UNIFORM STRUCTURES

A Thesis

Presented to

the Faculty of the Department of Mathematics

Kansas State Teachers College

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Saber N. El-Aydi

1974

Approved for the Major Department

Approved for the Graduate Council

ACKNOWLEDGMENTS

I am indebted to Dr. John Carlson for his guidance in the selection of the topic and in its preparation.

I would like also, to thank my wife and children for their loving patience.

Grateful thanks goes to Mrs. John Whitlock who typed the final format of this paper.

TABLE OF CONTENTS

Chapter Page

I. INTRODUCTION 1

II. FILTERS 3

III. UNIFORM STRUCTURES 12

A. Basic Concepts 12

B. Uniform Continuity 26

IV. UNIFORMIZATION PROBLEM 31

V. COMPLETENESS AND COMPLETION

OF UNIFORM SPACES 36

A. Completeness of uniform spaces 36

B. The Hausdorff uniform space

associated with a uniform space 46

C. Completion of uniform spaces 48

VI. SUGGESTIONS FOR FURTHER STUDY 53

BIBLIOGRAPHY 55

CHAPTER I

INTRODUCTION

The uniform space is one of the generalizations of a metric space. The uniform structure's concept lies between those of metric and topological structures, in the sense that every metric space is a uniform space and every uniform space is a topological space. The importance of uniform structures lies on the fact that uniform spaces preserve the main features of metric spaces. That is we can deal here with non-topological concepts of classical analysis such as completeness, total boundedness, Cauchy filters, uniform continuity...etc.

There are different approaches to uniform structures. One of them is defining uniform structures by means of filters, while another is defining uniform structures by a pseudometric or a family of pseudometrics. In 1940 TUKEY defined uniform structures by means of covers.

Now each element of a uniform structure is a neighborhood of the diagonal Δ . However not every neighborhood of Δ is a member of a uniform structure. On a set different uniform structures may be defined. But in a compact Hausdorff space the uniform structure is unique and it is the family of all neighborhoods of Δ . Two uniform structures may induce the same topology.

In chapter IV WEIL'S theorem is introduced. Now every pseudometric space is unformizable and every topological space admits a quasi-uniform structure. In chapter III it is shown that every open cover of a compact uniform space is a uniform cover.

In chapter V it is shown that the limit and the adherence of a Cauchy filter are equal in uniform spaces. If the uniform space is totally bounded, then every ultra filter is a Cauchy filter. In a quasi-uniform space the limit and the adherence of a Cauchy filter are equal if it is R_3. In a uniform space, the neighborhood filter is a minimal Cauchy filter, but this is not true in a quasi-uniform space. Now every compact uniform space is complete and every compact subspace of a complete Hausdorff uniform space is complete. In chapter V it is shown that a uniform space is compact if and only if it is complete and totally bounded. For every uniform space there is a Hausdorff uniform space associated with it. In chapter V it is shown that every uniform space has a completion. If the uniform space is pre-compact then its completion is compact.

CHAPTER II

FILTERS

In Euclidean space the following three results concerning sequences hold [10].

(1). A point $x \in X$ is a limit point of $A \subset X$ if and only if there exists a sequence of distinct points of A which converges to x.

(2). A point $x \in X$ is a cluster point (adherence point) of a sequence $\{x_n\}_1^\infty$ of elements of X if and only if there exists a subsequence $\{x_{nk}\}_1^\infty$ which converges to x.

(3). A function $f: X \rightarrow Y$ is continuous at $x \in X$ if and only if the sequence $\{x_n\}_1^\infty$ of elements of X converges to x implies the sequence $\{f(x_n)\}_1^\infty$ converges to f(x).

These results do not hold for topological spaces in general. They hold if the space is first countable.

The attempt to generalize sequences to more adequate concepts for topological spaces began in the beginning of this century. During the period from 1915 to 1940 two new concepts were devised; these were nets by E. H. Moore, H. L. Smith, J. L. Kelley, and others and filters by H. Cartan 1937, and others.

Since filters play an important role in the theory of uniform spaces, this chapter is devoted to a discussion of their basic properties.

DEFINITION 2.1 A filter $\mathcal{F}$ on a set X is a non-empty collection of subsets of X satisfying the following axioms:

F [1] $\emptyset \notin \mathcal{F}$

F [2] $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$,

F [3] $F_1 \subset F_2$, $F_1 \in \mathcal{F}$ implies $F_2 \in \mathcal{F}$.

DEFINITION 2.2 A non-empty collection $\mathcal{B}$ of subsets of a set X is called a filter base over X if it satisfies the following axioms:

B [1] $\emptyset \notin \mathcal{B}$

B [2] If $B_1, B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$.

DEFINITION 2.3 A non-empty collection $\mathcal{f}$ of subsets of X is called a filter subbase if it has the finite intersection property.

LEMMA 2.1 The family $\mathcal{F}$ which consists of all sets F such that $F \supset B$ for some $B \in \mathcal{B}$ where $\mathcal{B}$ is a filter base is a filter. $\mathcal{F}$ is said to be the filter generated by $\mathcal{B}$.

LEMMA 2.2 The family $\mathcal{B}$ which consists of all finite intersections of elements of a subbase filter $\mathcal{f}$ is a filter base.

EXAMPLE 2.1 Let X be a non-empty set, A a non-empty subset of X, then $\mathcal{F} = \{F \subset X \mid F \supset A\}$ is a filter on X.

EXAMPLE 2.2 In example 2.1, if A is a singleton set $\{x\}$ then $\mathcal{F}$ is called a discrete filter.

EXAMPLE 2.3 If A = X, then $\mathcal{F} = \{X\}$ and $\mathcal{F}$ is called the indiscrete filter on X.

EXAMPLE 2.4 Let $\{x_n\}_1^\infty$ be a given sequence, and set $F_i = \{x_i, x_{i+1}, \ldots\}$. Then $\mathcal{F} = \{F \subset X \mid F \supset F_i \text{ for some } i\}$ is the filter generated by the sequence $\{x_n\}_1^\infty$ and is called the elementary filter associated with the sequence.

EXAMPLE 2.5 Let X be a topological space, $x \in X$. Then the collection of all neighborhoods of x denoted by $\mathcal{N}_x$ is a filter on X called the neighborhood filter of x.

DEFINITION 2.4 A set $\mathcal{B} \subset \mathcal{N}_x$ is called a fundamental system of x if for every $N_x \in \mathcal{N}_x$ there exists $B \in \mathcal{B}$ such that $x \in B \subset N_x$.

EXAMPLE 2.6 Let (X,t) be a topological space. The fundamental system of neighborhoods of $x \in X$ is a filter base on X.

EXAMPLE 2.7 Let R denotes the real numbers and $x \in R$. Then $\{(x-\epsilon, x+\epsilon) \mid \epsilon > 0\}$ and $\{[x-\epsilon, x+\epsilon]\}$ are fundamental systems of neighborhoods of x and hence they are filter bases on R.

LEMMA 2.3 Let (X,t) be a topological space, $A \subset X$, $a \in \bar{A}$ and let $\mathcal{B}$ be the fundamental system of neighborhoods of a. Then $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$ is a filter base on A.

PROOF Let $a \in \bar{A}$. Since for each $B \in \mathcal{B}$, $B \cap A \neq \emptyset$ then $\emptyset \notin \mathcal{B}_A$ and axiom B[1] is satisfied.

Suppose $A_1, A_2 \in \mathcal{B}_A$. Then there exist $B_1, B_2 \in \mathcal{B}$ such that $A_1 = B_1 \cap A$, $A_2 = B_2 \cap A$. $A_1 \cap A_2 = (B_1 \cap A) \cap (B_2 \cap A) = (B_1 \cap B_2) \cap A \in \mathcal{B}_A$. Therefore axiom B[2] is satisfied and hence $\mathcal{B}_A$ is a filter base on A.

DEFINITION 2.4 A point $x \in X$ is called an adherence point or cluster point of a filter $\mathcal{F}$ on X, denoted by $x \in \text{adh}(\mathcal{F})$, if and only if $N_x \cap F \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $F \in \mathcal{F}$.

DEFINITION 2.5 A point $x \in X$ is called a limit point of a filter $\mathcal{F}$ on X, or $\mathcal{F}$ is said to converge to x if any only if $\mathcal{N}_x \subset \mathcal{F}$.

DEFINITION 2.6 A point $x \in X$ is called a limit point of a filter base $\mathcal{B}$ on X if the filter generated by $\mathcal{B}$ converges to x.

LEMMA 2.4 Let (X, t) be a topological space and let $\mathcal{F}$ be a filter on X, then $\text{adh}(\mathcal{F}) = \cap\{\overline{F} \mid F \in \mathcal{F}\}$.

PROOF (1) Let $x \in \text{adh}(\mathcal{F})$ then $N_x \cap F \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $F \in \mathcal{F}$. Hence $x \in \cap\{\overline{F} \mid F \in \mathcal{F}\}$.

(2) Let $x \in \cap\{\overline{F} \mid F \in \mathcal{F}\}$. Then $N_x \cap F \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $F \in \mathcal{F}$. Therefore $x \in \text{adh}(\mathcal{F})$.

THEOREM 2.1 Let (X, t) be a topological space, $x \in A$ and $A \subset X$, then $x \in \overline{A}$ if and only if there is a filter base on A which converges to x.

PROOF (1) Let $x \in \overline{A}$, then by lemma 2.3, $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}$ the fundamental system of $x\}$ is a filter base on A. For each $N_x \in \mathcal{N}_x$ there exists $B \in \mathcal{B}$ such that $B \subset N_x$. Since $B \cap A \subset B \subset N_x$, then $\mathcal{B}$ converges to x.

(2) Suppose that such a filter base $\mathcal{B}$ exists which converges to $x \in X$. Then for each $N_x \in \mathcal{N}_x$ there exists $B \in \mathcal{B}$ such that $B \subset N_x$. Since $B \subset A$ then $N_x \cap A \neq \emptyset$ for each $N_x \in \mathcal{N}_x$. This implies $x \in \overline{A}$.

This theorem is analogus to the corresponding result for sequences in Euclidean space.

THEOREM 2.2 Let (X, t) be a topological space, $x \in X$, then $x \in adh(\mathcal{F})$ if and only if there exists a filter $\mathcal{F}_x$ containing $\mathcal{F}$ which converges to x.

PROOF (1) Let $x \in adh(\mathcal{F})$. Now $F \cap N_x \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $F \in \mathcal{F}$. Set $\mathcal{B} = \{F \cap N_x \mid F \in \mathcal{F}$ and $N_x \in \mathcal{N}_x\}$. Thus $\emptyset \notin \mathcal{B}$ and hence axiom B[1] is satisfied. Let $B_1, B_2 \in \mathcal{B}$ then there exist $F_1, F_2 \in \mathcal{F}$ and $N_x^1, N_x^2 \in \mathcal{N}_x$ such that $B_1 = F_1 \cap N_x^1$, $B_2 = F_2 \cap N_x^2$. $B_1 \cap B_2 = (F_1 \cap N_x^1) \cap (F_2 \cap N_x^2) = (F_1 \cap F_2) \cap (N_x^1 \cap N_x^2)$. Since $\mathcal{F}$ and $\mathcal{N}_x$ are filters on X, then there exists $F \in \mathcal{F}$, $N_x \in \mathcal{N}_x$ such that $F \subset F_1 \cap F_2$, $N_x \subset N_x^1 \cap N_x^2$. Hence $F \cap N_x \in \mathcal{B}$ and $F \cap N_x \subset B_1 \cap B_2$. Thus axiom B[2] is satisfied and hence $\mathcal{B}$ is a filter base on X. Let $\mathcal{F}_x$ be the filter generated by $\mathcal{B}$. If $F \in \mathcal{F}$ and $N_x \in \mathcal{N}_x$, then $N_x \cap F \in \mathcal{B}$. Since $N_x \cap F \subset F$, then $F \in \mathcal{F}_x$. Hence $\mathcal{F}_x \subset \mathcal{F}$.

(2) Suppose that $\mathcal{F}$ is contained in a filter $\mathcal{F}_x$ which converges to x, then $N_x \in \mathcal{F}_x$ for each $N_x \in \mathcal{N}_x$ since $\mathcal{F} \subset \mathcal{F}_x$, then each $F \in \mathcal{F}$, $F \in \mathcal{F}_x$. Hence $N_x \cap F \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $F \in \mathcal{F}$. Thus $x \in adh(\mathcal{F})$.

This theorem is analogus to the corresponding result for sequences in Euclidean space.

DEFINITION 2.7 Let $\mathcal{F}$ be a filter on X and f a function from X into Y then $f(\mathcal{F}) = \{G \subset Y \mid G \supset f(F)$ for some $F \in \mathcal{F}\}$.

THEOREM 2.3 Let f: (X,t) (Y,s), then f is continuous if and only if $\mathcal{F}$ converges to x implies $f(\mathcal{F})$ converges to f(x).

PROOF (1) Let f be continuous then for each neighborhood

$N_{f(x)}$ of $f(x) = y \in Y$, there exists a neighborhood N_x of $x \in X$ such that $f(N_x) \subset N_{f(x)}$. If $\mathcal{F}$ converges to x, then $N_x \in \mathcal{F}$ and hence $N_{f(x)} \in f(\mathcal{F})$. Thus $f(\mathcal{F})$ converges to f(x).

(2) Assume that f is not continuous then if $x \in X$ there exists a neighborhood of f(x) such that no neighborhood N_x of x satisfies $f(N_x) \subset N_{f(x)}$. Hence $N_{f(x)} \notin f(\mathcal{F})$ and $f(\mathcal{F})$ does not converge to f(x). This contradicts the assumption of the theorem and hence f is continuous.

THEOREM 2.4 A topological space (X, t) is T_2 if and only if each filter on X converges to at most one point in X.

PROOF (1) Suppose (X, t) is T_2 and $\mathcal{F}$ a filter on X. If $\mathcal{F}$ converges to two distinct points $x, y \in X$. Then $\mathcal{N}_x \subset \mathcal{F}$ and $\mathcal{N}_y \subset \mathcal{F}$. Hence $N_x \cap N_y \neq \emptyset$ for each $N_x \in \mathcal{N}_x$ and $N_y \in \mathcal{N}_y$. This contradicts the assumption that X is T_2. Thus $\mathcal{F}$ does not converge to more than one point.

(2) Suppose X is not T_2. Then there exist two distinct points $x, y \in X$ such that for each $N_x \in \mathcal{N}_x$ and $N_y \in \mathcal{N}_y$, $N_x \cap N_y \neq \emptyset$. Set $\mathcal{B} = \{B \mid B = N_x \cap N_y$ for some $N_x \in \mathcal{N}_x$ and $N_y \in \mathcal{N}_y\}$. $\mathcal{B}$ is a filter base on X and the filter $\mathcal{G}$ generated by it contains both $\mathcal{N}_x$ and $\mathcal{N}_y$. Hence $\mathcal{G}$ converges to both x and y. This contradicts the assumption and hence X is T_2.

THEOREM 2.5 A topological space (X, t) is compact if and only if every filter on X has a non-empty adherence.

PROOF (1) Let (X, t) be a compact topological space and $\mathcal{F}$ a filter on X. By axiom F[2] , the class $\{F \mid F \in \mathcal{F}\}$

has the finite intersection property. Hence the class $\{\overline{F} \mid F \in \mathcal{F}\}$ has the finite intersection property. But since X is compact, then $\cap\{\overline{F} \mid F \in \mathcal{F}\} \neq \emptyset$.

By lemma 2.4, if $x \in \cap\{\overline{F} \mid F \in \mathcal{F}\}$, then $x \in$ adh $(\mathcal{F})$. Therefore adh $(\mathcal{F}) \neq \emptyset$.

(2) Let $\mathcal{S} = \{S_\alpha \mid \alpha \in \Lambda\}$ be a collection of closed sets with the finite intersection property. Then $\mathcal{S}$ is a subbase of a filter on X say $\mathcal{F}$. If $x \in$ adh $(\mathcal{F})$, then $x \in \bigcap_\alpha \{S_\alpha \mid S_\alpha \in \mathcal{S}\}$, otherwise there exists $S_j \in \mathcal{S}$ such that $x \notin S_j$. Thus $x \in S_j^c$. Since $S_j \in \mathcal{S}$ then $S_j \in \mathcal{F}$. Now S_j^c is a neighborhood of x and $S_j^c \cap S_j = \emptyset$ leads to a contradiction. Hence X is compact.

DEFINITION 2.8 A filter $\mathcal{F}$ on a set X is said to be an ultrafilter provided that no other filter on X properly contain $\mathcal{F}$.

EXAMPLE 2.8 Let $x \in X$, then the collection of subsets of X which contains x is an ultrafilter $\mathcal{F}$ on X.

PROOF By example 2.2, $\mathcal{F}$ is a filter on X. Suppose that there exists a filter $\mathcal{U}$ on X which properly contains $\mathcal{F}$. Therefore there exists $G \in \mathcal{U}$ such that $G \notin \mathcal{F}$. Since $\{x\} \in \mathcal{F}$ then $\{x\} \in \mathcal{U}$. This implies $G \cap \{x\} \neq \emptyset$; that is $x \in G$. Hence $G \in \mathcal{F}$ which is a contradiction. Thus $\mathcal{F}$ is an ultrafilter.

ZORN'S LEMMA If a non-empty partially ordered set X is such that every linearly ordered subset has an upper bound, then X contains a maximal element.

THEOREM 2.6 Let X be a non-empty set. Every filter on X is contained in an ultrafilter on X.

PROOF Let $\mathcal{F}$ be a filter on a set X. Let Γ be the class of all filters on X that contain $\mathcal{F}$. Γ is not empty since at least $\mathcal{F} \in \Gamma$. Partially order Γ by the inclusion relation $\subset$. Let ρ be a chain in Γ . Then $\mathcal{U} = \bigcup_{\alpha} \{ \mathcal{F}_\alpha \mid \mathcal{F}_\alpha \in \rho \}$ is an upper bound of ρ . Clearly $\mathcal{F}_\alpha \subset \mathcal{U}$ for each $\mathcal{F}_\alpha \in \rho$. Also axioms F[1] and F[3] can be easily verified for $\mathcal{U}$. Let $A, B \in \mathcal{U}$, then $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ for some $\mathcal{F}_i$, $\mathcal{F}_j \in \rho$. Either $\mathcal{F}_i \subset \mathcal{F}_j$ or $\mathcal{F}_j \subset \mathcal{F}_i$. In both cases A, B are elements of one of $\mathcal{F}_i$ and $\mathcal{F}_j$ and hence $A \cap B$ $\mathcal{U}$.

By Zorn's lemma Γ has a maximal element which by definition 2.8 is the ultra-filter containing $\mathcal{F}$.

THEOREM 2.7 Let X be a non-empty set. A filter $\mathcal{F}$ is an ultra-filter on X if and only if $A \cup B \in \mathcal{F}$, implies either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

PROOF (1) Assume that $\mathcal{F}$ is an ultra-filter on X. Suppose that $A \cup B \in \mathcal{F}$ such that $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$, where $A, B \subset X$. Set $\mathcal{U} = \{ Y \subset X \mid Y \cup B \in \mathcal{F} \}$, then $\mathcal{U}$ is not empty since $A \in \mathcal{U}$. It can be easily shown that $\mathcal{U}$ is a filter on X. If $F \in \mathcal{F}$, then $F \cup B \in \mathcal{F}$. Hence $F \in \mathcal{U}$ and thus $\mathcal{F} \subset \mathcal{U}$. Moreover $\mathcal{F}$ is a proper subset of $\mathcal{U}$ since $A \notin \mathcal{F}$ while $A \in \mathcal{U}$. This contradicts the assumption that $\mathcal{F}$ is an ultra-filter on X. Thus either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(2) Suppose that $\mathcal{F}$ is not an ultra-filter on X. Let $\mathcal{U}$ be an ultra-filter on X which properly contains $\mathcal{F}$. Then there exists $G \in \mathcal{U}$ such that $G \notin \mathcal{F}$. By the hypothesis of the theorem and since $X \in \mathcal{F}$, then $G^c \in \mathcal{F}$. Hence $G^c \in \mathcal{U}$ which is a contradiction. Hence $\mathcal{F}$ is an ultra-filter.

LEMMA 2.5 A filter $\mathcal{F}$ is an ultrafilter on a set X if and only if $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ implies $A \in \mathcal{F}$.

PROOF (1) Let $\mathcal{F}$ be an ultrafilter on X, then by theorem 2.7 either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. If $A^c \in \mathcal{F}$, one has a contradiction since $A \cap A^c = \emptyset$. Hence $A \in \mathcal{F}$.

(2) Suppose that $\mathcal{F}$ is not an ultrafilter. Then there exists an ultrafilter $\mathcal{G}$ which properly contains $\mathcal{F}$. Let $A \in \mathcal{G}$ such that $A \notin \mathcal{F}$. Now $A \cap G \neq \emptyset$ for all $G \in \mathcal{G}$ and thus $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Hence $A \in \mathcal{F}$ which is a contradiction.

LEMMA 2.6 If $\mathcal{F}$ is an ultrafilter on a set X, then for every $A \subset X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

This is a special case of theorem 2.7.

LEMMA 2.7 Let $\mathcal{F}$ be an ultrafilter on a set X, then $x \in \text{adh}\ (\mathcal{F})$ implies $x \in \lim\ (\mathcal{F})$.

PROOF Let $x \in \text{adh}\ (\mathcal{F})$. Suppose there exists $N_x \in \mathcal{N}_x$ such that $N_x \notin \mathcal{F}$. By lemma 2.6, $N_x^c \in \mathcal{F}$ and $N_x \cap N_x^c = \emptyset$ contradicts the assumption that $x \in \text{adh}\ (\mathcal{F})$. Hence $x \in \lim \mathcal{F}$.

THEOREM 2.8 A space is compact if and only if every ultrafilter on it converges.

This follows immediately from theorem 2.5 and lemma 2.7.

CHAPTER III

UNIFORM STRUCTURES

A. BASIC CONCEPTS

DEFINITION 3.1 Let X be a set and $U \subset X \times X$. Then $U^{-1} = \{(y, x) \mid (x, y) \in U\}$.

DEFINITION 3.2 Let X be a set and $U, V \subset X \times X$. Then $U \circ V = \{(x, y) \mid (x, z) \in U$ and $(z, y) \in V$ for some $z \in X\}$

DEFINITION 3.3 Let X be a set. A uniform structure for X is a non-empty collection $\mathcal{U}$ of subsets of X x X such that the following axioms are satisfied:

U [1] $\mathcal{U}$ is a filter on X x X,

U [2] $\Delta \subset U$, for every $U \in \mathcal{U}$, where $\Delta = \{(x, x) \mid x \in X\}$,

U [3] $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$,

U [4] For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The elements of $\mathcal{U}$ are called surroundings or uniformities.

DEFINITION 3.4 Let $(X, \mathcal{U})$ be a uniform space. $U \in \mathcal{U}$ is called symmetric if $U = U^{-1}$.

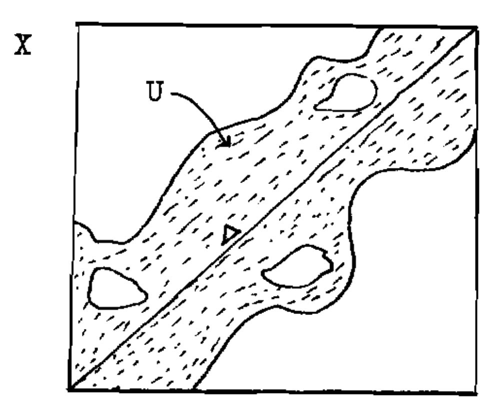


FIGURE 3.1

In Figure 3.1 a surrounding $U \subset X \times X$ surrounds the diagonal Δ .

LEMMA 3.1 Let $(X, \mathcal{U})$ be a uniform space. Then for each $U \in \mathcal{U}$, there exists a symmetric surrounding $V \in \mathcal{U}$, such that $V \circ V \subset U$.

PROOF Let $U \in \mathcal{U}$. By axiom U[4], there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$. Let $W = V \cap V^{-1}$. By axiom U[3], $V^{-1} \in \mathcal{U}$. $V \cap V^{-1} \in \mathcal{U}$ by axiom U[1]. Thus $W \in \mathcal{U}$. Now $W^{-1} = (V \cap V^{-1})^{-1} = V^{-1} \cap V = V \cap V^{-1} = W$. Therefore W is a symmetric surrounding. Since $W \subset V$, then $W \circ W \subset V \circ V \subset U$.

LEMMA 3.2 Let $(X, \mathcal{U})$ be a uniform space. If $U \in \mathcal{U}$, then $U \circ U^{-1}$ is a symmetric surrounding.

PROOF (1) Let $(x, y) \in U \circ U^{-1}$. Then $(x, \alpha) \in U$ and $(\alpha, y) \in U^{-1}$ for some $\alpha \in X$. Now $(y, \alpha) \in U$ and $(\alpha, x) \in U^{-1}$ implies $(y, x) \in U \circ U^{-1}$. Hence $(x, y) \in (U \circ U^{-1})^{-1}$.

(2) Let $(x, y) \in (U \circ U^{-1})^{-1}$. Then $(y, x) \in U \circ U^{-1}$. Hence $(y, \alpha) \in U$ and $(\alpha, x) \in U^{-1}$ for some $\alpha \in X$. $(x, \alpha) \in U$ and $(\alpha, y) \in U^{-1}$ implies $(x, y) \in U \circ U^{-1}$. Thus $U \circ U^{-1} = (U \circ U^{-1})^{-1}$.

LEMMA 3.3 Let X be a set. A non-empty collection $\mathcal{U}$ of subsets of $X \times X$ is a uniform structure on X if and only if it satisfies the following axioms:

U[1] $\mathcal{U}$ is a filter on $X \times X$,

U[2] $\Delta \subset U$ for every $U \in \mathcal{U}$

U′[3] For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, such that $V \circ V^{-1} \subset U$.

PROOF It suffices to prove that [U]′3 is equivalent to axioms U[3] and U[4]. If axioms U[3] and U[4] are satisfied, then for every $U \in \mathcal{U}$, we can choose a symmetric W, such that $W \circ W \subset U$ and hence $W \circ W^{-1} \subset U$. Thus axiom U[3] is satisfied.

Let U[3] be satisfied. Then given $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V^{-1} \subset U$. Let $(x, y) \in V$. Then since $(y, y) \in V^{-1}$, $(x, y) \in V \circ V^{-1}$. Thus $V \subset V \circ V^{-1} = (V \circ V^{-1})^{-1} \subset U^{-1}$. Since $V \in \mathcal{U}$, then by U[1], $U^{-1} \in \mathcal{U}$ and axiom U[3] is satisfied.

Let $W = V \cap V^{-1}$, then $W \subset V$ and $W \subset V^{-1}$. Therefore $W \circ W \subset V \circ V^{-1} \subset U$ and thus axiom U[4] is satisfied.

DEFINITION 3.5 A quasi-uniform structure is a non-empty collection $\mathcal{U}$ of subset of X x X which satisfies axioms U[1], U[2], and U[4].

Every uniform structure is a quasi-uniform structure, but the converse is not true in general.

EXAMPLE 3.1 Let X be a non-empty set and let $\mathcal{U} = \{X \times X\}$. Then $\mathcal{U}$ is a quasi-uniform structure and also a uniform structure. It is called the indiscrete uniform structure.

EXAMPLE 3.2 Let X be a non-empty set. Let $\mathcal{U}$ be all the subsets of X x X containing the diagonal Δ. Then $\mathcal{U}$ is a quasi-uniform structure and also a uniform structure. This is called the discrete uniform structure on X.

EXAMPLE 3.3 Let X be a non-empty set linearly ordered by the relation $\leq$. Set $V = \{(x, y) \mid x \leq y\}$. Then $\mathcal{U} = \{U \mid V \subset U \subset X \times X\}$ is a quasi-uniform structure for X, but it is not a uniform structure since axiom U[3] is not satisfied.

DEFINITION 3.6 A subfamily $\mathcal{U}_B$ of a uniform structure $\mathcal{U}$

on a set X is a base for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{U}_B$.

DEFINITION 3.7 A subfamily $\mathcal{U}_S$ of a uniform structure $\mathcal{U}$ is a subbase for $\mathcal{U}$ if the finite intersections of members of $\mathcal{U}_S$ is a base for $\mathcal{U}$.

THEOREM 3.1 A non-empty family $\mathcal{U}_B$ of subsets of X x X is a base for the uniform structure $\mathcal{U}$ on X if and only if the following axioms hold.

$U_B[1]$ $\mathcal{U}_B$ is a filter base,

$U_B[2]$ $\Delta \subset V$, for each $V \in \mathcal{U}_B$,

$U_B[3]$ If $V \in \mathcal{U}_B$, there exists $W \in \mathcal{U}_B$ such that $W \subset V^{-1}$,

$U_B[4]$ If $V \in \mathcal{U}_B$, there exists $W \in \mathcal{U}_B$ such that $W \circ W \subset V$.

PROOF (1) Let $\mathcal{U}_B$ be a base for the uniform structure $\mathcal{U}$ on a set X. Then by definition 3.6 $\mathcal{U}_B$ is a subfamily of $\mathcal{U}$, and $\mathcal{U} = \{U \mid U \supset V \text{ for some } V \in \mathcal{U}_B\}$. Axioms $U_B[1]$ and $U_B[2]$ can be easily verified. Let $V \in \mathcal{U}_B$, then $V \in \mathcal{U}$. Hence there exists $W \in \mathcal{U}$, such that $W \circ W \subset V$. Choose $U \in \mathcal{U}_B$, such that $U \subset W$. Then $U \circ U \subset W \circ W \subset V$ and hence axiom $U_B[4]$ is satisfied. Let $V \in \mathcal{U}_B$, then $V \in \mathcal{U}$. Choose $W \in \mathcal{U}$ such that $W \subset V^{-1}$. Now $W \supset V$ for some $U \in \mathcal{U}_B$ and therefore $U \subset V^{-1}$. Thus axiom $U_B[3]$ is satisfied.

(2) Let $\mathcal{U}_B$ be a non-empty family of subsets X x X which satisfies $U_B[1]$, $U_B[2]$, $U_B[3]$ and $U_B[4]$. Let $\mathcal{U} = \{U \mid U \supset V \text{ for some } V \in \mathcal{U}_B\}$. Then $\mathcal{U}$ satisfies axioms $U[1]$, $U[2]$ and $U[4]$. Let $U \in \mathcal{U}$, then $U \supset V$ for some $V \in \mathcal{U}_B$. By axiom $U_B[3]$, there exists $W \in \mathcal{U}_B$ such that $W \subset V^{-1}$. Now $U \supset V$ implies $U^{-1} \supset V^{-1} \supset W$ and hence $U^{-1} \in \mathcal{U}$. Thus axiom $U[3]$ is satisfied.

THEOREM 3.2 A family $\mathcal{U}_S$ of non-empty subsets of X x X is a subbase for a uniform structure $\mathcal{U}$ if and only if it satisfies the following axioms:

$U_S[1]$ $\Delta \subset S$ for each $S \in \mathcal{U}_S$,

$U_S[2]$ If $S \in \mathcal{U}_S$, then there exist sets $T_1, \ldots, T_m \in \mathcal{U}_S$ such that $T_1 \cap \ldots \cap T_m \subset S^{-1}$.

$U_S[3]$ If $S \in \mathcal{U}_S$, then there exist sets $T_1, \ldots, T_m \in \mathcal{U}_S$ such that $(T_1 \cap \ldots \cap T_m) \circ (T_1 \cap \ldots \cap T_m) \subset S$.

LEMMA 3.4 Let $(X, \mathcal{U})$ be a uniform space then,

(1) $(U^{-1})^{-1} = U$,

(2) If $U, V \in \mathcal{U}$ such that $U \subset V$, then $U^{-1} \subset V^{-1}$,

(3) If $U, V \in \mathcal{U}$, then $(U \cap V)^{-1} = U^{-1} \cap V^{-1}$,

(4) If $U, V \in \mathcal{U}$, then $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$.

LEMMA 3.5 Let $(X, \mathcal{U})$ be a uniform space. If $U_1, U_2 \in \mathcal{U}$, then $(U_1 \cap U_2)^n \subset U_1^n \cap U_2^n$.

PROOF Let $S = \{n \mid (U_1 \cap U_2)^n \subset U_1^n \cap U_2^n\}$. $\in$ S since $U_1 \cap U_2 \subset U_1 \cap U_2$. Suppose that $K \in S$. Let $(x, y) \in (U_1^k \cap U_2^k) \circ (U_1 \cap U_2)$ then $(x, \alpha) \in U_1^k \cap U_2^k$ and $(\alpha, y) \in U_1 \cap U_2$ for some $\alpha \in X$. Therefore $(x, y) \in U_1^{k+1} \cap U_2^{k+1}$. Hence $k+1 \in S$ and thus S is the set of all natural numbers.

THEOREM 3.3 Let $(X, \mathcal{U})$ be a uniform space. Then for each natural number n, $\mathcal{B}^n = \{V^n \mid V \in \mathcal{U}\}$ is a base for $\mathcal{U}$ on X.

PROOF (1) Since $\Delta \subset V$, then $\Delta \subset V \subset V^n$ and hence axiom $U_B[2]$ is satisfied.

(2) Let $V_1^n, V_2^n \in \mathcal{B}^n$ and let $V = V_1 \cap V_2$. By lemma 3.5 $V^n = (V_1 \cap V_2)^n \subset V_1^n \cap V_2^n$. Hence axiom $U_B[1]$ is satisfied.

(3) Let $V^n \in \mathcal{B}^n$. Choose $W \in \mathcal{U}$, such that $W \subset V^{-1}$. Hence $W^n \subset (V^{-1})^n = (V^n)^{-1}$.

(4) If $V^n \in \mathcal{B}^n$, then there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$. Thus $(W \circ W)^n = W^n \circ W^n \subset V^n$. Hence $\mathcal{B}^n$ is a filter base for a uniform structure $\mathcal{U}$ on X.

THEOREM 3.4 Let X be a set, $\mathcal{U}$ a quasi-uniform structure on X. Then $\mathcal{U}^* = \{U \cap V^{-1} | U, V \in \mathcal{U}\}$ is a base for a uniform structure on X.

PROOF (1) Since $\Delta \subset U$ and $\Delta \subset V^{-1}$, then $\Delta \subset U \cap V^{-1}$.

(2) Let $U_1 \cap V_1^{-1} \in \mathcal{U}^*$, $U_2 \cap V_2^{-1} \in \mathcal{U}^*$, then $(U_1 \cap V_1^{-1}) \cap (U_2 \cap V_2^{-1}) = (U_1 \cap U_2) \cap (V_1 \cap V_2)^{-1} \in \mathcal{U}^*$.

(3) Let $U \cap V^{-1} \in \mathcal{U}^*$, then $(U \cap V^{-1})^{-1} = U^{-1} \cap V = V \cap U^{-1} \in \mathcal{U}^*$.

(4) Let $U \cap V^{-1} \in \mathcal{U}^*$. Then there exists, $U_1, V_1 \in \mathcal{U}$ such that $U_1 \circ U_1 \subset U$, $V_1 \circ V_1 \subset V$. Hence $V_1^{-1} \circ V_1^{-1} \subset V^{-1}$. By lemma 3.5 $(U_1 \cap V_1^{-1}) \circ (U_1 \cap V_1^{-1}) \subset (U_1 \circ V_1) \cap (V_1^{-1} \circ V_1^{-1}) \subset U \cap V^{-1}$.

DEFINITION 3.8 Let $(X, \mathcal{U})$ be a uniform space. For each $x \in X$ set $U[x] = \{y | (x, y) \in U \text{ where } U \in \mathcal{U}\}$.

DEFINITION 3.9 Let $(X, \mathcal{U})$ be a uniform space. If $A \subset X$ then $U[A] = \{y | (x, y) \in U \text{ for some } x \in A, \text{ where } U \in \mathcal{U}\}$.

THEOREM 3.5 If $(X, \mathcal{U})$ is a uniform space, then the family of all subsets O of X such that for each $x \in O$ there is $U \in \mathcal{U}$ such that $U[x] \subset O$ is a topology on X called the uniform topology $t_{\mathcal{U}}$.

PROOF (1) Let $O_1, O_2 \in t_{\mathcal{U}}$, $x \in O_1 \cap O_2$. Then there exist $U, V \in \mathcal{U}$ such that $U[x] \subset O_1$ and $V[x] \subset O_2$. Therefore

$(U \cap V)[x] = U[x] \cap V[x] \subset O_1 \cap O_2$. Hence $O_1 \cap O_2 \in t_{\mathcal{U}}$.

(2) Let $\{O_i\}$ be a family of members of $t_{\mathcal{U}}$ and $x \in \bigcup_i \{O_i\}$. Then for each O_i, there exists $W_i \in \mathcal{U}$ such that $W_i[x] \subset O_i$. Hence $x \in \cup\{W_i[x]\} = (\cup W_i)[x] \subset \cup\{O_i\}$. Therefore $\cup\{O_i\} \in t_{\mathcal{U}}$ and $t_{\mathcal{U}}$ is a topology on X.

The topology $t_{\mathcal{U}}$ is precisely a generalization of the metric topology which is the family of all subsets O which contain a sphere about each of its points.

LEMMA 3.6 Let $(X, \mathcal{U})$ be a uniform space and $V \in \mathcal{U}$. If $y \in V[x]$, then $V[y] \subset V \circ V[x]$.

DEFINITION 3.10 Let X be a given set and let $\mathcal{N}_x$ be a collection of subsets of X for each $x \in X$. Then $\mathcal{N}_x = \cup\{N(x): x \in X\}$ is called a neighborhood system on X if it satisfies the following axioms:

N.1 For every $x \in X$, $\mathcal{N}_x \neq \emptyset$, and $x \in N_x$ for every $N_x \in \mathcal{N}_x$,

N.2 If $A \subset X$ and $A \supset N_x \in \mathcal{N}_x$, then $A \in \mathcal{N}_x$,

N.3 If $N_x^1 \in \mathcal{N}_x$, $N_x^2 \in \mathcal{N}_x$, then $N_x^1 \cap N_x^2 \in \mathcal{N}_x$,

N.4 If $N_x \in \mathcal{N}_x$, there exists $N_x^* \in \mathcal{N}_x$, $N_x^* \subset N_x$, such that for every $y \in N_x^*$, $N_x \in \mathcal{N}_y$.

THEOREM 3.6 Let $(X, \mathcal{U})$ be a uniform space, then the family $\mathcal{N} = \{U[x] \mid U \in \mathcal{U}, x \in X\}$ is a neighborhood system.

PROOF (1) Axiom N.1 is satisfied since $(x, x) \in U$ implies $x \in U[x]$.

(2) Let $N \supset U[x]$. Set $V = U \cup \{(x, y) \mid y \in N\}$. Since $U \subset V$, then $V \in \mathcal{U}$. If $a \in V[x]$, then $(x, a) \in V$. Hence $(x, a) \in U$ or $(x, a) \in \cup\{(x, y) \mid y \in N\}$. In both cases $a \in N$

and therefore $V[x] \subset N$. If $y \in N$, then $(x, y) \in V$ and hence $y \in V[x]$. Thus $N \subset V[x]$ and hence $N = V[x] \in \mathcal{N}(x)$.

(3) Let $U[x]$, $V[x] \in \mathcal{N}(x)$. Then $U[x] \cap V[x] = (U \cap V)[x] \in \mathcal{N}(x)$.

(4) Let $U[x] \in \mathcal{N}(x)$. Choose $V \in \mathcal{U}$, such that $V \circ V \subset U$. Hence $V[x] \subset U[x]$. If $y \in V[x]$, then by lemma 3.6 $V[y] \subset V \circ V[x] \subset U[x]$. Thus $U[x] \in \mathcal{N}(y)$.

The neighborhood system $\mathcal{N}$ induces the uniform topology $t_{\mathcal{U}}$ on X.

EXAMPLE 3.4 Let R be the set of real numbers. For each $\epsilon > 0$ define the set $V_\epsilon = \{(x, y) \mid |x-y| < \epsilon\}$. Then $\mathcal{U}_B = \{V_\epsilon \mid \epsilon > 0\}$ is a base for a uniform structure , called the additive or usual uniformity on R. The topology induced by $\mathcal{U}$ is the usual topology on R.

EXAMPLE 3.5 Let R be the set of real numbers. Then for each $a < b$ the sets $S_{ab} = \{(x, y) \mid \text{both } x, y < b \text{ or both } (x, y) > a\}$ is a subbase for a uniform structure which induces the usual topology on R.

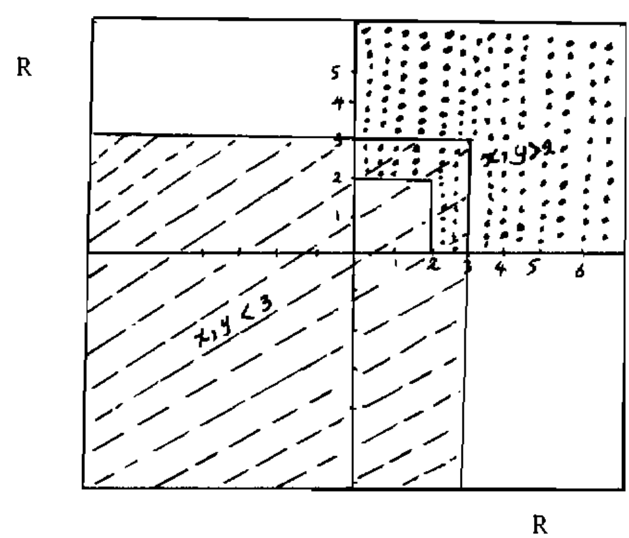


FIGURE 3.2

In figure 3.2 $S_{2,3} = \{(x, y)$ both $x, y > 2$ or both $x, y < 3\}$.

The uniform structures defined in examples 3.4 and 3.5 are different but they induce the same topology. For in example 3.4 where the uniform structure is the usual uniformity on R, there are surroundings such that $U[x_1] \cup \ldots \cup U[x_m] \neq R$ for all choices of finitely many points. While in example 3.5 for every surrounding U there are finitely many points $x_1, x_2, \ldots, x_m$ such that $U[x_1] \cup \ldots \cup U[x_m] = R$.

EXAMPLE 3.6 This is another example to show that different uniform structures may induce the same topology. Let $(N, \mathcal{U})$ be a uniform space, where N is the set of natural numbers and $\mathcal{U}$ is the discrete uniform structure defined in example 3.2. Then $\mathcal{U}$ induces the discrete topology on N.

Define $V_n = \{(x, y) \mid$ both $x, y > n$ or $x = y\}$ for $n \in N$. Set $\mathcal{V} = \{V \mid V \supset V_n$ for all $n \in N\}$. $\mathcal{V}$ is not the discrete uniform structure on N but it induces the discrete topology on N.

EXAMPLE 3.7 Let d_1 be the Euclidean metric on R, $d_2(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$. Then d_1, d_2 induce two different uniform structures $\mathcal{U}_1$, $\mathcal{U}_2$ while $\mathcal{U}_1$ and $\mathcal{U}_2$ induce the same topology on R. To show this let $U_\epsilon \in \mathcal{U}_2$, then $U_\epsilon = \{(x, y) \mid d_2(x, y) < \epsilon\}$. Since $d_2 \leq d_1$, then if $(x, y) \in V_\epsilon \in \mathcal{U}_1$, $d_1(x, y) < \epsilon$ implies $d_2(x, y) < \epsilon$ and hence $(x, y) \in U_\epsilon$. Therefore $V_\epsilon \subset U_\epsilon$ and hence $U_\epsilon \in \mathcal{U}_1$. Thus $\mathcal{U}_2 \subset \mathcal{U}_1$. To show that $\mathcal{U}_2$ is a proper subset of $\mathcal{U}_1$, let $U_1 = \{(x, y) \mid d_1(x, y) < 1\}$, then $U_1 \in \mathcal{U}_1$. Suppose

$x \geqslant o$, $y = 1 + x$, then $d_1(x, y) = 1$ and hence $(x, y) \notin U_1$. But $d_2(x, y) = \left| \frac{x}{1+x} - \frac{1+x}{2+x} \right| < 1$. Hence $U_1 \notin \mathcal{U}_2$ and $\mathcal{U}_2 \neq \mathcal{U}_1$.

The equivalence between d_1 and d_2 follows from the fact that $f: R \to (-1, 1)$ is a homeomorphism where $f(x) = \frac{x}{1+|x|}$.

Since $d_2(x, y) = d_1(f(x), f(y))$, then $d_1(x_n, x) \to o$ if and only if $d_2(x_n, x) \to o$.

EXAMPLE 3.8 On the set $\mathbf{Z}$ of integers define a uniform structure as follows: given a prime number p, let $V_n = \{(x, y) \in Z \times Z \mid x \equiv y \mod P^n\}$ for $n \in N$. Then the $\mathcal{U}_b = \{V_n \mid \text{for all } n \in N\}$ is a base for a uniform structure on Z, called the p-adic uniform structure.

THEOREM 3.7 Let $(X, \mathcal{U})$ be a uniform space, $A \subset X$. Then the interior of A relative to the topology is the set of all points x such that $U[x] \subset A$ for some $U \in \mathcal{U}$.

PROOF Let $B = \{x \mid U[x] \subset A \text{ for some } U \in \mathcal{U}\}$. Suppose that O is an open subset of A. Then $O = \{x \mid U[x] \subset O \text{ for some } U \in \mathcal{U}\}$. Clearly $O \subset B$. That is B contains every open subset of A. Let $x \in B$, then $U[x] \subset A$ for some $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$, such that $V \circ V \subset U$. Let $y \in V[x]$, then $V[y] \subset V \circ V[x] \subset U[x] \subset A$. Hence $y \in B$ and $V[x] \subset B$. Therefore B is an open subset of A and hence it is the interior of A.

THEOREM 3.8 Let $\mathcal{U}_B$ be a base for the uniform space $(X, \mathcal{U})$, $A \subset X$, then $\overline{A} = \cap \{V[A] \mid V \in \mathcal{U}_B\}$.

PROOF Let $x \in \overline{A}$. For each $V \in \mathcal{U}_B$, there exists a symmetric $W \in \mathcal{U}_B$, such that $W \circ W \subset V$. Now $W[x] \cap A \neq \emptyset$. Suppose $y \in W[x] \cap A$, then $y \in A$ and $y \in W[x]$. Hence

$x \in W[y] \subset V[y] \subset V[A]$. Thus $x \in \cap\{V[A] \mid V \in \mathcal{U}_B\}$. Let $x \in \cap\{V[A] \mid V \in \mathcal{U}_B\}$. Then for each $V \in \mathcal{U}_B$, there exists a symmetric $W \in \mathcal{U}_B$ with $W \subset V$. Since $x \in W[A]$, then for some $y \in A$, $(y, x) \in W$ and hence $(x, y) \in W$. Hence $y \in W[x] \subset V[x]$. Thus $V[x] \cap A \neq \emptyset$ for each $V \in \mathcal{U}_B$ which implies $x \in \bar{A}$. Therefore $\bar{A} = \cap\{V[A] \mid V \in \mathcal{U}_B\}$.

LEMMA 3.7 Let $(X, \mathcal{U})$ be a uniform space, $x \in X$ and let $U, V \in \mathcal{U}$. Then $U[x] \times V[x] \subset U^{-1} \circ V$.

LEMMA 3.8 Let $(X, \mathcal{U})$ be a uniform space, $U \in \mathcal{U}$, $V \in \mathcal{U}$ and V is symmetric. Then $V \circ U \circ V = \cup\{V[x] \times V[y] \mid (x, y) \in U\}$. Thus $V \circ U \circ V$ is a neighborhood of U in the product space $X \times X$.

PROOF (1) Let $(a, b) \in V \circ U \circ V$, then there exists $x, y \in X$, such that $(a, x) \in V$, $(x, y) \in U$ and $(y, b) \in V$. Since V is symmetric, then $a \in V[x]$ and $b \in V[y]$. Hence $(a, b) \in V[x] \times V[y]$, $(x, y) \in U$. Thus $V \circ U \circ V \subset \cup\{V[x] \times V[y] \mid (x, y) \in U\}$.

(2) Let $(a, b) \in \cup\{V[x] \times V[y] \mid (x, y) \in U\}$. Then $(a, b) \in V[x] \times V[y]$ for some $(x, y) \in U$. Hence $a \in V[x]$ which implies $(a, x) \in V$. $b \in V[y]$ implies $(y, b) \in V$. Thus $(a, b) \in V \circ U \circ V$. Therefore $V \circ U \circ V = \cup\{V[x] \times V[y] \mid (x, y) \in U\}$.

THEOREM 3.9 Let $(X, \mathcal{U})$ be a uniform space, and let $M \subset X \times X$. Then $\bar{M} = \cap\{V \circ M \circ V \mid V \in \mathcal{U}\}$.

PROOF Let $(x, y) \in \bar{M}$, then for each $V \in \mathcal{U}$, $V[x] \times V[y] \cap M \neq \emptyset$. For each $V \in \mathcal{U}$, there exists a symmetric $U \in \mathcal{U}$ such that $U \subset U \circ U \subset V$. Now $U[x] \times U[y] \cap M \neq \emptyset$ if and only if $(x, y) \in U[a] \times U[b] \subset V[a] \times V[b]$ for some $(a, b) \in M$, that is if and only if $(x, y) \in \cup\{V[a] \times V[b] \mid (a, b) \in M\}$. By lemma 3.8, it follows that $(x, y) \in \overline{M}$ if and only if $(x, y) \in \cap\{V \circ M \circ V \mid V \in \mathcal{U}\}$. Hence $\overline{M} = \cap\{V \circ M \circ V \mid V \in \mathcal{U}\}$.

LEMMA 3.9 The interiors of the surroundings in the uniform space $(X, \mathcal{U})$ form a base for the uniform structure $\mathcal{U}$; that is $\mathcal{U}$ has a base of open sets.

PROOF If $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$, such that $V \circ V \circ V \subset U$. By lemma 3.8, $V \circ V \circ V$ is a neighborhood of V. Therefore the interior of U contains V and hence int $(U) \in \mathcal{U}$. Thus the set of interiors of $U \in \mathcal{U}$ form a base of $\mathcal{U}$.

LEMMA 3.10 The closures of the surroundings in the uniform space $(X, \mathcal{U})$ form a base for the uniform structure $\mathcal{U}$.

PROOF Let $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$, such that $V \circ V \circ V \subset U$. By theorem 3.10 $\overline{V} \subset V \circ V \circ V \subset U$. Thus by definition 3.6 the lemma is established.

THEOREM 3.11 Every uniform structure has a base of symmetric surroundings.

PROOF If $U \in \mathcal{U}$, then by lemma 3.2 $V = U \cap U^{-1}$ is symmetric. Since $V \subset U$, the theorem is established.

LEMMA 3.11 Let $(X, \mathcal{U})$ be a uniform space, then for each $U \in \mathcal{U}$, $(\overline{U})^{-1} = \overline{(U^{-1})}$.

THEOREM 3.12 Every uniform structure $\mathcal{U}$ has a base of symmetric closed surroundings.

PROOF Let $U \in \mathcal{U}$, then by theorem 3.3, there exists $V \in \mathcal{U}$, such that $V \circ V \circ V \subset U$. By theorem 3.10 $\overline{V} = \cap\{W \circ V \circ W \mid$

$W \in \mathcal{U} \}$. Thus $\overline{V} \subset V \circ V \circ V$ and hence $\overline{V} \subset U$. By lemma 3.11 $(\overline{V})^{-1} = \overline{(V^{-1})}$. Hence $(\overline{V})^{-1}$ is a closed surrounding. Therefore $\overline{V} \cap \overline{V}^{-1}$ is a closed symmetric surrounding and contained in U. Hence the theorem is established.

LEMMA 3.10 Let $(X, \mathcal{U})$ be a uniform space, then each $U \in \mathcal{U}$ is a neighborhood of Δ. (However, not every neighborhood of Δ is necessarily an element of $\mathcal{U}$.)

PROOF (1) For each $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. Then for each $x \in X$ $V[x] \times V[x] \subset V \circ V \subset U$. Hence $\cup \{V[x] \times V[x] \mid x \in X\} \subset U$. But $\cup \{V[x] \times V[x] \mid x \in X\}$ is an open set in the product topology on $X \times X$ which contains Δ. Hence U is a neighborhood of Δ.

(2) Let $(R, \mathcal{U})$ be a uniform space where R is the set of real numbers, and $\mathcal{U}$ is the usual uniformity on R. The set $\{(x, y) \mid |x - y| < 1/(1 + |y|)\}$ is a neighborhood of Δ but not a member of $\mathcal{U}$.

THEOREM 3.14 Let $(X, \mathcal{U})$ be a uniform space, then the following are equivalent.

(1) X is T_2- space (2) $\cap \{U \mid U \in \mathcal{U}\} = \Delta$.

PROOF (1) Let X be T_2- space and assume that there exist two points $x, y \in X$, $x \neq y$ such that $(x, y) \in \cap \{U \mid U \in \mathcal{U}\}$. For each $U \in \mathcal{U}$, chose a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. Then $(x, y) \in V$. Hence $x \in V[y] \subset U[y]$ and $y \in V[x] \subset U[x]$. Thus $U[x] \cap W[y] \neq \emptyset$ for each U and $W \in \mathcal{U}$. Hence X is not T_2 which is a contradiction. Therefore, $\cap \{U \mid U \in \mathcal{U}\} = \Delta$.

(2) Assume $\cap \{U \mid U \in \mathcal{U}\} = \Delta$. If $x, y \in X$ and $x \neq y$,

then there exists $U \in \mathcal{U}$ such that $(x, y) \notin U$. This implies $y \notin U[x]$. Also $(y, x) \notin V$ for some $V \in \mathcal{U}$ and hence $x \notin V[y]$. Thus X is a T_2- space.

THEOREM 3.15 Let $(X, \mathcal{U})$ be a uniform space, $A \subset X$ such that $U[A] = A$ for some $U \in \mathcal{U}$, then A is both open and closed.

PROOF (1) Let $x \in \bar{A}$. Since $\bar{A} = \cap\{U[A] \mid U \in \mathcal{U}\}$, then $x \in U[A]$ which implies $x \in A$. Hence A is closed.

(2) Let $x \in A$, then $U[x] \subset A$ and A is open.

COROLLARY 3.1 Let $(X, \mathcal{U})$ be a uniform space, then for any $A \subset X$ and $U \in \mathcal{U}$, $\cup\{U^n[A] \mid n = 1, 2, \ldots\}$ is both open and closed.

PROOF Let $U[A] \cup U^2[A] \cup \ldots$ be such set. Then $U \circ [U[A] \cup U^2[A] \cup \ldots\} = U^2[A] \cup U^3[A] \cup \ldots \subset U[A] \cup U^2[A] \ldots$. Thus, by theorem 3.15, this set is both open and closed.

DEFINITION 3.11 Let $(X, \mathcal{U})$ be a uniform space. A uniform neighborhood of $A \subset X$ is a set which includes $U[A]$ for some $U \in \mathcal{U}$.

LEMMA 3.11 Every uniform neighborhood of A is a neighborhood of A.

THEOREM 3.16 Every neighborhood of a compact set is a uniform neighborhood.

PROOF Let N be a neighborhood of a compact set K. For each $x \in K$, choose $U_x \in \mathcal{U}$ such that $U_x \circ U_x [x] \subset N$. Since K is compact and is covered by $\{U_x [x] \mid x \in K\}$, then it has a finite subcover $\{U_{xi} [x_i] \mid i = 1, 2, \ldots, n\}$. $K \subset \bar{K} = \cap\{U [K] \mid U \in \mathcal{U}\} \subset W[K] \subset N$. Hence N is a uniform neighborhood of K.

EXAMPLE 3.9 The interval (o, 1) is not a uniform neighborhood of its subset of rational points with the usual uniform structure. Let $A = \{x \mid x \in Q, x \in (o, 1)\}$ where Q is the set of rationals. Assume that (o, 1) is a uniform neighborhood of A, then there exists $U_\epsilon = \{(x, y) \mid |x - y| < \epsilon\} (\epsilon > o)$ such that $A \subset U_\epsilon [A] \subset (o, 1)$. But this is impossible. Therefore (o, 1) is not a uniform neighborhood of its subset of rational points.

B. UNIFORM CONTINUITY

DEFINITION 3.12 Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces, then the function $f: X \to Y$ is called uniformly continuous if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $(a, b) \in U$ implies $(f(a), F(b)) \in V$.

DEFINITION 3.13 Let $f: X \to Y$, then define the function f x f denoted by f_2 from $X \times X \to Y \times Y$ by $f_2(a, b) = (f(a), f(b))$.

LEMMA 3.12 Let $f_2: X \times X \to Y \times Y$, then f is uniformly continuous if and only if $f_2^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are uniform structures on X and Y respectively.

EXAMPLE 3.10 The function $f: R \to R$ defined by $f(x) = x^3$, where R is the set of real numbers, is a homeomorphism of R onto itself which is not uniformly continuous with respect to the usual uniform structure.

LEMMA 3.13 If the function $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous, then f is continuous in the induced topologies.

EXAMPLE 3.11 The identity mapping of a uniform space onto itself is uniformly continuous.

EXAMPLE 3.12 Every mapping of a discrete uniform space into a uniform space is uniformly continuous.

DEFINITION 3.14 If $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniform structures on a set X, then $\mathcal{U}_1$ is said to be finer than $\mathcal{U}_2$ if $\mathcal{U}_1 \supset \mathcal{U}_2$. (or $\mathcal{U}_2$ is said to be coarser than $\mathcal{U}_1$).

THEOREM 3.17 Let $\mathcal{U}_1$, $\mathcal{U}_2$ be two uniform structures on X. Then $\mathcal{U}_1$ is finer than $\mathcal{U}_2$ if and only if the identity mapping i: $(X, \mathcal{U}_1) \rightarrow (X, \mathcal{U}_2)$ is uniformly continuous.

PROOF (1) Let $\mathcal{U}_1 \supset \mathcal{U}_2$ and consider i: $(X, \mathcal{U}_1) \longrightarrow (X, \mathcal{U}_2)$. Given $V \in \mathcal{U}_2$, then $i^{-1}(V) = V \in \mathcal{U}_1$ and hence i is uniformly continuous.

(2) Let i: $(X, \mathcal{U}_1) \rightarrow (X, \mathcal{U}_2)$ be uniformly continuous, then given $V \in \mathcal{U}_2$, $i^{-1}(V) = V \in \mathcal{U}_1$. Hence $\mathcal{U}_1 \supset \mathcal{U}_2$.

THEOREM 3.18 Let $\mathcal{U}_1$, $\mathcal{U}_2$ be two uniform structures on X. Then $\mathcal{U}_1$ is stronger than $\mathcal{U}_2$ if and only if the identity mapping i: $(X, \mathcal{U}_1) \rightarrow (X, \mathcal{U}_2)$ is continuous.

DEFINITION 3.15 A cover $\mathcal{C} = \{C_i\}$ of a uniform space $(X, \mathcal{U})$ is called a uniform cover, if there exists $U \in \mathcal{U}$, such that for each $x \in X$ $U[x] \subset C_i$ for some $C_i \in \mathcal{C}$. That is $\{U[x] \mid x \in X\}$ refines $\mathcal{C}$.

THEOREM 3.19 Let f: $(X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$. For each $V \in \mathcal{V}$ set $\mathcal{C}_V = \{f^{-1}(V[y]) \mid y \in Y\}$. Then f is uniformly continuous if and only if $\mathcal{C}_V$ is a uniform cover of X for each $V \in \mathcal{V}$.

PROOF (1) Suppose that f is uniformly continuous and $V \in \mathcal{V}$. If $y = f(x) \in Y$, then there is $U \in \mathcal{U}$, such that

$f(U[x]) \subset V[y]$ and so that $U[x] \subset f^{-1}(V[y])$. Hence $\mathcal{C}_V$ is a uniform cover of X.

(2) Conversely if $V \in \mathcal{V}$, then there exists a symmetric $W \in \mathcal{V}$ such that $W \circ W \subset V$. By hypothesis $\mathcal{C}_W$ is a uniform cover of X. Hence there exists $U \in \mathcal{U}$, such that $U[x]$ is included in some member of $\mathcal{C}_W$ for each $x \in X$. Let $(x_1, x_2) \in U$, then there exists $y \in Y$ with $f^{-1}(W[y]) \supset U[x_1]$. Hence $(f(x_1), f(x_2)) \in W[y] \times W[y] \subset W \circ W \subset V$ and therefore f is uniformly continuous.

THEOREM 3.20 Every open cover $\mathcal{C}$ of a compact uniform space $(X, \mathcal{U})$ is a uniform cover.

PROOF Let $\mathcal{C} = \{C_i\}$ be an open cover of X, then for each $x_i \in X$, choose $C_i \in \mathcal{C}$ with $x_i \in C_i$ and a surrounding V_i such that $V_i[x_i] \subset C_i$. For each $V_i \in \mathcal{U}$, choose a symmetric $U_i \in \mathcal{U}$ with $U_i \circ U_i \subset V_i$. The class $\{U_i[x_i] \mid x_i \in X\}$ is an open cover of X. Since X is compact there is a finite subcover $\{U_{ik}[x_{ik}] \mid k = 1, 2, \ldots, n\}$. Set $U = \bigcap_{k=1}^{n} U_{ik}$. Then $x \in X$ implies $x \in U_{ik}[x_{ik}]$ and hence $U[x] \subset U \circ U_{ik}[x_{ik}] \subset U_{ik} \circ U_{ik}[x_{ik}] \subset V_{ik}[x_{ik}] \subset C_{ik} \in \mathcal{C}$. Thus $\mathcal{C}$ is a uniform cover.

THEOREM 3.21 Every continuous function f from a compact uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{V})$ is uniformly continuous.

PROOF Given $V \in \mathcal{V}$, then for each $x \in X$, $f^{-1}(V[f(x)])$ is a neighborhood of x, since f is continuous and $V[f(x)]$ is a neighborhood of $f(x)$. Set $\mathcal{C}_V = \{f^{-1}(V[f(x)]) \mid x \in X\}$.

Then $\mathcal{T}_V$ is an open cover of X. Since X is compact then by theorem 3.20, $\mathcal{T}_V$ is a uniform cover. Hence by theorem 3.19 f is uniformly continuous.

THEOREM 3.22 A compact topological space has atmost one uniform structure which is compatible with it.

PROOF Let (X, t) be a topological space. Assume there exist two uniform structures $\mathcal{U}_1$, $\mathcal{U}_2$ on X compatible with t. Define the identity function i: $(X, \mathcal{U}_1) \to (X, \mathcal{U}_2)$. The family $\mathcal{T}_U = \{U[x_1] \mid x_1 \in X,\ U \in \mathcal{U}_2\}$ is an open cover of X. Also $\{i^{-1} U[x_1] \mid x_1 \in X\} = \{U[x_1]\}$ is an open cover of X. Then by theorems 3.20 and 3.19 i is uniformly continuous and hence it is continuous. Therefore by theorem 3.18 $\mathcal{U}_1$ is stronger than $\mathcal{U}_2$ which is a contradiction. Hence $\mathcal{U}_1 = \mathcal{U}_2$.

EXAMPLE 3.13 Consider the interval (o, ∞) with the usual uniformity. Then the collection $\mathcal{T} = \{(x - \frac{1}{x}, x] \mid x > 1\}$ is a cover for (o, ∞) but it is not a uniform cover.

EXAMPLE 3.14 Consider the interval (o, ∞) with the usual uniformity. Then the collection $\mathcal{T} = \{(x, \frac{1}{x}) \mid o < x < 1\}$ is a uniform cover.

DEFINITION 3.16 A pseudometric on a set X is a function d on X x X into R, the set of real numbers, satisfying for all x, y, z $\in$ X:

(1) $d(x, y) \geq o$ (2) $d(x, x) = o$

(3) $d(x, y) = d(y, x)$ (4) $d(x, z) \leq d(x, y) + d(y, z)$

LEMMA 3.14 Let d be a pseudometric, then for each positive number r, let $U_{d,r} = \{(x, y) \mid d(x, y) < r\}$. Then the family $\mathcal{U}_B = \{ U_{d,r} \mid r \in R^+ \}$ is a base for a uniform structure on X.

LEMMA 3.15 A non-empty family D of pseudometrics on X forms a uniform structure on X if it satisfies the following axioms:

(1) If $d_1, d_2 \in D$, then $d_1 \vee d_2 \in D$ where $d_1 \vee d_2 = \sup (d_1, d_2)$.

(2) If e is a pseudometric, and if for every $\epsilon > 0$, there exists $d \in D$ and $\delta > 0$, such that $d(x, y) \leq \delta$ implies $e(x, y) \leq \epsilon$ for all $x, y \in X$, then $e \in D$.

LEMMA 3.16 Let Φ be a çollection of covers of a set X such that (1) If $\mathcal{A}$, $\mathcal{B}$ are members of Φ, then there is a member of Φ which is a refinement of both $\mathcal{A}$ and $\mathcal{B}$, (2) if $\mathcal{A} \in \Phi$, then there is a member of Φ which is a star refinement of $\mathcal{A}$; and (3) if $\mathcal{B}$ is a cover of X and some refinement of $\mathcal{B}$ belongs to Φ, then $\mathcal{B} \in \Phi$. Then the family $\mathcal{U}_B = \{ U_{\mathcal{A}} \mid \mathcal{A} \in \Phi \}$ where $U_{\mathcal{A}} = \cup \{ G \times G \mid G \in \mathcal{A} \}$ is a base for a uniform structure on X. Φ is precisely the family of all uniform covers of X.

CHAPTER IV

UNIFORMIZATION PROBLEM

DEFINITION 4.1 A topological space (X, t) is said to be uniformizable if there is a uniform structure $\mathcal{U}$ on X compatible with it.

LEMMA 4.1 Let $(X, \mathcal{U})$ be a uniform space, $A \subset X$, then $\mathcal{U}_A = \{U \cap A \times A \mid U \in \mathcal{U}\}$ is a uniform structure on A called the relative uniform structure on A. $(A, \mathcal{U}_A)$ is called a subspace of $(X, \mathcal{U})$.

THEOREM 4.1 Every subspace A of a uniform space $(X, \mathcal{U})$ is uniformizable.

DEFINITION 4.2 Let (X, t) be a topological space, then it is called quasi-uniformizable if there is a quasi-uniform structure on X compatible with it.

THEOREM 4.2 Every topological space is quasi-uniformizable.

PROOF Let (X, t) be a topological space. Set $\mathcal{U}_S = \{0 \times 0 \cup 0^c \times X \mid 0 \in t\}$. Claim that $\mathcal{U}_S$ is a subbase for a quasi-uniform structure on X. To show this we need to verify axioms $U_S[1]$ and $U_S[3]$. Axiom $U_S[1]$ is satisfied since $\Delta \subset S$ for each $S \in \mathcal{U}_S$. Let $(x, y) \in S \circ S$, where $S \in \mathcal{U}_S$, then $(x, z) \in S$, $(z, y) \in S$ for some $z \in X$. Since $S = 0 \times 0 \cup 0^c \times X$, then there are two possibilities: (1) If $x \in 0$, then $y, z \in 0$ and hence $(x, y) \in 0 \times 0 \subset S$. Thus $S \circ S \subset S$.

(2) If $x \in 0^c$, since $y \in X$, then $(x, y) \in 0^c \times X \subset S$. Thus

$S \circ S \subset S$. Hence axiom $U_S[3]$ is satisfied.

The quasi-uniform structure $\mathcal{P}$ induced by $\mathcal{U}_S$ is called the Pervin-quasi uniform structure.

THEOREM 4.3 Every pseudometric space is uniformizable.

PROOF Let d be a pseudometric on a set X, t is the topology induced by d. Consider the sets $V_\epsilon = \{(x, y) \mid d(x, y) < \epsilon\}$ where $\epsilon > o$, then the family $\mathcal{U}_B$ of all sets V_ϵ is a base for a uniform structure $\mathcal{U}$ on X. Denote by $t_{\mathcal{U}}$ the uniform topology associated with $\mathcal{U}$. Then a set $O \subset X$ is open in $t_{\mathcal{U}}$ if and only if for each $x \in O$, there exists $U \in \mathcal{U}$ such that $x \in U[x] \subset O$. But $U \in \mathcal{U}$ if and only if $U \supset V_\epsilon$ for some $\epsilon > o$. Now $O \in t_{\mathcal{U}}$ if and only if $O \in t$ and thus $t_{\mathcal{U}} = t$.

DEFINITION 4.3 A topological space (X, t) is completely regular if for any closed set F and for any $x \notin F$, there exists a continuous function $f: X \to [o, 1]$ such that $f(x) = o$ and $f(F) = 1$.

DEFINITION 4.4 A diadic scale of open sets is a family $\{O_d\}$ where $d \in D$ the set of diadio rationals $d = \frac{m}{2^n}$, where $m = o, 1, \ldots, 2^n$, $n = 1, 2, 3, \ldots$, and $\overline{O_{d_i}} \subset O_{d_j}$ for each $d_i < d_j$.

LEMMA 4.2 A topological space (X, t) is completely regular if and only if for any closed set F and $x \notin F$, there exists a diadic scale of open sets $\{O_d\}$ $d \in D$ such that $x \in O_d \subset F^c$ for each $d \in D$.

PROOF (1) Let (X, t) be completely regular. Suppose that F is a closed subset of X and $x \notin F$. Then by definition

4.3 there exists a continuous function $f: X \to [o, 1]$ with $f(x) = o$ and $f(F) = 1$. Since f is continuous, then $f^{-1}[o, d_1) = O_{d1}$ is open in X, and contains x. If $d_1 < d_j$, then $[o, d_1) \subset [o, d_j]$. Hence $f^{-1}[o, d_1) \subset f^{-1}[o, d_j)$ which implies $O_{d1} \subset O_{dj}$. Let $a \in \overline{O_{d1}}$. Assume that $a \notin O_{dj}$, then $f(a) \geqslant d_j$. Let $d_1 < k < d_j$, then $f(a) \in (k, 1)$ which is open in $[o, 1]$. Hence $a \in f^{-1}(k, 1) = G$ where G is open in X. Hence $G \cap O_{d1} \neq \emptyset$. If $b \in G \cap O_{d1}$, then $f(b) \in (k, 1)$ and $f(b) \in (o, d_1)$ which contradicts our assumption. Thus $a \in O_{dj}$ and hence $\overline{O_{d1}} \subset O_{dj}$. Therefore $\{O_d\}$ is a diadic scale of open sets. Claim that $O_d \subset F^c$ for each $d \in D$, otherwise let $y \in O_d \cap F$. Since $y \in F$, then $f(y) = 1$. $y \in O_d$, then $f(y) = [o, d)$ and hence $f(y) = 1$ which is a contradiction. Therefore $O_d \subset F$.

(2) Define a function f such that $f(x) = o$ if $x \in O_d$ for each $d \in D$, and $f(x) = \sup\{d \mid x \notin O_d\}$ where $f: X \to [o, 1]$. Since $x \in O_d \subset F^c$ for each $d \in D$, then $f(x) = o$. $f(F) = \sup\{m/2^n \mid m = o, 1, \ldots, 2^n, n = 1, 2, \ldots\} = 1$. To show that f is continuous, consider the two kinds of open sets $[o, a)$ and $(b, 1]$. $f^{-1}[o, a) = \{x \mid f(x) < a\} = \bigcup_{d<a} \{O_d\}$ which is an open set in X, $f^{-1}(b, 1] = \{x \mid f(x) > b\} = \bigcup_{d>b} \{\overline{O}_d^c\}$ which is open in X. Hence f is a continuous function.

THEOREM 4.4 WEIL'S THEOREM

A topological space (X, t) is uniformizable if and only if it is completely regular.

PROOF (1) Suppose that (X, t) is uniformizable. Let $x \in O$ where $O \in t$. Hence there exists a uniform structure

$\mathcal{U}$ on X which induces t, with $U_1[x] \subset O$ for some $U \in \mathcal{U}$.
Select a sequence $\{U_{2^{-n}}\}$ $(n = 0, 1, \ldots)$ of symmetric surroundings such that $U_{2^{-(n+1)}} \circ U_{2^{-(n+1)}} \subset U_{2^{-n}}$ for every $n \in N$.
Define $U_d = U_{2^{-n_1}} \circ \cdots \circ U_{2^{-n_k}}$ for each positive diadic rational $d = 2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_k}$. Clearly $U_{d_1} \circ U_{d_1} \subset U_{d_1 + d_2}$ for each d_1, $d_2 \in D$ satisfying $d_1 + d_2 < 1$. If $0 < d_1 < d_2$ then $U_{d_1} \circ U_{d_2 - d_1} \subset U_{d_2}$ and hence $\overline{U_{d_1}[x]} \subset \text{int}(U_{d_2}[x])$. Define $O_d = \text{int}(U_d[x])$. Then $\overline{O_{d_1}} \subset O_{d_2}$.
Since $U_1[x] \subset O$, then $x \in O_d \subset O$ for each $d < 1$. Hence $\{O_d\}$ is a scale of open sets satisfying the condition of lemma 4.2 and thus (X, t) is completely regular.

(2) Suppose (X, t) is completely regular. Let C (X) be the collection of continuous real-valued functions on X. Define a set $S_{f,e} = \{(x, y) \in X \times X \mid |f(x) - f(y)| < e\}$ $(e > 0)$, where $f \in C(X)$. Then the family $\mathcal{U}_S$ of the sets $S_{f,e}$ is a subbase for a uniform structure $\mathcal{U}$ on X. To show this it suffices to verify axiom U_S [3], since clearly $\Delta \subset S_{f,e}$ and $S_{f,e}$ is symmetric. If $S_{f,e} \in \mathcal{U}_S$, let $\delta = \frac{e}{2}$, then $S_{f,\delta} \circ S_{f,\delta} \subset S_{f,e}$. Let $\mathcal{U}_B$ be the base generated by $\mathcal{U}_S$. Suppose F is a closed set in t, and $x \notin F$. Since X is completely regular there exists a $f \in C(X)$ with $f(F) = 1$ and $f(x) = 0$. If $Z \in S_{f,\frac{1}{2}}[x]$, then $|f(x) - f(Z)| < \frac{1}{2}$. Hence $|f(Z)| < \frac{1}{2}$ and thus $Z \notin F$. It follows that $S_{f,\frac{1}{2}}[x] \cap F = \emptyset$ and so F is closed in the uniform topology $t_{\mathcal{U}}$. Thus $t \subset t_{\mathcal{U}}$. Let $V \in \mathcal{U}_B$, then $V = S_{f_1,e_1} \cap \cdots \cap S_{f_n,e_n}$ where $S_{f_i,e_i} \in \mathcal{U}_S$. Hence

$V[x] = S_{f_1, e_1}[x] \cap \cdots \cap S_{f_n, e_n}[x]$.

$S_{f_k, e_k}[x] = \{y \mid |f_k(x) - f_k(y)| < e_k\} = f_k^{-1}(f_k(x) - e_k, f_k(x) + e_k)$, since $f_k(x) - e < f(y) < f_k(x) + e$. It follows that $S_{f_k, e_k}[x]$ is open in t and thus $V[x]$ is open in t. Hence $t_{\mathcal{U}} \subset t$ and thus $t_{\mathcal{U}} = t$.

COROLLARY 4.1 Every normal space is uniformizable.

This follows from the fact that every normal space is completely regular.

THEOREM 4.5 Every compact Hausdorff space is uniformizable.

Since every compact Hausdorff space is normal the result follows by corollary 4.1 .

CHAPTER V

COMPLETENESS AND COMPLETION OF UNIFORM SPACES

A. COMPLETENESS OF UNIFORM SPACES

DEFINITION 5.1 Let $(X,\mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. A subset A of X is said to be U-small if $A \times A \subset U$.

LEMMA 5.1 Let $(X,\mathcal{U})$ be a uniform space. If two subsets of X A and B are U-small, then $A \cup B$ is U^2-small, provided $A \cap B = \emptyset$.

PROOF Let $(a,b) \in (A \cup B) \times (A \cup B)$. Since $A \cap B \neq \emptyset$, there exists a $c \in A \cap B$. Then $(a, c) \in A \times A$ or $(a, c) \in B \times B$ and hence $(a, c) \in U$. Also $(c, b) \in U$ and thus $(a, b) \in U \circ U$. Therefore $A \cup B$ is U^2-small.

DEFINITION 5.2 A filter $\mathcal{F}$ in a uniform space $(X,\mathcal{U})$ is a Cauchy filter if and only if for each surrounding U, there is a $F \in \mathcal{F}$ which is U-small.

LEMMA 5.2 A filter $\mathcal{F}$ in a uniform space $(X,\mathcal{U})$ is a Cauchy filter if and only if for each $U \in \mathcal{U}$, there exists a point $x \in X$, such that $U[x] \in \mathcal{F}$.

PROOF (1) If $\mathcal{F}$ is a Cauchy filter in $(X,\mathcal{U})$, then for each $U \in \mathcal{U}$, there exists $F \in \mathcal{F}$ which is U-small. There exists a $x \in F$. Let $y \in F$, then $(x, y) \in U$ and $y \in U[x]$. Hence $F \subset U[x]$ which implies $U[x] \in \mathcal{F}$.

(2) Let $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$ with $V \circ V \subset U$. By hypothesis, there exists $x \in X$, such that

$V[x] \in \mathcal{F}$. $V[x] \times V[x] \subset V \circ V \subset U$ and hence $\mathcal{F}$ is a Cauchy filter.

LEMMA 5.3 An elementary filter $\mathcal{F}$ generated by a sequence $\{x_n\}_1^\infty$ is a Cauchy filter if and only if for every $U \in \mathcal{U}$ we can find a natural number N such that $(x_n, x_m) \in U$ for each $n, m > N$.

LEMMA 5.4 Let f be a function from X onto $(Y, \mathcal{V})$. Then the family $\mathcal{U} = \left\{ f_2^{-1}(V) \mid V \in \mathcal{V} \right\}$ is a uniform structure on X. Furthermore if $\mathcal{B}$ is a Cauchy filter base on Y, then $f^{-1}(\mathcal{B})$ is a Cauchy filter base on X.

THEOREM 5.1 Every convergent filter is a Cauchy filter.

PROOF Let $\mathcal{F}$ be a filter on a set X which converges to $x \in X$. Then $U[x] \in \mathcal{F}$ for each $U \in \mathcal{U}$. Hence by lemma 5.1, $\mathcal{F}$ is a Cauchy filter.

The converse of theorem 5.1 is not true in general; that is a Cauchy filter need not be convergent.

EXAMPLE 5.1 Consider the metric d on R defined by $d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|$. The sequence $\{1, 2, \ldots\}$ is a Cauchy sequence but does not converge. The elementary filter generated by the sequence is a Cauchy filter which does not converge.

EXAMPLE 5.2 Let X be a non-empty set. For each finite partition $\omega = \left\{ A_i \mid 1 \leq i \leq n \right\}$ of X, let $U_\omega = \bigcup_i A_i \times A_i$. Then the sets U_ω form a base for a uniform structure $\mathcal{U}$ on X. The topology induced by $\mathcal{U}$ is the discrete topology since for each $x \in X$, the sets $\{x\}$ and $\{x\}^c$ form a finite partition of X. Hence if $V = (\{x\} \times \{x\}) \cup (\{x\}^c \times \{x\}^c)$, then $V[x] = \{x\}$.

An important property for this uniform space is that every ultra filter $\mathcal{F}$ on X is a Cauchy filter with respect to $\mathcal{U}$. Furthermore if X is an infinite set and since it has the discrete topology, then it is not compact. Hence by theorem 2.8 there are ultrafilters on X which do not converge.

THEOREM 5.2 A uniformly continuous mapping preserves Cauchy filters.

PROOF Let f be a uniformly continuous mapping from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$ and $\mathcal{F}$ is a Cauchy filter on X. Given $V \in \mathcal{V}$, then $f_2^{-1}(V) \in \mathcal{U}$. Hence there exists $F \in \mathcal{F}$ with $F \times F \subset f_2^{-1}(V)$. This implies that $f_2(F \times F) = f(F) \times f(F) \subset V$. Since $f(F) \in f(\mathcal{F})$, then $f(\mathcal{F})$ is a Cauchy filter.

EXAMPLE 5.3 Let $V_\epsilon = \left\{ (x, y) \in R^+ \times R^+ \mid |x - y| < \epsilon \right\}$. Then the sets V $(\epsilon > 0)$ is a base for a uniform structure on R^+. Define a function $f\colon R^+ \longrightarrow R^+$ by $f(x) = \frac{1}{x}$. Consider the sequence $\left\{\frac{1}{n}\right\}_1^\infty$ in the domain of f. Then the filter $\mathcal{F}$ generated by the sequence is Cauchy. Now $f(\mathcal{F})$ is not a Cauchy filter since it is generated by the sequence $\{n\}_1^\infty$ which is not a Cauchy sequence.

LEMMA 5.5 If $\mathcal{F}$ is a Cauchy filter on X and if $\mathcal{F}'$ is a filter on X finer than $\mathcal{F}$, then $\mathcal{F}'$ is a Cauchy filter.

LEMMA 5.6 Let $\mathcal{U}$ and $\mathcal{V}$ be two uniform structures on X, such that $\mathcal{U}$ is finer than $\mathcal{V}$. If $\mathcal{F}$ is a Cauchy filter relative to $\mathcal{U}$, then it is a Cauchy filter relative to $\mathcal{V}$.

LEMMA 5.7 If $\mathcal{F}$ or $\mathcal{J}$ is a Cauchy filter on $(X, \mathcal{U})$ and if $F \cap G \neq \emptyset$ for each $F \in \mathcal{F}$ and $G \in \mathcal{J}$, then the family $\mathcal{G} = \{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{J}\}$ is a Cauchy filter on X.

THEOREM 5.3 Let $(X, \mathcal{U})$ be a uniform space. If $\mathcal{F}$ is a Cauchy filter on X, then $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

PROOF Let $a \in \text{adh}\, \mathcal{F}$ and $U \in \mathcal{U}$. Then there exists a symmetric $V \in \mathcal{U}$ with $V \circ V \subset U$. Since $\mathcal{F}$ is Cauchy, then there exists $F \in \mathcal{F}$, such that $F \times F \subset V$. Since $F \cap V[a] \neq \emptyset$, then let $b \in F \cap V[a]$ and $(a, b) \in V$. Suppose $x \in F$, then $(b, x) \in F \times F \subset V$. Thus $(a, x) \in V \circ V \subset U$. This implies $x \in U[a]$. Hence $F \subset U[a]$ and thus $U[a] \in \mathcal{F}$. Therefore $a \in \lim \mathcal{F}$. Clearly if $a \in \lim \mathcal{F}$, then $a \in \text{adh}\, \mathcal{F}$. Hence $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

EXAMPLE 5.4 Let $X = \{1, 2, 3, 4, 5\}$. Define $V_n = \{(x, y) \mid \text{both } x, y > n \text{ or } x = y\}$. Then the sets V_n form a base for a uniform structure on X. Let $\mathcal{F} = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, X\}$. Then $\mathcal{F}$ is a Cauchy filter on X. For example $V_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (4, 5), (5, 4)\}$ and $V_3[2] = \{2\} \in \mathcal{F}$. Also $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

DEFINITION 5.3 Let $(X, \mathcal{U})$ be a quasi-uniform space and $\mathcal{F}$ a filter on X. $\mathcal{F}$ is $\mathcal{U}$-Cauchy if for every $U \in \mathcal{U}$ there exists $x \in X$, such that $U[x] \in \mathcal{F}$.

In the case of a quasi-uniform space the adherence of a Cauchy filter is not necessarily equal to its limit. The following example will show this.

EXAMPLE 5.5 Let $X = \{1, 2, 3, 4, 5\}$ and $W = \{(x, y) | x \leq y\}$. Then W is a base for a quasi-uniform structure . Set $\mathcal{F} = \{X, \{2, 3, 4, 5\}\}$, then $\mathcal{F}$ is a Cauchy filter since $W[2] = \{2, 3, 4, 5\} \in \mathcal{F}$. Now adh $(\mathcal{F}) = X$, while $\lim \mathcal{F} = \{1, 2\}$.

A question arises if there are quasi-uniform spaces that are not uniform spaces in which the limit of every Cauchy filter equals its adherence.

DEFINITION 5.4 A quasi-uniform structure is R_3 if given $x \in X$ and $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V[x] \subset U[x]$.

THEOREM 5.4 Let $(X, \mathcal{U})$ be a R_3- quasi-uniform space. If $\mathcal{F}$ is a Cauchy filter on X, then $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

PROOF Let $x \in$ adh $(\mathcal{F})$, $U \in \mathcal{U}$. Since $\mathcal{U}$ is R_3, there exists a symmetric $V \in \mathcal{U}$, such that $V \circ V \circ V[x] \subset U[x]$. There exists $a \in X$ with $V[a] \in \mathcal{F}$. Since $V[a] \cap V[x] \neq \emptyset$, let $b \in V[a] \cap V[x]$. Suppose $y \in V[a]$, then $y \in V \circ V \circ V[x] \subset U[x]$ and hence $V[a] \subset U[x]$. Thus $U[x] \in \mathcal{F}$ and $x \in \lim \mathcal{F}$. Hence $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

THEOREM 5.5 Let (X, t) be a compact space. Then every ultrafilter on X is a Cauchy filter with respect to any uniform structure compatible with it.

PROOF Let $\mathcal{U}$ be a uniform structure compatible with t, and let $\mathcal{F}$ be an ultrafilter on X. By theorem 2.8 $\mathcal{F}$ converges and hence by theorem 5.1 $\mathcal{F}$ is a Cauchy filter.

Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{F}$ a filter on X. Define $\mathcal{F}^* = \{U[F] \mid F \in \mathcal{F}, U \in \mathcal{U}\}$.

LEMMA 5.8 $\mathcal{F}^*$ is a filter on X.

LEMMA 5.9 $\mathcal{F}^*$ is coarser than $\mathcal{F}$; that is $\mathcal{F}^* \subset \mathcal{F}$.

LEMMA 5.10 $\mathcal{F}^*$ is a Cauchy filter on $(X, \mathcal{U})$ if and only if $\mathcal{F}$ is a Cauchy filter on $(X, \mathcal{U})$.

PROOF (1) Let $\mathcal{F}^*$ be a Cauchy filter, then given $U \in \mathcal{U}$, there exists $V[F] \in \mathcal{F}^*$ such that $V[F] \times V[F] \subset U$. Since $F \subset V[F]$, then $F \times F \subset U$ and hence $\mathcal{F}$ is a Cauchy filter.

(2) Suppose $\mathcal{F}$ is a Cauchy filter. If $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$, such that $V \circ V \circ V \subset U$. There exists $F \in \mathcal{F}$ such that $F \times F \subset V$. Now $V[F] \times V[F] \subset V \circ V \circ V \subset U$. Hence $\mathcal{F}^*$ is Cauchy.

DEFINITION 5.5 A filter $\mathcal{F}$ on a set X is called an open filter if it has filter base of open sets.

LEMMA 5.11 $\mathcal{F}^*$ has an open filter base.

PROOF Let $U \in \mathcal{U}$, then by lemma 3.9, there exists an open surrounding $W \subset U$. Thus $W[F] \in \mathcal{F}^*$ and $W[F] \subset U[F]$. Since $W[F]$ is open, then the sets $W[F]$ form an open filter base for $\mathcal{F}^*$.

LEMMA 5.12 If $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters such that $\mathcal{F}$ is finer than $\mathcal{G}$, then $\mathcal{G}$ is finer than $\mathcal{F}^*$.

PROOF Let $U[F] \in \mathcal{F}^*$, then there are two cases:

Case (1) If $F \in \mathcal{G}$, then since $F \subset U[F]$, $U[F] \in \mathcal{G}$.
Hence $\mathcal{G}$ is finer $\mathcal{F}^*$.

Case (2) If $F \notin \mathcal{G}$. There exists a symmetric $V \in \mathcal{U}$, such that $V \circ V \subset U$. Since $\mathcal{G}$ is Cauchy, there exists $x \in X$ with $V[x] \in \mathcal{G}$, and consequently $V[x] \in \mathcal{F}$. Since $V[x] \cap F \neq \emptyset$, let $y \in V[x] \cap F$. Then $x \in V[y]$. Hence $V[x] \subset V \circ V[y] \subset U[y]$

and thus $U[y] \in \mathcal{G}$. Since $y \in F$, then $U[y] \subset U[F]$. Hence $U[F] \in \mathcal{G}$ and thus $\mathcal{G}$ is finer than $\mathcal{F}^*$.

LEMMA 5.13 If $\mathcal{F}$ is a Cauchy filter, then $\mathcal{F}^*$ is a minimal Cauchy filter on X.

LEMMA 5.14 If $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters, then $\mathcal{F}^* = \mathcal{G}^*$ if and only if $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter.

LEMMA 5.15 If $\mathcal{F}$ is a filter on X, then $(\mathcal{F}^*)^* = \mathcal{F}^*$.

THEOREM 5.6 Every neighborhood filter $\mathcal{N}_x$ for $x \in X$ is a minimal Cauchy filter.

PROOF Since $\lim \mathcal{N}_x \neq \emptyset$, then by theorem 5.1 $\mathcal{N}_x$ is a Cauchy filter. Assume there exists a Cauchy filter $\mathcal{F}$ on X which is properly contained in $\mathcal{N}_x$. There exists $U[x] \notin \mathcal{F}$. Choose a symmetric $V \in \mathcal{U}$, with $V \circ V \subset U$. Then $V[x] \notin \mathcal{F}$. Now $V[a] \in \mathcal{F}$ for some $a \in X$, which implies $V[a] \in \mathcal{N}_x$. Thus $x \in V[a]$ and hence $a \in V[x]$. Therefore $V[a] \subset V \circ V[x] \subset U[x]$ and consequently $U[x] \in \mathcal{F}$ which is a contradiction. Thus $\mathcal{N}_x$ is a minimal Cauchy filter.

DEFINITION 5.6 A uniform space $(X, \mathcal{U})$ is complete if and only if every Cauchy filter on X converges.

DEFINITION 5.7 A quasi-uniform space $(X, \mathcal{U})$ is complete if and only if every Cauchy filter has non-empty adherence.

DEFINITION 5.8 A quasi-uniform space $(X, \mathcal{U})$ is strongly complete if and only if every Cauchy filter converges.

LEMMA 5.16 In a uniform space completeness and strong completeness are equivalent.

PROOF Let $(X, \mathcal{U})$ be a uniform space. If $\mathcal{F}$ is a Cauchy filter on X, then by theorem 5.3 $\lim \mathcal{F} = \text{adh}\, \mathcal{F}$.

THEOREM 5.7 In a uniform space completeness is invariant under uniform isomorphism.

PROOF Let $(X, \mathcal{U})$ be a complete uniform space. Let f be a uniformly continuous function from $(X, \mathcal{U})$ onto $(Y, \mathcal{V})$. Let $\mathcal{F}$ be a Cauchy filter on $(Y, \mathcal{V})$. Then $f^{-1}(\mathcal{F})$ is a Cauchy filter on X which converges to some point $x \in X$. Now for each $V \in \mathcal{V}$, $f^{-1}(V[f(x)]) = f_2^{-1}(V)[x] \in f^{-1}(\mathcal{F})$. Hence $V[f(x)] \in \mathcal{F}$ for each $V \in \mathcal{V}$. Thus $f(x) \in \lim \mathcal{F}$ and hence $(Y, \mathcal{V})$ is complete.

EXAMPLE 5.6 On any set X, the discrete uniform structure $\{U \mid U \subset X \times X, U \supset \Delta\}$ is complete.

THEOREM 5.8 A closed subspace of a complete uniform space is complete.

PROOF Let $(X, \mathcal{U})$ be a complete uniform space and $(Y, \mathcal{U}_Y)$ a closed subspace of X, where $\mathcal{U}_Y = \{U \cap Y \times Y \mid U \in \mathcal{U}\}$. If $\mathcal{F}$ is a Cauchy filter on Y, then $\mathcal{F}$ is a Cauchy filter base on X. Since X is complete, $\mathcal{F}$ converges to a point $a \in X$. Since $\overline{F} \subset Y$, then $\text{adh}\, \mathcal{F} = \cap \overline{F} \subset Y$. Now $a \in \text{adh}\, \mathcal{F} \subset Y$ and hence Y is complete.

THEOREM 5.9 Every complete subspace of a Hausdorff uniform space is closed.

PROOF Suppose $(Y, \mathcal{U}_Y)$ is a complete subspace of the uniform Hausdorff space $(X, \mathcal{U})$. Let $y \in Y$, then $U[y] \cap Y \neq \emptyset$ for each $U \in \mathcal{U}$. The family of sets $\mathcal{F} = \{U[y] \cap Y \mid U \in \mathcal{U}\}$ is a Cauchy filter in Y. Hence $\mathcal{F}$ converges to $a \in Y$. Now $\mathcal{F}$ is a filter base in X and since $U[y]$ is an element of the filter in X generated by $\mathcal{F}$ for every $U \in \mathcal{U}$, then $y \in \lim \mathcal{F}$. By theorem 2.4 $a = y \in Y$. Hence Y is closed.

THEOREM 5.10 A compact subspace of a complete Hausdorff uniform space is complete.

THEOREM 5.11 Every compact uniform space is complete.

THEOREM 5.12 Let Y be a dense subspace of the uniform space $(X, \mathcal{U})$, such that every Cauchy filter in Y converges to a point in X, then X is complete.

PROOF Let $\mathcal{F}$ be a Cauchy filter on X. Consider the family $\mathcal{B} = \{ U[F] \cap Y \mid U \in \mathcal{U} \text{ and } F \in \mathcal{F} \}$. Then $\mathcal{F}$ is a Cauchy filter in Y which converges to a point $x \in X$. Let $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$, with $V \circ V \subset U$. Since $V[x] \cap (V[F] \cap Y) \neq \emptyset$, it follows that $V[x] \cap V[F] \neq \emptyset$. Therefore $U[x] \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. Hence $\mathcal{F}$ converges to x and X is complete.

EXAMPLE 5.7 Consider the space $(C^{*}(X), d)$ where $C^{*}(X)$ is the set of all bounded continuous real functions on X and d is a metric on it defined by $d(f, g) = \sup \{ |f(x) - g(x)| \mid x \in X \}$ then $C^{*}(X)$ is complete.

DEFINITION 5.9 A (quasi) uniform space $(X, \mathcal{U})$ is called totally bounded if for every $U \in \mathcal{U}$, there are finitely many sets $A_1, \ldots, A_n$ in X such that:

(1) $A_i \times A_i \subset U$ and (2) $\cup_i A_i = X$ for $1 \leq i \leq n$.

DEFINITION 5.10 A (quasi) uniform space $(X, \mathcal{U})$ is called precompact if for every $U \in \mathcal{U}$, there are finitely many points $x_1, \ldots, x_n \in X$, such that $\bigcup_i U[x_i] = X$.

LEMMA 5.17 Every totally bounded quasi-uniform space is precompact.

THEOREM 5.13 A uniform space is precompact if and only if it is totally bounded.

PROOF (1) If $(X,\mathcal{U})$ is a totally bounded uniform space, then by lemma 5.15 it is pre-compact.

(2) Suppose that $(X,\mathcal{U})$ is pre-compact. Let $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. There are finitely many points $x_1, \ldots, x_n \in X$, such that $\bigcup V[x_i] = X$ where $1 \leqslant i \leqslant n$. Now $V[x_i] \times V[x_i] \subset V \circ V \subset U$ and hence $(X,\mathcal{U})$ is totally bounded.

THEOREM 5.14 A uniform space is totally bounded if and only if every ultrafilter is a Cauchy filter.

PROOF (1) Suppose $(X,\mathcal{U})$ is pre-compact, then given $U \in \mathcal{U}$ there are finitely many points $x_1, \ldots, x_n \in X$, such that $\bigcup U[x_i] = X$ where $1 \leqslant i \leqslant n$. Let $\mathcal{F}$ be an ultra-filter on X. Since $X \in \mathcal{F}$, then at least one of the $U[x_i] \in \mathcal{F}$. Hence $\mathcal{F}$ is a Cauchy filter.

(2) Assume that $(X,\mathcal{U})$ is not pre-compact. Then set $\mathcal{B} = \{X - U[A] \mid A \text{ is a finite subset of } X\}$ for some $U \in \mathcal{U}$. Then $\mathcal{B}$ is a filter base on X. Let $\mathcal{F}$ be an ultrafilter on X containing $\mathcal{B}$. If $\mathcal{F}$ is a Cauchy filter, then $U[x] \in \mathcal{F}$ for some $x \in X$. But $X - U[x] \in \mathcal{F}$ which is a contradiction. Hence $\mathcal{F}$ is not a Cauchy filter on X.

THEOREM 5.15 A uniform space is totally bounded if and only if every filter is contained in a Cauchy filter.

THEOREM 5.16 A uniform space is compact if and only if it is totally bounded and complete.

PROOF (1) Suppose $(X,\mathcal{U})$ is a compact uniform space. Then by theorem 5.5 every ultrafilter is a Cauchy filter. Hence $(X,\mathcal{U})$ is totally bounded and complete.

(2) Assume that $(X,\mathcal{U})$ is totally bounded and complete. Hence every ultrafilter on X is a Cauchy filter which converges in X. Thus X is compact.

THEOREM 5.17 Every topological space has a totally bounded quasi-uniform structure compatible with it.

PROOF The Pervin quasi-uniform structure is totally bounded.

B. THE HAUSDORFF UNIFORM SPACE ASSOCIATED WITH UNIFORM SPACE

Let $(X,\mathcal{U})$ be a uniform space. Set $C = \cap \{V \mid V \in \mathcal{U}\}$.

LEMMA 5.18 (1) $C \supset \Delta$ (2) $C = C^2 = C^{-1}$

DEFINITION 5.10 Define a relation $\sim$ on a uniform space $(X,\mathcal{U})$ by $x \sim y$ if and only if $(x, y) \in C$.

LEMMA 5.19 The relation $\sim$ on X is an equivalence relation.

LEMMA 5.20 Let $\check{X}$ denotes the set of equivalence classes of $\sim$ on X. Then $\check{x} = [x] = \{y \in X \mid x \sim y\} = C[x]$.

LEMMA 5.21 Define $\check{V}$ by $(\check{x}, \check{y}) \in \check{V}$ if and only if there exists $x_0 \in \check{x}$, $y_0 \in \check{y}$, such that $(x_0, y_0) \in V$. Then $(\check{x}, \check{y}) \in \check{V}$ if and only if $(a, b) \in C \circ V \circ C$, for all $a \in \check{x}$ and $b \in \check{y}$.

LEMMA 5.22 Set $\check{\mathcal{U}}_B = \{\check{V} \mid V \in \mathcal{U}\}$, then $\mathcal{U}_B$ forms a uniform structure base on $\check{X}$.

THEOREM 5.18 If $\check{\mathcal{U}}$ is the uniform structure generated by $\check{\mathcal{U}}_B$, then $\check{\mathcal{U}}$ is Hausdorff. $(\check{X},\check{\mathcal{U}})$ is called the Hausdorff uniform space associated with $(X,\mathcal{U})$.

PROOF Suppose $(\check{x}, \check{y}) \in \check{U}$ for each $\check{U} \in \check{\mathcal{U}}$. Then $(x, y) \in C \circ V \circ C$ for each $x \in \check{x}$, $y \in \check{y}$ and $V \in \mathcal{U}$. Now $(x, y) \in \bigcap \{C \circ V \circ C \mid V \in \mathcal{U}\} \subset \bigcap \{V \circ V \circ V \mid V \in \mathcal{U}\} = C$, since $\{V^3 \mid V \in \mathcal{U}\}$ is a base for $\mathcal{U}$. Thus $(x, y) \in C$ and $\check{x} = \check{y}$. Hence $(\check{X}, \check{\mathcal{U}})$ is a Hausdorff space.

LEMMA 5.23 The function $\mathcal{P}$ from X onto $\check{X} = X/\sim$ defined by $\mathcal{P}(x) = \check{x}$ is uniformly continuous.

PROOF Let $\check{U} \in \check{\mathcal{U}}$. Since $U \subset C \circ U \circ C$, then $C \circ U \circ C \in \mathcal{U}$. Let $(x, y) \in C \circ U \circ C$, then $(x, x_0) \in C$, $(x_0, y_0) \in U$ and $(y_0, y) \in C$ for some $(x_0, y_0) \in U$. Now $x_0 \in \check{x}$, $y_0 \in \check{y}$, hence $(\check{x}, \check{y}) \in \check{U}$. Hence $\mathcal{P}$ is uniformly continuous.

DEFINITION 5.11 Let (X, t) be a topological space, Y is a set. Let f be a function from X onto Y. Then the quotient topology on Y is $Q = \{O \subset Y \mid f^{-1}(O) \text{ is open in } t\}$.

THEOREM 5.19 The uniform topology of $(\check{X}, \check{\mathcal{U}})$ coincides with its quotient topology under the mapping $\mathcal{P}$.

PROOF (1) Let $\check{t}$ denotes the uniform topology associated with $\check{\mathcal{U}}$ and Q denotes the quotient topology. By lemma 5.18 $\mathcal{P}$ is continuous. Let $O \in \check{t}$. Since $\mathcal{P}$ is continuous, then $\mathcal{P}^{-1}(O) \in t$. Hence $O \in Q$ and thus $\check{t} \subset Q$.

(2) Let $G \in Q$. Then $\mathcal{P}^{-1}(G) \in t$. Hence for each $x \in \mathcal{P}^{-1}(G)$ there exists $U \in \mathcal{U}$ with $U[x] \subset \mathcal{P}^{-1}(G)$. It follows that $\mathcal{P}(U[x]) \subset G$. There exists a symmetric $V \in \mathcal{U}$ such $V^3 \subset U$. Let $\check{y} \in \check{V}[\check{x}]$. Then $(\check{x}, \check{y}) \in \check{V}$ and thus $(x_0, y_0) \in C \circ V \circ C \subset V^3 \subset U$ for all $x_0 \in \check{x}$ and $y_0 \in \check{y}$. Hence $(x, y) \in U$ which implies $y \in U[x]$. Therefore $\mathcal{P}(y) =$

$\check{y} \in \mathcal{S}(U[x])$. Thus $\check{V}[\check{x}] \subset \mathcal{S}(U[x] \subset G.)$ It follows that $G \in \check{t}$ and hence $Q \subset \check{t}$. Thus $Q = \check{t}$.

C. COMPLETION OF UNIFORM SPACES

Let $\hat{X}$ be the set of all minimal Cauchy filters on a uniform space $(X, \mathcal{U})$. Define $\hat{V} = \{ (\mathcal{F}, \mathcal{G}) \; \mathcal{F}, \mathcal{G}$ are minimal Cauchy filters on X such that $\mathcal{F} \cap \mathcal{G}$ contains a V-small set $\}$.

LEMMA 5.24 Set $\hat{\mathcal{U}}_B = \{ \hat{V} \mid V \in \mathcal{U}$ and V is symmetric$\}$. Then $\hat{\mathcal{U}}_B$ forms a base for a uniform structure on X.

PROOF (1) Let $\mathcal{F} \in \hat{X}$. Then given a symmetric $V \in \mathcal{U}$, there exists $F \in \mathcal{F}$ which is a V-small. Hence $(\mathcal{F}, \mathcal{F}) \in \hat{V}$.

(2) Let $\hat{V}_1, \hat{V}_2 \in \hat{\mathcal{U}}_B$ and let $W = V_1 \cap V_2$. Then W is symmetric, which implies $\hat{W} \in \hat{\mathcal{U}}_B$. If $(\mathcal{F}, \mathcal{G}) \in \hat{W}$, then $\mathcal{F} \cap \mathcal{G}$ contains a W-small set and consequently $\mathcal{F} \cap \mathcal{G}$ contains a V_1-small set and a V_2-small set. Thus $(\mathcal{F}, \mathcal{G}) \in \hat{V}_1 \cap \hat{V}_2$ and hence $\hat{W} \subset \hat{V}_1 \cap \hat{V}_2$.

(3) From the definition each $\hat{V} \in \hat{\mathcal{U}}_B$ is symmetric and hence axiom $U_B[3]$ is satisfied.

(4) Let $\hat{V} \in \hat{\mathcal{U}}_B$, then there exists a symmetric $W \in \mathcal{U}$, such that $W^2 \subset V$. If $(\mathcal{F}, \mathcal{G}) \in \hat{W} \circ \hat{W}$, then $(\mathcal{F}, H) \in \hat{W}$, $(H, \mathcal{G}) \in \hat{W}$ for some $H \in \hat{X}$. Let F_1 be a W-small set such that $F_1 \in \mathcal{F} \cap H$ and F_2 be a W-small set such that $F_2 \in H \cap \mathcal{G}$. By lemma 5.1 $F_1 \cup F_2$ is a W^2-small set and hence V-small. Since $F_1 \cup F_2 \in \mathcal{F} \cap \mathcal{G}$, then $(\mathcal{F}, \mathcal{G}) \in \hat{V}$ and hence $\hat{W} \circ \hat{W} \subset \hat{V}$. Thus $\hat{\mathcal{U}}_B$ is a base for a uniform structure $\hat{\mathcal{U}}$ on $\hat{X}$.

LEMMA 5.25 $(\hat{X}, \hat{\mathcal{U}})$ is a Hausdorff space.

PROOF By theorem 3.14 it is sufficient to prove that $\cap \{ \hat{V} \mid V \text{ is symmetric} \} = \hat{\Delta}$. Let us assume the contrary, then suppose $(\mathcal{F}, \mathcal{G}) \in \cap \{\hat{V}\}$. Set $\mathcal{B} = \{ F \cup F' \mid F \in \mathcal{F}, F' \in \mathcal{G} \}$, then $\mathcal{B}$ is a filter base on X. Let H be the filter generated by $\mathcal{B}$. If $F'' \in H$, then $F'' \supset F \cup F'$ for some $F \in \mathcal{F}$, $F' \in \mathcal{G}$. Hence $F'' \supset F$, $F'' \supset F'$ and therefore H is coarser than $\mathcal{F}$ and $\mathcal{G}$. Now given any symmetric $V \in \mathcal{U}$, there is a V-small set $G \in \mathcal{F} \cap \mathcal{G}$ and hence $G \cup G = G \in H$. Thus H is a Cauchy filter. Since $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters, then $\mathcal{F} = \mathcal{G} = H$. Hence $\cap \{\hat{V}\} = \hat{\Delta}$ and thus $\hat{X}$ is Hausdorff.

LEMMA 5.26 Define a function $i: X \rightarrow \hat{X}$ by $i(x) = \mathcal{N}_x$ where $\mathcal{N}_x$ is the neighborhood filter of x. i is uniformly continuous.

PROOF By lemma 5.8 the neighborhood filter $\mathcal{N}_x$ of $x \in X$ is a minimal Cauchy filter and hence $\mathcal{N}_x \in \hat{X}$. Let $\hat{V} \in \hat{\mathcal{U}}$, then there exists an open symmetric $W \in \mathcal{U}$, such that $W^2 \subset V$. If $(x, y) \in W$ then $i_2(x, y) = (\mathcal{N}_x, \mathcal{N}_y)$. Since $W[y] \in \mathcal{N}_x$ then $W[y] \times W[y] \subset W \circ W \subset V$. Hence $W[y] \in \mathcal{N}_x \cap \mathcal{N}_y$ and thus $(\mathcal{N}_x, \mathcal{N}_y) \in \hat{V}$. This implies that i is uniformly continuous.

LEMMA 5.27 Let $(X, \mathcal{U})$ be a Hausdorff uniform space, then the function $i: X \rightarrow \hat{X}$ defined by $i(x) = \mathcal{N}_x$ is one to one. Furthermore $i^{-1}: i(X) \rightarrow (X)$ is uniformly continuous.

PROOF Assume for two distinct elements $x, y \in X$ that $i(x) = i(y)$; that is $\mathcal{N}_x = \mathcal{N}_y$. This contradicts the assumption that $(X, \mathcal{U})$ is Hausdorff. Thus i is one to one mapping. Let $U \in \mathcal{U}$. If $(\mathcal{N}_x, \mathcal{N}_y) \in \hat{U}$ then there exists $N \in \mathcal{N}_x \cap \mathcal{N}_y$ such that $N \times N \subset U$. Hence $(x, y) = i^{-1}(\mathcal{N}_x, \mathcal{N}_y) \in U$. Thus i^{-1} is uniformly continuous.

LEMMA 5.28 Let $(X, \mathcal{U})$ be a uniform space, then $i(X)$ is dense in X, where i is the function defined in lemma 5.26 .

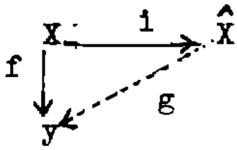
PROOF Let $\mathcal{F} \in \hat{X}$. If $U \in \mathcal{U}$, there exists an open symmetric $V \in \mathcal{U}$ such that $V^3 \subset U$. There exists $F \in \mathcal{F}$ with $F \times F \subset V$. By lemma 5.11 there exists an open set $O \in \mathcal{F}$ such that $O \times O \subset V[F] \times V[F] \subset V^3 \subset U$. If $x \in O$, then $x \in V[F]$. Hence $V[F] \in \mathcal{N}_x$. Thus $(\mathcal{F}, \mathcal{N}_x) \in \hat{U}$ and hence $\mathcal{N}_x \in \hat{U}(\mathcal{F})$. Therefore $\hat{U}(\mathcal{F}) \cap i(X) \neq \emptyset$ for each $\mathcal{F} \in \hat{X}$ and $\hat{U} \in \hat{\mathcal{U}}$. Hence $i(X)$ is dense in $\hat{X}$.

THEOREM 5.20 Let $(X, \mathcal{U})$ be a uniform space, then $(\hat{X}, \hat{\mathcal{U}})$ is complete.

PROOF Let $\mathcal{G}$ be a Cauchy filter on $i(X)$, then $i^{-1}(\mathcal{G}) = H$ is a Cauchy filter on X. $H^* = \{U[F] \mid F \in H,\ U \in \mathcal{U}\}$ is a minimal Cauchy filter on X coarser than H. Then $i(H^*) = \mathcal{F}$ is a Cauchy filter on $i(X)$ coarser than $\mathcal{G}$. If $U \in \mathcal{U}$, there exists an open symmetric $V \in \mathcal{U}$ with $V^3 \subset U$. Since H is Cauchy, there exists $F \in H$ such that $F \times F \subset V$. Hence $V[F] \times V[F] \subset V^3 \subset U$. Let $a \in V[F]$, then $V[F] \in \mathcal{N}_a$. Hence $(H^*, \mathcal{N}_a) \in \hat{U}$ and thus $\mathcal{N}_a \in \hat{U}[H^*]$. This implies $\hat{U}[H^*] \in \mathcal{G}$ and hence $\mathcal{G}$ converges to $H^* \in \hat{X}$. Therefore by theorem 5.12 $(\hat{X}, \hat{\mathcal{U}})$ is complete.

DEFINITION 5.12 When $(X, \mathcal{U})$ is a Hausdorff uniform space, then $(\hat{X}, \hat{\mathcal{U}})$ is said to be the completion of $(X, \mathcal{U})$.

LEMMA 5.29 Let f be a uniformly continuous mapping of X into a complete Hausdorff uniform space Y, then there is a unique uniformly continuous mapping $g: \hat{X} \rightarrow Y$ such that the diagram



commutes; that is $f = g \circ i$.

PROOF Define a mapping $g_o: i(X) \rightarrow Y$, such that $g_o(i(x)) = \lim f(\mathcal{N}x)$. Since f is uniformly continuous, hence it is continuous, then $\lim f(\mathcal{N}x) = f(\lim \mathcal{N}x) = f(x)$. Thus $g_o \circ i = f$. To show that g_o is uniformly continuous in $i(X)$, let U^* be a surrounding in Y. Since $f: X \rightarrow Y$ is uniformly continuous then there exists a symmetric surrounding V in X, such that if $(x, x') \in V$, then $(f(x), f(x'))\ U^*$. If $(i(x), i(x')) \in \hat{V}$, then $i(x)$, $i(x')$ have a neighborhood of both x, x' in common which is V-small and hence $(x, x') \in V$, which implies $(g_o(i(x)), g_o(i(x'))) \in U^*$ since $f(x) = g_o(i(x))$ and $f(x') \in g_o(i(x'))$. Hence g_o is uniformly continuous in $i(X)$. Since $i(X)$ is dense in $\hat{X}$ by lemma 5.28, then g_o can be extended to $g: \hat{X} \rightarrow Y$ such that $f = g \circ i$ and it is clear that g is the unique uniformly continuous mapping from $\hat{X}$ into Y.

LEMMA 5.30 The completion of a Hausdorff uniform space is unique; that is any two Hausdorff completions of a Hausdorff uniform space $(X, \mathcal{U})$ are uniformly isomorphic.

PROOF This follows immediately from lemma 5.29.

THEOREM 5.21 If $(X,\mathcal{U})$ is a totally bounded uniform space, then $(\hat{X},\hat{\mathcal{U}})$ is compact.

PROOF By theorem 5.20 $(\hat{X},\hat{\mathcal{U}})$ is complete. Let $\hat{U} \in \hat{\mathcal{U}}$, then there exists a symmetric $\hat{V} \in \hat{\mathcal{U}}$, such that $\hat{V} \circ \hat{V} \subset \hat{U}$. $\hat{W} = \hat{V} \cap (i\ (X) \times i\ (X)\ )$ is symmetric in the relative uniform structure on $i(X)$. Let $W = i_2^{-1}\ (\hat{W})$, then since $(X,\mathcal{U})$ is totally bounded, there are finitely many points $x_1, x_2, \ldots, x_n \in X$ such that $\bigcup \hat{W}\left[x_j\right] = X$ where $1 \leqslant j \leqslant n$. Hence $\bigcup \hat{W}\left[i\ (x_j)\right] = i\ (X)$ where $1 \leqslant j \leqslant n$. If $\mathcal{F} \in \hat{X}$, then $(\mathcal{F}, i\ (x)\ ) \in \hat{V}$ for some $i\ (x) \in i\ (X)$, since $i\ (X)$ is dense in X. Now $(i\ (x), i\ (x_j)\ ) \in \hat{W} \subset \hat{V}$ and hence $(\mathcal{F}, i\ (x_j)\ ) \subset \hat{V} \circ \hat{V} \subset \hat{U}$. Thus $\mathcal{F} \in \hat{U}\left[i\ (x_j)\right]$. Hence $\hat{X} = \bigcup \left\{\hat{U}\left[i\ (x_j)\right]\right\}$, where $1 \leqslant j \leqslant n$ and it follows that $(\hat{X},\hat{\mathcal{U}})$ is totally bounded. Therefore by theorem 5.16 $(\hat{X},\hat{\mathcal{U}})$ is compact.

CHAPTER VI

SUGGESTIONS FOR FURTHER STUDY

An interested problem is to characterize spaces with unique uniform structures. The following conditions are equivalent for any completely regular space X [5] :

(1) X admits a unique uniform structure,

(2) The stone-Čech compactification βX contains at most one point not in X.

(3) $|\beta X - X| \leq 1$.

(4) X has a unique compactification.

(5) Every function in $C^*(X)$ is uniformly continuous in every admissable structure on X.

(6) For any two normally seperated closed subsets of X at least one of them is compact. This is due to Doss (1949) [10].

The space of ordinals W and the Tychonoff plank T are examples of non compact spaces with unique uniform structures.

In 1959 Gál [10] proved that there is a one to one correspondence between all totally bounded uniform structures and all Hausdorff compactification that can be defined on a completely regular space.

An important theorem due to Shirota [5] which states that a completely regular space in which every closed discrete subspace has non-measurable cardinal admits a complete uniform structure if and only if it is real compact.

The concept of locally uniform spaces has been recently discussed. The interested is referred to James William [12].

If (X, t) is completely regular space and $\mathcal{U}_f = \vee \{ \mathcal{U} \mid \mathcal{U}$ is a compatible uniform structure.$\}$. One would like to have a description of $\mathcal{U}_f$.

The concept of fine spaces which are the spaces having the finest uniform structure compatible with the topology has been recently studied. Central results are Shirota's Theorem and Glicksberg's (1959). One would like to have a complete answer of the question "When is the product of fine spaces fine [7] .

Uniform structures on topological groups were first studied by Weil (1937). There are by now a number of texts devoted to the subject such as Pontrajagin (1939), and Montgomery and Zippin (1955). Now every topological group is completely regular and hence it is uniformizable.

A different approach to a uniform and quasi-uniform structure is due to Csaszar [3] who considered them as particular cases of syntopogenous structures.

BIBLIOGRAPHY

1. Bourbaki, Nicolas. General Topology. Addison-Wesley Publishing Company.

2. Carlson, John W. and Hicks, T. L. "Some Quasi-Uniform Space Examples." Journal of Mathematical Analysis and Applications. Vol. 39, No. 3 September 1972.

3. Czaszar, Akos. Foundations of General Topology. New York: The MacMillan Company.

4. Fairchild, W. and Tulcea, C. I. Topology. Philadelphia: W. B. Saunders Co., 1971.

5. Gillman. L. and Jerison, M. Rings of Continuous Functions. Princeton, New Jersey: D. Van Nostrand Company, Inc.

6. Gaal, Steven A. Point Set Topology. New York: Academic Press, 1964.

7. Isbell, J. R. Uniform Spaces. American Mathematical Society.

8. Kelley, John L. General Topology. New York: Academic Press, 1964.

9. Pervin, William J. Foundations of General Topology. New York: Academic Press.

10. Thron, Wolfgang J. Topological Structures. New York: Holt, Rinehart and Winston.

11. Wilansky, Albert. Topology for Analysis. Waltham, Massachusetts: Ginn Axerox Company - The College Division.

12. Williams, James. Locally Uniform Spaces. Transcript American Mathematical Society.