THE STRUCTURE OF ABELIAN GROUPS

A Thesis

Presented to

the Department of Mathematics

Emporia Kansas State College

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

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by

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August 1974

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Approved for the Major Department

Approved for the Graduate Council

ACKNOWLEDGMENTS

I wish to thank Dr. George Poole for his kind assistance and helpful suggestions. Appreciation is expressed to Dr. Marion Emerson and all my professors who instructed me at Emporia Kansas State College.

Finally, I thank my wife, Connie, for her patience and many hours spent typing this paper.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The theory of abelian groups is that branch of algebra which deals with groups that have the additional property of commutativity. Although this may not seem to be that striking a deviation from general group theory, this added property almost completely changes the methods and basic ideas of the study of these groups. Since every abelian group is, of course, a group, all the results that can be determined about general groups hold for abelian groups. However, there is an immense amount of knowledge that has been added by simply allowing the commutative property as part of the structure. This is why the study of abelian groups is such an interesting field.

One of the basic problems which confront group theorists is determining the structure of a given group and then classifying the group with others which have the same or similar structure. Thus, it can be said that the classification of groups means a scheme that tells when two systems are essentially the same. This idea manifests itself in trying to set up isomorphisms between two groups so that theorems that state when groups are isomorphic are of extreme importance. Another problem concerning the structure of groups is in stating the conditions which force a group to decompose into familiar subgroups or, hopefully, less complicated groups. In other words, the group theorists try to break down a group in hopes that it becomes a little more familiar.

It is the purpose of this paper to present the results that deal specifically with classifying and decomposing abelian groups. In the finite case, the problems have been resolved, where as for infinite abelian groups, the structure of only special cases has been determined. Although there is an immense amount of material concerning the structure of abelian groups, this paper is intended to provide a reasonable comprehensive summary of the main results concerning the structure of abelian groups. It is assumed that the reader has had some exposure to abstract algebra, set theory and use of transfinite tools, such as, Zorn's Lemma and the Axiom of Choice.

In Chapter II, a brief review of some of the more common terms and theorems in elementary group theory which will be used throughout the paper is presented. Some of the terms are basic and are not defined although a source is listed which explains the terms in depth. Likewise some of the theorems are stated without proofs. Finally, there are some definitions which may be new to the reader or are presented because they have been defined differently according to various authors. The heart of the paper begins in Chapter III where the theorems which ultimately classify all finite abelian groups are presented. In Chapter IV attention is focused on infinite abelian groups and a discussion of torsion and torsion free groups is presented as well as the classification theorems for the divisible, free and finitely generated groups.

From this point on, whenever the term group is used, it is understood that the group is abelian and, as is customary, that the binary operation is addition (+). Also the identity is 0 and the inverses of elements are the negatives. Note that there will be no distinction made in notation between the integer 0, the group identity

0 and the set containing only 0. The context will provide the distinction.

CHAPTER II

BASIC DEFINITIONS AND THEOREMS

Even though the term abelian group has been used a number of times already, a precise definition is given to avoid confusion.

DEFINITION 2.1. An abelian group G is a non-empty set with a binary operation + defined on the elements in G such that

(1) the operation is closed: a,b in G implies a+b is in G;

(2) there is an identity 0;

(3) each element has an inverse in G;

(4) the operation is associative: (a+b)+c = a+(b+c) for all a,b,c;

(5) the operation is commutative: a+b = b+a for all a,b.

The following table of notation is common and definitions or explanations, if needed, can be found in the book Infinite Abelian Groups by Fuchs [2] .

$na = a+a+a+\ldots+a$ (n times)	multiple of a
$\lvert G \rvert$	order of a group G
$B \leq A$	B is a subgroup of A
$B < A$	B is a proper subgroup of A
$a+B$	coset of a modulo B
$\lvert A : B \rvert$	index of B in A
A/B	quotient group
$\langle a \rangle$	cyclic group generated by a
$\langle S \rangle = \langle a_i \rangle_{i \in I}$	subgroup of A generated by $S = \{a_i\}$, a subset of A

$\sum n_i a_i$ n_i integers	linear combination of the a_i's
$\lvert\langle a\rangle\rvert$	order of an element a
$H \approx G$	H is isomorphic to G
C_n	cyclic group of order n
Z_n	group of integers modulo n
Z	group of integers
Q	group of rationals
$\{B_i\}_{i\in I}$	family of groups or subgroups

DEFINITION 2.2 A homomorphism $f: G \to H$ is a function from one group G into another H with

$f(a+b) = f(a)+f(b)$ for all a,b in G.

f is a monomorphism if f is one to one and an epimorphism if f is onto.

DEFINITION 2.3. An isomorphism is a homomorphism which is also a one-to-one and onto correspondence. An endomorphism is a homomorphism from one group into itself.

DEFINITION 2.4. If f is a homomorphism from G into H, then the kernel of f is the set $\ker(f) = \{x \in G: f(x) = 0\}$ and the image of f is the set $\mathrm{im}(f) = \{y \in H: y = f(x) \text{ for some } x \in G.\}$

The following theorems are presented without proofs which in most cases are quite easy and straightforward. If necessary, the reader may refer to the book The Theory of Groups by Rotman [7] for details of the proofs.

THEOREM 2.1. If S is a subset of a group G, then S is a subgroup of G if and only if

(1) $0 \in S$,

(2) $a \in S$ implies $-a \in S$,

(3) $a, b \in S$ implies $a+b \in S$.

THEOREM 2.2. If S is a subset of a group G, then S is a subgroup of G if and only if S is non-empty, and whenever $a, b \in S$, then $a-b \in S$.

Using this criteria, it is easy to check that ker(f) and im(f), defined in Definition 2.4, are subgroups of G and H respectively.

THEOREM 2.3. The intersection of any family of subgroups of G is a subgroup of G.

DEFINITION 2.5. Let S and T be non-empty subsets of a group G. Then $S+T = \{s+t : s \in S \text{ and } t \in T.\}$

THEOREM 2.4. (Lagrange) If S is a subgroup of a finite group G, then $|G : S| = |G|/|S|$, that is, the order of S divides the order of G.

COROLLARY 2.5. If G is a finite group such that $|G| = p$ for any prime, p, then G is cyclic.

COROLLARY 2.6. If G is a finite group and $a \in G$, then $|\langle a \rangle|$ divides $|G|$.

DEFINITION 2.6. A subgroup B of a group A is fully invariant in case B is carried into itself under every endomorphism of A.

THEOREM 2.7. (First Isomorphism Theorem) Let $f: G \to H$ be a homomorphism with $\ker(f) = K$. Then $G/K \cong \mathrm{im}(f)$.

This theorem is extremely important and shows that there is no significant difference between a quotient group and the image of a group under a homomorphism.

DEFINITION 2.7. The function $f: G \to G/K$ defined by $f(a) = a+K$ is called the natural homomorphism of G onto G/K, where K is any subgroup of G.

THEOREM 2.8. (Second Isomorphism Theorem) Let S and T be subgroups of G. Then $S \cap T$ is a subgroup of S and

$$S/(S \cap T) \approx (S+T)/T.$$

THEOREM 2.9. (Third Isomorphism Theorem) Let $K \leq H \leq G$ where K and H are subgroups of a group G. Then H/K is a subgroup of G/K and

$$(G/K) \ / \ (H/K) \approx G/H.$$

DEFINITION 2.8. If H and K are subgroups of G such that

(1) $H+K = G$ and

(2) $H \cap K = 0$,

then G is the (internal) direct sum of H and K and is denoted by $G = H \oplus K$.

DEFINITION 2.9. A subgroup H of G is called a direct summand of G is there is a $K \leq G$ such that $G = H \oplus K$. In this case, K is a complimentary direct summand or simply a compliment of H is G.

THEOREM 2.10. If $G = H \oplus K$, then $G/H \approx K$, that is, the compliment of H in G is unique up to isomorphism.

DEFINITION 2.10. If H and K are groups, the (external) direct sum of H and K, denoted by $H \oplus K$, is the set of all ordered pairs (h,k), where $h \in H$ and $k \in K$, with the binary operation

$(h,k) + (h',k') = (h+h', k+k')$.

Now it is clear that if $G = H \oplus K$ is an external direct sum, it is also an internal direct sum of $H \oplus 0$ and $0 \oplus K$. Thus there is no distinction in notation and since the two ideas yield isomorphic groups, the use of direct sum usually does not include either adjective internal or external. It is useful to extend the idea of direct sum to a family of subgroups.

DEFINITION 2.11. Let $\{B_i\}_{i \in I}$ be a family of subgroups such that

(1) $\sum_{i \in I} B_i = A$ (the B_i's generate A)

(2) for every $i \in I$, $B_i \cap \sum_{i \neq j} B_j = 0$.

Then A is a direct sum of its subgroups B_i.

Finally this review is concluded with some elementary properties of homomorphisms.

Let $f: G \to H$ be a homomorphism. Then

(1) $f(0) = 0$

(2) $f(na) = nf(a)$ for all integers n

(3) (f/A), the mapping f restricted to a subgroup A of G, is a homomorphism from A into H.

CHAPTER III

FINITE ABELIAN GROUPS

1. THE BASIS THEOREM

DEFINITION 3.1. Let p be a prime. A group G is p-primary (or is a p-group) in case every element in G has order a power of p.

THEOREM 3.1. (Primary Decomposition) Every finite abelian group G is a direct sum of p-primary groups.

Proof: For any prime p, let G_p be the set of all elements in G whose order is a power of p. Now $0 \in G_p$ is non-empty. Furthermore if a,b are in G_p, then $p^m a = 0$ and $p^n b = 0$ for some integers m and n. Thus $p^{mn}(a-b) = 0$ and so a-b is in G_p so that G_p is a subgroup. Now it suffices to show that $G = \sum_{p \in P} G_p$ as p ranges over all primes p which divide the order of G. The criteria of definition 2.11. is now used to establish this fact.

(1) To show $G = \sum G_p$, let $x \in G$ and assume $x \neq 0$. Furthermore assume that x has order n. By the fundamental theorem of arithmetic, $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ where the p_i are distinct primes and $e_i \geq 1$. Let $n_i = n/p_i^{e_i}$ for each i and observe that the greatest common divisor of the n_i is 1. Therefore there exists integers m_i such that $m_1 n_1 + m_2 n_2 + \ldots + m_k n_k = \sum m_i n_i = 1$ and hence $(\sum m_i n_i)x = \sum (m_i n_i)x = x$. For each i, $p_i^{e_i}(m_i n_i x) = m_i n x = 0$ and so $m_i n_i x$ is in G_{p_i}. Hence for any x in G, x can be written as an element of $\sum G_p$ so that $G \leq \sum G_p$ and since $\sum G_p \leq G$ clearly, $G = \sum G_p$.

(2) Let $x \in G_p \cap \sum_{p \neq q} G_q$. Since $x \in G_p$, $p^e x = 0$ for some e; since

$x \in \sum G_q$, $x = \sum x_q$, where each $x_q \in G_q$. Then $q^{e_q} x_q = 0$ for each prime q and some exponent e_q. Set $t = \prod q^{e_q}$ and then $tx = t\sum x_q = 0$. Now (t, p^e) has greatest common divisor 1 so that there exists integers a and b with $ap^e + bt = 1$. Hence $x = ap^e x + btx = 0$ so that $G_p \cap \sum_{p \neq q} G_q = 0$. Thus, $G = \sum G_p$.

DEFINITION 3.2. The subgroups G_p of G are called the primary components of G.

DEFINITION 3.3. Let G be an abelian group and m a positive integer. Then $mG = \{mx: x \in G\}$.

This section is directed toward establishing that every finite group is a direct sum of cyclic groups (Basis Theorem). Because of theorem 3.1., it is sufficient to consider only the special case of finite p-primary groups. The proof is based on the following lemma which is much more powerful than is needed since it will be stated in the infinite version. However, it is quite useful to demonstrate an application of Zorn's Lemma and also will be referred to when the infinite groups are considered.

LEMMA 3.2. Let G be a p-group and assume that a is an element of maximal order p^k (that is, there is no other element in G of larger order than a). Then $\langle a \rangle$ is a direct summand of G.

Proof: First Zorn's Lemma is used to obtain H, a subgroup of G, maximal with respect to $H \cap \langle a \rangle = 0$. Let $\mathcal{C}$ be the collection of all subgroups of G whose intersection with $\langle a \rangle$ is only 0. Then $\mathcal{C}$ is non-empty since 0 is in $\mathcal{C}$. Partially order the elements in $\mathcal{C}$ by set inclusion and let $\{H_i\}_{i \in I}$ be any chain in $\mathcal{C}$. It should be clear that this chain has an upper bound in $\mathcal{C}$, namely the set-theoretic union of the H_i's. Hence Zorn's Lemma is applied to $\mathcal{C}$ to obtain H and let $G' = H \oplus \langle a \rangle$.

Clearly $G' \subseteq G$ and to show $G \subseteq G'$ an indirect proof is used. Suppose G is not a subset of G', then there exists $x \in G$ such that $x \notin G'$. Furthermore, since $x \in G$, for some i, $p^i x \in G'$, $(p^i x \neq 0)$; otherwise $\langle H, x \rangle \oplus \langle a \rangle = 0$ and this would contradict the maximality of H. Assume $px \in G'$, then $px = h + na$ where $h \in H$ and $n \in Z$. Also $p^{k-1}(px) = p^{k-1}na + p^{k-1}h = 0$ by maximality of the order of a. Hence $p^{k-1}na = 0$ so that $p^{k-1}n$ must be divisible by p^k, that is $n = pj$ for some integer j. So $px = pja + h$ and $p(x-ja) = h$ is in H, however, $x-ja$ is not in H.

Now $\langle H, x-ja \rangle \cap \langle a \rangle \neq 0$ since H is maximal in this property. Let ra be in the intersection. Thus $ra = h' + s(x-ja)$ where $h' \in H$ and $sx \in H \oplus \langle a \rangle$. Also $(s,p) = 1$ since $p(x-ja) \in H$ and $H \cap \langle a \rangle = 0$. Since sx, px are in G' and $(s,p) = 1$, then $x \in G'$, a contradiction. Thus $G \subseteq G'$ and so $G = H \oplus \langle a \rangle$.

THEOREM 3.3. (Basis Theorem) Every finite group G is a direct sum of cyclic groups.

Proof: Because of lemma 3.2., the proof is trivial. Assume G is p-primary (theorem 3.1.) and if in G an element of maximal order, a, is choosen, then $G = H \oplus \langle a \rangle$ where H is determined as in the proof of the lemma. Next, apply the same process to H which is of smaller order than G. Continuing in this manner, G can be represented as a direct sum of cyclic groups.

2. FUNDAMENTAL THEOREM OF FINITE ABELIAN GROUPS

It has been shown that every finite group is a direct sum of p-primary groups and furthermore is a sum of primary cyclic groups. However, the basic question of when two finite groups are isomorphic has still not been resolved. To do this a unique factorization theorem, analogous to the fundamental theorem of arithmetic is needed, where primary cyclic groups would correspond to prime numbers. This theorem does exist and is called the fundamental theorem of finite abelian groups. The following series of definitions and theorems will lead to this theorem.

DEFINITION 3.3. Let G be a group. Then the n-socle of G, denoted by $G[n]$, is the set of all elements g in G such that $ng = 0$.

COROLLARY 3.4. The n-socle of G is a fully invariant subgroup.

Proof: Let $a,b \in G[n]$. Then $na = nb = 0$ and hence $n(a-b) = 0$ so that $a-b \in G[n]$ and $G[n]$ is a subgroup of G. Also if $f: G \to G$ is a homomorphism and $f(a) \in f(G[n])$, then $na = 0$ and $n(f(a)) = f(na) = f(0) = 0$ by properties of homomorphisms. Hence $f(a) \in G[n]$ and thus $f(G[n]) \subseteq G[n]$ so that $G[n]$ is fully invariant.

The next definition is motivated by a desire to find a way to count the number of cyclic subgroups of a fixed order p^n of a finite p-primary group.

DEFINITION 3.4. If G is a finite p-primary group and if $n \geq 0$ is an integer, then $\bigcup(n,G) = d\left(\frac{p^nG \cap G[p]}{p^{n+1}G \cap G[p]}\right)$ where $d(H)$ is the dimension of H as a vector space over Z_p. Notice for $H = \frac{p^nG \cap G[p]}{p^{n+1}G \cap G[p]}$, $pH = 0$ and hence H is called an elementary p-primary group and it is easy to see that any two decompositions of H into a direct sum of cyclic groups have the same number of summands, denoted by $d(H)$.

Because of the technical nature of this definition, an illustration of its meaning is provided in the following example.

Let $G = C_p \oplus C_{p^3} \approx Z_p \oplus Z_{p^3}$ and if $p = 2$, then $G = Z_2 \oplus Z_8$.

Now $G[2] = \{(0,0), (1,0), (1,4), (0,4)\}$

$$2G = 2Z_2 \oplus 2Z_8 = 0 \oplus 2Z_8 = \{(0,2), (0,4), (0,6), (0,0)\}$$

$$4G = 0 \oplus 4Z_8 = \{(0,0), (0,4)\}$$

$$8G = 0 \oplus 8Z_8 = \{(0,0)\}.$$

So $U(0,G) = d\left(\dfrac{2^0 G \cap G[2]}{2G \cap G[2]}\right) = d\left(\dfrac{G[2]}{2G \cap G[2]}\right) = d\left(\dfrac{\{(0,0), (0,4), (1,0), (1,4)\}}{\{(0,0), (0,4)\}}\right)$

Now if $\{(0,0), (0,4)\} = H$, then the cosets of the factor group are

$$(0,0) + H = H \; ; \; (0,4) + H = H$$

$$(1,0) + H = (1,0) + H \; ; \; (1,4) + H = (1,0) + H.$$

Thus the quotient group has only two elements and hence is a vector space over Z_2 of dimension 1 since every vector space over a field has dimension equal to the number of copies of the field.

Now $U(1,G) = d\left(\dfrac{2^1 G \cap G[2]}{2^2 G \cap G[2]}\right) = d\left(\dfrac{\{(0,0), (0,4)\}}{\{(0,0), (0,4)\}}\right) = 0$

$$U(2,G) = d\left(\frac{2^2 G \cap G[2]}{2^3 G \cap G[2]}\right) = d\left(\frac{\{(0,0), (0,4)\}}{\{(0,0)\}}\right) = 1$$

$$U(3,G) = d\left(\frac{2^3 G \cap G[2]}{2^4 G \cap G[2]}\right) = d\left(\frac{\{(0,0)\}}{\{(0,0)\}}\right) = 0,$$

and so on.

In this example, $U(n,G)$ gave the number of cyclic summands of G of order p^{n+1} which is what was desired.

THEOREM 3.5. Let G be a finite p-primary group. Any two decompositions of G into direct sums of cyclic groups have the same number of summands of each order. In fact, the number of cyclic summands of order p^{n+1} is $U(n,G)$.

Proof: Let $G = \sum C_i$, where each C_i is a cyclic subgroup of G.

It is to be shown that the number of C_i of order p^{n+1} is $U(n,G)$ and to do so the following notation is used

$$G = \sum C_i = \sum Cp \oplus \sum Cp^2 \oplus \dots \oplus \sum Cp^t.$$

If there are no summands of order p^k, then $\sum Cp^k = 0$.

$$\text{Now } G[p] = \sum Cp[p] \oplus \sum Cp^2[p] \oplus \dots \oplus \sum Cp^t[p].$$
$$= \sum Cp \oplus \sum pCp^2 \oplus \dots \oplus \sum p^{t-1}Cp^t.$$
$$\text{Also } p^nG = p^n\sum Cp \oplus p^n\sum Cp^2 \oplus \dots \oplus p^n\sum Cp^t$$
$$= 0 \oplus 0 \oplus \dots \oplus p^n\sum Cp^{n+1} \oplus \dots \oplus p^n\sum Cp^t, \ n < t.$$

Then for all $n < t$

$$G[p] \cap p^nG = \sum p^nCp^{n+1} \oplus \sum p^{n+1}Cp^{n+2} \oplus \dots \oplus \sum p^{t-1}Cp^t$$
$$= \sum p^nCp^{n+1} \oplus G[p] \cap p^{n+1}G.$$

Thus $G[p] \cap p^nG/G[p] \cap p^{n+1}G \approx \sum p^nCp^{n+1}$ by theorem 2.10. Therefore $U(n,G) = d\left(\dfrac{G[p] \cap p^nG}{G[p] \cap p^{n+1}G}\right)$ is the number of cyclic summands of order p^{n+1}. Furthermore, since $U(n,G)$ is defined solely in terms of G and does not depend on any particular decomposition of G, this number is the same for any two decompositions.

COROLLARY 3.6. Let G and H be finite p-primary groups. Then $G \approx H$ if and only if $U(n,G) = U(n,H)$ for all $n \geq 0$.

Proof: Suppose $f: G \to H$ is an isomorphism. Now $G = \sum C_i$ where each C_i is cyclic by theorem 3.3. By theorem 3.5., $U(n,G)$ is the number of cyclic summands of order p^{n+1}. $H = f(G) = f(\sum C_i) = \sum f(C_i)$ and $f(C_i) \approx C_i$ for all i under the isomorphism. So for each n, there are $U(n,G)$ summands $f(C_i)$ of H of order p^{n+1}. But this number is precisely $U(n,H)$.

Conversely: If $U(n,G) = U(n,H)$ for all $n \geq 0$, then G is isomorphic to H because they have the same type of direct sum decomposition into cyclic groups. Hence any decomposition of H is a decomposition of G and the groups are clearly isomorphic.

LEMMA 3.7. Let G and H be finite groups and $f: G \to H$ be a homomorphism. Then for each prime p, $f(G_p) \subset H_p$.

Proof: Let $x \in G_p$ for a fixed prime p. Then $p^n x = 0$ for some integer n. Then $p^n f(x) = f(p^n x) = f(0) = 0$ by properties of a homomorphism. Hence $f(x) \in H_p$ so that $f(G_p) \subset H_p$.

The last two theorems are merely a restatement of theorem 3.5. and the corollary 3.6. in terms of general finite groups instead of p-primary groups. One can easily see that since every finite group is a direct sum of p-primary groups (theorem 3.1.), the theorems are essentially complete.

THEOREM 3.8. Let G and H be finite abelian groups. Then $G \approx H$ if and only if $G_p \approx H_p$ for all primes p.

THEOREM 3.9. (Fundamental Theorem of Finite Abelian Groups) Let G be a finite group. Then any two decompositions of G into a direct sum of primary cyclic groups have the same number of summands of each order.

This theorem concludes the presentation on finite groups. It is interesting to note that much of the early studies of group theory dealt almost exclusively with finite groups. In fact, the primary decomposition theorem and basis theorem were known to be proven in the 19^{th} century.

CHAPTER IV

INFINITE ABELIAN GROUPS

1. INTRODUCTION

In the early part of the 20^{th} century, the attention of researchers in group theory was directed to infinite abelian groups. At this time the structure of countable torsion groups was developed by H. Prüfer (1923), H. Ulm (1933) and L. Zippin (1935) [2]. In the theory of torsion-free groups the structure problem has been resolved only for special cases of torsion-free groups.

DEFINITION 4.1. Let G be an arbitrary abelian group. Then T denotes the set of all elements in G of finite order.

DEFINITION 4.2. A group G is torsion in case G = T and torsion-free in case T = 0, that is, G contains no elements of finite order other than 0.

THEOREM 4.1. T is a fully invariant subgroup of G and the factor group G/T is torsion-free.

Proof: T is a subgroup. Clearly T is non-empty since $0 \in T$. Let $a, b \in T$. Then $na = mb = 0$ for some positive integers m and n. Then $mn(a-b) = 0$ so that $a-b \in T$ and T is a subgroup of G.

T is fully invariant. Let $f: G \to G$ be an endomorphism and suppose $f(a) \in f(T)$. Then $na = 0$ for some integer n and $nf(a) = f(na)$ $= f(0) = 0$. Hence $f(a) \in T$ and $f(T) \subseteq T$.

G/T is torsion-free. It suffices to show that T is the only element of finite order in G/T. Let $a+T$ be an element in G/T of finite order m. Then $m(a+T) = T$ and $ma \in T$ so that there exists an integer n with $n(ma) = 0$. Hence $a \in T$ and $a+T = T$ so T is the only element of finite order in G/T.

Now the study of abelian groups can be split into three parts: 1) the classification of torsion groups, 2) the classification of torsion-free groups, and 3) the study of how the two are put together to form an arbitrary group.

As previously noted, much work has been done in the first of these parts and the following two theorems have counterparts in the theory of finite groups.

THEOREM 4.2. Any torsion group is a direct sum of p-primary groups.

Proof: As in theorem 3.1., let G_p be the primary component of a torsion group G. The proof of this theorem follows the proof of theorem 3.1. That is, the G_p generate G as p ranges over the primes and the intersection of G_p and $\cup G_q$ for $p \neq q$ is only 0.

THEOREM 4.3. Let G and H be torsion groups. Then $G \approx H$ if and only if $G_p \approx H_p$ for all primes p.

Proof: (Since this was not proven for the finite case, a proof is included here to take care of both situations.) Let $f: G \to H$ be an isomorphism. Then as in theorem 3.7., it is easy to show $f(G_p) \subset H_p$ and $g(H_p) \subset G_p$ where $g: H \to G$ is the inverse of f.

Then $f_p = (f/G_p)$ and $g_p = (g/H_p)$ are isomorphisms from G_p to H_p and H_p to G_p respectively. Hence $G_p \approx H_p$.

Conversely if $f_p: G_p \to H_p$ is an isomorphism for each p; then define $f: G \to H$ by $f(x_p) = f_p(x_p)$ and f then is an isomorphism.

Now the study of torsion groups reduces to the study of p-primary groups. It may not be clear to the reader that infinite torsion groups exist, however, they do since any direct sum of a finite group over an infinite index set is torsion and clearly infinite. One of the more interesting torsion infinite groups is $Z(p^{\infty})$ which plays a very important role in the next few sections. $Z(p^{\infty}) = \langle c_1 \mid pc_1 = 0;\ pc_2 = c_1;\ pc_3 = c_2;\ \ldots \rangle$ is an ascending union of finite cyclic groups. It is clearly torsion since some power of p will annihilate any element and also is infinite since the generators c_i are infinite.

2. DIVISIBLE GROUPS

Besides the groups that are direct sums of cyclic groups, another important class of groups are the divisible groups. In an abelian group, any element can be multiplied by an integer but "dividing" by an integer is a different story. The result may not exist in the particular structure or if it does, it may not be unique. The most common examples of divisible groups are the rationals and the real numbers. Although it is not obvious, $Z(p^{\infty})$ is also divisible. It shall be shown that every group is a direct sum of a divisible group and a reduced group.

DEFINITION 4.3. A group G is divisible if $nG = G$ for every integer $n \neq 0$ or equivalently if for each $x \in G$ and non-zero integer n, there exists $y \in G$ with $ny = x$.

DEFINITION 4.4. A group G is reduced if it has no non-trivial subgroups which are divisible.

It should be clear that a subgroup of a divisible group is not necessarily divisible since the subgroup Z of Q is not divisible. To better understand the concept of divisibility and to establish some results that will be helpful in the proofs of the theorems which follow, the following elementary consequences of divisibility are presented.

LEMMA 4.4. A quotient or homomorphic image of a divisible group is divisible.

Proof: Suppose $f: G \to H$ is a homomorphism from a divisible group G into H. Let $x \in f(G)$. Then $x = f(a)$ for some $a \in G$. Let n be any integer and since G is divisible, there exists $b \in G$ with $nb = a$. Then $nf(b) = f(nb) = f(a) = x$ and since $f(b) \in f(G)$, then x is divisible by n.

LEMMA 4.5. G is divisible if and only if $G = pG$ for each p.

This should be clear since for any integer n, n can be written as a product of primes and so $nG = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} G = G$.

LEMMA 4.6. If $\{G_i\}$ $i \in I$ is a family of divisible groups, then their direct sum $\sum G_i$ is divisible.

Proof: For any integer n, it is clear that $n \sum G_i \subseteq \sum G_i$. To show the reverse inclusion, let $x \in \sum G_i$. Then $x = \sum a_i$ where each $a_i \in G_i$. Now since the G_i are divisible, there exists $b_i \in G_i$ with $nb_i = a_i$ for each i. Let $b = \sum b_i$. Then $nb = n \sum b_i = \sum nb_i = \sum a_i = x$ and hence $x \in n \sum G_i$. Thus, $n \sum G_i = \sum G_i$ and the direct sum is divisible.

LEMMA 4.7. If G is divisible and H is a direct summand of G, then H is divisible.

Proof: Let $a \in G = H \oplus K$. Then a is divisible by n and hence $nb = a$ for some $b \in G$. Thus $n(h+k) = h'+k'$ where $a = h'+k'$ and $b = h+k$.

So $nh-h' = k'-nk = 0$ since $H \cap K = 0$. Hence $h' = nh$ and H is divisible.

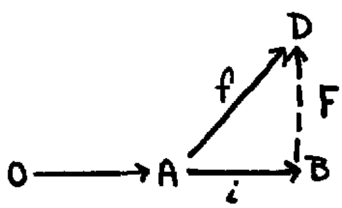
LEMMA 4.8. If G is torsion-free divisible, then $nx = a$ has a unique solution.

Proof: Suppose $nx = a$ and $ny = a$ for non-zero elements x and y in G. Then $n(x-y) = 0$ and since G is torsion-free, $x-y = 0$ or $x = y$.

With these results, the goal of classifying divisible groups is resumed.

DEFINITION 4.5. A group D is said to be injective if, given A a subgroup of B and a homomorphism f from A to D, f can be extended to a homomorphism F from B into D, and $f = F \circ i$ where i is the inclusion map.

The following diagram illustrates this definition:



The next theorem shows that divisible groups are exactly the groups which have this property.

THEOREM 4.9. A group G is divisible if and only if G is injective.

Proof: Suppose G is divisible and the above diagram is given. Consider $\mathcal{S} = \{(S,h) \mid S$ is a subgroup of B containing A and $h: S \to G$ extends $f\}$. Now $\mathcal{S}$ is non-empty since (A,f) is in $\mathcal{S}$. Partially order $\mathcal{S}$ by decreeing $(S_1,h_1) \leq (S_2,h_2)$ if S_1 is a subset of S_2 and, h_2 restricted to S_1, $(h_2 \mid S_1) = h_1$. Let $\{(S_i, h_i)\}_{i \in I}$ be a chain in $\mathcal{S}$ and it is to be shown that (S,h) where $S = \bigcup S_i$ and $h: S \to G$ defined by $h(s) = h_i(s)$ where $s \in S$ is in S_i for some i, is an upper bound for the chain.

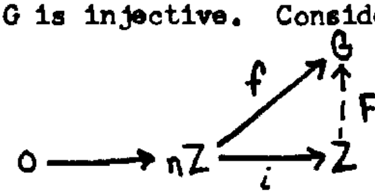
Clearly S is a subgroup of B and contains A. Also h extends f since each h_i extends f. Then $(S,h) \in \mathcal{S}$ and since S contains each S_i of the chain and h restricted to each S_i is exactly h_i, $(S_i,h_i) \leq (S,h)$ for each i. Thus the chain has an upper bound. Hence Zorn's Lemma may be applied to obtain a maximal pair (S_0, h_0).

It suffices to show that $S_0 = B$. Suppose there exists $b \in B$ such that $b \notin S_0$. Define $S' = S_0 + \langle b \rangle$. Let k be the smallest positive integer such that $kb \in S_0$. Now, every element y in S' has a unique expression $y = s_0 + tb$ where $0 \leq t \leq k$ since if $y = s_0 + tb = s_0' + t'b$, then $s_0 - s_0' = b(t'-t) \in S_0$. Thus $b(t'-t) = 0$ or $t' = t$ and hence $s_0 = s_0'$.

Let $c = kb$ and since $c \in S_0$, $h(c)$ is defined, and there exists $x \in G$ with $kx = h(c)$ since G is divisible. Define $h' : S' \to G$ by $h'(s_0 + tb) = h(s_0) + tx$. Then h' is a homomorphism and h' extends h. Then $(S_0,h_0) < (S',h')$, a contradiction of the maximal pair (S_0,h_0).

Now if $kb \notin S_0$ except when $k = 0$, then define $h' : S' \to G$ by $h'(s + tb) = h(s_0) + rx$ for any fixed $x \in G$ and $r \geq 0$ and again h' is a homomorphism extending h. Thus the same contradiction is demonstrated. Hence $S_0 = B$.

Conversely, suppose G is injective. Consider the diagram:



where $f(nz) = ng$ for $g \in G$, arbitrary but fixed. Then clearly f is a homomorphism.

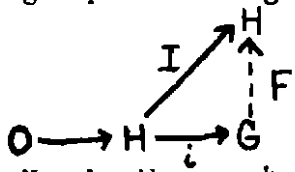
So by the injective property of G, there exists $F: Z \to G$ such that F extends f, that is $F \circ i = f$.

Hence $n(F(1)) = F(n) = F \circ i(n) = f(n) = g$ and so g is divisible by n. Therefore G is divisible.

With this result it is possible to show that a divisible subgroup of a group is a direct summand.

THEOREM 4.10. If H is a divisible group, then H is a direct summand of every group containing it.

Proof: Suppose G is any group containing H. Consider the diagram



where I is the identity map. Now by theorem 4.9., there exists F: $G \to H$ which extends I. Hence $F \circ i = I$ and so $F(a) = a$ for each $a \in G$. Then the proof is simply to show that $G = H \oplus \ker(F)$.

(i) $H \cap \ker(F) = 0$. Let $x \in H \cap \ker(F)$. Then $F(g) = x$ for some $g \in G$ on one hand, while $F(x) = 0$ on the other.

Then $x = F(g) = F(F(g)) = F(x) = 0$. Hence $x = 0$.

(ii) Let $x \in G$. Then $x = F(x) + x - F(x)$ and $F(x)$ is in H while $x - F(x)$ is in $\ker(F)$ since $F(x - F(x)) = F(x) - F(F(x)) = F(x) - F(x) = 0$. Thus $G = H \oplus \ker(F)$ and H is a direct summand.

THEOREM 4.11. Every group G is a direct sum of a divisible group D and a reduced group $R (G = D \oplus R)$.

Proof: Given a group G, consider D the subgroup generated by all divisible subgroups of G. Then D is divisible by lemma 4.6., and is called the maximal divisible subgroup. Hence by theorem 4.10., D is a direct summand of G and the complimentary summand R of D is reduced since it could not have any divisible subgroups.

Now theorem 4.11. reduces the classification of abelian groups to that of the divisible and reduced cases. Furthermore in the case of the divisible groups, the classification is completely known and presented in the following theorem. It shows that the only divisible groups are direct sums of the rationals Q and $Z(p^{\infty})$ for various primes p.

THEOREM 4.12. A divisible group G is a direct sum of groups each isomorphic to the rational numbers Q or $Z(p^{\infty})$ for some prime p.

Proof: Let T be the torsion subgroup of G and it will be shown that T is divisible. Let $x \in T$ and let n be an integer. Since G is divisible, there exists y in G with $ny = x$. Since $x \in T$, then $kx = 0$ for some k and hence $k(ny) = kx = 0$ and so $y \in T$. Thus for any integer n, there exists a solution in T to the equation $ny = x$ so T is divisible.

Then by theorem 4.10., T is a direct summand of G so that $G = T \oplus F$ where F must be isomorphic to G/T by theorem 2.10., and hence torsion-free by theorem 4.1. Furthermore F is divisible by lemma 4.7. Now the summands T and F will be studied separately.

The discussion of F will be carried out in standard vector space theory. Let x be any element in F and n a non-zero integer. Since F is divisible and torsion-free, there is a unique element y in F with $ny = x$, lemma 4.8. Thus the expression $(1/n)x = y$ is meaningful, as is $(p/q)x$ where p/q is any rational number, that is, $(p/q)x = py$, where $x = qy$. With this definition of scalar multiplication of elements in F by rational numbers, F becomes a vector space over the field Q. It is a routine exercise to check the vector space axioms. Thus, from a result in vector space theory, F is isomorphic to a direct sum of copies of Q over an index set I whose cardinality is the number of elements in a basis for F. Hence $F \approx \sum_{i \in I} Q_i$.

Now the divisible torsion group T, by theorem 4.2., is a direct sum of p-primary groups T_p and each direct summand T_p must again be divisible, Lemma 4.7. So for the remainder of the proof, assume that T itself is a p-primary group. Zorn's Lemma is used to show that T is a direct sum of groups isomorphic to $Z(p^{\infty})$.

Consider the collection of all subgroups of T which are isomorphic to $Z(p^{\infty})$. Since it is to be shown that T is a direct sum of such subgroups, it is necessary to consider only independent sets of such subgroups. Let B be the set of all collections of independent sets of subgroups of T isomorphic to $Z(p^{\infty})$. Hence each element in B is a collection of independent sets of subgroups which may be partially ordered by set inclusion. The proof that every chain in B has an upper bound in B is straightforward and so that Zorn's Lemma is applied to B to obtain a maximal independent set of subgroups of T isomorphic to $Z(p^{\infty})$, say $\left\{S_i\right\}_{i \in I}$.

Let $S = \sum S_i$ and the proof is completed by showing that S = T. Now S is divisible by Lemma 4.6., and so since S is a subgroup of T, $T = S \oplus R$ by theorem 4.10. So it must be shown that R = 0. In an indirect manner, assume $R \neq 0$ and let $x_1 \in R$ such that x_1 has order p.

Using the divisibility of R, there exists x_2 such that $px_2 = x_1$ and x_3 such that $px_3 = x_2$ and in general x_{n+1} with $px_{n+1} = x_n$. Then there is an obvious way of defining an isomorphism from the subgroup of R generated by the x_n's and $Z(p^{\infty})$, that is, $f(x_i) = C_i$, for each i where the C_i are the generators of $Z(p^{\infty})$. Hence R contains a subgroup isomorphic to $Z(p^{\infty})$, a contradiction. Thus R = 0 and T = S so that $T = \sum_{k \in K_p} Z(p^{\infty})_p$.

In the proof of the previous theorem, two sets of cardinal numbers were used: one for the number of rational summands and another for every prime, p, giving the number of summands of $Z(p^{\infty})$ for each p-primary summand of T. These cardinal numbers are invariants and form a complete set of invariants from which the divisible group D can be uniquely constructed. Hence, any divisible group D decomposes as follows:

$$D = \sum_{i \in I} Q_i \oplus \sum_{p \in P} \sum_{k \in K} Z(p^{\infty})_k .$$

The concept of a free group which has properties that are in a sense dual to those of divisible groups, will be discussed in the next section.

3. FREE ABELIAN GROUPS

DEFINITION 4.6. F is a free abelian group on $\{x_k\}_{k \in K}$ in case F is a direct sum of infinite cyclic groups Z_k, where $Z_k = \langle x_k \rangle$.

It should be clear that every non-zero element x of a free group F on $\{x_k\}$ has a unique representation $x = \sum_{k \in K} m_k x_k$ for non-zero integers m_k, since if $x = \sum m_k x_k$ and $x = \sum n_k x_k$, then $\sum (m_k - n_k) x_k = 0$ and hence $m_k = n_k$ for each k. So each element does have a unique expression, this result is stated in the following theorem.

THEOREM 4.13. If F is a free group on $\{x_k\}_{k \in K}$, then every non-zero element x in F is a unique linear combination of the x_k's, $x = \sum m_k x_k$ for non-zero integers m.

DEFINITION 4.7. The set of $\{x_k\}_{k \in K}$ is called a free set of generators of F and F_m will denote the free group of m free generators, that is, $F_m = \sum_{k \in K} Z_k$ and m is the cardinality of the set K.

THEOREM 4.14. The free groups F_m and F_n are isomorphic if and only if m = n for the cardinals m and n.

Proof: Suppose $m = n$. Then F_m and F_n are direct sums of the same number of infinite cyclic groups Z_k and are clearly isomorphic.

Conversely suppose $F_m \approx F_n$. Then let p be any prime and consider the quotient group F_m/pF_m. This group becomes a vector space over Z_p when a scalar multiplication is defined on F_m/pF_m by $n(x+pF_m) = nx+pF_m$ where n is in Z_p and the coset $x+pF_m$ is in F_m/pF_m and, the vector space axioms are verified. Hence F_m/pF_m has a basis which is claimed to be the set $\left\{x_i+pF_m\right\}$ where the x_i are the generators of F_m. It is clear that this set spans the vector space so it only needs to be shown that the set is independent.

Let $\sum m_i(x_i+pF_m)$ be in pF_m, the zero of F_m/pF_m, where each m_i is in Z_p and not all $m_i x_i$ are in pF_m. Then $m_i x_i$ is in pF_m and hence $m_i x_i = p(n_i x_i)$ where $pn_i = m_i$ for each i, that is, each m_i is divisible by p. Hence, each $m_i x_i$ is in pF_m, a contradiction, and so F_m/pF_m has a basis, $\left\{x_i+pF_m\right\}$. Thus, F_m/pF_m has dimension m and since the dimension of isomorphic vector spaces is an invariant, $m = n$.

DEFINITION 4.8. The rank of a free group F is the cardinal number associated with the number of elements in the set of free generators of F.

THEOREM 4.15. A set $\mathfrak{X} = \left\{x_i\right\}_{i\in I}$ of generators of a free group F is a free set of generators if and only if every mapping $f: \mathfrak{X} \to A$ where A is any group can be extended to a unique homomorphism $h: F \to A$.

Proof: Let $\mathfrak{X} = \left\{x_i\right\}_{i\in I}$ be a set of free generators of F. If $f: x_i \to a_i$ is any mapping from $\mathfrak{X}$ into A, then define $h: F \to A$ by $h(x) = h(\sum n_i x_i) = \sum n_i a_i$ where x has the unique representation as a linear combination of the x_i's, $x = \sum_{i\in I} n_i x_i$, by theorem 4.8. This unique representation is precisely why h is well-defined. To show h is a homomorphism, if $x = \sum n_i x_i$ and $y = \sum m_i x_i$ are in F, then

$$\begin{aligned} h(x+y) &= h(\sum n_i x_i + \sum m_i x_i) \\ &= h(\sum (n_i + m_i) x_i) \\ &= \sum (n_i + m_i) a_i = \sum n_i a_i + \sum m_i a_i . \end{aligned}$$

Thus h is a homomorphism and since h is defined in terms of f, h is unique for a given mapping f.

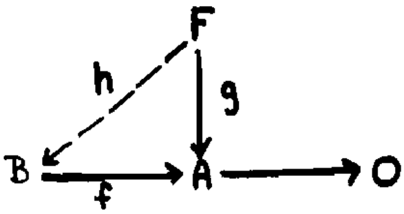
Conversely, assume that $\underline{\bar{X}} \subseteq F$ has the property that every mapping $f: \underline{\bar{X}} \to A$ can be extended to a homomorphism $h: F \to A$. Then let A be a free group with a free set of generators $\{y_i\}_{i \in I}$, where the index I is the same as that for $\underline{\bar{X}}$. Then the map $f: X \to A$ defined by $f(x_i) = y_i$, for each i, can be extended to a homomorphism $h: F \to A$ and furthermore h is obviously an isomorphism. Thus F is isomorphic to A and hence the set $\underline{\bar{X}}$ is a free set of generators of F.

THEOREM 4.16. Every abelian group G is a quotient of a free abelian group.

Proof: First it will be demonstrated that given any set $\underline{\bar{X}}$, there exists a free group F with $\underline{\bar{X}}$ as its basis. If $\underline{\bar{X}}$ is a set containing a single element x, an infinite cyclic group Z_x can be constructed that has x as its generator. In general, $F = \sum_{x \in X} Z_x$ and in particular for the group G, $F = \sum_{a \in G} Z_a$. Then F is free and G is a basis for F. The identity function $I: G \to G$ can be extended to a homomorphism $h: F \to G$ by theorem 4.15. Now h is clearly onto so that G is a quotient of F by theorem 2.7.

The next theorem shows that a free group F has the projective property, which is the dual to the injective property.

DEFINITION 4.8. A group F is projective if to each diagram



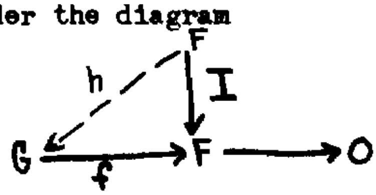
there exists a homomorphism $h: F \to B$ with $f \circ h = g$, where $f: B \to A$ is a homomorphism onto A and $g: F \to A$ is also a homomorphism.

THEOREM 4.17. If a group is free, then it is projective.

Proof: Let $f: B \to A$ be a homomorphism onto A and suppose F is free and $g: F \to A$ is also a homomorphism. Let $\overline{X} = \{x_i\}_{i \in I}$ be a basis for F. Since f is onto, for each i, there exists an element b_i in B with $f(b_i) = g(x_i)$. Then define a mapping $h': \overline{X} \to B$ by $h'(x_i) = b_i$. Then by theorem 4.15., there exists $h: F \to B$ where h is a homomorphism extending h', that is, $h(x_i) = h'(x_i) = b_i$. Furthermore $f \circ h = g$ since on the set of generators $\{x_i\}$ of F, $f \circ h(x_i) = f(b_i) = g(x_i)$. Hence F is projective.

COROLLARY 4.18. Let G be a group and let $f: G \to F$ be onto, where F is free. Then $G = \ker(f) \oplus S$, where $S \approx F$.

Proof: Consider the diagram



where I is the identity map. Since F is free, it is projective by theorem 4.17., and there exists a homomorphism $h: F \to G$ with $f \circ h = I$. Now h is one-to-one and so $S = \text{im}(h)$ is isomorphic to F. Claim that $G = \ker(f) \oplus S$. This is the same situation that was present in theorem 4.10., and so the mechanics will not be repeated and the proof is complete.

Another way of stating the above corollary is that G/K is free implies that K is a direct summand of G since quotients and homomorphic images are similar.

One might suspect that the converse of theorem 4.17. is true because of its similarity to the injective property of divisible groups. It can be proven. However, to do so, it must first be established that a subgroup of a free group is again free, the next theorem.

THEOREM 4.19. Every subgroup H of a free group F is free.

Proof: Let $\{x_i\}_{i \in I}$ be a basis of F, that is, $F = \sum \langle x_i \rangle$. Assume that the set K is well-ordered in some way (that every non-empty set can be well-ordered is an axiom of set theory called Zermelo's version of the axiom of choice and is equivalent to Zorn's Lemma).

For each $k \in K$, let $F_k = \sum_{j<k} \langle x_j \rangle$ and $H_k = H \cap F_k$. Thus $F = \bigcup F_k$ and $H = \bigcup H_k$. Also $H_k = F_k \cap H_{k+1}$ so that $H_{k+1}/H_k = H_{k+1}/H_{k+1} \cap F_k \approx (H_{k+1} + F_k)/F_k \subseteq F_{k+1}/F_k \approx Z$.

The first isomorphism is a result of theorem 2.8., where as the second is set up by the mapping of $x_{k+1} + F_k$ to 1, where $x_{k+1} + F_k$ is in F_{k+1}/F_k and 1 is the generator of Z. Since H_{k+1}/H_k is free, by corollary 4.18., H_k is a direct summand of H_{k+1}. So $H_{k+1} = H_k$ or $H_{k+1} = H_k \oplus \langle h_k \rangle$ where $\langle h_k \rangle \approx Z$. Then for each k an h_k is obtained which may or may not be 0. Claim that H is free on the set of h_k's.

Let H* be the set generated by the h_k's. Since $F = \bigcup F_k$, each $h \in H$ is in some F_{k+1}. Define $u: F \rightarrow K$ by $u(h) = k$ where $h \in F_{k+1}$ and $h \notin F_k$, $k < k+1$. Assume $H^* \neq H$ and consider $\{u(h): h \in H \text{ and } h \notin H^*\}$. Then there is a least such element j of the set K, for K is well-ordered. Choose h' in H with $u(h') = j$ and $h' \notin H^*$.

Then $h' \in H \cap F_{j+1}$ so that $h' \in H_{j+1} = H_j \oplus \langle h_j \rangle$, and $h' = a + mh_j$ where $a \in H_j$ and m is an integer.

CASE I Suppose $m = 0$ (that is, $H_{j+1} = H_j$). Then $h' = a$ and so $h' \in H_j$. Thus $\mu(h') < j$, a contradiction.

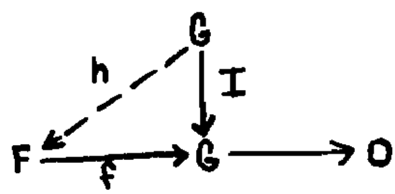
CASE II Suppose $m \neq 0$. Then $a = h' - mh_j$ is in H. Also a is not in H* since this would imply that h' is in H*. However $\mu(a) < j$ in this case, again, a contradiction.

Hence H = H*. All that is needed now is to show that linear combinations of the h_k's are unique, that is, if $\sum m_i h_{ki} = 0$, then each $m_i = 0$. Assume $m_i \neq 0$ for some i. Then $m_i h_{ki}$ is in $\langle h_{ki} \rangle \cap \bar{H}_{ki}$, a contradiction. So H is free on the h_k's.

With this theorem the converse of theorem 4.17. will now be established.

THEOREM 4.20. A group G that has the projective property is free.

Proof: Consider the diagram



where I is the identity map and $f: F \to G$ is a homomorphism of a free group F onto G which exists by virtue of theorem 4.16. Now by the projective property of G, there exists a homomorphism $h: G \to F$ with $f \circ h = I$. Now h is one-to-one and G is then isomorphic to a subgroup of F. Hence, by theorem 4.19., G is free.

With these results, attention is returned to divisible groups to present two results which ultimately prove the converse of theorem 4.10.

THEOREM 4.21. Every group G can be imbedded in a divisible group.

Proof: It is clear that Z can be imbedded into a divisible group, namely Q. Hence every free group F can be imbedded into a direct sum of copies of Q since F is a direct sum of infinite cyclic groups which are isomorphic to Z. Now given any group G, by theorem 4.16., $G \approx F/N$ for some free group F and subgroup N of F. Hence $G \approx F/N \subset \sum Q/N$ and the last group is divisible by lemma 4.4.

COROLLARY 4.22. A group G is divisible if and only if it is a direct summand of every group containing it.

Proof: Necessity is precisely theorem 4.10., as was noted earlier. To prove the sufficiency, first imbed G into a divisible group D via theorem 4.21. Then G is divisible since every direct summand of a divisible group is divisible, lemma 4.7.

4. FINITELY GENERATED ABELIAN GROUPS

DEFINITION 4.8. A group G is finitely generated in case there is a finite subset $\mathfrak{X}$ of G such that, the subgroup of G generated by the set $\mathfrak{X}$ is G.

It is clear that every finite group G is finitely generated, but in this section, it will be shown that the theorems that were proven in Chapter III concerning finite groups can now be proven for finitely generated groups. In particular, a Basis Theorem and a Fundamental Theorem for finitely generated groups will be established.

THEOREM 4.23. Every finitely generated torsion-free group G is free.

Proof: The proof is shown by induction on n, where $G = \langle x_1, \ldots, x_n \rangle$, that G is free. If n = 1, then $G = \langle x \rangle$ and since G is torsion-free, G is clearly an infinite cyclic group or 0 if x = 0. Thus G is free.

Induction hypothesis: Assume that for any group G generated by n-1 elements and also torsion-free, G is free. Define $\langle x_n \rangle_* = \{ y \in G:$

$my \in \langle x_n \rangle$ for some $m \neq 0\}$. Then it is easy to check that $\langle x_n \rangle_*$ is a subgroup of G and that $G/\langle x_n \rangle_*$ is torsion-free. Also $G/\langle x_n \rangle_*$ is generated by $\{x_1 + \langle x_n \rangle_*, x_2 + \langle x_n \rangle_*, \ldots, x_{n-1} + \langle x_n \rangle_*\}$, that is, n-1 elements and so by the induction hypothesis, $G/\langle x_n \rangle_*$ is free. By corollary 4.18., $G = \langle x_n \rangle_* \oplus K$ where K is isomorphic to $G/\langle x_n \rangle_*$ and hence is free. Thus it only needs to be shown that $\langle x_n \rangle_*$ is isomorphic to Z.

If y is in $\langle x_n \rangle_*$, then $my = kx_n$ for some $m \neq 0$. Define $f: \langle x_n \rangle_* \to Q$ by $f(y) = k/m$ where $my = kx_n$. Now f is well-defined since elements in $\langle x_n \rangle_*$ are in G and hence have unique representation. Also f is a homomorphism since if $my = kx_n$ and $m'y' = k'x_n$, then $mm'(y+y') = (km'+k'm)x_n$ and so $f(y+y') = (km'+k'm)/mm' = k/m + k'/m' = f(y) + f(y')$. Finally f is one-to-one, for if y is in ker(f), then $my = kx_n$ for some $m \neq 0$ and $0 = f(y) = k/m$. Hence $my = 0$ and so $y = 0$. That is, $\ker(f) = 0$.

Thus $\langle x_n \rangle_*$ is isomorphic to a subgroup H of Q. Let $H = \langle a_1/b_1, \ldots, a_t/b_t \rangle$ and let $b = \prod b_i$. Then define $f^*: H \to Z$ by $f^*(h) = by$. Again f^* is a well-defined homomorphism which is one-to-one so that H and hence $\langle x_n \rangle_*$ is isomorphic to a subgroup of Z. Thus $\langle x_n \rangle_*$ is free.

THEOREM 4.24. (Basis Theorem) Every finitely generated group G is a direct sum of cyclic groups.

Proof: G/T is a finitely generated torsion-free group by theorem 4.1., and hence G/T is free by theorem 4.23. So $G \approx T \oplus K$ where K is again free by corollary 4.18., and theorem 2.10. Hence K is a direct sum of infinite cyclic groups. Now T is a finite group since it is finitely generated and each generator has finite order. Therefore, T is a direct sum of cyclic groups by the basis theorem for finite groups, theorem 3.3.

THEOREM 4.25. (Fundamental Theorem of Finitely Generated Groups) Every finitely generated group G is a direct sum of primary and infinitely cyclic groups, and the number of summands of each kind depends only on G.

Proof: Now $G \approx T \oplus K$ where K is free. The uniqueness for T is precisely the fundamental theorem for finite groups, theorem 3.8.; the uniqueness of the number of infinite cyclic summands in theorem 4.14. Then the presentation on finitely generated groups is complete.

5. TORSION GROUPS: PURE SUBGROUPS

The main theorem in this section is a result known as Kulikov's theorem, that every torsion group G contains a basic subgroup. To establish this result, a very useful concept in abelian group theory, that of pure subgroups must be investigated. This notion is generally attributed to H. Prüfer and is an intermediate step between subgroups and direct summands. The value of these subgroups is their usefulness in proving the existence of direct summands.

DEFINITION 4.9. A subgroup H of a group G is pure in G if h in H and $h = ny$ for some integer n and y in G, imply the existence of h' in H with $h = nh'$. In other words, if an element of H is divisible by n in G, it must be divisible by n in H also.

For example, the subgroup $H = \{0,2\}$ of $G = Z_4 = \{0,1,2,3\}$ is not pure in G since 2 is a multiple of 2 in G but not in H. The following consequences are presented to help the reader understand more fully the concept of purity.

LEMMA 4.26. Any direct summand is pure.

This is clear since for H a subgroup of G to be divisible, its elements have to be divisible by every integer n. Consequently, every element in H is divisible by an integer n in H whenever they are divisible by n in G.

LEMMA 4.28. A pure subgroup of a divisible group is divisible.

Proof: Let x be in H a pure subgroup of a divisible group G. Then x is divisible by every integer n in G since G is divisible and hence for any n, there exists y in H with ny = x since H is pure in G. Thus, H is divisible.

LEMMA 4.29. The torsion subgroup of a group is pure.

Proof: Let x be in the torsion subgroup T of a group G and suppose x is divisible by n in G, that is ny = x for some y in G. Since x is in T, x has finite order, that is, mx = 0 for some integer m. Now m(ny) = mx = 0 so that y is in T also. Thus, T is pure in G.

LEMMA 4.30. Every ascending union of pure subgroups is pure.

Proof: Let $\{S_i\}_{i\in I}$ be an ascending chain of pure subgroups, that is, $S_k \subseteq S_{k+1}$ for all k in I. Consider $S = \cup S_i$ and let x be in S such that x is divisible by n in the group that contains the S_i's. Then there exists y in G with ny = x. Let j be the smallest such index that $x \in S_j$ and $x \notin S_{j-1}$. Now since S_j is pure in G, and x is divisible by n in G, there exists y' in S_j with ny' = x. Clearly then y' is in S and so S is pure in G.

LEMMA 4.31. Purity is transitive, that is, if K is pure in H and H is pure in G, then K is pure in G.

Proof: Suppose $k \in K$ is divisible by n in G, that is, there exists

$g \in G$ with $ng = k$. Now since $K \leq H$, $k = h$ for some $h \in H$ and so h is divisible by n in G. Furthermore, since H is pure in G, there exists $h' \in H$ such that $nh' = h = k$. Hence K is divisible by n in H and since K is pure in H, there exists $k' \in K$ with $nk' = k$. Thus K is pure in G.

Since many proofs to follow deal with quotient groups and cosets as elements in these quotient groups, the following convention is adopted: If G/S is a group and x is in G, then $\bar{x}$ is the corresponding element in G/S to x, that is, $\bar{x}$ represents the coset $x + S$.

LEMMA 4.32. Let S be pure in G and $\bar{y}$ be in G/S. Then there exists an x in G corresponding to $\bar{y}$ in G/S having the same order as $\bar{y}$.

Proof: Suppose $f: G \to G/S$ is any homomorphism onto G/S. Then if $\bar{y}$ has infinite order, then any element z such that $f(z) = \bar{y}$ will suffice. If $\bar{y}$ has finite order n, then first choose any z in G with $f(z) = \bar{y}$.

Then nz is in S and since S is pure in G, there exists h in S with $nz = nh$ (that is, nz is divisible by n). Let $x = z-h$ and x has the desired properties.

THEOREM 4.33. Let G be a group and H a pure subgroup of G; such that G/H is a direct sum of cyclic groups. Then H is a direct summand of G.

Proof: Suppose $f: G \to G/H$ is any homomorphism onto G/H. And suppose $G/H = \sum_{i \in I} \langle \bar{y}_i \rangle$. Then for each generator $\bar{y}_i$, there exists x_i in G with $f(x_i) = \bar{y}_i$ for each i and the order of x_i is the order of $\bar{y}_i$ by lemma 4.32. (An application of the axiom of choice is employed in the selection of the x_i for the index set I is taken to be well-ordered.) Let K be the subgroup of G generated by the x_i's. It suffices to show that $G = H \oplus K$.

(i) $G = H+K$. Let t be any element in G and suppose $f(t) = \bar{t}$ in G/H. Then $\bar{t} = \sum a_i \bar{y}_i$ for integral coefficients a_i. Then $f(t - \sum a_i x_i) = \bar{t} - \sum a_i \bar{y}_i = 0$ in G/H. Hence $t - \sum a_i x_i$ is in H and $t = (t - \sum a_i x_i) + \sum a_i x_i$ is in H+K.

(ii) $H \cap K = 0$. Let w be in $H \cap K$. Then $w = \sum b_i x_i$ since w is in K and further $\sum b_i \bar{y}_i = 0$ since w is in H. If $\bar{y}_i$ has infinite order, this means that $b_i = 0$; if y_i has finite order n_i, then b_i is a multiple of n_i. In either case, $a_i x_i = 0$ and so $w = 0$.

LEMMA 4.34. Let T be pure in G. If $T \subset S \subset G$, and S/T is pure in G/T, then S is pure in G.

Proof: Let s be in S and suppose $s = nx$ where x is in G. It must be proven that s is divisible by n in S. Let $\bar{s}$ and $\bar{x}$ be the corresponding cosets in G/T. Then $\bar{s} = n\bar{x}$, and by purity of S/T in G/T, $\bar{s} = n\bar{y}$ where $\bar{y}$ is in S/T. Let y be an element in S that maps to $\bar{y}$. Then $s = ny+t$ for some t in T. Hence $t = ny-nx$, so by purity of T in G, there exists t' in T with $t = ny-nx = nt'$. Thus $s = n(y-t')$ and since $T \subset S$, y-t' is in S, so S is pure in G.

LEMMA 4.35. A p-primary group G which is not divisible contains a pure cyclic subgroup.

Proof: First the fact that if the p-socle of G, $G[p]$, is not divisible, then there exists a y in G, such that $\langle y \rangle$ is pure, is proven. So let $x \in G[p]$ and assume that x is divisible by p^k and not p^{k+1}. Let $p^k y = x$ and claim that $\langle y \rangle$ is pure. It is sufficient to only check powers of p and multiples of y of the form $p^i y$ since if x in G has order n and $(m,n) = 1$, then x is divisible by n. Suppose $p^n z = p^i y$. Then $z = p^{i-n}y$ which is in $\langle y \rangle$ if $i-n \leq k+1$. If $i-n > k+1$, then $z = 0$, otherwise, x would be divisible by p^m where m is greater than k, a contradiction. Hence $\langle y \rangle$ is pure.

Next it is shown that if $G[p]$ is divisible, then G is divisible, and this contradiction completes the proof. Assume that every x in $G[p]$ is divisible by every power of p. The proof is by induction on k that if $p^k x = 0$, for any x in G, then x is divisible by p. If $k = 1$, then $px = 0$ and so x is in $G[p]$ and hence x is divisible by p.

The induction hypothesis states that if $p^k x = 0$, then x is divisible by p. Then suppose $p^{k+1}x = 0$. If $y = p^k x$, then y is in $G[p]$ and hence is divisible by p so that there exists z in G with $p^{k+1}z = y = p^k x$. Then $p^k(pz-x) = 0$ and by the induction hypothesis, there exists w in G, with $pw = pz-x$. Therefore, $x = p(z-w)$ and hence x in G is in pG. Hence by lemma 4.5., G is divisible. So by the above remarks, the proof is complete.

DEFINITION 4.10. A subset $\mathfrak{X}$ of non-zero elements of a group G is independent in case $\sum m_i x_i = 0$ implies each $m_i x_i = 0$, where x_i is in $\mathfrak{X}$ and m_i is an integer.

DEFINITION 4.11. A subset $\mathfrak{X}$ of G is pure-independent if $\mathfrak{X}$ is independent and $\langle \mathfrak{X} \rangle$ is pure in G.

LEMMA 4.36. Let G be a p-primary group. If $\mathfrak{X}$ is a maximal pure-independent subset of G (that is, $\mathfrak{X}$ is contained in no larger such subset), then $G/\langle \mathfrak{X} \rangle$ is divisible.

Proof: By lemma 4.35., if it is assumed that $G/\langle \mathfrak{X} \rangle$ is not divisible, then it contains a pure cyclic subgroup, $\langle \bar{y} \rangle$. Now since

$\langle X \rangle$ is pure in G, there exists y in G with the order of y and $\bar{y}$ the same by lemma 4.32. Now $X^* = \{X, y\}$ will be pure-independent. First of all, $\langle X \rangle \subset \langle X^* \rangle \subset G$ clearly and so $\langle \bar{X}^* \rangle / \langle X \rangle$ is isomorphic to $\langle \bar{y} \rangle$, which is pure in $G/\langle X \rangle$. Thus $\langle X^* \rangle$ is pure in G by lemma 4.34. Secondly, suppose $my + \sum m_i x_i = 0$, where $x_i \in X$ and m_i, m are integers. In $G/\langle x \rangle$, this equation becomes $m\bar{y} = 0$ so $my = o$ since y and $\bar{y}$ have the same order. Furthermore, since X is independent, each $m_i x_i = 0$, and so X^* is pure-independent, a contradiction of the maximality of X. Thus $G/\langle X \rangle$ is divisible.

DEFINITION 4.12. Let G be a torsion group. A subgroup B of G is a basic subgroup of G in case

(1) B is a direct sum of cyclic groups,

(2) B is pure in G,

(3) G/B is divisible.

Now, by using the previous lemmas, it can be shown that every torsion group contains a basic subgroup. The basic subgroup, B, allows the study of torsion groups to reduce to an extension problem of a direct sum of cyclic groups by a divisible group since B is a direct sum of cyclic groups and G/B is divisible.

THEOREM 4.37. (Kulikov) Every torsion group contains a basic subgroup.

Proof: If G is divisible, then B = 0 is a basic subgroup. If G is not divisible, then G contains pure-independent sets by lemma 4.35. Now purity is preserved in ascending unions and so is independence. Thus pure-independence is preserved. Therefore, a straightforward application of Zorn's Lemma to the collection of all pure-independent subsets of G yields a maximal pure-independent subset $\overline{X}$ of G. Then $B = \langle \overline{X} \rangle$ is a direct sum of cyclic groups since $\langle \overline{X} \rangle = \sum \langle x_i \rangle$ where $x_i \in \overline{X}$ follows immediately from the independence of the set $\overline{X}$. Thus by lemma 4.36., B is a basic subgroup.

This section is concluded with another lemma concerning the behavior of purity with respect to homomorphisms or quotient groups.

LEMMA 4.38. Let S be a pure subgroup of G with $nS = 0$. Then $(S+nG)/nG$ is pure in G/nG.

Proof: Assume that $\bar{x}$ in $(S+nG)/nG$ is divisible by m in G/nG so that $\bar{x} = m\bar{y}$ where $\bar{y}$ is in G/nG. Then let x and y correspond to $\bar{x}$ and $\bar{y}$ such that x is in S. Thus x and my differ by an element in nG, that is $x = my+nz$. If r is the greatest common divisor of m and n, then divide m and n by r to obtain $m = rm'$ and $n = rn'$. Now the greatest common divisor of m' and n' is 1 so that there exists integers a and b with $am' + bn' = 1$. Since $x = my + nz = r(m'n + n'z)$, x is a multiple of r in G. Thus there exists s in S with $x = rs$. Hence $x = rs = r(am'+bn')s = mas + nbs = mas$ since $nS = 0$. Now converting back to elements in the factor groups, there is $\bar{x} = m(a\bar{s})$ and so $\bar{x}$ is divisible by m in $S+nG/nG$.

6. Torsion Groups of Bounded Order

DEFINITION 4.13. A group G is of bounded order if it is torsion and there is a fixed upper bound to the orders of the elements.

Thus there must exist a positive integer n such that $nx = 0$ for all x or more simply $nG = 0$. Of course, any finite group is of bounded order, but an infinite torsion group can also be of bounded order. Take, for example, the direct sum of an infinite number of finite cyclic groups each of which has order 2.

It will be proven that any group of bounded order is a direct sum of cyclic groups. This, in a way, is the most satisfactory generalisation of theorems 3.4. and 4.25. The next lemma demonstrates an easy way to obtain a pure cyclic subgroup. Since it is not obvious that a cyclic summand can be constructed in a given group of bounded order, this will amply illustrate the advantage of a pure subgroup as a substitute for a direct summand.

LEMMA 4.39. Let G be a p-primary group with $p^rG = 0$ for some r. Let x be an element of order p^r in G. Then $\langle x \rangle$ is pure in G.

Proof: As in the proof of lemma 4.35., it is necessary to check only powers of p and multiples of x of the form p^ix. Suppose, then, that $p^ix = p^jy$ for some y in G (that is, p^ix is divisible by p^j in G). It must be shown that p^ix is divisible by p^j in $\langle x \rangle$. If $j \leq i$, then $y = p^{i-j}x$ which is in $\langle x \rangle$. If $j > i$, we have $0 = p^ry = p^{r-j}(p^ix)$ and so x has order p^{r-j+i}, a contradiction that x has order p^r. Thus $\langle x \rangle$ is pure in G.

At this point, enough information has been accumulated to show that from a finite group of bounded order, a cyclic direct summand of the group can be constructed. Lemma 4.39. gives a pure cyclic subgroup $\langle \bar{x} \rangle$ and by induction $G/\langle \bar{x} \rangle$ is a direct sum of cyclic groups; hence, by theorem 4.33., $\langle \bar{x} \rangle$ is a direct summand. However, this procedure does not lend itself to the infinite case since no inductive assumption can verify that $G/\langle \bar{x} \rangle$ is a direct sum of cyclic groups. Instead Kulikov's Theorem will be used together with the basic subgroup B, which was generated by a maximal independent-pure set.

THEOREM 4.40. A group of bounded order is a direct sum of cyclic groups.

Proof: Suppose G is of bounded order such that nG = 0 for some n. Then theorem 4.37. is applied to obtain B a basic subgroup of G. Thus, G/B is divisible and hence G/B = n(G/B). This last group contains the single coset 0+B = B, so that G/B = B and hence G = B. Then by definition of a basic subgroup, G = B is a direct sum of cyclic groups.

In the theory of finite groups, it was determined when two finite p-groups are isomorphic in terms of the number of cyclic summands of order p^{n+1}. The problem of when two infinite p-primary groups that are direct sums of cyclic groups are isomorphic will now be resolved. It is interesting that the answer is essentially the same as for the finite case.

DEFINITION 4.14. If G is p-primary, consider the vector space over Z_p : $G\{n\} = \dfrac{p^nG \cap G[p]}{p^{n+1}G \cap G[p]}$. Then $U(n,G)$ is the dimension of $G\{n\}$ as a vector space over Z_p and is called the n^{th} Ulm invariant.

Then if G and H are p-primary groups, G and H have the same Ulm invariants in case $G\{n\}$ and $H\{n\}$ have the same dimension for each $n \geq 0$. Notice that for infinite p-primary groups, $U(n,G)$ may be infinite.

THEOREM 4.41. Let G be a p-primary group that is a direct sum of cyclic groups. The number of summands of G isomorphic to the cyclic group of order p^{n+1} is the dimension of $G\{n\}$. More over, if H is p-primary and a direct sum of cyclic groups, then $G \approx H$ if and only if they have the same Ulm invariants.

Note: The proof of this theorem is essentially the same as the proofs of theorem 3.4. and 3.5. since the allowance that $\bigcup(n,G)$ may be infinite offers no obstacles.

Next it is shown that there does exist a situation when a pure subgroup is necessarily a direct summand. Before this is shown, a lemma will be presented which will aid in the proof of the theorem.

LEMMA 4.42. Let S and T be subgroups of G with $S \cap T = 0$ and suppose $S+T/T$ is a direct summand of G/T. Then S is a direct summand of G.

Proof: (This proof is set-theoretic.) Let R/T be the complimentary summand to $S+T/T$ in G/T. Then $(S+T)+R = G$ and $R \cap (S+T) = T$. It suffices to show that $G = S \oplus R$. Since $T \subset R$, $S+R = S+T+R = G$ and hence S and R generate G. Also $(R \cap S) \subset R \cap (S+T) = T$ and hence $R \cap S \subset T \cap S = 0$. Thus $G = R \oplus S$.

THEOREM 4.43. Let G be a group and S a pure subgroup of bounded order. Then S is a direct summand of G.

Proof: Suppose $nS = 0$ for some n. Then by lemma 4.38., $(S+nG)/nG$ is pure in G/nG. Also G/nG is clearly of bounded order since $n(G/nG) = nG$. Hence G/nG is a direct sum of cyclic groups by theorem 4.40. Then by theorem 4.33. $(S+nG)/nG$ is a direct summand of G/nG. Next $S \cap nG = nS = 0$ so that we may now apply lemma 4.42., with nG playing the role of T. Hence S is a direct summand of G.

As a special case of this theorem, consider the torsion subgroup T of any group G. Now T is always pure by lemma 4.29., and hence T is a direct summand of G if T is also of bounded order. Furthermore, since every divisible subgroup is a direct summand, it can now be said that T is a direct summand of G if T is a direct sum of a divisible group and a group of bounded order. These remarks are now stated in the form of a theorem.

THEOREM 4.44. Let G be a group and T its torsion subgroup. Then T is a direct summand if (i) T is of bounded order or (ii) T is a direct sum of a divisible group and a group of bounded order.

The final result in this section deals with groups which are indecomposable, that is, they cannot be written as a direct sum except in the trivial way, $G = G \oplus 0$.

THEOREM 4.45. An indecomposable group cannot be mixed, that is, it is either torsion or torsion-free.

Proof: Assume that G is an indecomposable mixed group. Then the torsion subgroup T is not divisible since this would force T to be a direct summand by theorem 4.10. So by lemma 4.34., T contains a pure cyclic subgroup, say $\langle x \rangle$. Now since x is in T, x has finite order and thus $\langle x \rangle$ is of bounded order. Also $\langle x \rangle$ is pure in G by lemma 4.31. Hence by theorem 4.43., $\langle x \rangle$ is a direct summand, a contradiction. Hence G is not mixed, so G is either torsion or torsion-free.

Recall that in theorem 4.12., it was shown that the torsion subgroup of a divisible group, G, is divisible and isomorphic to copies of $Z(p^{\infty})$. Thus if an indecomposable torsion group G is divisible, then it is isomorphic to $Z(p^{\infty})$; where as if it is reduced, it is a cyclic group and so all indecomposable torsion groups have been determined. However, the classification of torsion-free indecomposable groups is quite a different story and in fact, an unsolved problem.

7. TORSION-FREE GROUPS

DEFINITION 4.15. The rank of a torsion-free group G is the number of elements in a maximal independent subset of G.

Note that a free abelian group is torsion-free and its rank is the number of infinite cyclic summands, or the cardinality of the index set of the set of generators. It is easy to see that the two notions of rank agree for these groups. In this section, the torsion-free groups of rank 1, that is groups such as the integers Z and rationals Q will be classified. At the present time, there is not even an adequate classification of groups of finite rank and so only the groups of rank 1 will be considered.

LEMMA 4.46. Every torsion-free group G can be imbedded in a vector space V over Q.

First G is imbedded in a divisible group D by theorem 4.16., and then consider the natural homomorphism from D onto D/T where T is the torsion subgroup of D. Now D/T is torsion-free and is isomorphic to copies of Q.

LEMMA 4.47. A torsion-free group G has rank at most r if and only if G can be imbedded in an r-dimensional vector space over Q.

With these lemmas, the study of rank 1 torsion-free groups begins by realizing that they are isomorphic to a subgroup of Q. The following are non-isomorphic subgroups of Q.

G_1: all rationals whose denominator is square-free,

G_2: all rationals of the form $m/2^k$, that is, dyadic rationals,

G_3: all rationals whose decimal expression if finite (that is, whose denominators are powers of 10). Together with Z and Q, these groups are all non-isomorphic subgroups of Q and one might observe that they can all be expressed by the numbers allowed in the denominators of elements of the groups.

Let $p_1, p_2, \ldots p_n, \ldots$ be the sequence of primes.

DEFINITION 4.16. A characteristic is a sequence $(k_1, k_2, \ldots k_n, \ldots)$ where each k_n is a non-negative integer or ∞.

If G is a subgroup of Q and if x is in G and $x \neq 0$, then x determines a characteristic in the following way: in the n^{th} component of the characteristic of x, place the highest power of the prime p_n that divides x in G, that is, the largest non-negative integer k such that there is an element y in G with $p_n{}^k y = x$. If there is no largest such k, set $k = \infty$. Some more advanced students of group theory might recognize that k_n is the p_n-height of x. The concept of height will be discussed in greater detail in the next section.

It is convenient to write each non-zero integer as a formal infinite product of primes $\prod p_i{}^{a_i}$ where p_i ranges over all primes p and $a_i \geq 0$. Let $m = \prod p_i{}^{a_i}$ and $n = \prod p_i{}^{b_i}$ be given integers. If x in G has characteristic $(k_1, k_2, \ldots, k_n, \ldots)$, then the definition of characteristic states that there exists y in G satisfying $my = nx$ if and only if $a_i \leq k_i + b_i$ for all i.

It will now be demonstrated how to determine a characteristic for the element $x = 1$ in $G = Z \subset Q$. For $p_1 = 2$, the largest non-negative integer k, such that, there exists y in G satisfying $2^k(\bar{y}) = 1$ is $k = 0$. Likewise for $p_2 = 3$, the largest k satisfying $3^k(y) = 1$ is $k = 0$ and for $p_3 = 5$, $5^0(y) = 1$, has solution $y = 1$. Continuing for each prime p_i, k_i is 0 and hence the characteristic of 1 in Z is $(0,0,0,\ldots,0,\ldots)$. The characterists of 2 in Z is $(1,0,0,\ldots,0,0,\ldots)$ and for 12 in Z is $(2,1,0,0,\ldots,0,\ldots)$.

If, however, $G = Q$, then for $x = 1$ in Q, x has characteristic $(\infty, \infty, \infty, \ldots, \infty, \ldots)$ since Q is a divisible group. Thus it is clear that distinct non-zero elements of the same group G may give rise to distinct characteristics. So the following definition is given:

DEFINITION 4.17. Two characteristics are equivalent if

(i) they have ∞ in the same coordinates and

(ii) they differ in at most a finite number of coordinates.

Then this definition of equivalence is an equivalence relation (that is, it satisfies the reflexive, symmetric and transitive properties); an equivalence class of characteristics is called a type. The next theorem states that the characteristics of distinct elements of a subgroup of Q are equivalent.

LEMMA 4.48. Let G be a subgroup of Q and let x and x' be non-zero elements of G. Then the characteristics of x and x' are equivalent.

Proof: First if $x' = mx$ for some integer, then the characteristics of x and x' are equivalent because:

(i) x' is divisible by every power of the prime p_i that divides x (plus only a finite number more) and

(ii) x' is divisible by every power of p_i if and only if x is. Hence their characteristics have ∞ in the same coordinates and differ in at most a finite number of coordinates.

Now for the more general case, since G is a subgroup of Q, there are integers m and n with $mx = nx'$. The characteristic of x is equivalent to the characteristic of $mx = nx'$ which is equivalent to that of x'.

DEFINITION 4.18. As a result of this lemma, if G is a torsion-free group of rank 1, then define the type of G, denoted by t(G), as the type of any non-zero element of G.

THEOREM 4.49. Let G and G' be torsion-free groups of rank 1, then $G \cong G'$ if and only if t(G) = t(G').

Proof: Suppose $f: G \to G'$ is an isomorphism. Then if x is in G, the characteristic of x and f(x) are equivalent. In fact, they are the same for if ∞ is in the n^{th} coordinate slot of the characteristic of x, then there is no largest power of the prime p_n which divides x and so there could not be a largest power of p_n that divides f(x) under the isomorphism. Likewise, if k is any non-zero integer in the i^{th} slot of the characteristic of x, k is in the i^{th} slot of the characteristic of f(x) due to the fact that divisibility is preserved under a homomorphism, lemma 4.4. Hence t(G) = t(G').

Conversely, assume t(G) = t(G') where G and G' are torsion-free groups of rank 1 and hence are subgroups of Q. If g and g' are non-zero elements in G and G' respectively, then their characteristics $(k_1, k_2, \ldots, k_n, \ldots)$ and $(k'_1, k'_2, \ldots, k'_n \ldots)$ differ in only a finite number of coordinates. Set the notation $\infty - \infty = 0$ and define the rational number l by $l = \prod p_i^{k_i - k'_i}$. Notice that $k_i - k'_i = 0$ for almost all i.

Define $f: G \to Q$ by $f(x) = \mu x$ where $\mu = lg/g'$ and note that f is a homomorphism since $f(x+y) = \mu(x+y) = \mu x + \mu y = f(x)+f(y)$. Now any rational number x is in G if and only if there are integers $m = \prod p_i^{a_i}$ and $n = \prod p_i^{b_i}$ where $mx = ng$ and $a_i \leq b_i + k_i$ for all i; likewise a rational number y is in G if and only if there are integers m and n with $my = ng'$ and $a_i \leq b_i + k'_i$ for all i.

Claim that $f(G) \subset G'$. Let x be in G, then $mx = ng$ and $a_i \leq b_i + k_i$;

hence $m(\mu x) = n(\mu g) = nlg'$. Since $a_i \leq (b_i+k_i-k'_i)+k'_i$, it follows that $\mu x = f(x)$ is in G'. In a similar manner, if $h: G' \to Q$ is defined by $h(x') = \mu^{-1}x'$, then it can be shown that $h(G') \subset G$. Therefore, f and h are inverses so that $G \approx G'$.

The final theorem of this section shows that for any type, t, there is a group of rank 1 whose type is exactly t. By taking any representative characteristic from t, say $(k_1, k_2, \ldots, k_n, \ldots)$, define G to be the subgroup of Q generated by all rationals of the form $1/m$, where, for all n, $p_n^{\,t}$ divides m if and only if $t \leq k_n$. It is clear that G is torsion-free as a subgroup of Q and that the maximal independent sets of elements in Q are the singleton sets so that G has rank 1. Also the element 1 in G has the given characteristic for the largest power of p_n that divides 1 is exactly k_n for each n, by definition of m above. Hence, the following theorem has been proven.

THEOREM 4.50. If t is a type, then there exists a group G of rank 1 with $t(G) = t$.

So for torsion-free groups of rank 1, the importance of the characteristic has been demonstrated as well as the fact that all torsion-free groups of rank 1 are subgroups of Q.

8. ULM'S THEOREM

The main theorem presented in this section is Ulm's Theorem and it accomplishes the complete classification of countable reduced torsion groups. The theorem does not state that a countable reduced torsion group looks like a particular group as is the case of the divisible groups where theorem 4.12. classifies the divisible groups as a direct sum of copies of Q and $Z(p^{\infty})$. Rather, a complete set of invariants is defined so that it is possible to determine when two such groups are isomorphic. Again, by theorem 4.2., any torsion group decomposes into its p-primary components and so throughout this section, assume that G and H are countable reduced p-primary groups.

The theorem uses both the cardinal and ordinal numbers in a very essential way. Definition 4.14. of the n^{th} Ulm invariant used only the natural numbers and so the n^{th} Ulm invariant, $U(n,G)$, is a function from the natural numbers to the cardinal numbers. This definition can be extended to the transfinite ordinals in the following manner.

Let $G_n = p^n G$ $(n = 0,1,2,3,\ldots)$. Then $G_{n+1} = pG_n$; $G_w = \bigcap_{n<w} G_n$ where w is a limit ordinal and again, $G_{w+1} = pG_w$. Thus, for any ordinal α, $G_{\alpha+1} = pG_{\alpha}$, and if α is a limit ordinal $G_{\alpha} = \bigcap_{\beta<\alpha} G_{\beta}$. Hence the chain $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_w \supset G_{w+1} \ldots$ is a decreasing chain of subgroups.

DEFINITION 4.19. The first ordinal λ such that $G_{\lambda} = 0$ is called the length of G (Note: There does exist such a λ for every p-primary countable reduced group.).

Now, in order to emphasize the use of the ordinal numbers, definition 4.14. is stated in slightly different notation.

DEFINITION 4.20. For each ordinal α, define $f_G(\alpha) = \dim\left(\frac{G_{\alpha}[p]}{G_{\alpha+1}[p]}\right)$. Then $f_G(\alpha)$ is called the α^{th} Ulm invariant and f_G is a function from the ordinals to the cardinals.

It has already been shown that if G is a direct sum of cyclic groups, then $f_G(n)$ is the number of cyclic summands of G isomorphic to the cyclic group of order p^{n+1}. Thus direct sums of cyclic groups are completely characterized by the Ulm invariants (theorem 4.41.).

DEFINITION 4.21. Let x be in G. The height of x in G, $ht(x)$, is α if x is in G_α but not in $G_{\alpha+1}$.

Thus this definition assigns to each non-zero element x in G a well-defined ordinal less than λ, the length of G. As for the element 0, it is desirable to write $ht(0) = \infty$ with the understanding that ∞ exceeds any ordinal. The following lemma states some fundamental inequalities concerning height which follow immediately from the definition.

LEMMA 4.51. Let x,y be in G and p be a fixed prime.

(a) If $ht(x) < ht(y)$, then $ht(x+y) = ht(x)$.

(b) If $ht(x) = ht(y)$, then $ht(x+y) \geq ht(x)$.

(c) If $x \neq 0$, then $ht(px) > ht(x)$.

The proof of the lemma is not difficult and is left to the reader.

THEOREM 4.52. (Ulm's Theorem) Two countable reduced p-primary groups G and H are isomorphic if and only if $f_G(\alpha) = f_H(\alpha)$ for each ordinal α, that is, they have the same Ulm invariants.

The proof that the condition is necessary, that is, if G and H are isomorphic, then they have the same Ulm invariants, is the easy direction since the dimension of isomorphic vector spaces is an invariant. The other direction is quite complicated so a brief outline will be given and then additional definitions and important lemmas will be presented in order to make the proof as clear as possible.

The idea of the proof is roughly this: Choose two sequences in G and H, say $\{0 = x_0, x_1, x_2, x_3, \ldots\}$ and $\{0 = y_0, y_1, y_2, y_3, \ldots\}$ where $G = \langle x_i \rangle$ and $H = \langle y_i \rangle$. This can be done since G and H are countable. Now suppose $U \leq G$ and $V \leq H$ are finite subgroups with $\phi : U \to V$ a height preserving isomorphism. Thus for any x in U, $ht_G(x) = ht_H(\phi(x))$. The heart of the proof then lies in being able to extend ϕ to larger subgroups U* and V* of G and H respectively.

DEFINITION 4.22. Let S be a subgroup of G and x be an element in G. Then x is proper with respect to S if $ht(x) \geq ht(x+s)$ for all s in S, that is, x has maximal height in the coset x+S.

It is easy to see that in this case, $ht(x+s) = \min\{ht(x), ht(s)\}$ for each s in S since $ht(x) \geq ht(x+s)$.

LEMMA 4.52. Let $\phi: U \to V$ be a height preserving isomorphism between subgroups U and V of G and H respectively. Suppose that x and y are proper with respect to U and V respectively, and also that px is in U and py is in V with $ht(x) = ht(y)$. Then the map $\theta: U+Zx \to V+Zy$ defined by $\theta(u+nx) = \phi(u)+ny$ is a height preserving isomorphism that extends ϕ.

Proof: To show that θ is well-defined suppose $u+nx = u'+mx$ or $u-u' = (m-n)x$ which is in $U \cap Zx$. Now p is the order of x+U since px is in U and so $(m-n) = sp$ for some integer s. Thus $\theta(u-u') = s\theta(px) = spy = (m-n)y$. Hence $\phi(u)+ny = \phi(u')+my$, that is $\theta(u+nx) = \theta(u'+mx)$.

Now θ is onto since ϕ is onto and $\theta(x) = y$.

θ is one-to-one: Suppose $u+nx$ is in $\ker\theta$, that is, $\theta(u+nx) = \phi(u)+ny = 0$. Thus $ny = -\phi(u)$ which is in V. Thus p divides n so $u+nx$ is in U. Hence $u+nx$ is in $U \cap \ker\theta = \ker\phi = 0$. Hence $\ker\theta = 0$ and θ is one-to-one.

Finally to show θ is height preserving: consider $\theta(u+nx) = \phi(u)+ny$. If p divides n, then $u+nx$ is in U and so $ht(u+nx) =$

$ht(\phi(u+nx)) = ht(\theta(u+nx))$ since ϕ is height preserving. Therefore assume that p and n are relatively prime where $1 = tn + up$ and t and p are relatively prime. Then let $u_0 = tu - upx$, which is in U, and hence $t(u+nx) = u_0+x$. Now $ht(u+nx) = ht(t(u+nx)) = \min\{ht(u), ht(x)\}$ since t and p are relatively prime and x is proper with respect to U. Similarly $ht(\phi(u)+ny) = ht(t(\phi(u)+ny)) = \min\{ht(\phi(u_0)), ht(y)\}$. Now since $ht(\phi(u_0)) = ht(u_0)$ and $ht(x) = ht(y)$, it is clear that $ht(u+nx) = ht(\theta(u+nx))$ so that θ is height preserving.

If it were not for the fact that x in G-U may not be proper with respect to U, lemma 4.52. would be quite valuable in the proof of Ulm's Theorem almost immediately. This problem can be easily solved since in the extension process, U will be finite at each step and so it will be possible to find $x' = x+u$ in x+U with x' proper in U and px in U. Now the problem is to find a y in H-V which has the desired properties stated in the lemma, that is, y is proper with respect to V, $py = \phi(px)$ and $ht(x) = ht(y)$. The next two lemmas show that such a y can be found.

Before lemma 4.53. can be stated, the following notation is needed. Let $U \leq G$. Then $U_\alpha = U \cap G_\alpha$; $p^{-1}G_{\alpha+2} = \{x \mid px \text{ is in } G_{\alpha+2}\}$; $U^*_\alpha = U_\alpha \cap p^{-1}G_{\alpha+2}$. So for any x in U_α , px is in $U_{\alpha+1}$. Usually this is all that can be said, however, there are some elements that are carried "past" $U_{\alpha+1}$ and this set is called U^*_α . Now for any x in U^*_α , px may be written as px' where x' is in $G_{\alpha+1}$ and thus px = px'. Since x is in G_α and x' is in $G_{\alpha+1}$, then x-x' is in $G_\alpha[p]$ and is used to define the homomorphism $T: U^*_\alpha \rightarrow G_\alpha[p]/G_{\alpha+1}[p]$, that is, $T(x) = (x-x')+G_{\alpha+1}[p]$ is the mapping that takes x in U^*_α to x-x' in $G_\alpha[p]$ followed by the natural homomorphism from $G_\alpha[p]$ into $G_\alpha[p]/G_{\alpha+1}[p]$

Now ker(T) is exactly $U_{\alpha+1}$ so that $T^*: U^*_\alpha / U_{\alpha+1} \rightarrow G_\alpha[p]/G_{\alpha+1}[p]$ is a monomorphism.

LEMMA 4.53. The statements (a) T^* is not onto and (b) there exists w in $G_\alpha[p]$ such that ht(w) = α and w is proper with respect to U, are equivalent.

Proof: (a) implies (b). Suppose $w+G_{\alpha+1}[p]$ is not in the range of T^*. Then w is not in $G_{\alpha+1}[p]$ and so ht(w) = α since w is in $G_\alpha[p]$. To show w is proper with respect to U, suppose the contrary. Then there exists u in U with $ht(u-w) > \alpha$. Since $u-w$ is in $G_{\alpha+1}$, $u-w = pt$ where t is in G_α. Since pw = 0, $pu = p(u-w) = p^2t$ which is in $G_{\alpha+2}$ so that u is in U^*_α. Now, applying the definition of T^* to the coset $u+U_{\alpha+1}$, $T^*(u+U_{\alpha+1}) = w+G_{\alpha+1}[p]$, a contradiction since $w+G_{\alpha+1}[p]$ was assumed to be not in the range of T^*. Thus w is proper with respect to U.

(b) implies (a). Suppose w is in $G_\alpha[p]$, ht(w) = α and w is proper with respect to U. Then $w+G_{\alpha+1}[p]$ is not in the range of T^* since, if it were, there would exist x in U and y in $G_{\alpha+1}$ such that $p(x-y) = 0$ and $w+G_{\alpha+1}[p] = (x-y) + G_{\alpha+1}[p]$. Hence $ht(x-w) > \alpha$, a contradiction that w is proper with respect to U.

The following situation is set up to help the reader understand the proof of Ulm's Theorem and to demonstrate the need of the next lemma.

Suppose U and V are finite subgroups of G and H respectively and x is in G-U. Assume that px is in U (if px is not in U, but $p(p^n x)$ is in U, then redefine x as $p^n x$). Consider the elements in x+U and suppose that $\{x+u_1, x+u_2, \ldots, x+u_k\}$ are the elements in x+U with maximal height α. These elements can be found since U is finite.

Among these, find one, say $x+u_t$, such that $ht(p(x+u_t))$ is maximal. Now redefine x as $x+u_t$.

Thus (1) x is in G-U, (2) px is in U, (3) $ht(x) = \alpha$, (4) x is proper with respect to U, (5) ht(px) is maximal, and finally (6) $\phi(px) = z$ where $\phi: U \to V$ is a height preserving isomorphism. Now, with this situation given, it is necessary to find y in H-V with py = z, $ht(y) = \alpha$ and y is proper with respect to V so that lemma 4.52. can be applied to extend ϕ.

LEMMA 4.54. Given the situation just described, it is possible to find y in H-V with $py = z = \phi(px)$, $ht(y) = \alpha$ and y is proper with respect to V.

Proof: Two cases must be considered.

Case I $ht(z) = \alpha+1$. Now neither z nor px are zero since ht(0) $= \infty$ which is larger than any cardinal. In this case, any element y in H_α with py = z will suffice to prove the theorem.

First $ht(y) = \alpha$ since if $ht(y) > \alpha$, then $ht(py) = ht(z) > \alpha+1$, a contradiction.

Second, y is not in V since if it were, then $\phi(w) = y$ where w is in U. Thus px = pw since $\phi(pw) = py = \phi(px)$. Also x-w is not in U lest x be in U. Furthermore, $ht(x-w) = \alpha$ since $ht(w) = \alpha$ and x is proper with respect to U. But $ht(px-pw) = ht(0) = \infty$ which is greater than $\alpha+1$. This is a contradiction of the maximality of ht(px). So y is not in V. Finally to show y is proper with respect to V, assume that it is not, that is, suppose $ht(y+v) \geq \alpha+1$ where v is in V and $\phi(u) =$ v for some u in U. Since y is not in V, then $y+v \neq 0$ and so $ht(py+pv) \geq \alpha+2$. Therefore $ht(px+pu) \geq \alpha+2$. Now v must have height at least α and so does u since ϕ is height preserving. Then $ht(x+u) = \alpha$ which again contradicts the maximality of ht(px). Hence y is proper with respect to V.

Case II ht(z) $> \alpha+1$. ht(px) $> \alpha+1$ means that px = pw for some w in $G_{\alpha+1}$. Then x-w is in $G_\alpha[p]$ and ht(x-w) = α so that x-w is proper with respect to U since ht(x) = α and ht(w) $\geq \alpha+1$. Now part b of lemma 4.53., is satisfied by x-w so that part a is also true. Since $U^*_\alpha / U_{\alpha+1}$ is finite and T* is not onto, part a of lemma 4.53., then the dimension of $U^*_\alpha / U_{\alpha+1}$ as a vector space over Z_p is strictly less than the Ulm invariant $f_G(\alpha)$. Since ϕ is a height preserving isomorphism, U_α is mapped on V_α, likewise U^*_α is mapped onto V^*_α and $U^*_\alpha / U_{\alpha+1}$ is mapped onto $V^*_\alpha / V_{\alpha+1}$. Therefore the dimension of $V^*_\alpha / V_{\alpha+1}$ is less than $f_G(\alpha)$ which equals $f_H(\alpha)$ by hypothesis.

Applying lemma 4.53. again, there exists an element y_1 in H with $py_1 = 0$, ht(y_1) = α and y_1 is proper with respect to V. Next noting that ht(z) $> \alpha+1$, z = py_2 where y_2 is in $H_{\alpha+1}$. Taking $y = y_1+y_2$, y has the properties that py = z, ht(y) = α and y is proper with respect to V.

Now the proof of the sufficient condition of Ulm's Theorem can be given. Thus, if G and H are countable reduced p-primary groups such that they have the same Ulm invariants, then they are isomorphic.

Proof: Since both G and H are countable, let $G = \{0 = x_0, x_1, x_2, \ldots\}$ and $H = \{0 = y_0, y_1, y_2, \ldots\}$.

Step 1: Let U = V = 0 and ϕ: U→V be a height preserving isomorphism. Assume x_1 satisfies the hypothesis of lemma 4.54. Then there exists y_{k_1} in H with the properties described in the conclusion of lemma 4.54. Now by lemma 4.52., ϕ can be extended to a height preserving isomorphism ϕ_1: $U_1 \to V_1$ where $U_1 = \langle 0, x_1 \rangle$ and $V_1 = \langle 0, y_{k_1} \rangle$.

Step 2: Assume that y_{k_2} is the first element in H not in V_1 and satisfies the hypothesis of lemma 4.54 where the height preserving isomorphism is the inverse of ϕ_1 obtained in step 1, that is, $\phi_1^{-1}: V_1 \to U_1$. By lemma 4.54., there exists x_{k_2} in G with the properties stated in the conclusion of the lemma. By lemma 4.52., ϕ_1^{-1} can be extended to $\phi_2: V_2 \to U_2$ where $V_2 = \langle V_1, y_{k_2}\rangle$ and $U_2 = \langle U_1, x_{k_2}\rangle$.

Step 3: Assume x_{k_3} is the first element in G not in U_2 and satisfies the hypothesis of lemma 4.54., where $\phi_2^{-1}: U_2 \to V_2$ is the height preserving isomorphism. Then there exists y_{k_3} in H and so again ϕ_2^{-1} can be extended to $\phi_3: U_3 \to V_3$ where $U_3 = \langle U_2, x_{k_3}\rangle$ and $V_3 = \langle V_2, y_{k_3}\rangle$.

Step 4: Assume y_{k_4} is the first element in H not in V_3 and proceed as in step 2.

Using this alternation between G and H, that is, in the 2n-1 step, consider the n^{th} element of G and in the $2n^{th}$ step consider the n^{th} element in H, the isomorphism between G and H can be established in the following manner.

First of all, recall that ϕ_1 is a function from a subset, U_1, of G onto a subset, V_1, of H so that ϕ_1 is a subset of the cartesian product of G and H, $G \times H$. Also ϕ_2 is a function from a subset, V_2, of H onto a subset, U_2, of G so that ϕ_2 is a subset of the cartesian product, $H \times G$. Now the inverse of ϕ_2, ϕ_2^{-1}, is a subset of $G \times H$ and contains ϕ_1 because U_2 contains U_1 and V_2 contains V_1. ϕ_3 is a subset of $G \times H$ which contains ϕ_2^{-1} and ϕ_4^{-1} is a subset of $G \times H$ which contains ϕ_3. Thus, $\phi_1 \subset \phi_2^{-1} \subset \phi_3 \subset \phi_4^{-1} \subset \ldots \subset \phi_{2n-1} \subset \phi_{2n}^{-1} \subset \ldots$ is an increasing chain of subsets of $G \times H$. Now in order to obtain the isomorphism between G and H, define the mapping ϕ^* from G into H as the union of the set of all ϕ_{2n-1}'s with the set of all ϕ_{2n}^{-1}'s, that is $\phi^* = (\bigcup_{n\in N} \phi_{2n-1}) \cup (\bigcup_{n \in N} \phi_{2n}^{-1})$.

Now any element x_k in G is a member of some subset U_k of G by induction and thus is in the domain of some function ϕ_k. Hence the domain of ϕ^* is all of G. Furthermore, because of the alternation between G and H in the steps of the proof, any element y_t in H is a member of some subset V_t of H and thus is in the domain of some function ϕ_t. Hence y_t is in the range of ϕ_t^{-1} and so the range of ϕ^* is all of H.

9. EXTENSIONS OF ULM'S THEOREM

At the time it first appeared, Ulm's Theorem was considered to be the most striking result yet obtained in the theory of abelian groups. Since that time much work has been done in extending Ulm's Theorem to larger classes of reduced p-primary groups. This section will state some of the more important generalizations of Ulm's Theorem. No attempt will be made at proving these extensions since the proofs are in general quite difficult, however, a reference is given where the details can be found.

Kolettis [5] extended Ulm's Theorem to direct sums of countable reduced p-primary groups. The general idea of the proof of Ulm's Theorem for direct sums of such groups was to get a canonical decomposition of such a group that is uniquely determined by the Ulm invariants of the group. More specifically, Kolettis proved the following.

THEOREM 4.55. Let the reduced primary group G be an uncountable direct sum of countable groups and let $f_G(\alpha)$ be the α^{th} Ulm invariant of G. Then $G = \sum K_\beta$ where the summation is over those ordinals β such that $f_G(\beta) \neq 0$ and for such a β, the group K_β is the unique reduced countable primary group whose Ulm invariants are given by:

$f_{K_\alpha}(\alpha) = 0$ if $\alpha > \beta$ or if $\alpha < \beta$ and $f_G(\alpha) < \aleph_0$; $f_{K_\beta}(\alpha) = \aleph_0$ if $\alpha < \beta$ and $f_G(\alpha) > \aleph_0$; $f_{K_\beta}(\beta) = \inf(f_G(\beta), \aleph_0)$.

The next extension of Ulm's Theorem that will be mentioned involves a class of groups called totally projective groups. First the concept of nice subgroups will be defined and then the totally projective groups.

DEFINITION 4.23. A subgroup A of G is nice if every coset of A has a representative of the same height, that is, for each g in G, ht(g+A) = ht(g+a) for some a in A, where the two heights are computed in G/A and in G respectively.

DEFINITION 4.24. A reduced p-group G is totally projective if it has a system L of nice subgroups such that

(a) 0 is in L

(b) the subgroup generated by any subset of L is in L

(c) if S is in L and A/S is countable, then there exists B in L with $B \supseteq A$ and B/A countable.

Now countable reduced p-groups are totally projective, since if G is such a group, simply let $L = \{0, G\}$. More generally if $G = \sum_{i \in I} G_i$ with each G_i countable, then $L = \{\sum_{i \in J} G_i,\ J \subseteq I\}$ is a nice system of subgroups satisfying the conditions that make G totally projective.

Thus totally projective groups contain the two classes of groups for which Ulm's Theorem has been proven. Next, Ulm's Theorem was extended to totally projective groups of length less than Ωw by Parker and Walker [6] and then to all totally projective groups by Hill [3]. Walker [9] presented a proof that simplifies Hill's proof for totally projective groups by giving one that is in essence the same as the proof of Ulm's Theorem for the countable case presented in the previous section. After some preliminary lemmas and definitions, Walker defined the α^{th} Ulm invariant of G relative to A and then proved the main theorem which is essentially the same as lemma 4.52.

DEFINITION 4.25. Let A be a subgroup of G and α be an ordinal. The α^{th} Ulm invariant of G relative to A is

$$f_{G,A}(\alpha) = \dim((p^{\alpha} G)[p]/A(\alpha)).$$

THEOREM 4.56. Let G/A and H/B be totally projective with A and B nice subgroups such that $f_{G,A}(\alpha) = f_{H,B}(\alpha)$ for each α. Then any height preserving isomorphism $\emptyset : A \to B$ extends to an isomorphism $G \to H$.

With this theorem, Ulm's Theorem for totally projective groups becomes quite simple. Notice that this theorem is quite a bit more powerful than lemma 4.52. in the previous section, however, the very same approach was used in the proof to obtain this result.

THEOREM 4.57. Two totally projective groups are isomorphic if and only if they have the same Ulm invariants.

The most recent extension of Ulm's Theorem was completed by Warfield [10] , who extended Ulm's Theorem to a class of mixed modules called the KT-module.

DEFINITION 4.26. If λ is a limit ordinal, then M is a λ-elementary KT-module if $p^{\lambda} M \approx R$, where R is a discrete valuation ring, regarded as a module over itself, and $M/p^{\lambda} M$ is a reduced torsion totally projective module.

The theorem that Warfield proved is the following.

THEOREM 4.58. If A and B are KT-modules, then A and B are isomorphic if and only if for all ordinals α and λ, $f_A(\alpha) = f_B(\alpha)$ and $h_A(\lambda) = h_B(\lambda)$ where $h_A(\lambda) = \dim (V_\lambda(A))$ where $V_\lambda(A) = p^\lambda A/p^{\lambda+1}A+T_\lambda$ where T_λ is the torsion submodule of $p^\lambda A$.

It is easy to see that this last extension of Ulm's Theorem takes the reader clearly beyond the scope of this paper but was presented to give a more complete picture of the work that has resulted from the study of abelian groups.

SUMMARY AND SUGGESTIONS FOR FURTHER STUDY

As stated in the introduction, this paper is by no means an exhaustive treatment of the theory of abelian groups. However, it is fairly complete in the study of the general classification and decomposition of theorems without being too technical or abstract. The material covered was intended to convey a considerable amount of information concerning the basic ideas, methods and fundamental results of abelian group theory. There are as many unanswered and unsolved problems in this area as one might expect.

The interested reader has many directions of further study. One might be in following the progress of Ulm's Theorem and examining the different proofs of the same theorem. There is room for a very detailed exploration in this area. The final theorem in section 9, Chapter IV, was concerned with an algebraic structure other than the group structure. It might be interesting to see which results or theorems presented in this paper could be extended to other structures such as the modules or commutative rings. Also there are many results dealing specifically again with abelian groups which have not been presented in this paper, as well as whole new concepts. For example, nice subgroups were defined in section 9, Chapter IV, however very little was established about them. It might be interesting to try to prove whether direct summands are nice or if finite subgroups are nice, and so on.

Thus it is easy to see that there is as much to consider for the interested reader as has been covered in this paper.

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