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This paper presents an introduction to Čech closure spaces. The set of all Čech closure operators on a set is closed under the operations of union and composition. An association between Čech closure operators on a finite set and zero-one relation matrices is used to present matrix operations corresponding to union and composition of Čech closure operators. Finitely generated Čech closure operators are defined, and it is shown that the set of all finitely generated Čech closure operators on a set, partially ordered in a natural way, yeilds a uniquely complemented, distributive, and complete lattice and is therefore a Boolean algebra. A Čech closure operator generates a semi-topology and an underlying topology; relationships between these are studied. Several separation properties are generalized to Čech closure spaces and studied in this broader context.

ČECH CLOSURE SPACES

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CHAPTER I

INTRODUCTION

Čech closure spaces were introduced by Čech [3]. For each Čech closure space there exists an underlying topological space that can be defined in a natural way. Some familiarity with the rudiments of topology on the part of the reader is assumed.

The set of all Čech closure operators on a non-empty set is closed under union and composition; while the composition of two topological closure operators is not necessarily a topological closure operator, it is a Čech closure operator. The set of all Čech closure operators on a set, partially ordered by inclusion, yields a complete lattice.

A finitely generated Čech closure space is a generalization of a finite Čech closure space. The set of all finitely generated Čech closure operators on a set, partially ordered by inclusion, yields a uniquely complemented distributive complete lattice.

Sharp, in [5], and Bonnett and Porter, in [2], represent finite topological spaces using zero-one matrices. Matrix characterizations of many topological properties are given in these papers. Čech closure operators on a finite set can also be represented by zero-one matrices; matrix operations corresponding to union and composition of Čech closure operators are defined.

In this paper, several mild separation properties are extended to Čech closure spaces and characterized, for finite spaces, in terms of the matrix associated with the closure operator. It is shown that finitely generated Čech closure spaces satisfying certain of these separation properties are topological spaces. Therefore some of the separation properties for Čech closure spaces carry over to the underlying topological space.

A Čech closure operator also generates a semi-topology; that is, a collection of sets that satisfies the axioms for a topology, except for the union axiom.

Čech closure operators of finite degree are defined and studied. It is of interest to note that Čech closure spaces of finite degree provide a generalization of topological spaces.

CHAPTER II

BASIC DEFINITIONS AND RESULTS

DEFINITION 2.1. A mapping $c: P(X) \to P(X)$ is called a Čech closure operator provided it satisfies the following three axioms:

(C1) $c(\emptyset) = \emptyset$

(C2) $A \subset c(A)$ for all $A \subset X$

(C3) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subset X$.

Then c, together with the underlying set X, is called a Čech closure space and is denoted by (X,c). If c also satisfies:

(C4) $c(c(A)) = c(A)$ for all $A \subset X$,

then (X,c) is a topological space.

DEFINITION 2.2. Let (X,c) be a Čech closure space. A subset A of X is called closed provided $A = c(A)$. A subset A of X is called open provided its complement X-A is closed. Let $t(c) = \{O: X-O = c(X-O)\}$.

LEMMA 2.1. Let (X,c) be a Čech closure space, and $A \subset B \subset X$. Then c(A) is contained in c(B).

PROOF. $c(A) \subset c(A) \cup c(B) = c(A \cup B) = c(B)$ since $A \cup B = B$.

LEMMA 2.2. Let (X,c) be a Čech closure space, and $A \subset X$. If c(A) is contained in A, then A is closed.

THEOREM 2.3. Let (X,c) be a Čech closure space. Then t(c) is a topology on X (t(c) is called the underlying topology of (X,c)).

PROOF. Clearly, X and $\emptyset$ are members of t(c). Suppose O and Q are members of t(c). $X-(O \cap Q) = (X-O) \cup (X-Q) = c(X-O) \cup c(X-Q) =$ $= c((X-O) \cup (X-Q)) = c(X-(O \cap Q))$. Now consider an arbitrary collection of sets $\{O_\alpha : \alpha \in \Lambda\}$, each a member of t(c). For each $\alpha \in \Lambda$, $X-O_\alpha$ is closed and $\bigcap \{X-O_\alpha : \alpha \in \Lambda\}$ is contained in $X-O_\alpha$. Lemma 2.1 then implies that $c(\bigcap \{X-O_\alpha : \alpha \in \Lambda\})$ is contained in $c(X-O_\alpha) = X-O_\alpha$ for every $\alpha \in \Lambda$. Hence

$c(\bigcap\{X-O_\alpha: \alpha\in\Lambda\})$ is contained in $\bigcap\{X-O_\alpha: \alpha\in\Lambda\}$ and by lemma 2.2, $\bigcap\{X-O_\alpha: \alpha\in\Lambda\} = X-\bigcup\{O_\alpha: \alpha\in\Lambda\}$ is closed.

Consider these examples of Čech closure spaces.

EXAMPLE 2.A. Let $X = \{1,2,3,4\}$. Define $c(1) = \{1,2\}$, $c(2) = \{1,2\}$, $c(3) = \{2,3\}$, $c(4) = \{3,4\}$. For all A contained in X, let

$$c(A) = \begin{cases} \emptyset \text{ if } A = \emptyset \\ \bigcup\{c(a): a\in A\} \text{ otherwise.} \end{cases}$$

By the definition of c(A), (C1) and (C2) are satisfied. Let A and B be subsets of X. Then $c(A\cup B) = \bigcup\{c(x): x\in A\cup B\} = (\bigcup\{c(x): x\in A\}) \bigcup (\bigcup\{c(x): x\in B\}) = c(A)\bigcup c(B)$. Thus (X,c) is a Čech closure space.

EXAMPLE 2.B. Let N represent the natural numbers. For all elements n of N, let $c(n) = \{n, n+1\}$. For any A contained in N, define

$$c(A) = \begin{cases} \emptyset \text{ if } A = \emptyset \\ \bigcup\{c(n): n\in A\} \text{ otherwise.} \end{cases}$$

EXAMPLE 2.C Let X be any infinite set. For all A contained in X, let

$$c(A) = \begin{cases} \emptyset \text{ if } A = \emptyset \\ A \text{ if } A \text{ is finite} \\ X \text{ otherwise.} \end{cases}$$

Notice that example 2.C, in addition to being a Čech closure space, is a topological space.

DEFINITION 2.3. Let (X,c) be a Čech closure space. If c(A) = A for every set A contained in X, c is called the discrete closure operator on X. If c(A) = X for every set a contained in X, c is called the trivial closure operator on X.

DEFINITION 2.4. In a Čech closure space (X,c), c is finitely generated provided for any subset A of X, $c(A) = \bigcup\{c(a): a\in A\}$. (X,c) is then called a finitely generated Čech closure space.

THEOREM 2.4. Every finite Čech closure space is finitely generated.

THEOREM 2.5. Let X be a non-empty set and $e: X \rightarrow P(X)$ be a mapping such that $x \in e(x)$ for each $x \in X$. Let $c(A) = \bigcup\{e(a): a \in A\}$ for all $A \subset X$. Then (X,c) is a finitely generated Čech closure space.

Essentially, a finitely generated Čech closure operator is determined by its action on singleton sets.

DEFINITION 2.5. Let c and d be Čech closure operators on a set X, and A be a subset of X. Then define:

$(c \cup d)(A) = c(A) \bigcup d(A)$, and

$(c \circ d)(A) = c(d(A))$.

THEOREM 2,6. Let c and d be Čech closure operators on a set X. Then $(c \cup d)$ and $(c \circ d)$ are Čech closure operators on X.

PROOF. Let A and B be contained in X. Easily $(c \cup d)(\phi) = \phi$ and $A \subset (c \cup d)(A)$. Now $(c \cup d)(A \cup B) =$

$$= c(A \cup B) \bigcup d(A \cup B) =$$
$$= c(A) \bigcup c(B) \bigcup d(A) \bigcup d(B) =$$
$$= (c \cup d)(A) \bigcup (c \cup d)(B).$$

Clearly $(c \circ d)(\phi) = \phi$ and $A \subset (c \circ d)(A)$. By definition,

$$(c \circ d)(A \cup B) =$$
$$= c(d(A \cup B)) =$$
$$= c(d(A) \bigcup d(B)) =$$
$$= (c \circ d)(A) \bigcup (c \circ d)(B).$$

THEOREM 2.7 In the set of all Čech closure operators $\{c_\alpha\}$ on a set X, the operation $\cup$ is associative, commutative, and has an identity, while the operation $\circ$ is associative and has an identity.

PROOF. Since $\cup$ is defined in terms of set unions, it inherits commutativity and associativity. Let c_0 be the discrete closure operator on X, and d any Čech closure operator on X. Then for any subset A of X, $(d \cup c_0)(A) = d(A) \cup c_0(A) = d(A) \cup A = d(A)$. Now consider c, d, and e, elements of $\{c_\alpha\}$. $(c \circ (d \circ e))(A) = c(d(e(A))) = ((c \circ d) \circ e)(A)$. Again, let c_0 be the discrete closure operator, and d any element of $\{c_\alpha\}$. For any subset A of X, $(c_0 \circ d)(A) = c_0(d(A)) = d(A) = d(c_0(A)) =$ $= (d \circ c_0)(A)$.

THEOREM 2,8. If c and d are Čech closure operators on a set X, and A is a subset of X, then $(c \cup d)(A)$ is contained in $(c \circ d)(A)$.

PROOF. By definition, $(c \cup d)(A) = c(A) \cup d(A)$. A is contained in d(A), so by lemma 2,2, c(A) is contained in $c(d(A)) = (c \circ d)(A)$. Now d(A) is contained in c(d(A)); hence $(c \cup d)(A)$ is contained in $(c \circ d)(A)$.

COROLLARY 2.9. If c and d are Čech closure operators on a set X, and A is a subset of X that is closed under d, or d(A) is closed under c, then $(c \cup d)(A) = (c \circ d)(A)$.

While union of two Čech closure operators commutes, composition does not; the union of two Čech closure operators does not, in general, give the same result as their composition, in either order. The following theorem shows, however, that $t(c \circ d) = t(d \circ c) = t(c \cup d)$.

THEOREM 2.10. Let (X,c) and (X,d) be Čech closure spaces. Then $t(c \circ d) = t(d \circ c) = t(c) \cap t(d) = t(c \cup d)$.

PROOF. Let O be a member of $t(c \circ d)$. Then $(c \circ d)(X-O) = X-O$, which is contained in d(X-O). Lemma 2.1 implies, then, that $(c \circ d)(X-O) =$ $= d(X-O)$ and, hence, O is a member of t(d). Then $c(X-O) = c(d(X-O)) =$ $= X-O$ and O is a member of t(c). Thus $t(c \circ d)$, and similarly $t(d \circ c)$, are contained in $t(c) \cap t(d)$. Now let Q be a member of $t(c) \cap t(d)$. $X-Q =$ $= c(X-Q) = d(X-Q)$; hence $(c \circ d)(X-Q) = c(X-Q) = X-Q = d(X-Q) =$ $= (d \circ c)(X-Q)$.

Let O be a member of $t(c \cup d)$. $(c \cup d)(X-O) = c(X-O) \cup d(X-O) = X-O$

and thus $c(X-O) = d(X-O) = X-O$. Therefore O is a member of $t(c) \bigcap t(d)$. Now let Q be a member of $t(c) \bigcap t(d)$. Then $c(X-Q) = d(X-Q) = X-Q = $ $= c(X-Q) \bigcup d(X-Q) = (c \cup d)(X-Q)$, and $t(c \cup d) = t(c) \bigcap t(d)$.

DEFINITION 2.6. Let $\{c_\alpha : \alpha \in \Lambda\}$ be a collection of Čech closure operators on a set X, and let B be a subset of X. Then $(\bigcup \{c_\alpha : \alpha \in \Lambda\})(B) = \bigcup \{c_\alpha(B) : \alpha \in \Lambda\}$.

THEOREM 2.11. If c_α is a Čech closure operator on a set X for each element α of Λ, then $(\bigcup \{c_\alpha : \alpha \in \Lambda\})$ is a Čech closure operator on X.

PROOF. The proof parallels the proof of theorem 2.6.

DEFINITION 2.7. Let (X,c) be a Čech closure space. Then, if there exists a smallest natural number n such that $c^n(A) = c^{n+1}(A)$ for all A contained in X, then c is said to be of degree n, and (X,c) is said to be of finite degree.

THEOREM 2.12. A closure operator is Kurotowski if and only if it is of degree one.

Clearly, there exist finitely generated Čech closure operators that are not of finite degree (consider example 2.B). The following theorem is due to Sharp [5].

THEOREM 2.13. If t is a topology on a set X then the family of complements of members of t is also a topology on X (called the dual topology with respect to t).

PROOF. The proof is similar to that of theorem 2.3.

CHAPTER III

MATRIX REPRESENTATION

Shapp, in [5], showed that an $n \times n$, zero-one, reflexive and transitive relation matrix $T = [t_{ij}]$ can be associated with each topology on a finite set with cardinality n in the following way:

$$t_{ij} = \begin{cases} 1 & \text{if } j \in \{\bar{x}\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus there is a one-to one correspondence between the topologies on a finite set X and the quasi-orderings of X [5].

The same kind of correspondence between Čech closure operators on a finite set and zero-one reflexive matrices can be established as follows:

DEFINITION 3.1. Let (X,c) be a Čech closure space where $|X| = n$. Define the $n \times n$ matrix $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in c(x_i) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, there is a one-to one correspondence between the reflexive relations on a finite set X and the Čech closure operators on X.

Some of the notation developed by Bonnett and Porter, in [2], extends easily to Čech closure spaces. Let X be a finite set. Then with each element x_i of X, associate the vector $\epsilon_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$, where δ_{ij} is the Kronecker delta and $|X| = n$. With each subset A of X, associate the vector $A_v = \sum (\epsilon_i : x_i \in A)$. Clearly, then, if A is the matrix associated with a Čech closure operator c on a finite set X, then A_i, the i^{th} row of A, is $(c(x_i))_v$.

Throughout this paper, "I" will be used to denote the identity matrix.

THEOREM 3.1. Let (X,c) be a finite Čech closure space and A its associated matrix. Let B be a subset of X. Then $(c(B))_v = B_v \cdot A$, where the matrix multiplication is with respect to Boolean arithmetic.

PROOF. Since $c(B) = \bigcup \{c(x_i): x_i \in B\}$, the theorem follows if $(c(x_i))_v = A_i = \epsilon_i \cdot A$. Let u_j denote the j^{th} entry of $\epsilon_i \cdot A$. Then $u_j = \sum_{k=1}^{n} \delta_{ik} a_{kj} = a_{ij}$; hence $\epsilon_i \cdot A = A_i$.

COROLLARY 3.2. Let (X,c) be a finite Čech closure space and A the associated matrix. A subset B of X is closed if and only if $B_v = B_v \cdot A$, where the matrix multiplication is with respect to Boolean arithmetic.

THEOREM 3.3. A reflexive, $n \times n$, zero-one matrix T corresponds to a topology on a finite set if and only if $T^2 = T$, where the matrix multiplication is with respect to Boolean arithmetic [5].

THEOREM 3.4. Let (X,c) and (X,d) be finite Čech closure spaces with associated matrices A and B, respectively. Then:

(1) the matrix associated with $(X, c \cup d)$ is $A+B$, and

(2) the matrix associated with $(X, c \circ d)$ is BA,

where matrix addition and multiplication is with respect to Boolean arithmetic.

PROOF. (1). The theorem follows if $(A+B)_i = ((c \cup d)(x_i))_v$ for any element x_i of X. Let x_i be an element of X. $(c \cup d)(x_i) =$
$= c(x_i) \bigcup d(x_i) = \{x_j: a_{ij} = 1\} \bigcup \{x_j: b_{ij} = 1\}$; hence $(A+B)_i =$
$= ((c \cup d)(x_i))_v$.

(2). The theorem follows if $(BA)_i = ((c \circ d)(x_i))_v$ for any element x_i of X. Let x_i be an element of X. $(c \circ d)(x_i) = c(d(x_i)) =$
$= \{x_k: x_k \in c(x_j);\ x_j \in d(x_i) \text{ for some } x_j \in X\}$. $(BA)_i =$
$= (\sum_j b_{ij}a_{j1}, \sum_j b_{ij}a_{j2}, \ldots, \sum_j b_{ij}a_{jn})$.

$$\sum_j b_{ij}\, a_{jk} = \begin{cases} 1 \text{ if } b_{ij} = 1 \text{ and } a_{jk} = 1 \text{ for some } 1 \leq j \leq n \\ 0 \text{ otherwise.} \end{cases}$$

That is,

$$\sum_j b_{ij}\, a_{jk} = \begin{cases} 1 \text{ if } x_j \in d(x_i) \text{ and } x_k \in c(x_j) \text{ for some } x_j \in X \\ 0 \text{ otherwise.} \end{cases}$$

Hence $(BA)_i = ((c \circ d)(x_i))_v$.

THEOREM 3.5. Let (X,c) be a finite Čech closure space with associated matrix A. Then:

(1) c is of finite degree n,

(2) $t(c^n) = t(c^k) = t(c)$; $1 \leq k \leq n$,

(3) A^n is the matrix associated with $t(c)$.

PROOF. (1). Since X is finite, $n = \inf \{m: c^m(A) = c^{m+1}(A) \text{ for all } A \subset X\}$ exists.

(2). Let O be a member of $t(c)$. Then $X-O = c(X-O) = c^2(X-O) = \ldots = c^n(X-O)$. Now let O be a member of $t(c^n)$. $c^n(X-O) = X-O = c(c^n(X-O)) = c(X-O)$; hence $t(c) = t(c^n)$.

The proof of (3) follows from theorem 3.4.

The following theorem is due to Sharp [5].

THEOREM 3.6. If T is the matrix corresponding to a topology t, then T^T (the transpose of T) is the matrix corresponding to the dual topology with respect to t.

DEFINITION 3.2. Let (X,c) be a finite Čech closure space and A the matrix associated with it. Then the dual Čech closure space with respect to (X,c) is the Čech closure space associated with the transpose of A, and is denoted by (X,c^T).

THEOREM 3.7. Let (X,c), (X,c^T), and (X,d) be finite Čech closure spaces. Then:

(1) c and c^T have the same degree,

(2) $t(c)$ and $t(c^T)$ are dual topologies,

(3) $(c \circ d)^T = d^T \circ c^T$.

PROOF. (1) and (2). Let A be the matrix associated with (X,c), and let n be the degree of c. A^n, then, is the matrix associated with t(c), and $(A^n)^T = (A^T)^n$ is the matrix associated with $t(c^T)$.

(3). Let B be the matrix associated with (X,d). $(BA)^T = A^T B^T$, where matrix multiplication is with respect to Boolean arithmetic, and thus (3) holds.

CHAPTER IV

THE LATTICE OF ČECH CLOSURE OPERATORS

DEFINITION 4.1. For any pair of Čech closure operators c and d on a set X, $c < d$ provided $c(A) \subset d(A)$ for all sets A contained in X.

DEFINITION 4.2. Let X be a non-empty set. Then define L(X) as the set of all Čech closure operators on X, and C(X) as the set of all finitely generated Čech closure operators on X.

THEOREM 4.1. Let X be a non-empty set. Then $(L(X), <)$ and $(C(X), <)$ are partially ordered (reflexive, anti-symmetric, and transitive) sets.

PROOF. The theorem follows from definition 4.1.

DEFINITION 4.3. Let X be a non-empty set and c and d be elements of L(X). Then let $c \vee d = \text{l.u.b.}\{c,d\}$ and $c \wedge d = \text{g.l.b.}\{c,d\}$.

THEOREM 4.2. Let X be a non-empty set and c_α a Čech closure operator on X for each $\alpha \in \Lambda$. Define d by $d(A) = \bigcup\{c_\alpha(A): \alpha \in \Lambda\}$ for each $A \subset X$. Then:

(1) $d = \bigvee\{c_\alpha: \alpha \in \Lambda\}$, and

(2) $(L(X), \vee, \wedge)$ and $(C(X), \vee, \wedge)$ are complete lattices.

PROOF. That d is a Cech closure operator is shown in theorem 2.11; d is clearly an upper bound of $\{c_\alpha: \alpha \in \Lambda\}$. Let e be any element of L(X) that is an upper bound of $\{c_\alpha: \alpha \in \Lambda\}$. Then $c_\alpha(A) \subset e(A)$ for any $\alpha \in \Lambda$ and subset A of X; thus $d < e$. Now $\bigwedge\{c_\alpha: \alpha \in \Lambda\} =$ $= \sup\{e: e \in L(X) \text{ and } e < c_\alpha \text{ for each } \alpha \in \Lambda\}$. Hence $(L(X), \vee, \wedge)$ is a complete lattice.

Since the least upper bound of a collection of finitely generated Čech closure operators is finitely generated, one shows that $(C(X), \vee, \wedge)$

is a complete lattice in a similar manner.

The symbols L(X) and C(X) will also be used to denote the lattice of Čech closure operators on a non-empty set X and the lattice of finitely generated Čech closure operators on X, respectively.

The operations $\vee$ and $\wedge$ have the following properties in any lattice [1]. Let c, d, and e be elements of L(X). Then:

(1) $c \wedge c = c$; $c \vee c = c$,

(2) $c \wedge d = d \wedge c$; $c \vee d = d \vee c$,

(3) $(c \wedge d) \wedge e = c \wedge (d \wedge e)$; $(c \vee d) \vee e = c \vee (d \vee e)$,

(4) $c \wedge (c \vee d) = c \vee (c \wedge d) = c$, and

(5) $c < d \Leftrightarrow c \wedge d = c$ and $c \vee d = d$.

EXAMPLE 4.A. Although the union of two Čech closure operators results in a Čech closure operator, the analogous result need not hold for their intersection. Let $X = \{1,2,3\}$ and let $c(\phi) = \phi$, $c(1) = \{1,2\}$, $c(2) = \{2\}$, $c(3) = \{3\}$, $d(\phi) = \phi$, $d(1) = \{1\}$, $d(2) = \{2\}$, and $d(3) = \{2,3\}$. Also, let $(c \cap d)(A) = c(A) \cap d(A)$ for any subset A of X. Clearly, $(c \cap d)(\phi) = \phi$ and $A \subset (c \cap d)(A)$. Let $A = \{1\}$ and $B = \{3\}$. Then $(c \cap d)(A \cup B) = c(A \cup B) \cap d(A \cup B) = X$, but $(c \cap d)(A) \cup (c \cap d)(B) = (c(1) \cap d(1)) \cup (c(3) \cap d(3)) = \{1,3\}$. Hence $(c \cap d)$ does not satisfy (C3); therefore $(c \cap d)$ is not a Čech closure operator.

From theorem 4.2 it follows that $c \vee d = c \cup d$. If c and d are elements of C(X) for some non-empty set X, then $c \wedge d$ can also be easily determined.

THEOREM 4.4. Let X be a non-empty set, and c and d be elements of C(X). Define $e(x) = c(x) \cap d(x)$ for any element x of X. Then $(c \wedge d)(A) = \bigcup \{e(x) : x \in A\}$.

PROOF. Define $e(A) = \bigcup \{e(x) : x \in A\}$ for any subset A of X. Then by theorem 2.5, e is a Čech closure operator on X. Clearly, $e < c$ and $e < d$; hence $e < c \wedge d$. Now let $y \in (c \wedge d)(x)$ for some element x of X. Then $y \in c(x) \cap d(x) = e(x)$. Since e is finitely generated, the theorem follows.

DEFINITION 4.4. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$, zero-one matrices. Define the matrix operations $\wedge$ and $\vee$ as follows:

(1) $A \wedge B = [a_{ij} \wedge b_{ij}]$, and

(2) $A \vee B = [a_{ij} \vee b_{ij}]$.

If A and B are matrices associated with Čech closure operators c and d, respectively, on a non-empty finite set X, theorem 3.4 implies that $A \vee B$ is the matrix associated with $c \vee d$.

THEOREM 4.5. Let (X,c) and (X,d) be Čech closure spaces with associated matrices A and B, respectively. Then the matrix associated with $(X, c \wedge d)$ is $A \wedge B$,

PROOF. This theorem follows from theorem 4.4.

THEOREM 4.6. Let X be a non-empty set. Then C(X) is a distributive lattice and hence a modular lattice.

PROOF Let A be a subset of X, and c, d, and e be elements of C(X).

$(c \wedge (d \vee e))(A) =$

$= \bigcup \{c(a) \cap (d(a) \cup e(a)) \colon a \in A\} =$

$= \bigcup \{(c(a) \cap d(a)) \cup (c(a) \cap e(a)) \colon a \in A\} =$

$= ((c \wedge d) \vee (c \wedge e))(A)$.

Then $(c \vee d) \wedge (c \vee e) = ((c \vee d) \wedge c) \vee ((c \vee d) \wedge e) =$ $= c \vee ((c \wedge e) \vee (d \wedge e)) = (c \vee (c \wedge e)) \vee (d \wedge e) = c \vee (d \wedge e)$ by properties of $\wedge$ and $\vee$ and the first part of this proof.

DEFINITION 4.5. Let X be a non-empty set, and $c \in C(X)$. For each $x \in X$, let $e(x) = \{x\} \cup X - c(x)$. Define $c'(A) = \bigcup \{e(x) \colon x \in A\}$ for any set A contained in X.

THEOREM 4.7. Let X be a non-empty set and $c \in C(X)$. Then:

(1) $c' \in C(X)$,

(2) c' is a complement of c in $C(X)$, and

(3) $C(X)$ is uniquely complemented.

PROOF. (1). $c' \in C(X)$ by theorem 2.5.

(2). Let A be a subset of X. $(c \vee c')(A) = c(A) \cup (\bigcup\{c(x): x \in A\}) =$
$= c(A) \cup (\bigcup\{\{x\} \cup (X - c(x)): x \in A\}) = X$. $(c \wedge c')(A) =$
$= \bigcup\{c(a) \cap c'(a): a \in A\} = \bigcup\{c(a) \cap (\{a\} \cup X - c(a)): a \in A\} = A$.

(3) Every distributive and complemented lattice is uniquely complemented.

COROLLARY 4.8. $C(X)$ is Boolean algebra, if X is a non-empty set.

DEFINITION 4.6. Let c be a Čech closure operator on a finite set X, and consider the associated matrix $A = [a_{ij}]$. Define $A' = [a'_{ij}]$ by

$$a'_{ij} = \begin{cases} 1 \text{ if } i=j \\ 1 \text{ if } i \neq j \text{ and } a_{ij} = 0 \\ 0 \text{ if } i \neq j \text{ and } a_{ij} = 1 \end{cases}$$

THEOREM 4.9. Let (X,c) be a finite Čech closure space with associated matrix A. Then A' is the matrix associated with c', the complement of c.

Clearly, all the members of the lattice of topologies on a finite set X are contained by $C(X)$, although the lattice of topologies is not a sublattice of $C(X)$.

THEOREM 4.10. If $|X| = n$, $C(X)$ has $2^{n(n-1)}$ elements.

PROOF. Each reflexive, zero-one $n \times n$ matrix represents a Čech closure operator on X.

THEOREM 4.11. If X is finite, $C(X)$ is an atomic lattice. If $|X| = n$,

$C(X)$ has $n(n-1)$ atoms.

CHAPTER V

SEMI-TOPOLOGIES

DEFINITION 5.1. Let X be a non-empty set and $\check{t}$ a subset of P(X). Then $\check{t}$ is called a semi-topology provided:

(S1) $\emptyset \in t$; $X \in t$, and

(S2) $O \in t$ and $Q \in t$ implies $O \cap Q \in t$.

DEFINITION 5.2. Let (X,c) be a Čech closure space. A subset B of X is called c-closed provided B = c(A) for some set A contained in X. A subset O of X is called c-open provided X-O is a c-closed set. Let $\check{t}(c)$ denote the collection of all c-open sets in (X,c).

THEOREM 5.1. Let (X,c) be a Cech closure space. Then $\check{t}(c)$ is a semi-topology.

PROOF. Clearly, X and $\emptyset$ are elements of $\check{t}(c)$. Let O and Q be elements of $\check{t}(c)$. Then there exist subsets A and B of X such that $X-O = c(A)$ and $X-Q = c(B)$. $X-(O \cap Q) = (X-O) \bigcup (X-Q) = c(A) \bigcup c(B) = $ $= c(A \cup B)$; hence $O \cap Q$ is an element of $\check{t}(c)$.

COROLLARY 5.2. Let (X,c) be a Čech closure space. Then $\check{t}(c)$ contains t(c).

EXAMPLE 5.A. Distinct Čech closure operators can generate the same semi-topology. Let $X = \{1,2,3\}$ and $c(1) = \{1,2\}$, $c(2) = \{1,2\}$, and $c(3) = \{1,3\}$; let $d(1) = \{1,3\}$, $d(2) = \{1,2\}$, and $d(3) = X$. Then $t(c) = t(d) = \{\emptyset, \{3\}, \{2\}, X\}$.

EXAMPLE 5.B. Not every semi-topology is generated by a Čech closure operator. Let $X = \{1,2,3,4\}$ and $\check{t} = \{\emptyset, X, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$. $(X,\check{t})$ is a semi-topological space; assume $\check{t} = \check{t}(c)$ for some Čech closure operator c on X. Then X- $\{1,3,4\}$ = c(A) for some A contained in X; thus c(2) = $\{2\}$. Similarly, c(1) = $\{1\}$. Since $\{1,3\}$ and $\{2,3\}$ are elements of $\check{t}$, it follows that c(4) = $\{4\}$. Then, since $\{1,4\}$ belongs to $\check{t}$, $\{2,3\}$ = = c(A) for some set A contained in X. A $\neq \{2\}$; if A = $\{2,3\}$ we would have that $\check{t}$ is the discrete topology on X, so A must equal $\{3\}$. Then c(1) = $\{1\}$, c(2) = $\{2\}$, c(3) = $\{2,3\}$, and c(4) = $\{4\}$. Thus $\check{t}(c) = \{\emptyset, X, \{2\}, \{3\}, \{4\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}\}$, which does not equal $\check{t}$. Therefore $\check{t}$ is not generated by a Čech closure operator.

EXAMPLE 5.C. Consider the semi-topology $\check{t} = \{\emptyset, X, \{1\}, \{2\}\}$ where $X = \{1,2,3\}$. Define $\bar{c}(A) = \{x: O \in \check{t}$ and $x \in O$ implies $O \cap A \neq \emptyset\}$. Then $\bar{c}(1) = \{1,3\}$, $\bar{c}(2) = \{2,3\}$, $\bar{c}(3) = \{3\}$ and $\check{t}(\bar{c})$ = $= \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}$ is a semi-topology which contains $\check{t}$.

That $\bar{c}$ as defined above is a Kurotowski closure operator and that $\check{t}$ is contained in $\check{t}(\bar{c})$ is the content of the next two theorems.

THEOREM 5.3. Let $(X,\check{t})$ be a semi-topological space and define, for any A contained in X, $\bar{c}(A) = \{x: O \in \check{t}$ and $x \in O$ implies $O \cap A \neq \emptyset\}$. Then $\bar{c}$ is a Kurotowski closure operator.

PROOF. That $\bar{c}(\emptyset) = \emptyset$ and $A \subset \bar{c}(A)$ is evident. $\bar{c}(A \cup B)$ = $= \{x: O \in \check{t}$ and $x \in O$ implies $O \cap (A \cup B) \neq \emptyset\}$ = $= \{x: O \in \check{t}$ and $x \in O$ implies $(O \cap A) \cup (O \cap B) \neq \emptyset\} = \bar{c}(A) \cup \bar{c}(B)$. Let x be an element of $\bar{c}(\bar{c}(A))$ for some non-empty set A contained in X, and let O be a member of $\check{t}$ such that x is an element of O. Then $O \cap \bar{c}(A) \neq \emptyset$. Now suppose x is not an element of $\bar{c}(A)$. Then there is a Q, belonging to $\check{t}$, such that x is an element of Q and $A \cap Q = \emptyset$. Since x is an element of $\bar{c}(\bar{c}(A))$, there exists some element y of X such that y is an element of $(O \cap Q) \cap \bar{c}(A)$. Then y is an element of Q and $\bar{c}(A)$, which implies

$A \cap Q \neq \phi$. Hence $\bar{c}(\bar{c}(A)) = \bar{c}(A)$.

THEOREM 5.4. Let $(X, \check{t})$ be a semi-topological space and $\bar{c}$ be defined as above. Then $\check{t}$ is contained in $\check{t}(\bar{c}) = t(\bar{c})$.

PROOF. Let O be a member of $\check{t}$, and suppose x is not an element of X-O. Then x is an element of O. Now $O \cap X\text{-}O = \phi$; hence x is not an element of $\bar{c}(X\text{-}O)$. Thus $\bar{c}(X\text{-}O)$ is contained in X-O, so $\bar{c}(X\text{-}O) = X\text{-}O$ and O is a member of $\check{t}(\bar{c})$.

COROLLARY 5.5. Let (X,c) be a Čech closure space. Then $t(c) \subset \check{t}(c) \subset t(\bar{c})$.

COROLLARY 5.6. Let (X,c) be a Čech closure space and A be a subset of X. Then $A \subset \bar{c}(A) \subset c(A) \subset \overline{A}$.

CHAPTER VI

SEPARATION PROPERTIES

Several mild separation properties are useful in the context of Čech closure spaces.

DEFINITION 6.1. Let (X,c) be a Čech closure space. Then:

(1) (X,c) is called T_1 if for each $x \in X$, $c(x) = \{x\}$,

(2) (X,c) is called T_0 if for each $x,y \in X$, $x \neq y$, either $x \notin c(y)$ or $y \notin c(x)$,

(3) (X,c) is called R_1 if for each $x,y \in X$, $c(x) \cap c(y) = \emptyset$ or $c(x) = c(y)$,

(4) (X,c) is called symmetric (R_0) if for each $x,y \in X$, $x \in c(y)$ implies $y \in c(x)$,

(5) (X,c) is called T_Y if for each $x,y \in X$, $x \neq y$, $c(x) \cap c(y)$ equals either the empty set or a singleton set,

(6) (X,c) is called T_{YS} if for each $x,y \in X$, $x \neq y$, $c(x) \cap c(y)$ equals either $\emptyset$, $\{x\}$, or $\{y\}$,

(7) (X,c) is called T_F if for each $x \in X$ and disjoint set F, either $x \notin c(F)$ or $c(x) \cap F = \emptyset$, and

(8) (X,c) is called T_{FF} if for each pair of disjoint sets F and G, either $c(F) \cap G = \emptyset$ or $F \cap c(G) = \emptyset$.

A discussion of most of the above separation axioms in the context of topological spaces is found in Bonnett and Porter [2].

THEOREM 6.1. The following series of implications hold for Čech closure spaces:

(1) $T_1 \Rightarrow R_1 \Rightarrow R_0$,

(2) $T_1 \Rightarrow T_{YS} \Rightarrow T_Y \Rightarrow T_0$, and

(3) $T_1 \Rightarrow T_{FF} \Rightarrow T_F \Rightarrow T_0$.

THEOREM 6.2. Let (X,c) be a finitely generated symmetric Čech closure space. Then $0 \in t(c)$ if and only if $X-0 \in t(c)$.

PROOF. Let $0 \in t(c)$. Then $X-0 = c(X-0)$. Let $x \in c(0)$. Then there exists a $y \in 0$ such that $x \in c(y)$. Assume $x \in X-0$. This implies that $y \in c(x) \subset c(X-0) = X-0$, a contradiction. Hence $c(0) = 0$ and $X-0 \in t(c)$.

COROLLARY 6.3. In a finite symmetric topological space, a set is open if and only if it is closed.

THEOREM 6.4. Every finitely generated R_1 Čech closure space is a topological space.

PROOF. Let (X,c) be a finitely generated R_1 Čech closure space and A be a subset of X. Let $x \in c(c(A))$. There exists a $t \in c(A)$ such that $x \in c(t)$, and an $a \in A$ such that $t \in c(a)$. Since R_1 implies R_0, $t \in c(x)$, which implies $c(x) = c(a)$. Thus $x \in c(A)$ and $c(c(A)) = c(A)$.

THEOREM 6.5. Every finitely generated T_F Čech closure space is a topological space.

PROOF. Let (X,c) be a finitely generated Čech closure space, and suppose that (X,c) is not a topological space. Then there exist elements x_i, x_j, and x_k of X such that $x_i \in c(x_j)$, $x_j \in c(x_k)$, and $x_i \notin c(x_k)$. Now $x_j \notin \{x_i, x_k\}$; hence either $x_j \notin c(\{x_i, x_k\})$ or $c(x_j) \cap \{x_i, x_k\} = \phi$. But $x_j \in c(x_k)$ and $x_i \in c(x_j)$. Therefore (X,c) is a topological space.

COROLLARY 6.6. Every finitely generated T_{FF} Čech closure space is a topological space.

THEOREM 6.7. Let (X,c) be a finite Čech closure space with associated matrix A. The following pairs of statements are equivalent:

(A) (X,c) is R_0.

(A*) A is symmetric.

(B) (X,c) is R_1.

(B*) Two rows of A are either equal or disjoint.

(C) (X,c) is T_0.

(C*) A is anti-symmetric.

(D) (X,c) is T_1.

(D*) A = I.

(E) (X,c) is T_Y.

(E*) The intersection of two distinct rows of A is either empty or a singleton.

(F) (X,c) is T_{YS}.

(F*) The intersection of the i^{th} and j^{th} rows of A, where $i \neq j$, is either $\emptyset$, a_{ii}, or a_{jj}.

(G) (X,c) is T_F.

(G*) For each i, either the i^{th} row or the i^{th} column of A-I is zero.

(H) (X,c) is T_{FF}.

(H*) A-I or $(A-I)^T$ has at most one non-zero row.

PROOF. The proofs of A through F follow immediately from the definitions. The proofs of G and H follow from theorem 6.5 and theorems 3.2 and 3.5 in [2], respectively.

THEOREM 6.8. Let (X,c) be a finitely generated Čech closure space.

(1) If (X,c) is T_1 then t(c) is T_1.

(2) If (X,c) is T_F then t(c) is T_F.

(3) If (X,c) is T_{FF} then t(c) is T_{FF}.

(4) If (X,c) is R_1 then t(c) is R_1.

PROOF. All parts follow from the fact that the separation axioms stated assure that (X,c) is a topological space.

EXAMPLE 6.A. A finitely generated Čech closure space can be T_0 and the underlying topology may not be T_0. Let $X = \{x_1, x_2, x_3\}$ and let c be represented by $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Then by theorem 6.7, (X,c) is T_0, yet $t(c)$ is the trivial topology on X.

EXAMPLE 6.B. A finitely generated Čech closure space can be T_{YS} (and hence T_Y) and the underlying topology not be T_Y (and hence not T_{YS}). Let $X = \{x_1, x_2, x_3\}$ and let c be represented by $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Then (X,c) is T_{YS} by theorem 6.7, but $t(c)$ is the trivial topology on X.

CHAPTER VII

SUMMARY

An introduction to Čech closure spaces has been presented, and an attempt at discovering some fundamental properties of Čech closure spaces was made in this paper.

There is a one-to-one correspondence between the reflexive relations on a non-empty finite set X and the Čech closure operators on X; there is a one-to-one correspondence between the reflexive and transitive relations on X and the topologies on X [5].

The lattice of all finitely generated Čech closure operators on a non-empty set (and, consequently, the lattice of all Čech closure operators on a finite set) is a Boolean algebra. The relationship between the lattice of Čech closure operators and the lattice of topological closure operators on a fixed set is an area for further investigation.

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