r_{yh.c}^2}}$$

where h becomes the variables in the regression equation which affects the coefficient, c becomes the variables in the regression equation which are kept constant, k is the number of variables in c, and n is the number of variables in the regression equation. The T distribution has n-k-3 degrees of freedom.

CONCLUSION

CONCLUSION

The T test has greatly aided the statistician who does not have a great deal of time, or material, or money to spend on gathering statistics. The T test has allowed him accuracy even though the number of data is quite small.

The examples that were used were centered around track and field which is not considered a major science. It was demonstrated that the T test has a great deal of application outside the basic sciences and pure mathematics.

The T distribution is easy to use and is limited only by the imagination of the statistician performing the tests.

REFERENCES

1. $\phi(X)$ is the normal distribution of x.

2. The expectation of a probability function F(x)

   $= \sum_{x} xF(x).$

3. The moments of a function are the expectations of X^k where k is an integer designating the number of the moment. The moment generating function is the expectation of (e^{tx}).

4. The Jacobian linear transformation is a method of changing variables in one equation into another equal number of dependent variables and creating a new equation which is equal to the original equation only with new variables. A matrix is set up as follows and the determinant is determined.

$$\begin{bmatrix} \frac{ds_1}{dy_1} & \cdots\cdots\cdots & \frac{ds_1}{dy_n} \\ \vdots & & \vdots \\ \frac{ds_n}{dy_1} & \cdots\cdots\cdots & \frac{ds_n}{dy_n} \end{bmatrix}$$

5. The determinant of a two by two matrix is a*d-b*c where a,b,c, and d are in a matrix as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. If the random variables $X_1,\ldots,X_n$ form a random sample from a discrete distribution or a continuous distribution for which the

probability function of the probability density function is $f(x/\theta)$, where the parameter θ belongs to some parameter space Ω, and θ is either a real-valued parameter or a vector then for any observed vector $\underline{x}=(x_1,\ldots\ldots,x_n)$ in the sample, the value of the joint p.d.f. of the joint p.f. is the likelihood function, denoted as $f_n(\underline{x}/\theta)$. The maximum likelihood function has a maximum likelihood estimator if for each possible observed vector $\underline{x}$, there is a $\delta(\underline{x})\in\Omega$ which denotes a value of $\theta\in\Omega$ which maximizes the maximum likelihood function and $\hat{\theta} = \delta(\underline{x})$ then $\hat{\theta}$ is the maximum likelihood estimator.

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APPENDIX

TABLE 2 (*continued*)[a]

x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
·1	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
·7	·7580	·7611	·7642	·7673	·7703	·7734	·7764	·7794	·7823	·7852
·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·90147
1·3	·90320	·90490	·90658	·90824	·90988	·91149	·91309	·91466	·91621	·91774
1·4	·91924	·92073	·92220	·92364	·92507	·92647	·92785	·92922	·93056	·93189
1·5	·93319	·93448	·93574	·93699	·93822	·93943	·94062	·94179	·94295	·94408
1·6	·94520	·94630	·94738	·94845	·94950	·95053	·95154	·95254	·95352	·95449
1·7	·95543	·95637	·95728	·95818	·95907	·95994	·96080	·96164	·96246	·96327
1·8	·96407	·96485	·96562	·96638	·96712	·96784	·96856	·96926	·96995	·97062
1·9	·97128	·97193	·97257	·97320	·97381	·97441	·97500	·97558	·97615	·97670
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	·9^{2}0097	·9^{2}0358	·9^{2}0613	·9^{2}0863	·9^{2}1106	·9^{2}1344	·9^{2}1576
2·4	·9^{2}1802	·9^{2}2024	·9^{2}2240	·9^{2}2451	·9^{2}2656	·9^{2}2857	·9^{2}3053	·9^{2}3244	·9^{2}3431	·9^{2}3613
2·5	·9^{2}3790	·9^{2}3963	·9^{2}4132	·9^{2}4297	·9^{2}4457	·9^{2}4614	·9^{2}4766	·9^{2}4915	·9^{2}5060	·9^{2}5201
2·6	·9^{2}5339	·9^{2}5473	·9^{2}5604	·9^{2}5731	·9^{2}5855	·9^{2}5975	·9^{2}6093	·9^{2}6207	·9^{2}6319	·9^{2}6427
2·7	·9^{2}6533	·9^{2}6630	·9^{2}6736	·9^{2}6833	·9^{2}6928	·9^{2}7020	·9^{2}7110	·9^{2}7197	·9^{2}7282	·9^{2}7365
2·8	·9^{2}7445	·9^{2}7523	·9^{2}7599	·9^{2}7673	·9^{2}7744	·9^{2}7814	·9^{2}7882	·9^{2}7948	·9^{2}8012	·9^{2}8074
2·9	·9^{2}8134	·9^{2}8193	·9^{2}8250	·9^{2}8305	·9^{2}8359	·9^{2}8411	·9^{2}8462	·9^{2}8511	·9^{2}8559	·9^{2}8605
3·0	·9^{2}8650	·9^{2}8694	·9^{2}8736	·9^{2}8777	·9^{2}8817	·9^{2}8856	·9^{2}8893	·9^{2}8930	·9^{2}8965	·9^{2}8999
3·1	·9^{3}0324	·9^{3}0646	·9^{3}0957	·9^{3}1260	·9^{3}1553	·9^{3}1836	·9^{3}2112	·9^{3}2378	·9^{3}2636	·9^{3}2886
3·2	·9^{3}3129	·9^{3}3363	·9^{3}3590	·9^{3}3810	·9^{3}4024	·9^{3}4230	·9^{3}4429	·9^{3}4623	·9^{3}4810	·9^{3}4991
3·3	·9^{3}5166	·9^{3}5335	·9^{3}5499	·9^{3}5658	·9^{3}5811	·9^{3}5959	·9^{3}6103	·9^{3}6242	·9^{3}6376	·9^{3}6505
3·4	·9^{3}6631	·9^{3}6752	·9^{3}6869	·9^{3}6982	·9^{3}7091	·9^{3}7197	·9^{3}7299	·9^{3}7398	·9^{3}7493	·9^{3}7585
3·5	·9^{3}7674	·9^{3}7759	·9^{3}7842	·9^{3}7922	·9^{3}7999	·9^{3}8074	·9^{3}8146	·9^{3}8215	·9^{3}8282	·9^{3}8347
3·6	·9^{3}8409	·9^{3}8469	·9^{3}8527	·9^{3}8583	·9^{3}8637	·9^{3}8689	·9^{3}8739	·9^{3}8787	·9^{3}8834	·9^{3}8879
3·7	·9^{3}8922	·9^{3}8964	·9^{4}0039	·9^{4}0426	·9^{4}0799	·9^{4}1158	·9^{4}1504	·9^{4}1838	·9^{4}2159	·9^{4}2468
3·8	·9^{4}2765	·9^{4}3052	·9^{4}3327	·9^{4}3593	·9^{4}3848	·9^{4}4094	·9^{4}4331	·9^{4}4558	·9^{4}4777	·9^{4}4988
3·9	·9^{4}5190	·9^{4}5385	·9^{4}5573	·9^{4}5753	·9^{4}5926	·9^{4}6092	·9^{4}6253	·9^{4}6406	·9^{4}6554	·9^{4}6696

[a] Abridged from Table II of A. Hald, "Statistical Tables and Formulas," 1952, Wiley, New York.

TABLE 5. *Percentiles of the Student's t Distribution (Section 1.2.7)**

ν \ q	60%	75%	90%	95%	97.5%	99%	99.5%	99.95%
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.941
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	6.859
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.405
8	.262	.706	2.397	1.860	2.306	2.896	3.355	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291

* Table 5 is taken from Table III of R. A. Fisher and F. Yates (1963): "Statistical Tables for Biological, Agricultural and Medical Research," published by Oliver and Boyd, Edinburgh, and used by permission of the authors and publishers.

APPENDIX II

TABLE 2. *Cumulative N(0, 1) (Section 1.2.5)*[a]

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
− ·0	·5000	·4960	·4920	·4880	·4840	·4801	·4761	·4721	·4681	·4641
− ·1	·4602	·4562	·4522	·4483	·4443	·4404	·4364	·4325	·4286	·4247
− ·2	·4207	·4168	·4129	·4090	·4052	·4013	·3974	·3936	·3897	·3859
− ·3	·3821	·3783	·3745	·3707	·3669	·3632	·3594	·3557	·3520	·3483
− ·4	·3446	·3409	·3372	·3336	·3300	·3264	·3228	·3192	·3156	·3121
− ·5	·3085	·3050	·3015	·2981	·2946	·2912	·2877	·2843	·2810	·2776
− ·6	·2743	·2709	·2676	·2643	·2611	·2578	·2546	·2514	·2483	·2451
− ·7	·2420	·2389	·2358	·2327	·2297	·2266	·2236	·2206	·2177	·2148
− ·8	·2119	·2090	·2061	·2033	·2005	·1977	·1949	·1922	·1894	·1867
− ·9	·1841	·1814	·1788	·1762	·1736	·1711	·1685	·1660	·1635	·1611
−1·0	·1587	·1562	·1539	·1515	·1492	·1469	·1446	·1423	·1401	·1379
−1·1	·1357	·1335	·1314	·1292	·1271	·1251	·1230	·1210	·1190	·1170
−1·2	·1151	·1131	·1112	·1093	·1075	·1056	·1038	·1020	·1003	·09853
−1·3	·09680	·09510	·09342	·09176	·09012	·08851	·08691	·08534	·08379	·08226
−1·4	·08076	·07927	·07780	·07636	·07493	·07353	·07215	·07078	·06944	·06811
−1·5	·06681	·06552	·06426	·06301	·06178	·06057	·05938	·05821	·05705	·05592
−1·6	·05480	·05370	·05262	·05155	·05050	·04947	·04846	·04746	·04648	·04551
−1·7	·04457	·04363	·04272	·04182	·04093	·04006	·03920	·03836	·03754	·03673
−1·8	·03593	·03515	·03438	·03362	·03288	·03216	·03144	·03074	·03005	·02938
−1·9	·02872	·02807	·02743	·02680	·02619	·02559	·02500	·02442	·02385	·02330
−2·0	·02275	·02222	·02169	·02118	·02068	·02018	·01970	·01923	·01876	·01831
−2·1	·01786	·01743	·01700	·01659	·01618	·01578	·01539	·01500	·01463	·01426
−2·2	·01390	·01355	·01321	·01287	·01255	·01222	·01191	·01160	·01130	·01101
−2·3	·01072	·01044	·01017	$\cdot 0^{2}9903$	$\cdot 0^{2}9642$	$\cdot 0^{2}9387$	$\cdot 0^{2}9137$	$\cdot 0^{2}8894$	$\cdot 0^{2}8656$	$\cdot 0^{2}8424$
−2·4	$\cdot 0^{2}8198$	$\cdot 0^{2}7976$	$\cdot 0^{2}7760$	$\cdot 0^{2}7549$	$\cdot 0^{2}7344$	$\cdot 0^{2}7143$	$\cdot 0^{2}6947$	$\cdot 0^{2}6756$	$\cdot 0^{2}6569$	$\cdot 0^{2}6387$
−2·5	$\cdot 0^{2}6210$	$\cdot 0^{2}6037$	$\cdot 0^{2}5868$	$\cdot 0^{2}5703$	$\cdot 0^{2}5543$	$\cdot 0^{2}5386$	$\cdot 0^{2}5234$	$\cdot 0^{2}5085$	$\cdot 0^{2}4940$	$\cdot 0^{2}4799$
−2·6	$\cdot 0^{2}4661$	$\cdot 0^{2}4527$	$\cdot 0^{2}4396$	$\cdot 0^{2}4269$	$\cdot 0^{2}4145$	$\cdot 0^{2}4025$	$\cdot 0^{2}3907$	$\cdot 0^{2}3793$	$\cdot 0^{2}3681$	$\cdot 0^{2}3573$
−2·7	$\cdot 0^{2}3467$	$\cdot 0^{2}3364$	$\cdot 0^{2}3264$	$\cdot 0^{2}3167$	$\cdot 0^{2}3072$	$\cdot 0^{2}2980$	$\cdot 0^{2}2890$	$\cdot 0^{2}2803$	$\cdot 0^{2}2718$	$\cdot 0^{2}2635$
−2·8	$\cdot 0^{2}2555$	$\cdot 0^{2}2477$	$\cdot 0^{2}2401$	$\cdot 0^{2}2327$	$\cdot 0^{2}2256$	$\cdot 0^{2}2186$	$\cdot 0^{2}2118$	$\cdot 0^{2}2052$	$\cdot 0^{2}1988$	$\cdot 0^{2}1926$
−2·9	$\cdot 0^{2}1866$	$\cdot 0^{2}1807$	$\cdot 0^{2}1750$	$\cdot 0^{2}1695$	$\cdot 0^{2}1641$	$\cdot 0^{2}1589$	$\cdot 0^{2}1538$	$\cdot 0^{2}1489$	$\cdot 0^{2}1441$	$\cdot 0^{2}1395$
−3·0	$\cdot 0^{2}1350$	$\cdot 0^{2}1306$	$\cdot 0^{2}1264$	$\cdot 0^{2}1223$	$\cdot 0^{2}1183$	$\cdot 0^{2}1144$	$\cdot 0^{2}1107$	$\cdot 0^{2}1070$	$\cdot 0^{2}1035$	$\cdot 0^{2}1001$
−3·1	$\cdot 0^{3}9676$	$\cdot 0^{3}9354$	$\cdot 0^{3}9043$	$\cdot 0^{3}8740$	$\cdot 0^{3}8447$	$\cdot 0^{3}8164$	$\cdot 0^{3}7888$	$\cdot 0^{3}7622$	$\cdot 0^{3}7364$	$\cdot 0^{3}7114$
−3·2	$\cdot 0^{3}6871$	$\cdot 0^{3}6637$	$\cdot 0^{3}6410$	$\cdot 0^{3}6190$	$\cdot 0^{3}5976$	$\cdot 0^{3}5770$	$\cdot 0^{3}5571$	$\cdot 0^{3}5377$	$\cdot 0^{3}5190$	$\cdot 0^{3}5009$
−3·3	$\cdot 0^{3}4834$	$\cdot 0^{3}4665$	$\cdot 0^{3}4501$	$\cdot 0^{3}4342$	$\cdot 0^{3}4189$	$\cdot 0^{3}4041$	$\cdot 0^{3}3897$	$\cdot 0^{3}3758$	$\cdot 0^{3}3624$	$\cdot 0^{3}3495$
−3·4	$\cdot 0^{3}3369$	$\cdot 0^{3}3248$	$\cdot 0^{3}3131$	$\cdot 0^{3}3018$	$\cdot 0^{3}2909$	$\cdot 0^{3}2803$	$\cdot 0^{3}2701$	$\cdot 0^{3}2602$	$\cdot 0^{3}2507$	$\cdot 0^{3}2415$
−3·5	$\cdot 0^{3}2326$	$\cdot 0^{3}2241$	$\cdot 0^{3}2158$	$\cdot 0^{3}2078$	$\cdot 0^{3}2001$	$\cdot 0^{3}1926$	$\cdot 0^{3}1854$	$\cdot 0^{3}1785$	$\cdot 0^{3}1718$	$\cdot 0^{3}1653$
−3·6	$\cdot 0^{3}1591$	$\cdot 0^{3}1531$	$\cdot 0^{3}1473$	$\cdot 0^{3}1417$	$\cdot 0^{3}1363$	$\cdot 0^{3}1311$	$\cdot 0^{3}1261$	$\cdot 0^{3}1213$	$\cdot 0^{3}1166$	$\cdot 0^{3}1121$
−3·7	$\cdot 0^{3}1078$	$\cdot 0^{3}1036$	$\cdot 0^{4}9961$	$\cdot 0^{4}9574$	$\cdot 0^{4}9201$	$\cdot 0^{4}8842$	$\cdot 0^{4}8496$	$\cdot 0^{4}8162$	$\cdot 0^{4}7841$	$\cdot 0^{4}7532$
−3·8	$\cdot 0^{4}7235$	$\cdot 0^{4}6948$	$\cdot 0^{4}6673$	$\cdot 0^{4}6407$	$\cdot 0^{4}6152$	$\cdot 0^{4}5906$	$\cdot 0^{4}5669$	$\cdot 0^{4}5442$	$\cdot 0^{4}5223$	$\cdot 0^{4}5012$
−3·9	$\cdot 0^{4}4810$	$\cdot 0^{4}4615$	$\cdot 0^{4}4427$	$\cdot 0^{4}4247$	$\cdot 0^{4}4074$	$\cdot 0^{4}3908$	$\cdot 0^{4}3747$	$\cdot 0^{4}3594$	$\cdot 0^{4}3446$	$\cdot 0^{4}3304$

95th Percentile

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161·4	199·5	215·7	224·6	230·2	234·0	236·8	238·9	240·5	241·9	243·9	245·9	248·0	249·1	250·1	251·1	252·2	253·3	254·3
2	18·51	19·00	19·16	19·25	19·30	19·33	19·35	19·37	19·38	19·40	19·41	19·43	19·45	19·45	19·46	19·47	19·48	19·49	19·50
3	10·13	9·55	9·28	9·12	9·01	8·94	8·89	8·85	8·81	8·79	8·74	8·70	8·66	8·64	8·62	8·59	8·57	8·55	8·53
4	7·71	6·94	6·59	6·39	6·26	6·16	6·09	6·04	6·00	5·96	5·91	5·86	5·80	5·77	5·75	5·72	5·69	5·66	5·63
5	6·61	5·79	5·41	5·19	5·05	4·95	4·88	4·82	4·77	4·74	4·68	4·62	4·56	4·53	4·50	4·46	4·43	4·40	4·36
6	5·99	5·14	4·76	4·53	4·39	4·28	4·21	4·15	4·10	4·06	4·00	3·94	3·87	3·84	3·81	3·77	3·74	3·70	3·67
7	5·59	4·74	4·35	4·12	3·97	3·87	3·79	3·73	3·68	3·64	3·57	3·51	3·44	3·41	3·38	3·34	3·30	3·27	3·23
8	5·32	4·46	4·07	3·84	3·69	3·58	3·50	3·44	3·39	3·35	3·28	3·22	3·15	3·12	3·08	3·04	3·01	2·97	2·93
9	5·12	4·26	3·86	3·63	3·48	3·37	3·29	3·23	3·18	3·14	3·07	3·01	2·94	2·90	2·86	2·83	2·79	2·75	2·71
10	4·96	4·10	3·71	3·48	3·33	3·22	3·14	3·07	3·02	2·98	2·91	2·85	2·77	2·74	2·70	2·66	2·62	2·58	2·54
11	4·84	3·98	3·59	3·36	3·20	3·09	3·01	2·95	2·90	2·85	2·79	2·72	2·65	2·61	2·57	2·53	2·49	2·45	2·40
12	4·75	3·89	3·49	3·26	3·11	3·00	2·91	2·85	2·80	2·75	2·69	2·62	2·54	2·51	2·47	2·43	2·38	2·34	2·30
13	4·67	3·81	3·41	3·18	3·03	2·92	2·83	2·77	2·71	2·67	2·60	2·53	2·46	2·42	2·38	2·34	2·30	2·25	2·21
14	4·60	3·74	3·34	3·11	2·96	2·85	2·76	2·70	2·65	2·60	2·53	2·46	2·39	2·35	2·31	2·27	2·22	2·18	2·13
15	4·54	3·68	3·29	3·06	2·90	2·79	2·71	2·64	2·59	2·54	2·48	2·40	2·33	2·29	2·25	2·20	2·16	2·11	2·07
16	4·49	3·63	3·24	3·01	2·85	2·74	2·66	2·59	2·54	2·49	2·42	2·35	2·28	2·24	2·19	2·15	2·11	2·06	2·01
17	4·45	3·59	3·20	2·96	2·81	2·70	2·61	2·55	2·49	2·45	2·38	2·31	2·23	2·19	2·15	2·10	2·06	2·01	1·96
18	4·41	3·55	3·16	2·93	2·77	2·66	2·58	2·51	2·46	2·41	2·34	2·27	2·19	2·15	2·11	2·06	2·02	1·97	1·92
19	4·38	3·52	3·13	2·90	2·74	2·63	2·54	2·48	2·42	2·38	2·31	2·23	2·16	2·11	2·07	2·03	1·98	1·93	1·88
20	4·35	3·49	3·10	2·87	2·71	2·60	2·51	2·45	2·39	2·35	2·28	2·20	2·12	2·08	2·04	1·99	1·95	1·90	1·84
21	4·32	3·47	3·07	2·84	2·68	2·57	2·49	2·42	2·37	2·32	2·25	2·18	2·10	2·05	2·01	1·96	1·92	1·87	1·81
22	4·30	3·44	3·05	2·82	2·66	2·55	2·46	2·40	2·34	2·30	2·23	2·15	2·07	2·03	1·98	1·94	1·89	1·84	1·78
23	4·28	3·42	3·03	2·80	2·64	2·53	2·44	2·37	2·32	2·27	2·20	2·13	2·05	2·01	1·96	1·91	1·86	1·81	1·76
24	4·26	3·40	3·01	2·78	2·62	2·51	2·42	2·36	2·30	2·25	2·18	2·11	2·03	1·98	1·94	1·89	1·84	1·79	1·73
25	4·24	3·39	2·99	2·76	2·60	2·49	2·40	2·34	2·28	2·24	2·16	2·09	2·01	1·96	1·92	1·87	1·82	1·77	1·71
26	4·23	3·37	2·98	2·74	2·59	2·47	2·39	2·32	2·27	2·22	2·15	2·07	1·99	1·95	1·90	1·85	1·80	1·75	1·69
27	4·21	3·35	2·96	2·73	2·57	2·46	2·37	2·31	2·25	2·20	2·13	2·06	1·97	1·93	1·88	1·84	1·79	1·73	1·67
28	4·20	3·34	2·95	2·71	2·56	2·45	2·36	2·29	2·24	2·19	2·12	2·04	1·96	1·91	1·87	1·82	1·77	1·71	1·65
29	4·18	3·33	2·93	2·70	2·55	2·43	2·35	2·28	2·22	2·18	2·10	2·03	1·94	1·90	1·85	1·81	1·75	1·70	1·64
30	4·17	3·32	2·92	2·69	2·53	2·42	2·33	2·27	2·21	2·16	2·09	2·01	1·93	1·89	1·84	1·79	1·74	1·68	1·62
40	4·08	3·23	2·84	2·61	2·45	2·34	2·25	2·18	2·12	2·08	2·00	1·92	1·84	1·79	1·74	1·69	1·64	1·58	1·51
60	4·00	3·15	2·76	2·53	2·37	2·25	2·17	2·10	2·04	1·99	1·92	1·84	1·75	1·70	1·65	1·59	1·53	1·47	1·39
120	3·92	3·07	2·68	2·45	2·29	2·17	2·09	2·02	1·96	1·91	1·83	1·75	1·66	1·61	1·55	1·50	1·43	1·35	1·25
∞	3·84	3·00	2·60	2·37	2·21	2·10	2·01	1·94	1·88	1·83	1·75	1·67	1·57	1·52	1·46	1·39	1·32	1·22	1·00

TABLE 6 (continued)*

97.5th Percentile

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	647·8	799·5	864·2	899·6	921·8	937·1	948·2	956·7	963·3	968·6	976·7	984·9	993·1	997·2	1001	1006	1010	1014	1018
2	38·51	39·00	39·17	39·25	39·30	39·33	39·36	39·37	39·39	39·40	39·41	39·43	39·45	39·46	39·46	39·47	39·48	39·49	39·50
3	17·44	16·04	15·44	15·10	14·88	14·73	14·62	14·54	14·47	14·42	14·34	14·25	14·17	14·12	14·08	14·04	13·99	13·95	13·90
4	12·22	10·65	9·98	9·60	9·36	9·20	9·07	8·98	8·90	8·84	8·75	8·66	8·56	8·51	8·46	8·41	8·36	8·31	8·26
5	10·01	8·43	7·76	7·39	7·15	6·98	6·85	6·76	6·68	6·62	6·52	6·43	6·33	6·28	6·23	6·18	6·12	6·07	6·02
6	8·81	7·26	6·60	6·23	5·99	5·82	5·70	5·60	5·52	5·46	5·37	5·27	5·17	5·12	5·07	5·01	4·96	4·90	4·85
7	8·07	6·54	5·89	5·52	5·29	5·12	4·99	4·90	4·82	4·76	4·67	4·57	4·47	4·42	4·36	4·31	4·25	4·20	4·14
8	7·57	6·06	5·42	5·05	4·82	4·65	4·53	4·43	4·36	4·30	4·20	4·10	4·00	3·95	3·89	3·84	3·78	3·73	3·67
9	7·21	5·71	5·08	4·72	4·48	4·32	4·20	4·10	4·03	3·96	3·87	3·77	3·67	3·61	3·56	3·51	3·45	3·39	3·33
10	6·94	5·46	4·83	4·47	4·24	4·07	3·95	3·85	3·78	3·72	3·62	3·52	3·42	3·37	3·31	3·26	3·20	3·14	3·08
11	6·72	5·26	4·63	4·28	4·04	3·88	3·76	3·66	3·59	3·53	3·43	3·33	3·23	3·17	3·12	3·06	3·00	2·94	2·88
12	6·55	5·10	4·47	4·12	3·89	3·73	3·61	3·51	3·44	3·37	3·28	3·18	3·07	3·02	2·96	2·91	2·85	2·79	2·72
13	6·41	4·97	4·35	4·00	3·77	3·60	3·48	3·39	3·31	3·25	3·15	3·05	2·95	2·89	2·84	2·78	2·72	2·66	2·60
14	6·30	4·86	4·24	3·89	3·66	3·50	3·38	3·29	3·21	3·15	3·05	2·95	2·84	2·79	2·73	2·67	2·61	2·55	2·49
15	6·20	4·77	4·15	3·80	3·58	3·41	3·29	3·20	3·12	3·06	2·96	2·86	2·76	2·70	2·64	2·59	2·52	2·46	2·40
16	6·12	4·69	4·08	3·73	3·50	3·34	3·22	3·12	3·05	2·99	2·89	2·79	2·68	2·63	2·57	2·51	2·45	2·38	2·32
17	6·04	4·62	4·01	3·66	3·44	3·28	3·16	3·06	2·98	2·92	2·82	2·72	2·62	2·56	2·50	2·44	2·38	2·32	2·25
18	5·98	4·56	3·95	3·61	3·38	3·22	3·10	3·01	2·93	2·87	2·77	2·67	2·56	2·50	2·44	2·38	2·32	2·26	2·19
19	5·92	4·51	3·90	3·56	3·33	3·17	3·05	2·96	2·88	2·82	2·72	2·62	2·51	2·45	2·39	2·33	2·27	2·20	2·13
20	5·87	4·46	3·86	3·51	3·29	3·13	3·01	2·91	2·84	2·77	2·68	2·57	2·46	2·41	2·35	2·29	2·22	2·16	2·09
21	5·83	4·42	3·82	3·48	3·25	3·09	2·97	2·87	2·80	2·73	2·64	2·53	2·42	2·37	2·31	2·25	2·18	2·11	2·04
22	5·79	4·38	3·78	3·44	3·22	3·05	2·93	2·84	2·76	2·70	2·60	2·50	2·39	2·33	2·27	2·21	2·14	2·08	2·00
23	5·75	4·35	3·75	3·41	3·18	3·02	2·90	2·81	2·73	2·67	2·57	2·47	2·36	2·30	2·24	2·18	2·11	2·04	1·97
24	5·72	4·32	3·72	3·38	3·15	2·99	2·87	2·78	2·70	2·64	2·54	2·44	2·33	2·27	2·21	2·15	2·08	2·01	1·94
25	5·69	4·29	3·69	3·35	3·13	2·97	2·85	2·75	2·68	2·61	2·51	2·41	2·30	2·24	2·18	2·12	2·05	1·98	1·91
26	5·66	4·27	3·67	3·33	3·10	2·94	2·82	2·73	2·65	2·59	2·49	2·39	2·28	2·22	2·16	2·09	2·03	1·95	1·88
27	5·63	4·24	3·65	3·31	3·08	2·92	2·80	2·71	2·63	2·57	2·47	2·36	2·25	2·19	2·13	2·07	2·00	1·93	1·85
28	5·61	4·22	3·63	3·29	3·06	2·90	2·78	2·69	2·61	2·55	2·45	2·34	2·23	2·17	2·11	2·05	1·98	1·91	1·83
29	5·59	4·20	3·61	3·27	3·04	2·88	2·76	2·67	2·59	2·53	2·43	2·32	2·21	2·15	2·09	2·03	1·96	1·89	1·81
30	5·57	4·18	3·59	3·25	3·03	2·87	2·75	2·65	2·57	2·51	2·41	2·31	2·20	2·14	2·07	2·01	1·94	1·87	1·79
40	5·42	4·05	3·46	3·13	2·90	2·74	2·62	2·53	2·45	2·39	2·29	2·18	2·07	2·01	1·94	1·88	1·80	1·72	1·64
60	5·29	3·93	3·34	3·01	2·79	2·63	2·51	2·41	2·33	2·27	2·17	2·06	1·94	1·88	1·82	1·74	1·67	1·58	1·48
120	5·15	3·80	3·23	2·89	2·67	2·52	2·39	2·30	2·22	2·16	2·05	1·94	1·82	1·76	1·69	1·61	1·53	1·43	1·31
∞	5·02	3·69	3·12	2·79	2·57	2·41	2·29	2·19	2·11	2·05	1·94	1·83	1·71	1·64	1·57	1·48	1·39	1·27	1·00