THE GEOMETRY OF THE IMAGINARY SPHERE

A Thesis

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by

Allen L. Ringer

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Thomas Bonner

Approved for the Major Department

Harold E. Truitt

Approved for the Graduate Council

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CHAPTER ONE

Introduction

Modern technology has made great advances since the end of World War II. The areas of plastics and medical science have the greatest notice, but increasing success in thermonuclear energy will be among the most significant in history. Suddenly modern man has made great accomplishments to improve his own efficiency and increase his capabilities on earth, under the sea, and in space.

The speed of modern advances in technology are a direct result of the advances of mathematics. The creation of new abstract mathematical spaces that have been developed and studied logically has provided the basis for the other sciences and engineering to move on. As physicists and chemists found new properties for existing material and developed new substances and principles, mathematics already had encountered spaces with the characteristics of many of them. Mathematicians have created abstract, logical spaces and discovered the characteristics of these spaces simply as a form of mental exercise or just to see what would result. Spaces have been created, developed and studied without models. When applications did arise, practical study was minimized with the understanding of the abstract space. In no small sense, mathematics can take a great deal of credit for the modern technological boom.

The evolution of mathematics into an abstract science has covered centuries and volumes of seemingly unproductive but useful work. The end result has been the beautiful logical world of mathematics defined by Bertrand Russell as "the subject in which we never know what we are talking about, nor whether what we are saying is true."[1]

The mathematical area of geometry has led in this evolution. From the earliest of times and civilizations, it seems that more interest has centered on geometry related areas.[2] A great many of the principles of plane goemetry and other areas were discovered and assumed without logical foundation. Euclid, finally, began the organization of geometry into a logical science.[3] He organized geometry into a deductive system based on a few axioms (or postulates) which he considered to be self-evident or unmistakably true propositions. Following Euclid's work, other geometers began to refine and simplify his system. The result was nearly twenty centuries of work, some of which was shown to be incorrect, and a sudden break-through for abstract geometry. This led to the organization of all mathematics into a purely abstract logical science.

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[1]Eric Temple Bell, Mathematics, Queen and Servant of Science (New York: McGraw-Hill Book Company, Inc., 1951), p. 17.

[2]David Eugene Smith, History of Mathematics, Vol. I: General Survey of the History of Elementary Mathematics (New York: Dover Publications, Inc., 1958), pp. 549-570.

[3]Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart and Winston, 1964), p. 21.

The refinement process on Euclid's system centered on checking the list of self-evident statements to determine whether any of them could be logically deduced from the others. Those that could were removed from the list and called theorems. The list was finally reduced to five statements.

The five statements that remained were sufficient to develop all of the theorem properties of geometry. But the question remained, were they all necessary? To answer this question, most mathematicians were satisfied that the first four statements could be called axioms, but the fifth remaining statement continued to be a puzzle. The statement, "If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced infinitely, meet on that side on which the angles are together less than two right angles,"[4] could have several possible attractions. Possibly, simply, its length or the concept it communicated caused it to be suspect. The concept was acceptable, but mathematicians believed that it was possible to resolve a proof of this statement from the others.

In the effort to prove the fifth remaining Euclidean axiom, no acceptable proof was found using only the first four axioms. It was discovered that, using this axiom and the other four, other properties could be proven. If one of these other properties were substituted for the fifth axiom,

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[4]Harold E. Wolfe, Introduction to Non-Euclidean Geometry (New York: Holt, Rinehart and Winston, 1945), p. 4.

the property could not be proven. But if accepted, the substituted property would allow for the proof of the fifth axiom. It was finally decided that the fifth axiom and the other properties were, logically, equivalent. With the first four accepted statements and either the fifth axiom or one of its equivalent statements, geometry was a complete system. But without one of the statements the system was not complete.

The question could have easily rested resolved at this point. But, perhaps, out of spite and frustration at so much time and effort spent without significant results, a great experiment occurred that caused the area of abstract, logical mathematics to open up to discovery.

The first four axioms and the fifth axiom or one of its equivalent statements could be, intuitively, accepted as being true on a plane surface. One of these equivalent statements has come to be called the parallel postulate. The parallel postulate states that, "through a given point, not on a given line, one, and only one line can be drawn that does not intersect the given line."[5]

It was apparent, at this point, that five axioms would have to be accepted without proof. While the axioms were obviously true, the notion grew that since they were accepted without proof, they might not actually have to be true. An experiment was attempted.

At the beginning of the nineteenth century, after so much frustrated effort around the fifth axiom, it is only

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[5]G. Y. Rainich and S. M. Dowdy, Geometry for Teachers (New York: John Wiley and Sons, Inc., 1968), p. 34.

natural that it received the force of the new venture. Since the axioms had to be accepted as true, perhaps by accepting a false statement as true there would be a break-down that could then justify the truth of the axioms. This concept could be an indirect proof of the truth of the axioms. Since it was an effort to prove the fifth axiom or one of its equivalent statements, a false statement or negation could be substituted which would lead to a contradiction of one of the other axioms and thus prove the desired statement. The parallel postulate became the axiom that was to be contradicted.

The logical negation of through a given point, not on a given line, there is one, and only one, line that does not intersect the given line would have to be, through a given point, not on a given line, there is no line that does not intersect the given line or there is more than one line that does not intersect the given line. The structure of the negation of the parallel postulate made it necessary for the indirect proof to be done in two parts.

The statements would have to be separately substituted for the parallel postulate to give two separate contradictions. The first statement, through a given point, not on a given line, there is more than one line that does not intersect the given line, when reaching a contradiction would prove that there was at most one non-intersecting line. The second part of the indirect proof would begin by substituting, through a given point, not on a given line, there are no lines that do not intersect the given line. When a contradiction was reached using this statement, thus proving it false, it would be shown that there was at least one non-intersecting line.

The two parts of the indirect proof would, separately, show that there was at most one non-intersecting line and at least one non-intersecting line. By putting the two together, it would then be determined that, with the given conditions, there was one, and only one, non-intersecting line. The parallel postulate would have been proven.

The proof was outlined and was, logically, correct. The process of constructing the actual proof began. At this point, the problem began to break-down again. In their efforts, the mathematicians were logically developing and proving other statements that were not true on the Euclidean plane, but they could not get the desired contradiction. They began to realize that something more important had taken place. Two new consistent spaces had been developed.

Several important decisions and realizations, now took place. The result has been modern abstract mathematics. The five axioms of Euclid, that were left, were logically independent. That is, one of them could not be proven from the other four. The five remaining axioms were complete. By using only the five independent axioms, all of the Euclidean properties could be developed. Thus any set of axioms, discriptive of an undefined set of elements, only had to be independent and complete to develop a logical space. Truth was not a factor. The two sets of axioms used to develop the indirect proof of the parallel postulate were independent and complete, since two consistent spaces had developed. Geometry, now, had its first two non-Euclidean spaces. Abstract, logical mathematics was here without physical application. Other abstract spaces were developed and continue to come into existence, without models, due to acceptance of these facts.

The two new geometries were to be called Hyperbolic Geometry and Elliptic Geometry. Hyperbolic Geometry was the result of the first set of axioms including, "through a given point, not on a given line, there is more than one line that does not intersect the given line."[6] Elliptic Geometry came from the second set of axioms whose characteristic fifth property was, "through a given point, not on a given line, there are no lines that do not intersect the given line."[7]

With the creation of abstract, logical spaces has come a need for abstract models of the spaces. The models provide a more practical understanding of the logical system and an earlier recognition of what system has application to a physical or practical situation.

One model space that has been suggested by Lambert is a sphere with an imaginary radius.[8] The sphere or hemisphere with a real radius is known to be a model of Elliptic Geometry.[9] If the radius of a sphere is changed to an imaginary number, a new model is developed that is suspected to be a model of a hyperbolic system. The problem is to develop the

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[6]Wolfe, Non-Euclidean Geometry, p. 66.

[7]Ibid. p. 174.

[8]Rainich and Dowdy, Geometry for Teachers, p. 141.

[9]Wolfe, Non-Euclidean Geometry, p. 178.

study of the sphere of imaginary radius and demonstrate, directly, that it is a model for Hyperbolic Geometry. Can such a space be defined and developed and a direct proof provided that satisfies the characteristic postulate of Hyperbolic Geometry?

Solving this problem requires the development of the space and defining of points and lines in the space. The points and lines must be shown to be well defined. This will be done by demonstrating that they satisfy two axioms characteristic of points and lines in the hyperbolic system. A demonstration that two distinct points determine one and only one line, and that two distinct lines intersect in, at most, one point will satisfy this requirement. Next, the direct proof that this space satisfies the characteristic postulate of Hyperbolic Geometry: through a given point, not on a given line, there is more than one line that does not intersect the given line.

In the work to solve these problems are examples and properties which illustrate methods of working on and with the imaginary sphere. These examples and proofs may be useful in developing an understanding of the sphere of imaginary radius and make it more useful in non-Euclidean study.

CHAPTER TWO

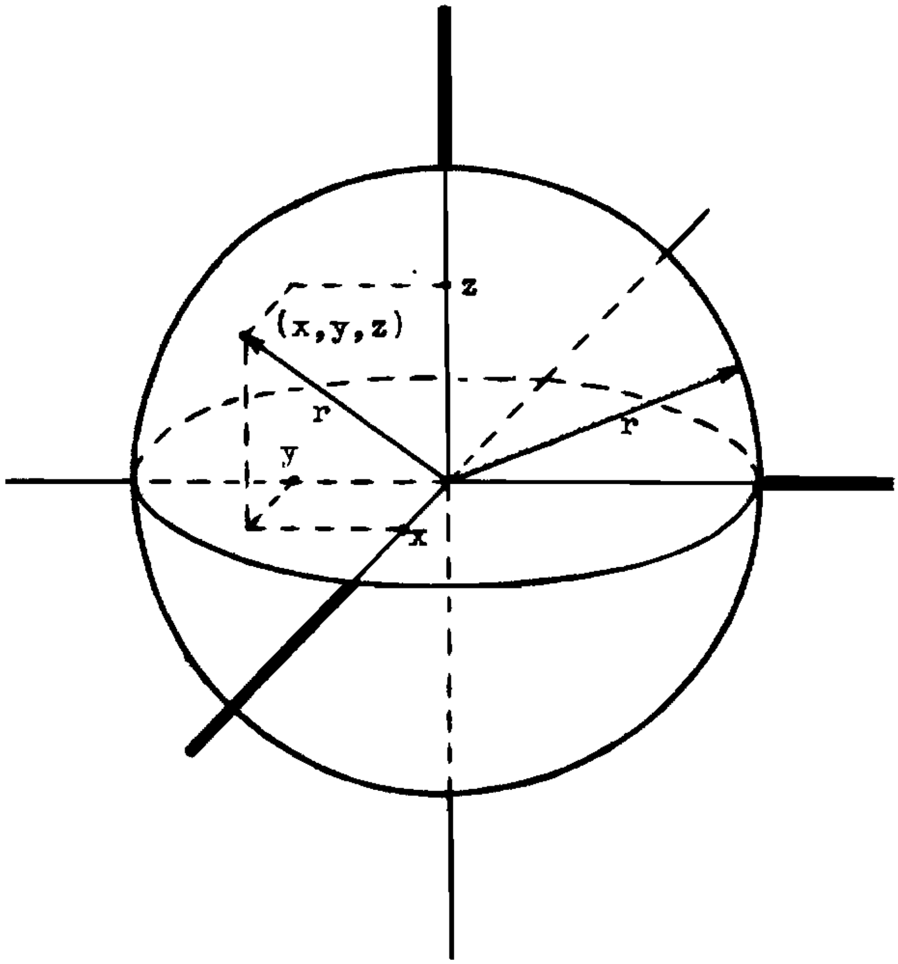
The Sphere of Imaginary Radius

The equation of a real sphere in a three dimensional real number space has the form $x^2 + y^2 + z^2 = r^2$ where r is a real number and the radius of the sphere. The set of all ordered triples, (x, y, z), that satisfy the equation for a specified r form a sphere of radius r with its center at the origin of the x, y, and z axes. Each ordered triple, (x, y, z), satisfying the equation, represents a point on the surface of a sphere for a given r (Fig. 1). Lines are restricted to be great circles on the surface of the sphere. Great circles on the sphere may be determined by the intersection of the sphere with planes passing through the origin.[10]

For a point P to be on a line, or great circle, it must satisfy two equations. It must satisfy the equation of the sphere, to be on the surface, and it must satisfy the equation of the plane, to be on the intersection. To determine if a point is on a line, use may be made of vectors and their properties. Let Q be a plane passing through the origin and let P be a point (x, y, z) on the surface of the sphere. If P is on the line determined by plane Q, it is sufficient for it to be in the plane since it is known to be on the sphere. Now consider a vector $\vec{V}$ that is perpendicular to plane Q. Let $\vec{V}$ = (A, B, C) where A, B, and C are real numbers. Since $\vec{V}$ is

[10]Rainich and Dowdy, <u>Geometry for Teachers</u>, p. 142.

The Real Sphere



$$x^2 + y^2 + z^2 = r^2$$

Figure 1

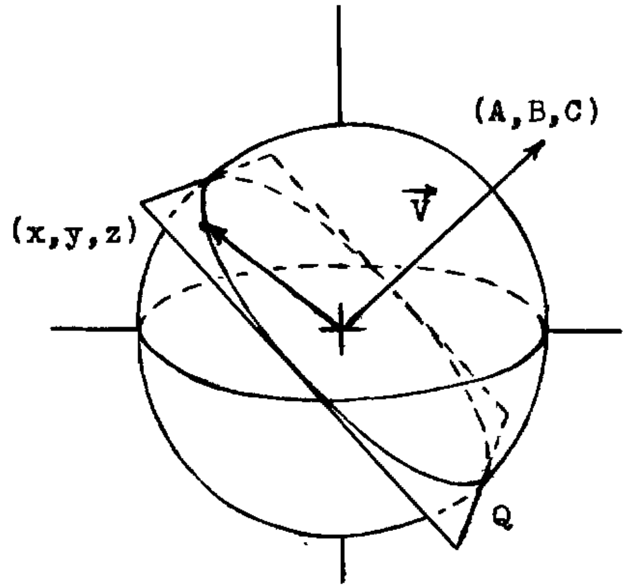
perpendicular to plane Q, the plane contains all standard position vectors perpendicular to $\vec{V}$. $P = (x, y, z)$ can be considered as a standard vector (x, y, z). Therefore if $(x, y, z) \cdot \vec{V} = 0$, the vector (x, y, z) is perpendicular to $\vec{V}$ and is contained in plane Q. That would mean that P is contained in plane Q and, thus, on the line that is determined by Q. If $P = (x, y, z)$ on the surface, and $\vec{V} = (A, B, C)$, then $Ax + By + Cz = 0$ indicates P is on the line determined by Q (Fig. 2). In other words, a great circle, or line, on a sphere in an x, y, z coordinate space is the set of points (x, y, z) that satisfy $x^2 + y^2 + z^2 = r^2$ and $Ax + By + Cz = 0$ at the same time for predetermined real numbers r, A, B, and C.[11]

One characteristic of lines on a sphere, of course, is that any two lines have two points of intersection diametrically opposite each other on the surface of the sphere (Fig. 3). All lines intersect so there could be no non-intersecting lines through a given point not on a given line.[12] Since two intersecting lines should determine one, and only one, point, and lines and points as defined do not satisfy this, further restriction of the surface is necessary. If the surface is reduced, the lines and point descriptions can remain unchanged. The points of intersection of all lines are diametrically opposite so half of them could be taken out by reducing the surface to a hemisphere. This is done by restricting one of the

[11] Ibid.

[12] Ibid. p. 143.

The Real Sphere and Intersecting Plane



$Ax + By + Cz = 0$

Figure 2

Two Intersecting Lines

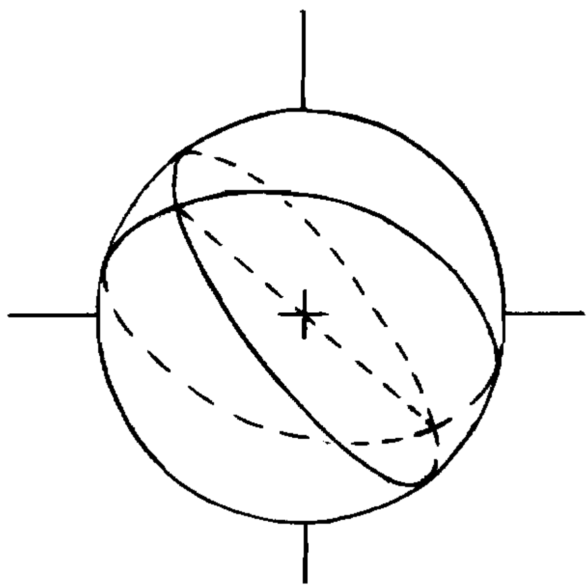


Figure 3

variables x, y, or z. For example, if z were limited to being a real number greater than zero, the number of solution points (x, y, z) for $x^2 + y^2 + z^2 = r^2$ is reduced leaving only the upper half of the sphere minus the great circle edge of the half-sphere (Fig. 4). Thus, if $z > 0$, $x^2 + y^2 + z^2 = r^2$ produces a surface on which the defined lines intersect in, at most, one place.[13] On this redefined surface, it would be possible, however, to be given a line and a point not on the line and find a line that would not intersect the given line (Fig. 5). To keep this from happening, more refinement on x, y, and z is necessary. If $z \geq 0$ when $x > 0$, $z > 0$ when $x < 0$, $z \geq 0$ when $x = 0$ and $y \geq 0$, and $z > 0$ when $x = 0$ and $y < 0$. Half of the edge of the hemisphere is included in the surface. Now all lines will intersect and they will intersect in only one point (Fig. 6). The space is a model for Elliptic Geometry. Without including half of the edge of the hemisphere, the model would be, in a sense, another space for Euclidean Geometry.

The concept of the sphere with imaginary radius will be developed directly from the real sphere. It is important to understand that no visual concept or illustration of the sphere is presented at this time to allow for an analytical, abstract approach to the problem. The sphere will be presented later in a complex number space to aid in understanding. Hopefully, such an approach will inhance the versatility and beauty of abstract mathematics in any system and illustrate that a model is not necessary for mathematical development.

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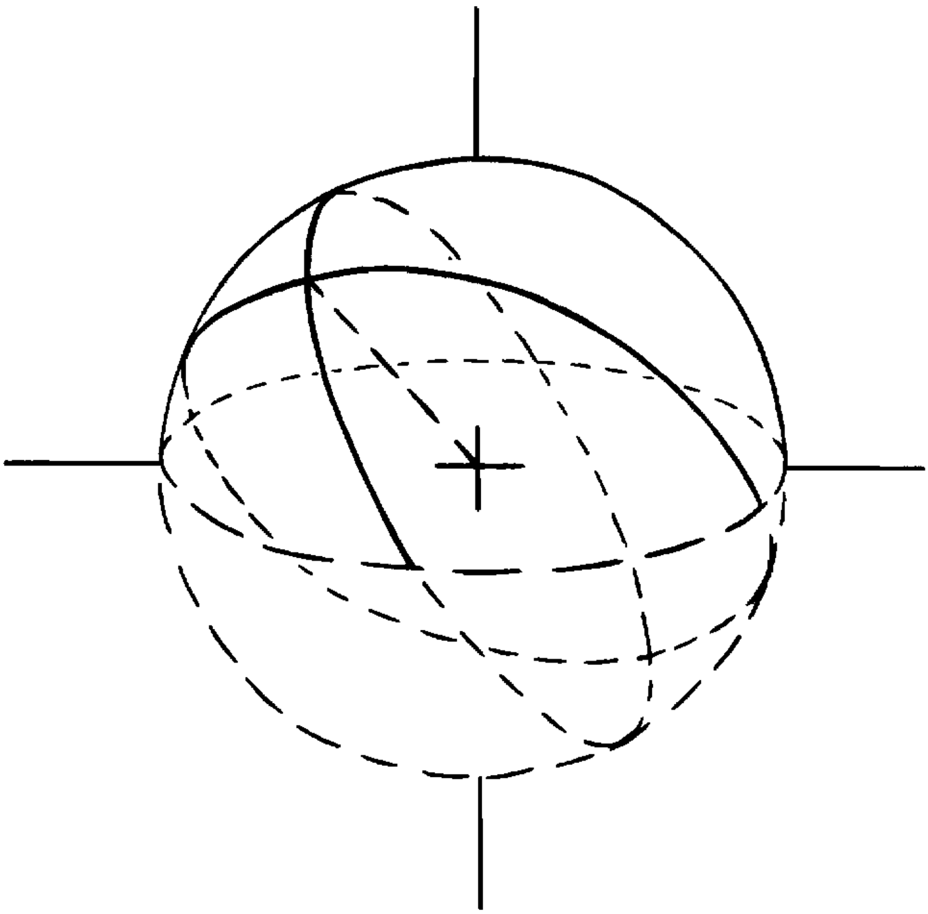
[13] Ibid.

The Upper Hemisphere



$z > 0$

Figure 4

Non-Intersecting Lines

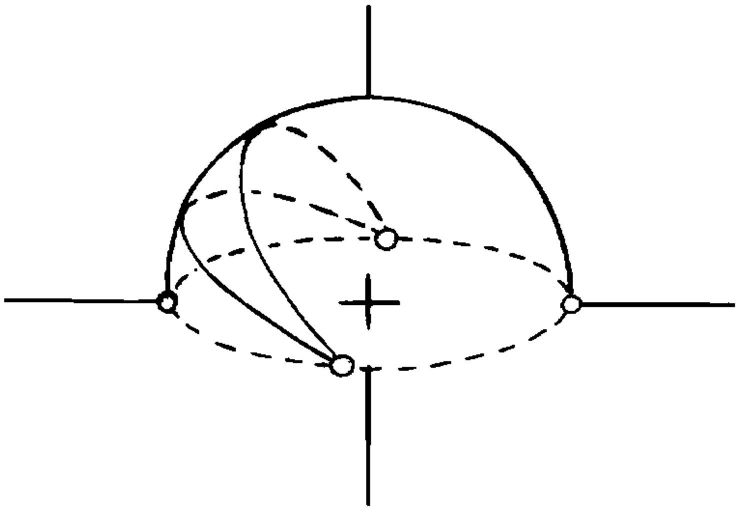


Figure 5

Hemisphere with Half-Edge

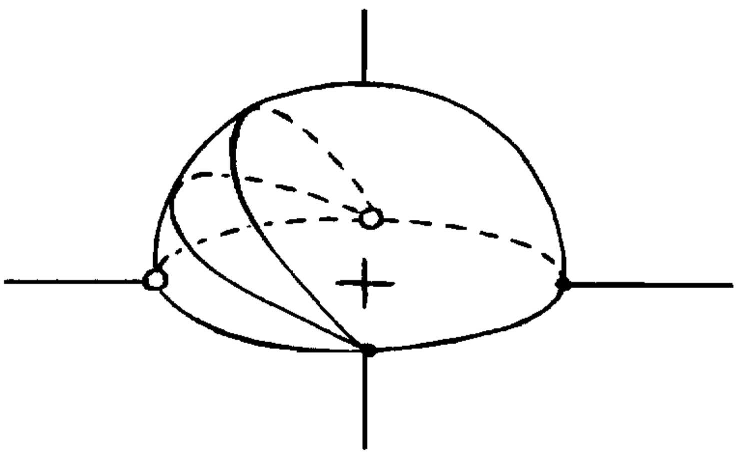


Figure 6

The development of the sphere of imaginary radius must begin with an understanding of what is meant by an imaginary radius. A real sphere is produced by the equation $x^2 + y^2 + z^2 = r^2$ where x, y, z, and r are real numbers and r is the radius of the sphere about the x, y, z origin. The new sphere is to have a purely imaginary radius. This is accomplished by replacing r, in the original equation, with an imaginary number. Let the radius be equal to ri where r is a fixed positive real number.[14] For example, 3i may be the radius of an imaginary sphere.

At this point, replacing r in $x^2 + y^2 + z^2 = r$ with ri yields:

$$x^2 + y^2 + z^2 = (ri)^2$$
$$x^2 + y^2 + z^2 = r^2i^2$$
$$x^2 + y^2 + z^2 = -r^2 \quad \text{OR}$$
$$x^2 + y^2 + z^2 + r^2 = 0,$$

and the first difficulty in the new space has appeared. x^2, y^2, z^2, and r^2 are all real numbers. They are all non-negative as a result of the squaring. Therefore, only if they all equal zero does $x^2 + y^2 + z^2 + r^2 = 0$ have solutions for x, y, and z. But r has been predetermined to be a positive real number, indicating that it is not zero. The equation as it is, then, has no solutions for x, y, or z since they are still real numbers.

To get around this difficulty, two logical proposals are offered. First, x, y, and z could be changed from real to imaginary numbers. But this would accomplish nothing more than a move backwards to the equation of the real sphere. A

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[14] *Ibid.* p. 144.

second possibility would be to change one real variable to an imaginary variable. By changing one real variable x, y, or z to an imaginary variable and letting the others remain real, the effect of the imaginary radius could be off set and a solution could be found. Change z to zi where z is a real number. By substituting zi for z in $x^2 + y^2 + z^2 + r^2 = 0$, the result is:

$$x^2 + y^2 + (zi)^2 + r^2 = 0,$$
$$x^2 + y^2 + z^2 i^2 + r^2 = 0,$$
$$x^2 + y^2 - z^2 + r^2 = 0, \text{ OR}$$
$$x^2 + y^2 + r^2 = z^2$$

The imaginary sphere is determined by the equation $x^2 + y^2 + r^2 = z^2$ where r is a fixed positive real number.[15] The surface of the imaginary sphere is defined to be the set of ordered triples, (x, y, z), of real numbers that are solutions of the equation for a specified r. Just as on the real sphere where the ordered triple solutions represented points, the ordered triples, (x, y, z) satisfying $x^2 + y^2 + r^2 = z^2$ will represent points on the surface of the imaginary sphere of radius ri. For example, let ri = 3i. The sphere is $x^2 + y^2 + 9 = z^2$. The ordered triple $(2, \sqrt{3}, 4)$ names a point on the sphere because by substitution $2^2 + (\sqrt{3})^2 + 9 = 4^2$ or $4 + 3 + 9 = 16$. This set of ordered triples could be used to determine several more points on the sphere. $(-2, \sqrt{3}, 4)$, $(-2, -\sqrt{3}, 4)$, $(-2, -\sqrt{3}, -4)$, $(2, -\sqrt{3}, 4)$, $(2, -\sqrt{3}, -4)$, $(2, \sqrt{3}, -4)$, $(-2, \sqrt{3}, -4)$ would all be related points on the surface of the sphere.

So far, an imaginary sphere has been developed and

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[15] *Ibid.* p. 144.

points on this surface have been defined using the real sphere as a guide. Lines will be defined, next, in the same manner. The concept of the surface, points and lines may have to be refined later as more characteristics are discovered to reach the ultimate objective of creating a hyperbolic space.

On the real sphere, a line was determined by a plane through the origin of the x, y, z coordinate space and was the set of all points, (x, y, z) satisfying $x^2 + y^2 + z^2 = r^2$ and $Ax + By + Cz = 0$, if (A, B, C) was a vector perpendicular to the plane. On the imaginary sphere lines would be sets of points (x, y, z) satisfying $x^2 + y^2 + r^2 = z^2$ and a form of $Ax + By + Cz = 0$. Since on the imaginary sphere z was called zi, the z in the equation of the plane must have the same notation. Now the equation of the plane appears as $Ax + By + Czi = 0$. A close look at this new equation reveals an interesting fact. All of the variables are real numbers but the imaginary term i is present on the left side of the equation and absent on the right side. The only way real numbers A, B, and C could cause this to happen would be if C were to equal zero. $Ax + By + Czi = 0$ could be possible if C were zero, but the problem would be limited to a single plane of vectors to work with, (A, B, 0). This single plane of vectors would limit the space too much. To open the space up and allow for more lines, a second plan is proposed and will be used. If C is not zero, the problem remains to get the imaginary number, i, out of the left hand term Czi. The way to do this is to let C be an imaginary number also. Change the real number C to an imaginary number Ci where C is a real number. Substitution into the vector equation now produces:

$$\begin{aligned} Ax + By + (Ci)zi &= 0, \\ Ax + By + Czi^2 &= 0, \\ Ax + By - Cz &= 0, \quad \text{OR} \\ Ax + By &= Cz \end{aligned}$$

which could be called the equation of a plane in the imaginary space.[16]

The development of the imaginary sphere and the definition of points and lines has, to this point, been an intuitive venture using the real sphere as a guide. It is unwise to assume, after looking at the real sphere, that all is in order at this time. A few examples could demonstrate that problems exist and give a clue as to what can be done to correct them.

Consider the imaginary sphere of radius $3i$ determined by $x^2 + y^2 + 9 = z^2$, its intersection with the plane $-9x + 2\sqrt{3}y = 6z$ where $A = -9$, $B = 2\sqrt{3}$, and $C = 6$, and the intersection with the line $-5x + 2\sqrt{3}y = 4z$. The two plane equations together give $4z + 5x = 6z + 9x$ or $-4x = 2z$ or $x = \frac{-z}{2}$. Substituting into the equation of the second plane for x produces $\frac{5z}{2} + 2\sqrt{3}y = 4z$, solving for y:

$$\begin{aligned} 5z + 4\sqrt{3}y &= 8z \\ 4\sqrt{3}y &= 3z \\ y &= \frac{3z}{4\sqrt{3}} \\ y &= \frac{\sqrt{3}z}{4} \end{aligned}$$

Now, $x = \frac{-z}{2}$ and $y = \frac{\sqrt{3}z}{4}$ is substituted into the equation of the sphere to find the points of intersection.

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[16] *Ibid.* p. 144.

$$\left(\frac{-z}{2}\right)^2 + \left(\frac{\sqrt{3}\,z}{4}\right)^2 + 9 = z^2$$

$$\frac{z^2}{4} + \frac{3z^2}{16} + 9 = z^2$$

$$4z^2 + 3z^2 + 144 = 16z^2$$

$$7z^2 + 144 = 16z^2$$

$$144 = 9z^2$$

$$16 = z^2$$

$$4 = z \quad \text{OR}$$

$$-4 = z$$

If $z = 4$, then $x = -2$ and $y = \sqrt{3}$ or if $z = -4$, $x = 2$, and $y = -\sqrt{3}$. This means that there are two points of intersection of these two lines on this sphere. That must not happen, but it is easy to correct. Simply impose the condition that z is greater than zero. Now only the point $(-2, \sqrt{3}, 4)$ remains as the intersection of the two lines on the surface. The surface of the imaginary sphere will, of course, have to be redefined with the new condition, $z > 0$.[17]

The surface of the sphere and the points are now defined in fairly good order. It is not known whether or not all lines, as they are defined will produce points on the sphere. The two lines used earlier did intersect the sphere to determine points of intersection. Using the same sphere, try to find an intersection with $y = 3z$ where $A = 0$, $B = 1$, and $C = 3$. Substituting into $x^2 + y^2 + 9 = z^2$ for the set of points in the intersection gives $x^2 + (3z)^2 + 9 = z^2$, $x^2 + 9z^2 + 9 = z^2$, or $x^2 = -8z^2 - 9$. The left side is greater

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[17]*Ibid.* p. 144.

than zero and the right side is less than zero. No solution can exist to this situation, indicating that not all planes $Ax + By = Cz$ intersect $x^2 + y^2 + r^2 = z^2$.

To determine the condition, thus far, not imposed on the definition of a line, consider two planes $Ax + By = Cz$ and $Bx - Ay = Cz$ and the sphere $x^2 + y^2 + r^2 = z^2$. The two lines result from the vectors (A, B, Ci) and $(B, -A, Ci)$. Only under the condition $A = B = 0$ could these vectors be parallel or, otherwise, not arbitrary. Such conditions will not be a part of this presentation. Assuming that the three equations have a common solution (x, y, z) where $z > 0$, the emphasis of this demonstration is to determine the property of A, B, and C necessary for an intersection.

$Ax + By = Cz$ and $Bx - Ay = Cz$ produce

$Ax + By = Bx - Ay$. Squaring both sides yields:

$(Ax + By)^2 = (Bx - Ay)^2$. Adding equal numbers to both sides:

$$(Ax + By)^2 + (Bx - Ay)^2 = (Bx - Ay)^2 + (Ax + By)^2,$$

then expanding and reducing the right side produces:

$$(Ax + By)^2 + (Bx - Ay)^2 = B^2x^2 - 2ABxy + A^2y^2 + A^2x^2 + 2ABxy + B^2y^2,$$

$$(Ax + By)^2 + (Bx - Ay)^2 = A^2x^2 + B^2x^2 + A^2y^2 + B^2y^2,$$

$$(Ax + By)^2 + (Bx - Ay)^2 = (A^2 + B^2)x^2 + (A^2 + B^2)y^2,$$

OR

$$(Ax + By)^2 + (Bx - Ay)^2 = (A^2 + B^2)(x^2 + y^2).$$

At this point, it should be noticed that from the equation of first line $(Ax + By)^2 = C^2z^2$ and in the equation of the sphere $x^2 + y^2 = z^2 - r^2$. Making the proper substitution into the above statement results with:

$$C^2z^2 + (Bx - Ay)^2 = (A^2 + B^2)\ (z^2 - r^2).$$

expanding the right side leaves:

$$C^2z^2 + (Bx - Ay)^2 = (A^2 + B^2)z^2 - (A^2 + B^2)r^2,$$

and algebraic addition gives:

$$(Bx - Ay)^2 + (A^2 + B^2)r^2 = (A^2 + B^2)z^2 - C^2z^2.$$

Factoring the right expression produces:

$$(Bx - Ay)^2 + (A^2 + B^2)r^2 = (A^2 + B^2 - C^2)z^2.$$

Careful examination of this last statement provides two important points. The z is a squared factor of the right member. This means that two possible z solutions would work, allowing for two points of intersection, (x, y, z), and $(x, y, -z)$. To keep this from happening, the single solution, $z > 0$, concept is reinforced.

The most important point, at this time, however, is to notice that the left member of the statement is the sum of squares of real numbers. It is, therefore, greater than zero while the right side is the product of z^2 and $(A^2 + B^2 - C^2)$. For the right side to also be positive, $A^2 + B^2 - C^2$ must be greater than zero. If $A^2 + B^2 - C^2 > 0$, then $A^2 + B^2 > C^2$.

This condition, $A^2 + B^2 > C^2$ is a necessary condition for $Ax + By = Cz$ and $Bx - Ay = Cz$ to intersect $x^2 + y^2 + r^2 = z^2$. Now it must be shown that $A^2 + B^2 > C^2$ assures that a plane will intersect the sphere.

If $A^2 + B^2 > C^2$ and $z > 0$, are there solutions for x and y in both $x^2 + y^2 + r^2 = z^2$ and $Ax + By = Cz$? If $A^2 + B^2 > C^2$, then $(A^2 + B^2)z^2 > C^2z^2$. In the equation of the sphere $z^2 = x^2 + y^2 + r^2$, so by substitution:

$$(A^2 + B^2)\ (x^2 + y^2 + r^2) > C^2z^2 \quad \text{OR}$$

$$A^2x^2 + A^2y^2 + A^2r^2 + B^2x^2 + B^2y^2 + B^2r^2 > C^2z^2 \ .$$

$C^2z^2 = (Ax + By)^2$ so:

$$A^2x^2 + A^2y^2 + A^2r^2 + B^2x^2 + B^2y^2 + B^2r^2 > (Ax + By)^2.$$

$$A^2x^2 + A^2y^2 + A^2r^2 + B^2x^2 + B^2y^2 + B^2r^2 > A^2x^2 + 2ABxy + B^2y^2$$

$$A^2y^2 + B^2x^2 + A^2r^2 + B^2r^2 > 2ABxy$$

$$A^2y^2 - 2ABxy + B^2x^2 > -A^2r^2 - B^2r^2$$

$$(Ay - Bx)^2 > -r^2(A^2 + B^2)$$

The left member of this inequality is greater than zero and the right member is less than zero. There will always be solutions for x and y in this situation. Thus, $A^2 + B^2 > C^2$ is a necessary and sufficient condition for $Ax + By = Cz$ to intersect $x^2 + y^2 + r^2 = z^2$ for real numbers A, B, C, and r.[18]

The imaginary sphere has now been developed and points and lines in the space have been defined. The sphere of imaginary radius is the set of ordered real number triples, (x, y, z), $z > 0$, satisfying the equation $x^2 + y^2 + r^2 = z^2$ where r is a positive real number. Lines are determined by the intersection of the imaginary sphere with planes having equations of the form $Ax + By = Cz$ where A, B, and C are real numbers and $A^2 + B^2 > C^2$.

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[18] Ibid. p. 145.

CHAPTER THREE

Points and Lines

The points of the sphere of imaginary radius are defined as ordered triples of real numbers, (x, y, z), $z > 0$, satisfying the equation $x^2 + y^2 + r^2 = z^2$ where r is a positive real number. Lines in the space of the imaginary sphere are defined as the set of points in the intersection of $x^2 + y^2 + r^2 = z^2$ and $Ax + By = Cz$ where A, B, C, and r are real numbers, $r > 0$, and $A^2 + B^2 > C^2$.

The development of this space has been intuitive and, hopefully, thorough. However, inconsistencies may still remain in the definitions. A check on the conditions imposed in the definitions of points and lines, in the space of the imaginary sphere, would be to see if they are well defined lines and points. A method proposed to do this is to determine whether or not they satisfy two axioms characteristic of points and lines in Hyperbolic Geometry.

The first axiom to be checked states that two distinct points determine one, and only one, line. Along with the proof that this axiom is satisfied will come a method of determining the equation of a line between two arbitrary points.

The second axiom, that the points and lines will be checked against is, two lines intersect in, at most, one point. Examples are provided to aid in understanding the problems and the methods of the proofs.

As an example that two distinct points on the imaginary sphere determine one, and only one, line, consider the points $(2, \sqrt{3}, 4)$ and $(-3\sqrt{2}, 3, 6)$ on the surface of $x^2 + y^2 + 9 = z^2$, where $r = 3$, the radius is $3i$ and $z > 0$. There are, actually, three points in the space through which the plane containing the line must pass, $(2, \sqrt{3}, 4)$, $(-3\sqrt{2}, 3, 6)$, and $(0, 0, 0)$. Euclidean Geometry could state that three points determine one and only one plane and the demonstration would be complete. This property is still true, in the sense that lines are defined as planes intersecting the sphere, but further justification is necessary with the absence of proof of this theorem in the space of the imaginary sphere.

If there is a line passing through $(2, \sqrt{3}, 4)$ and $(-3\sqrt{2}, 3, 6)$, it is contained in a plane of the form $Ax + By = Cz$ where $A^2 + B^2 > C^2$ and both points satisfy the equation. The problem is to determine solutions for A, B, and C under these conditions and show that the solutions are unique.

Substitution of the two points into the equation of the plane provides two equations of planes to work with, $2A + \sqrt{3}B = 4C$ and $-3\sqrt{2}A + 3B = 6C$. The second equation gives a solution of B in terms of A and C, $B = 2C + \sqrt{2}A$. Substitution of this solution for B into the first equation produces:

$$2A + \sqrt{3}(2C + \sqrt{2}A) = 4C,$$

$$2A + 2\sqrt{3}C + \sqrt{6}A = 4C,$$

$$2A + \sqrt{6}A = 4C - 2\sqrt{3}C,$$

$$(2 + \sqrt{6})A = (4 - 2\sqrt{3})C,$$

In terms of C, $A = \left(\dfrac{4 - 2\sqrt{3}}{2 + \sqrt{6}}\right)C.$

Substitution of this solution for A into $B = 2C + \sqrt{2}A$ produces:

$$B = 2C + \sqrt{2}\left(\frac{4 - 2\sqrt{3}}{2 + \sqrt{6}}\right)C,$$

$$B = 2C + \left(\frac{4\sqrt{2} - 2\sqrt{6}}{2 + \sqrt{6}}\right)C, \quad \text{OR}$$

$$B = \left[2 + 2\sqrt{2}\left(\frac{2 - \sqrt{3}}{2 + \sqrt{6}}\right)\right]C\,.$$

Now, A and B have parametric solutions in terms of C,

$$A = \left(\frac{4 - 2\sqrt{3}}{2 + \sqrt{6}}\right)C \quad \text{and} \quad B = \left[2 + 2\sqrt{2}\left(\frac{2 - \sqrt{3}}{2 + \sqrt{6}}\right)\right]C.$$

This does not appear to resolve the question of a unique line through $(2, \sqrt{3}, 4)$, and $(-3\sqrt{2}, 3, 6)$, but remember that $Ax + By = Cz$ comes from a vector (A, B, C), in standard position, perpendicular to the plane through the origin containing (x, y, z). The relation between A, B, and C is thus given in the vector form:

$$\left(\left(\frac{4 - 2\sqrt{3}}{2 + \sqrt{6}}\right)C,\ \left[\ 2 + 2\sqrt{2}\left(\frac{2 - \sqrt{3}}{2 + \sqrt{6}}\right)\right]C,\ \ C\right)\ .$$

Any real replacement of C would result in an acceptable vector, but if two different substitutions would be made, one would be a scalar multiple of the other. The result is that one vector would be scalar multiple of the other and the two vectors would be parallel. Since the vectors are parallel, they are perpendicular to the same plane through the origin. Any real substitution of C is acceptable but the solution plane would be unique. Therefore, there is only one line containing $(2, \sqrt{3}, 4)$ and $(-3\sqrt{2}, 3, 6)$. Let $C = 2 + \sqrt{6}$ and the equation of the plane is:

$$(4 - 2\sqrt{3})x + [2(2 + \sqrt{6}) + 2\sqrt{2}\ (2 - \sqrt{3})]\ y = (2 + \sqrt{6})z \quad \text{OR}$$

$$(4 - 2\sqrt{3})x + (4 + 4\sqrt{2})y = (2 + \sqrt{6})z.$$

The example illustrates the following proof that two distinct, arbitrary, points on the sphere of imaginary radius determine one, and only one, line.

Assume that two points, P_1 and P_2 are on the surface of $x^2 + y^2 + r^2 = z^2$ where r is a positive real number, $z > 0$. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. If there is a line through P_1 and P_2, it is the intersection of the sphere with a plane of the form $Ax + By = Cz$ where $A^2 + B^2 > C^2$ and P_1 and P_2 are both solutions to the equation. Substitution P_1 and P_2 into the equation for x, y, and z will provide two equations to solve for A, B, and C. Such a substitution gives $Ax_1 + By_1 = Cz_1$ and $Ax_2 + By_2 = Cz_2$. When the first equation is solved for A in terms of B and C,

$$Ax_1 = Cz_1 - By_1 \quad \text{OR} \quad A = (Cz_1 - By_1)\frac{1}{x_1} .$$

Replacement of A in the second equation $Ax_2 + By_2 = Cz_2$ with $(Cz_1 - By_1)\frac{1}{x_1}$ yields $\left((Cz_1 - By_1)\frac{1}{x_1}\right) x_2 + By_2 = Cz_2$. Expanding and solving this equation for B in terms of C produces:

$$\left((Cz_1 - By_1)\frac{1}{x_1}\right) x_2 + By_2 = Cz_2,$$

$$(Cz_1 - By_1)\frac{x_2}{x_1} + By_2 = Cz_2,$$

$$\frac{Cz_1x_2}{x_1} - B\left(\frac{y_1x_2}{x_1}\right) + By_2 = Cz_2,$$

$$By_2 - B\left(\frac{y_1x_2}{x_1}\right) = Cz_2 - C\left(\frac{z_1x_2}{x_1}\right),$$

$$B\left(y_2 - \frac{y_1x_2}{x_1}\right) = \left(z_2 - \frac{z_1x_2}{x_1}\right)C,$$

$$B\left(\frac{y_2x_1 - y_1x_2}{x_1}\right) = \left(\frac{z_2x_1 - z_1x_2}{x_1}\right)C, \quad \text{OR}$$

$$B = \left(\frac{z_2x_1 - z_1x_2}{y_2x_1 - y_1x_2}\right) C$$

Since $A = (Cz_1 - By_1)\frac{1}{x_1}$, substitution for B its solution in terms of C gives:

$$A = \left(Cz_1 - \left[\left(\frac{z_2x_1 - z_1x_2}{y_2x_1 - y_1x_2}\right) C\right] y_1\right) \frac{1}{x_1} \quad ,$$

$$A = \left(\frac{z_1}{x_1}\right) C - \left(\frac{(z_2x_1 - z_1x_2)\ y_1}{(y_2x_1 - y_1x_2)\ x_1}\right) \quad C \quad ,$$

$$A = \left(\frac{z_1}{x_1} - \frac{(z_2x_1 - z_1x_2)\ y_1}{(y_2x_1 - y_1x_2)\ x_1}\right) \quad C \quad ,$$

$$A = \left(\frac{z_1y_2x_1 - z_1y_1x_2 - z_2x_1y_1 + z_1x_2y_1}{y_2x_1^2 - y_1x_1x_2}\right) C \ ,$$

$$A = \left(\frac{z_1y_2x_1 - z_2x_1y_1}{y_2x_1^2 - y_1x_1x_2}\right) C \quad , \quad \text{OR}$$

$$A = \left(\frac{z_1y_2 - z_2y_1}{y_2x_1 - y_1x_2}\right) C$$

for A in terms of C.

Now, A and B have been solved for in terms of C. As a vector, (A, B, C), the solution appears as:

$$\left(\left(\frac{z_1y_2 - z_2y_1}{y_2x_1 - y_1x_2}\right) C \quad , \quad \left(\frac{z_2x_1 - z_1x_2}{y_2x_1 - y_1x_2}\right) C \quad , \quad C\right), \text{ OR}$$

$$\left(\left(\frac{z_1y_2 - z_2y_1}{y_2x_1 - y_1x_2}\right) \quad , \quad \left(\frac{z_2x_1 - z_1x_2}{y_2x_1 - y_1x_2}\right), \quad 1\right) C$$

Any real substitution for C would produce an acceptable vector since any two different substitutions would simply result in parallel vectors. The vector (A, B, C) is perpendicular to the plane passing through P_1 and P_2. Any two parallel vectors are both perpendicular to the same plane that passes through the origin. Therefore, for any real number C, the solution is unique. There is one, and only one, line passing through P_1 and P_2.

Let $C = (y_2x_1 - y_1x_2)$ and the vector solution is:

$$(z_1y_2 - z_2y_1\ ,\ z_2x_1 - z_1x_2\ ,\ y_2x_1 - y_1x_2).$$

This solution provides a formula for determining the equation of the plane passing through two given points, (x_1, y_1, z_1) and (x_2, y_2, z_2). The equation of the plane that determines the unique line between these two points is:

$$(z_1y_2 - z_2y_1)x + (z_2x_1 - z_1x_2)y = (y_2x_1 - y_1x_2)z.$$

The next problem in showing that the points and lines in the space of the imaginary sphere are well defined is to show that two lines intersect in, at most, one point. One example of the concept was developed in the last chapter in the effort to refine the definition of the sphere of imaginary radius. Another example, illustrating the method of proof to be used, would be the sphere with radius 3i determined by $x^2 + y^2 + 9 = z^2$, $z > 0$, and plane $2\sqrt{6}x + 4y = 5z$ and $4x - 6\sqrt{6}y = -2\sqrt{6}z$. The intersection of the sphere and the two planes should be one point.

Solving the second plane equation for x determines:

$$4x - 6\sqrt{6}y = -2\sqrt{6}z,$$

$$4x = 6\sqrt{6}y - 2\sqrt{6}z, \quad \text{OR}$$

$$x = \frac{3\sqrt{6}}{2}y - \frac{\sqrt{6}}{2}z\ .$$

Substitution of the solution for x into the first plane equation, $2\sqrt{6}x + 4y = 5z$, and solving for y gives:

$$2\sqrt{6}\left(\frac{3\sqrt{6}}{2}y - \frac{\sqrt{6}}{2}z\right) + 4y = 5z,$$

$$18y - 6z + 4y = 5z,$$

$$22y = 11z, \quad \text{OR}$$

$$y = \frac{z}{2}$$

Now, by substituting this solution of y into the solution of x in terms of y and z produces:

$$x = \frac{3\sqrt{6}}{2}y - \frac{\sqrt{6}}{2}z$$

$$x = \left(\frac{3\sqrt{6}}{2}\right)\left(\frac{z}{2}\right) - \frac{\sqrt{6}}{2}z$$

$$x = \frac{3\sqrt{6}}{4}z - \frac{2\sqrt{6}}{4}z$$

$$x = \frac{\sqrt{6}}{4}z \quad \text{and} \quad y = \frac{z}{2}$$

By substituting these solutions for x and y in terms of z into the equation of the sphere, the point of intersection (x, y, z) is determined:

$$\left(\frac{\sqrt{6}}{4}z\right)^2 + \left(\frac{z}{2}\right)^2 + 9 = z^2,$$

$$\frac{6}{16}z^2 + \frac{z^2}{4} + 9 = z^2,$$

$$\frac{6}{16}z^2 + \frac{4}{16}z^2 + 9 = z^2,$$

$$\frac{5}{8}z^2 + 9 = z^2,$$

$$9 = \frac{3z^2}{8},$$

$$24 = z^2,$$

$$\sqrt{24} = z,$$

$$2\sqrt{6} = z, \quad z > 0, \text{ and}$$

$$y = \frac{z}{2},$$

$$y = \frac{2\sqrt{6}}{2},$$

$$y = \sqrt{6},$$

and
$$x = \frac{\sqrt{6}}{4} z,$$

$$x = \frac{\sqrt{6}}{4}(2\sqrt{6}),$$

$$x = \frac{6}{2},$$

$$x = 3, \quad \text{OR}$$

$$(x, y, z) = (3, \sqrt{6}, 2\sqrt{6})$$

In this example, the two given lines are seen to intersect on the sphere in exactly one point. The following example will show that for some pairs of lines there is not even one point of intersection on the sphere.

Consider the sphere of radius 3i again, with lines determined by $12x - 4y = 12z$ and $4x - 2y = 4z$. Both of the equations satisfy the condition $A^2 + B^2 > C^2$ and therefore determine lines in the space. But their point of intersection does not exist. To show this, solve the second equation for y in terms of x and z.

$$4x - 2y = 4z$$

$$4x - 4z = 2y$$

$$2x - 2z = y$$

Substitution into the first equation, $12x - 4y = 12z$, for y yields:

$$12x - 4(2x - 2z) = 12z,$$

$$12x - 8x + 8z = 12z, \quad \text{OR}$$

$$4x = 4z,$$

$$x = z$$

Now $y = 2z - 2z$ or $y = 0$ and $x = z$

Substitution on the equation of the sphere results in $z^2 + 0^2 + 9 = z^2$ or $9 = 0$. No real numbers, z, exist that can produce this situation. Therefore, no intersection (x, y, z) for these two lines, determined by $12x - 4y = 12z$ and $4x - 2y = 4z$, exists.

The preceding examples indicate a fact which will now be proven. Namely, not all lines intersect in the space of the imaginary sphere, but when they do, their intersection must be one point. Assume that $Ax + By = Cz$ and $A'x + B'y = C'z$, where A, B, C, A', B', and C' are real numbers such that $A^2 + B^2 > C^2$ and $A'^2 + B'^2 > C'^2$, intersect on the sphere $x^2 + y^2 + r^2 = z^2$, $z > 0$ and r is a positive real number. Their intersection will be the point(s), (x, y, z), that satisfy all the equations.

To begin finding the intersection, solve the first line equation for x in terms of y and z.

$$Ax + By = Cz$$

$$Ax = Cz - By$$

$$x = \frac{Cz - By}{A}$$

Replacing x in the second line equation allows for a solution of y in terms of z.

$$A'\left(\frac{Cz - By}{A}\right) + B'y = C'z$$

$$\frac{A'C}{A}z - \frac{A'B}{A}y + B'y = C'z$$

$$B'y - \frac{A'B}{A}y = C'z - \frac{A'C}{A}z$$

$$(AB' - A'B)y = (C'A - A'C)z$$

$$y = \left(\frac{C'A - A'C}{AB' - A'B}\right) z$$

Placing this solution for y into the previous solution for x, $x = \frac{Cz - By}{A}$ gives a solution for x in terms of z.

$$x = \frac{Cz - B\left(\frac{C'A - A'C}{AB' - A'B}\right)z}{A}$$

$$x = \frac{C}{A} z - \left(\frac{BC'A - A'CB}{A^2B' - A'BA}\right) z \quad ,$$

$$x = \left(\frac{C}{A} - \frac{BC'A - A'CB}{A^2B' - A'BA}\right) z \ ,$$

$$x = \left(\frac{CAB' - CA'B - BC'A + A'CB}{A^2B' - A'BA}\right) z \ , \quad \text{OR}$$

$$x = \left(\frac{CB' - BC'}{AB' - A'B}\right) z \ .$$

Having solved for x and y in terms of z, substitutions can now be made into $x^2 + y^2 + r^2 = z^2$ to solve for z:

$$\left[\left(\frac{CB' - BC'}{AB' - A'B}\right)z\right]^2 + \left[\left(\frac{C'A - A'C}{AB' - A'B}\right) z\right]^2 + r^2 = z^2,$$

$$r^2 = 1 - \left(\frac{CB' - BC'}{AB - A B}\right)^2 - \left(\frac{C'A - A'C}{AB' - A'B}\right)^2 z^2$$

$$r^2 = \left(\frac{(AB' - A'B)^2 - (CB' - BC')^2 - (AC' - A'C)^2}{(AB' - A'B)^2}\right) z^2$$

$$\left(\frac{r^2 (AB' - A'B)^2}{(AB' - A'B)^2 - (CB' - BC')^2 - (AC' - A'C)^2}\right) = z^2$$

$$\sqrt{\frac{r^2 (AB' - A'B)^2}{(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2}} = z$$

This solution, since $z > 0$, is unique if it exists. This solution exists if $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. If this number is not greater than zero the lines do not intersect. This could provide a relatively quick check to see if two lines intersect. If $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 \leq 0$, the lines are non-intersecting.

Since $(AB' - A'B)$ is a factor of z, the solution

$$x = \left(\frac{CB' - BC'}{AB' - A'B}\right) z \quad \text{and} \quad y = \left(\frac{C'A - A'C}{AB' - A'B}\right) z, \text{ where}$$

$$z = \sqrt{\frac{r^2 (AB' - A'B)^2}{(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2}}$$ can be reduced. The result is:

$$x = (CB' - B'C)\, z \quad \text{and} \quad y = (C'A - A'C)\, z \quad \text{where}$$

$$z = \sqrt{\frac{r^2}{(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2}} \;.$$

The solution of z is unique which makes the solutions of x and y unique. Therefore, if $Ax + By = Cz$ and $A'x + B'y = C'z$ intersect, the intersection is, at most, one point.

It has been shown that on the sphere with imaginary radius, two distinct points determine one, and only one, line, and two lines intersect in, at most, one point. The points and lines of the space are well defined.

CHAPTER FOUR

Non-Intersecting Lines

The sphere of imaginary radius has been developed and the points and lines have been shown to be well defined. Besides the definitions of points and lines, two other properties have been developed. One that will be useful at this time is if two lines determined by $Ax + By = Cz$ and $A'x + B'y = C'z$ intersect on the sphere $x^2 + y^2 + r^2 = z^2$, then $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. This condition is a fairly easy check to see if two lines intersect without actually trying to find the point of intersection.

If the imaginary sphere, as defined, is a model of Hyperbolic Geometry, it must satisfy the characteristic property of Hyperbolic Geometry. It must be shown that through a given point, not on a given line, there is more than one line that does not intersect the given line. To indicate that this characteristic holds in this space, an example will be presented first. This will provide an intuitive acceptance that the proof is possible and an outline of the proof itself.

Consider the sphere of radius $2i$ given by $x^2 + y^2 + 4 = z^2$, the line determined by $x + y = -z$ where $A = 1$, $B = 1$, and $C = -1$, and the point $(1, 2, 3)$. The point $(1, 2, 3)$ is on the surface of the sphere since it satisfies $x^2 + y^2 + 4 = z^2$. The plane $x + y = -z$ intersects the sphere of radius $2i$ since it satisfies $A^2 + B^2 > C^2$.

The point (1, 2, 3) is not on the line determined by $x + y = -z$. If there is a line through (1, 2, 3) that does not intersect the line $x + y = -z$ on the surface of $x^2 + y^2 + 4 = z^2$, let it have the form $A'x + B'y = C'z$. The point (1, 2, 3) must satisfy the equation of this line or $A' + 2B' = 3C'$. In terms of B' and C', $A' = 3C' - 2B'$.

It has been shown that a necessary condition for two lines $Ax + By = Cz$ and $A'x + B'y = C'z$ to intersect is that $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. If this condition is not satisfied or if $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 \leqslant 0$, the lines will not intersect on the sphere. Consider the case where $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$, $A = 1$, $B = 1$, $C = -1$, and $A' = 3C' - 2B'$. Making the proper substitutions produces:

$$[B' - (3C' - 2B')]^2 - [-B' - C']^2 - [C' - (3C' - 2B')(-1)]^2 = 0$$

$$(3B' - 3C')^2 - (-B' - C')^2 - (4C' - 2B')^2$$

$$(9B'^2 - 18C'B' + 9C'^2) - (B'^2 + 2C'B' + C'^2) - (16C'^2 - 16C'B' + 4B'^2) = 0,$$

$$4B'^2 - 4C'B' - 8C'^2 = 0$$

This last equation can be solved for B' in terms of C' by application of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$B' = \frac{4C' \pm \sqrt{16C'^2 - 4 \cdot 4 \cdot (-8C'^2)}}{8}$$

$$B' = \frac{4C' \pm \sqrt{16C'^2 + 128C'^2}}{8}$$

$$B' = \frac{4C' \pm \sqrt{144C'^2}}{8}$$

$$B' = \frac{4C' \pm 12C'}{8} = \frac{C' \pm 3C'}{2}$$

$$B' = \frac{C' + 3C'}{2} = \frac{4C'}{2} = 2C'$$

OR

$$B' = \frac{C' - 3C'}{2} = \frac{-2C'}{2} = -C'$$

Two solutions for B' in terms of C' have been determined. Remembering that $A' = 3C' - 2B'$, the following solutions for A' can be found in terms of C'.

$$A' = 3C' - 2(2C') = 3C' - 4C' = -C'$$

$$A' = 3C' - 2(-C') = 3C' + 2C' = 5C'$$

With the corresponding pairs of solutions for A' and B' in terms of C' the following two vectors (A', B', C') have been found:

$$(-C', 2C', C') \text{ and } (5C', -C', C').$$

It has been demonstrated that any real substitution of C or C' is acceptable in this situation of parametric vector solutions. Let $C' = 1$. Two vectors $(-1, 2, 1)$ and $(5, -1, 1)$ have been determined which produce the two planes $-x + 2y = z$ and $5x - y = z$ which contain $(1, 2, 3)$ and do not intersect $x + y = -z$ on the sphere of radius $2i$, $x^2 + y^2 + 4 = z^2$.

A further demonstration of the fact that $-x + 2y = z$ and $5x - y = z$ do not intersect $x + y = -z$ would be to try to find the points of intersection on $x^2 + y^2 + 4 = z^2$. Consider first $-x + 2y = z$ and $x + y = -z$. Adding them together reveals $3y = 0$ or $y = 0$. If $y = 0$, then $x = -z$. Substitution of these values into the equation of the sphere produces:

$$(-z)^2 + (0)^2 + 4 = z^2 \ ,$$
$$z^2 + 4 = z^2 \ ,$$
$$4 = 0 \ .$$

No solution exists for the intersection of these two lines on the sphere.

Next consider $5x - y = z$ and $x + y = -z$ and their point of intersection on $x^2 + y^2 + 4 = z^2$. Adding them together gives $6x = 0$ or $x = 0$. If $x = 0$, then $y = -z$ which results in:

$$(0)^2 + (-z)^2 + 4 = z^2 \ ,$$
$$z^2 + 4 = z^2 \ ,$$
$$4 = 0 \ .$$

Again, no solution exists.

It has been shown that on the sphere through the given point, (1, 2, 3) on the sphere $x^2 + y^2 + 4 = z^2$, not on the line determined by $x + y = -z$ there is more than one line that does not intersect the given line.

The space of the imaginary sphere, as it has been developed, may therefore be a model of a Hyperbolic Geometry. Its lines and points have been well defined and it seems to satisfy the characteristic property of a hyperbolic space. This last condition will be shown at this time. It will be shown that through an arbitrary given point not on a given line there is more than one line that does not intersect the given line.

The proof of this characteristic will not be presented fully at this time. Rather, the proof will be outlined for the case when the given point (x_1, y_1, z_1) has a non-zero first coordinate. A full presentation is presented in Appendix I.

Consider the sphere of radius ri, $r > 0$, specified by $x^2 + y^2 + r^2 = z^2$, $z > 0$, and the line determined by $Ax + By = Cz$, $A^2 + B^2 > C^2$. Let (x_1, y_1, z_1) be a point P on the sphere but not on $Ax + By = Cz$. If there is a line that does not intersect $Ax + By = Cz$, it is determined by a plane that has the form $A'x + B'y = C'z$. By letting $A'x + B'y = C'z$ contain point P:

$$A'x_1 + B'y_1 = C'z_1$$
$$A'x_1 = C'z_1 - B'y_1$$
$$A' = \frac{C'z_1 - B'y_1}{x_1} \quad ; \; x_1 \neq 0.$$

If $Ax + By = Cz$ intersects $A'x + B'y = C'z$ on the sphere, it is known that $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. But these two lines do not intersect on the sphere so the condition $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 \leqslant 0$ is imposed. More specifically, consider the condition $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$.

By making the proper substitution for A' the equation becomes:

$$\left[AB' - \left(\frac{C'z_1 - B'y_1}{x_1}\right)B\right]^2 - \left[CB' - C'B\right]^2 - \left[AC' - \left(\frac{C'z_1 - B'y_1}{x_1}\right)C\right]^2 = 0$$

where A, B, C, x_1, y_1, and z_1 are all predetermined real numbers.

Expanding and simplifying this equation results in the following quadratic equation involving B' and C' as unknown values:

$$pB'^2 + 2mC'B' + nC'^2 = 0 \quad \text{where:}$$

$$p = (Ax_1 + By_1)^2 - (x_1^2 - y_1^2)C^2$$
$$m = Bz_1(-Ax_1 - By_1) + Cy_1(Cz_1 - Ax_1) + CBx_1^2$$
$$n = B^2(z_1^2 - x_1^2) - (Ax_1 - Cz_1)^2$$

If $p \neq 0$, the quadratic equation $pB'^2 + 2mC'B' + nC'^2 = 0$ can be solved for B' in terms of C' using the quadratic formula. The case where $p = 0$ is presented in the Appendix.

$$B' = \frac{-2mC' \pm \sqrt{4m^2C'^2 - 4pnC'^2}}{2p}$$

$$B' = \left(\frac{-m \pm \sqrt{m^2 - pn}}{p}\right)C'$$

for p, m, and n as defined above.

This means that in terms of C' there are two possible solutions for B'.

$$B' = \left(\frac{-m + \sqrt{m^2 - pn}}{p}\right)C'$$

OR

$$B' = \left(\frac{-m - \sqrt{m^2 - pn}}{p}\right)C'$$

Since $A' = \dfrac{C'z_1 - B'y_1}{x_1}$, there are also two solutions for A' in terms of C'. By substitution of the solutions of B':

$$A' = \frac{C'z_1 - \left(\dfrac{-m + \sqrt{m^2 - pn}}{p}\right)C'y_1}{x_1},$$

$$A' = \left(\frac{pz_1 + my_1 - y_1\sqrt{m^2 - pn}}{px_1}\right)C' \qquad \text{AND}$$

$$A' = \frac{C'z_1 - \left(\frac{-my_1 - y_1\sqrt{m^2 - pn}}{p}\right)C'y_1}{x_1} ,$$

$$A' = \left(\frac{pz_1 + my_1 + y_1\sqrt{m^2 - pn}}{px_1}\right)C' .$$

These solutions, put into parametric vector form, gives:

$$\left(\left(\frac{pz_1 + my_1 - y_1\sqrt{m^2 - pn}}{px_1}\right)C' , \quad \left(\frac{-m + \sqrt{m^2 - pn}}{p}\right)C' , \quad C'\right)$$

AND

$$\left(\left(\frac{pz_1 + my_1 + y_1\sqrt{m^2 - pn}}{px_1}\right)C' , \quad \left(\frac{-m - \sqrt{m^2 - pn}}{p}\right)C' , \quad C'\right)$$

For p, m, and n as defined before in terms of specified real numbers A, B, C, x_1, y_1, and z_1.

As discussed in Chapter Three, these solutions are unique. Although different real numbers could be substituted for C' in each vector solution, if different numbers were substituted in the same solution, the result would be two parallel vectors which would both be perpendicualr to the same plane passing through the origin. For this reason any real substitution for C' will produce acceptible solution vectors. Let $C' = px_1$, where $p = (Ax_1 + By_1)^2 - (x_1^2 + y_1^2)C^2$. The vector solutions are then:

$$((pz_1 + my_1 - y_1\sqrt{m^2 - pn}), \quad (-mx_1 + x_1\sqrt{m^2 - pn}), \ px_1)$$

AND

$$((pz_1 + my_1 + y_1\sqrt{m^2 - pn}), \quad (-mx_1 - x_1\sqrt{m^2 - pn}), \ px_1)$$

These two direction vectors determine two distinct lines:

$$(pz_1 + my_1 - y_1\sqrt{m^2 - pn})x + (-mx_1 + x_1\sqrt{m^2 - pn})y = (px_1)z$$

AND

$$(pz_1 + my_1 + y_1\sqrt{m^2 - pn})x + (-mx_1 - x_1\sqrt{m^2 - pn})y = (px_1)z,$$

where:

$$p = (Ax_1 + By_1)^2 - (x_1^2 - y_1^2)C^2$$
$$m = Bz_1(-Ax_1 - By_1) + Cy_1(Cz_1 - Ax_1) + CBx_1^2$$
$$n = B^2(z_1^2 - x_1^2) - (Ax_1 - Cz_1)^2.$$

Both of these distinct lines contain (x_1, y_1, z_1) and neither of them intersects $Ax + By = Cz$. Therefore, in the space of the sphere of radius ri determined by $x^2 + y^2 + r^2 = z^2$, through a given point, (x_1, y_1, z_1), not on a given line, there is more than one line that does not intersect the given line.

The sphere of imaginary radius has been developed and points and lines have been defined. The points and lines have been shown to be well defined and satisfy the characteristics of intersection and determining lines. A method of determining the line through two given points has been presented and a necessary condition for two lines to intersect has been discovered. It has been demonstrated that the sphere of imaginary radius satisfies the characteristic property of Hyperbolic Geometry: through a given point, not on a given line, there is more than one line that does not intersect the given line. At this point it may be presumed that the sphere of imaginary radius is a model of a Hyperbolic Geometry.

Examples have been supplied to aid in understanding as the work, thus far, has progressed, but no attempt has been made to see what the imaginary sphere looks like.

CHAPTER FIVE

Illustrations

The sphere of imaginary radius has been developed and shown to satisfy the characteristic axiom of Hyperbolic Geometry. All of the work to this point has been abstract and without reference to illustrations.

At this time, a presentation is offered to aid in understanding the sphere, lines, and properties worked with earlier with the use of graphical illustrations. This presentation is not intended as a proof but rather as a different presentation of already proven properties.

The sphere of imaginary radius was, initially, defined as the set of points satisfying $x^2 + y^2 + r^2 = z^2$ where ri was the radius. This equation can be transformed by subtracting $x^2 + y^2$ from both sides, $r^2 = z^2 - x^2 - y^2$, and dividing both sides by r^2 into $1 = \frac{z^2}{r^2} - \frac{x^2}{r^2} - \frac{y^2}{r^2}$. This equation has the same form as a real hyperboloid with two branches in standard form, $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,[19] where $c^2 = a^2 = b^2 = r^2$. Since the radius and one variable are actually imaginary numbers, the sphere can be illustrated in a complex number space. The axes of this space are x, y, zi and the sphere appears as in Figure 7.

Lines were defined as the set of points satisfying

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[19]Richard E. Johnson and Fred L. Kiokemeister, Calculus with Analytic Geometry (Boston: Allyn and Bacon, Inc., 1960), p. 496.

The Sphere of Imaginary Radius

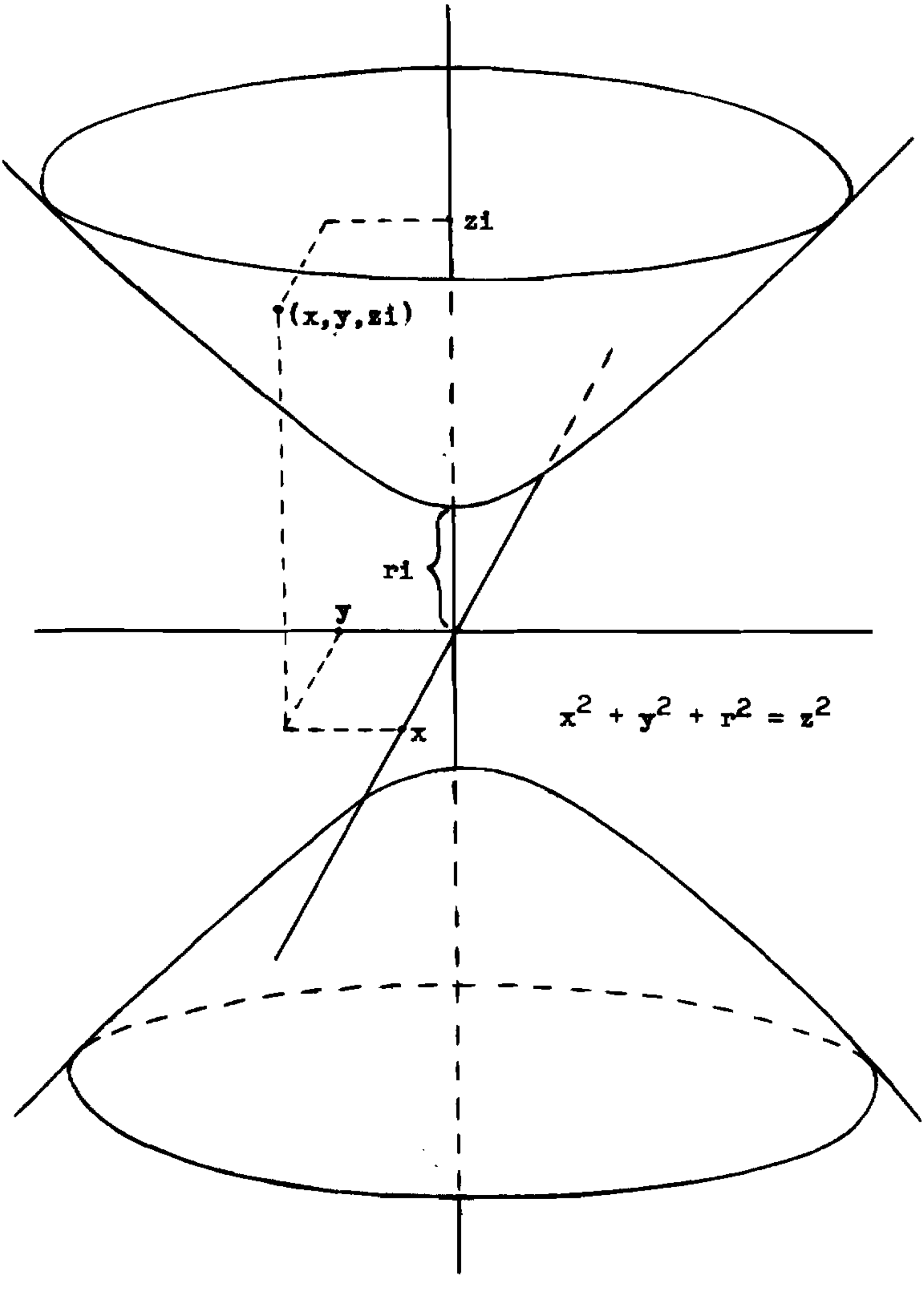


Figure 7

$x^2 + y^2 + r^2 = z^2$ and $Ax + By = Cz$. (A, B, Ci) was a vector perpendicular to a plane intersecting the sphere. The intersection of this plane and the sphere is the line of the space (Fig. 8). The line is actually a hyperbola on the intersecting plane.

Two distinct planes intersecting the sphere determine two distinct lines (Fig. 9). But the intersection of the two lines is two points. One is on the top branch and the other is on the lower branch.

The two points of intersection of two lines were not to be allowed. This problem was resolved by imposing the condition that $z > 0$. The effect is that the lower branch is cut out of the space. Only the top branch was left and the intersection of two distinct lines was then at most, one point.

Notice that this space also has the Euclidean property that three points determine one, and only one plane, and thus one line on the sphere. Two distinct points on the sphere and the origin, then, determine a unique line in the space.

The next problem to arise was the realization that not all planes intersect the sphere. A condition on the lines that intersect the sphere was needed. The condition was that for a plane, $Ax + By = Cz$, to intersect the sphere $A^2 + B^2 > C^2$.

Surrounding the hyperboloid is an asymptotic cone. This cone is determined by the equation $x^2 + y^2 = z^2$. Any vector, (A, B, C) having the property that $A^2 + B^2 < C^2$ will fall inside the cone. If $A^2 + B^2 = C^2$, the vector is on the cone and if $A^2 + B^2 > C^2$, the vector is on the exterior of the cone.

The Imaginary Sphere and Intersecting Plane

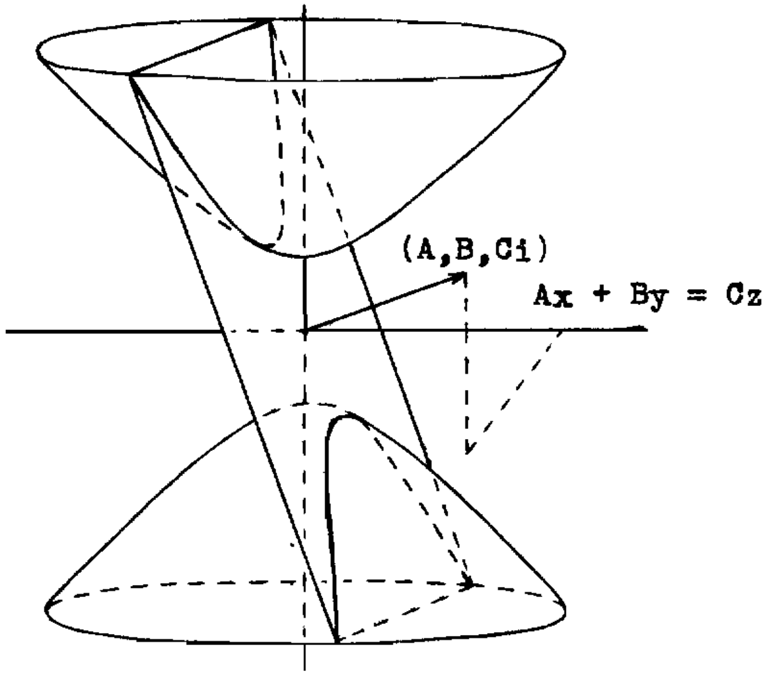


Figure 8

Two Intersecting Planes

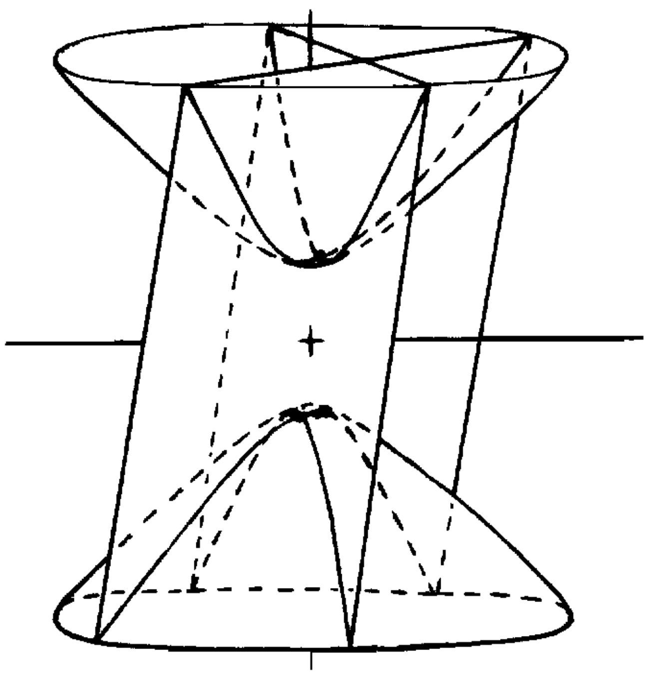


Figure 9

Since the plane containing the line is perpendicular to (A, B, C), when $A^2 + B^2 < C^2$ the vector is on the interior of the cone and the plane falls on the exterior. When $A^2 + B^2 = C^2$, the vector is on the cone and the corresponding plane is tangent, in a sense, to the cone. This plane does not intersect the sphere. Finally, when $A^2 + B^2 > C^2$, (A, B, C) is on the exterior of the cone and the plane is on the interior (Fig. 10). Since the cone acts asymptotically to the sphere, any plane on the interior of the cone intersects the sphere. $A^2 + B^2 > C^2$ is a necessary and sufficient condition for the plane $Ax + By = Cz$ to intersect the imaginary sphere.

The cone about the sphere of imaginary radius is important in the discussion of non-intersecting lines. When two planes intersect, a line is determined by their intersection (Fig. 11). The line must intersect the sphere for there to exist a point of intersection of the lines determined by the planes and the imaginary sphere.

Consider planes $Ax + By = Cz$ and $A'x + B'y = C'z$ determined by vectors (A, B, C) and (A′, B′, C′). It was determined, abstractly, that for these planes to have an intersection on the sphere $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. The line of intersection can be found by the first solving the two equations for x and y in terms of z.

$$\begin{aligned} Ax + By &= Cz \\ A'x + B'y &= C'z \end{aligned}$$

for x

$$\begin{aligned} AB'x + BB'y &= CB'x \\ A'Bx + BB'y &= C'Bz \\ \hline (AB' - A'B)x &= (CB' - C'B)z \end{aligned}$$

for y

$$\begin{aligned} A'Ax + A'By &= A'Cz \\ A'Ax + AB'y &= AC'z \\ \hline (AB' - A'B)y &= (AC' - A'C)z \end{aligned}$$

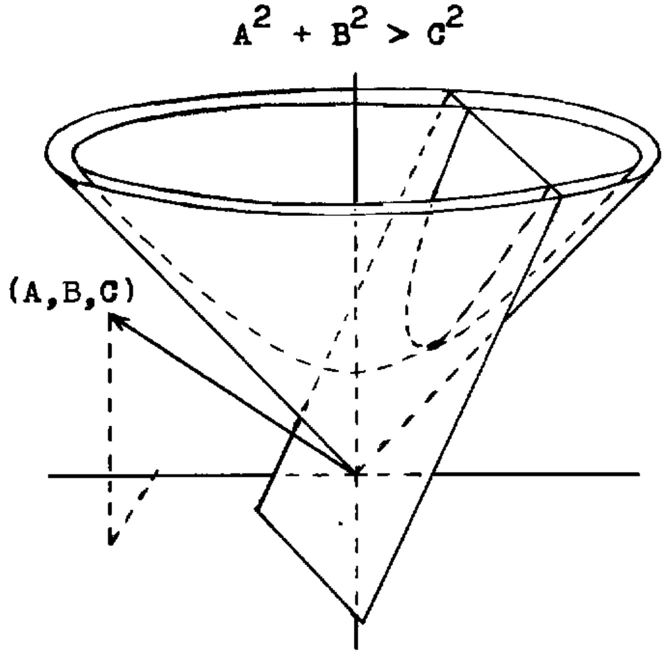


Figure 10

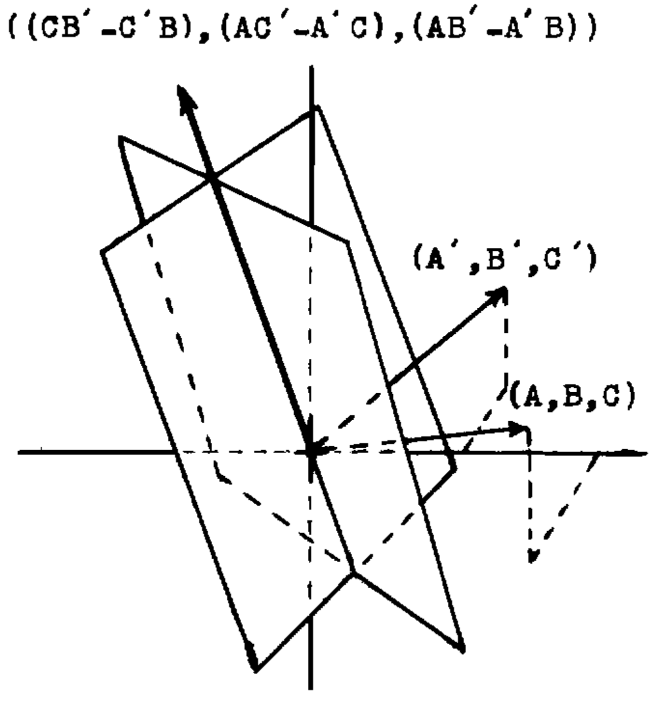


Figure 11

$$x = \frac{(CB' - C'B)}{(AB' - A'B)} z \qquad y = \frac{(AC' - A'C)}{(AB' - A'B)} z$$

A vector solution could be used to indicate the direction of this line of intersection:

$$\left(\frac{(CB' - C'B)}{(AB' - A'B)} z, \quad \frac{(AC' - A'C)}{(AB' - A'B)} z, \quad z \right) .$$

Let $z = (AB' - A'B)$ and the direction vector becomes:

$$((CB' - C'B), \quad (AC' - A'C), \quad (AB' - A'B)) .$$

For the line to intersect the sphere, this vector must fall on the interior of the cone, $x^2 + y^2 = z^2$, that surrounds the sphere. It has already been demonstrated that for (A, B, C) to be on the interior of the cone $A^2 + B^2 < C^2$. Applying this to the vector above, only when $(CB' - C'B)^2 + (AC' - A'C)^2 < (AB' - A'B)^2$ or,

$$0 < (AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2$$

does the vector fall within the cone and produce an intersection with the sphere (Fig. 12). When $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$, $Ax + By = Cz$ and $A'x + B'y = C'y$ intersect on the cone and no intersection exists with the sphere (Fig. 13). Finally, when the line of intersection is on the exterior of the cone, $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 < 0$, there will be no intersection on the imaginary sphere.

The proof that the sphere satisfied the hyperbolic property, through a given point not on a given line there is more than one line that does not intersect the given line, began with the existence of $Ax + By = Cz$ and (x_1, y_1, z_1) not on $Ax + By = Cz$. Effort was then applied to find $A'x + B'y = C'z$ such that (x_1, y_1, z_1) was on this line and $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$. The effect was to find a

$$(AB'-A'B)^2 - (CB'-C'B)^2 - (AC'-A'C)^2 > 0$$

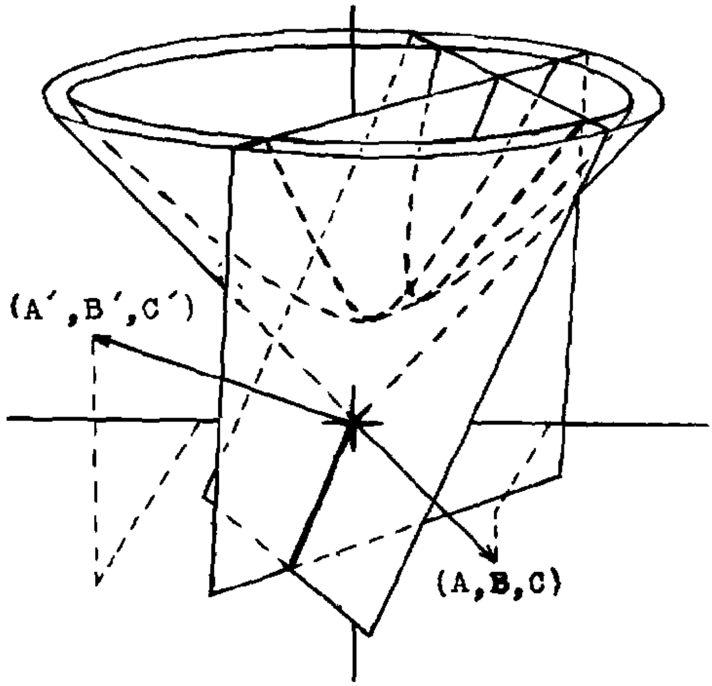


Figure 12

$$(AB'-A'B)^2 - (CB'-C'B)^2 - (AC'-A'C)^2 = 0$$

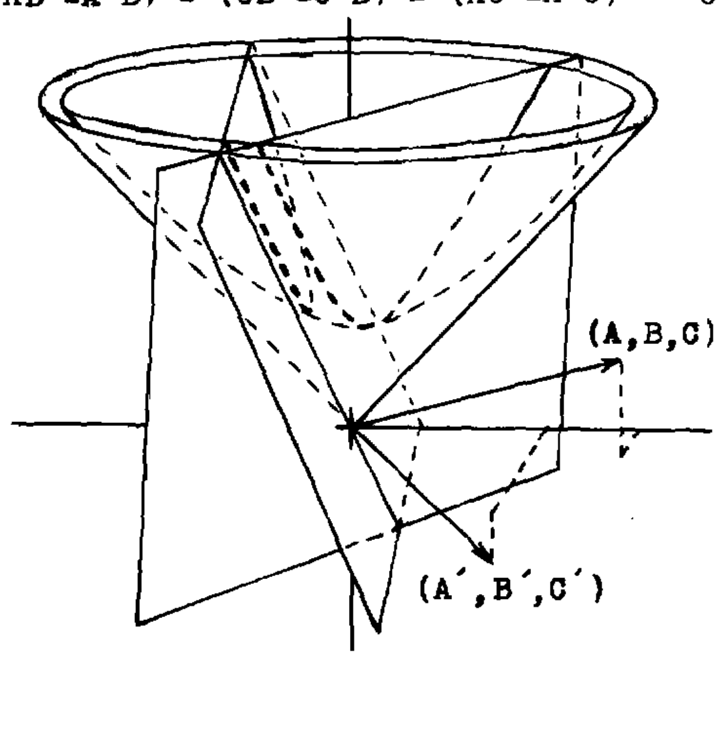
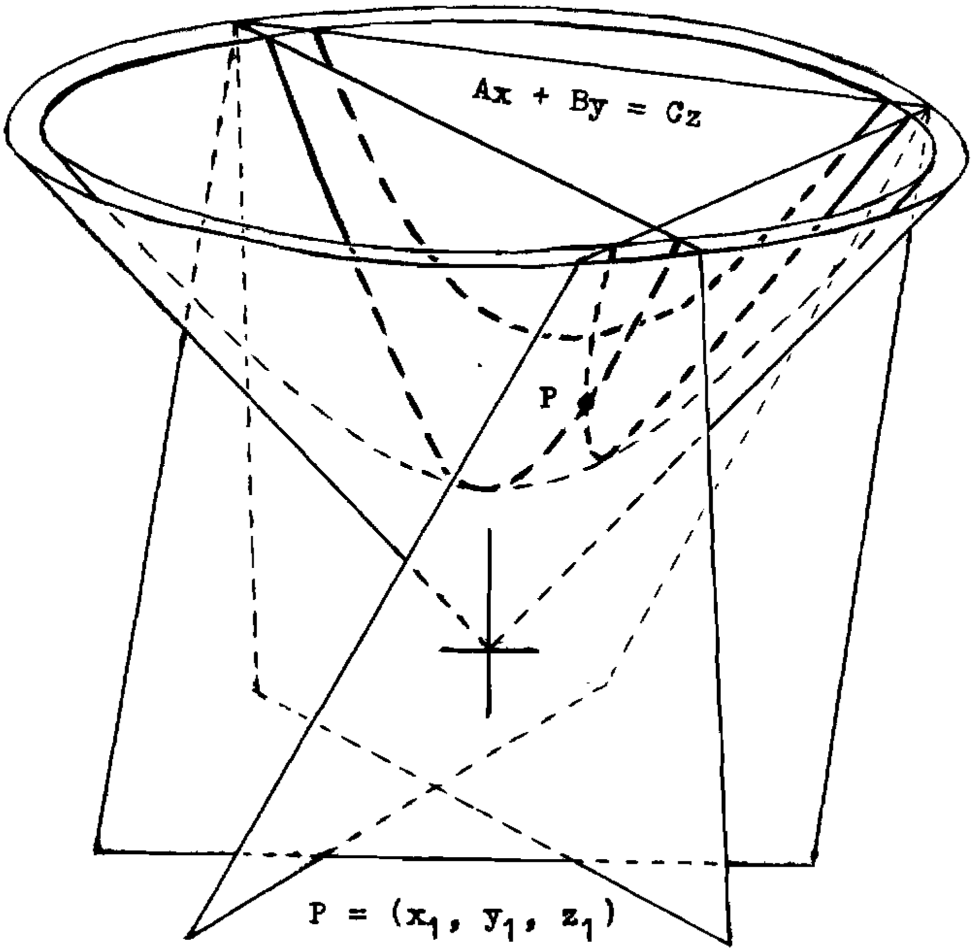


Figure 13

plane containing (x_1, y_1, z_1) and intersecting $Ax + By = Cz$ on the asymptotic cone surrounding the imaginary sphere. Notice, (Fig. 14), that $Ax + By = Cz$ has two lines of intersection with the cone and that it is possible to determine two planes, containing (x_1, y_1, z_1), that intersect $Ax + By = Cz$ on the cone.

Any planes through (x_1, y_1, z_1) that intersect $Ax + By = Cz$ on the interior of the cone will intersect on the surface of the sphere. Their corresponding line will than intersect $Ax + By = Cz$. Planes through (x_1, y_1, z_1) intersecting $Ax + By = Cz$ on the cone or its exterior will determine non-intersecting lines. Moving from the interior of the cone outward, it is, intuitively, acceptable that the planes through (x_1, y_1, z_1) intersecting $Ax + By = Cz$ on the cone are the first non-intersecting lines. These lines are called parallel in a given sense, or direction, to $Ax + By = Cz$.

The Hyperbolic Characteristic



Through a given point, P, not on a given line, determined by $Ax + By = Cz$, there is more than one line that does not intersect the given line.

Figure 14

CHAPTER SIX

Summary

The sphere of imaginary radius has been shown, directly, to be a model of Hyperbolic Geometry. The process to generate the proof has involved several concepts and resulted in the discovery of several ideas that make working on the imaginary sphere easier.

The concept of the real sphere and some of its characteristics were presented and developed in a very cursory manner to provide an analogy for the development of the imaginary sphere. The intersection of lines and methods of changing these characteristics were briefly discussed along with their effect on the geometry of the sphere. The sphere of imaginary radius was developed using the real sphere as a pattern. The sphere of radius ri, $r > 0$ was determined to be the set of ordered triples of real numbers (x, y, z) satisfying $x^2 + y^2 + r^2 = z^2$ where $z > 0$. Lines were defined as the set of points satisfying the equation of the imaginary sphere and $Ax + By = Cz$ where (A, B, C) was a vector such that $A^2 + B^2 > C^2$.

The points and lines were shown to be well defined by proving that they satisfied two conditions. It was shown that two points determine one, and only one line, and that two distinct lines determine, at most, one point. From these two proofs came several important points of information. A method of determining the equation of the plane containing points

(x_1, y_1, z_1) and (x_2, y_2, z_2) on $x^2 + y^2 + r^2 = z^2$ was found to be $(z_1y_2 - z_2y_1)x + (z_2x_1 - z_1x_2)y = (y_2x_1 - y_1x_2)z$. A condition for the intersection of two lines on the imaginary sphere specified by $Ax + By = Cz$ and $A'x + B'y = C'z$ was that $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. A demonstration and formula, if desired, for finding the point of intersection of two lines on the sphere of radius ri was presented. If the lines are determined by planes $Ax + By = Cz$ and $A'x + B'y = C'z$ the point of intersection (x, y, z) is:

$$x = (CB' - C'B)z, \qquad y = (C'A - A'C)z, \quad \text{where}$$

$$z = \sqrt{\frac{r^2}{(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2}}$$

The sphere of imaginary radius was shown to be a model of Hyperbolic Geometry by demonstrating that it satisfied the characteristic statement of that geometry. It was proven that on the imaginary sphere there is, through a given point not on a given line, more than one line that does not intersect the given line.

All of the work thus far described, has been done without the aid of illustrations other than numerical examples. These examples were provided simply to aid in understanding the concepts and proofs. This approach was to demonstrate the abstractness of mathematical creation. Mathematics does not depend on physical application for its creation. A mathematical space depends only on logic, not on truth. The abstractness of mathematics is its real beauty. Its logical foundation sets it apart from other sciences and allows it to be pure. The development of the imaginary sphere has been a conscious attempt to work with the sphere of imaginary radius without the aid of a visual representation. This abstract approach has shown that the work can be accomplished without a physical model. The properties discussed were discovered without the aid of any illustration of the sphere and perhaps were easier to see in the analytical sense.

After the sphere of imaginary radius was developed and proven to satisfy the hyperbolic characteristic of non-intersecting lines, a graphical presentation of what the imaginary sphere looks like was offered. This presentation attempted to follow as closely as possible the actual development and work with the sphere. Its purpose was to assist in accepting and understanding the analytical work and was not intended to be a part of the formal development and proofs.

The direct approach of this presentation opened the way for more work in the same line. The imaginary sphere has been shown to be a model of a hyperbolic space without the use of transformations of any other indirect method. Work could continue on in a direct manner to demonstrate that other hyperbolic properties are exemplified on the imaginary sphere. In this vein, a development of such concepts as what is meant by acute and right angles, betweeness of points and measures could be done and proof provided for such hyperbolic theorems on the imaginary sphere as:

(1) If t is any line and P is any point not on t, then there are always two lines through P which do not intersect t, which make equal

acute angles with the perpendicular from P to t, and which are such that every line passing through P lying within the angle containing that perpendicular intersects t, while every other line through P does not.[20]

(2) If a straight line is parallel through a given point in a given sense to a given line, it is, at each of its points, the parallel in the given sense to the given line.[21]

(3) In a trirectangular quadrilateral, the fourth angle is acute.[22]

The proofs of these and other hyperbolic properties by methods introduced in this thesis are left for further study.

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[20] Wolfe, Non-Euclidean Geometry, p. 67.

[21] Ibid. p. 68.

[22] Ibid. p. 79.

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APPENDIX I

Prove: On the sphere of radius ri, determined by $x^2 + y^2 + r^2 = z^2$, through a given point not on a given line there is more than one line that does not intersect the given line.

Given: Sphere $x^2 + y^2 + r^2 = z^2$, $z > 0$, line L, $Ax + By = Cz$ and point P, (x_1, y_1, z_1) not on line L.

Let $A'x + B'y = C'z$ be a line containing P, (x_1, y_1, z_1).

<u>Case I</u>: Let $x_1 \neq 0$, then $A'x_1 + B'y_1 = C'z_1$,

$$A'x_1 = C'z_1 - B'y_1,$$

$$A' = \frac{C'z_1 - B'y_1}{x_1}.$$

If $Ax + By = Cz$ intersects $A'x + B'y = C'z$ then $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 > 0$. If the condition $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 \leq 0$ is imposed, $Ax + By = Cz$ and $A'x + B'y = C'z$ will not intersect on the sphere of radius ri; consider, specifically, the case where $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$. Substitution of $\dfrac{C'z_1 - B'y_1}{x_1}$ for A' gives:

$$\left[AB' - \left(\frac{(C'z_1 - B'y_1)}{x_1}\right)B\right]^2 - \left[CB' - C'B\right]^2 - \left[AC' - \left(\frac{(C'z_1 - B'y_1)}{x_1}\right)C\right]^2 = 0$$

$$\left[\frac{Ax_1B' - (Bz_1C' - By_1B')}{x_1}\right]^2 - \left[CB' - BC'\right]^2 - \left[\frac{Ax_1C' - (Cz_1C' - Cy_1B')}{x_1}\right]^2 = 0$$

At this point, let $d = Ax_1$, $e = Bz_1$, $f = By_1$, $g = Cz_1$ and $h = Cy_1$. Proper substitution gives:

$$\left(\frac{dB' - (eC' - fB')}{x_1}\right)^2 - (CB' - BC')^2 - \left(\frac{dC' - (gC' - hB')}{x_1}\right)^2 = 0,$$

$$\left(\frac{(d + f)B' - eC'}{x_1}\right)^2 - (CB' - BC')^2 - \left(\frac{(d - g)C' + hB'}{x_1}\right)^2 = 0$$

Now let $j = d + f$, and $k = d - g$. The result is:

$$\left(\frac{jB' - eC'}{x_1}\right)^2 - (CB' - BC')^2 - \left(\frac{kC' + hB'}{x_1}\right)^2 = 0,$$

$$\frac{(jB' - eC')^2}{x_1^2} - (CB' - BC')^2 - \frac{(kC' + hB')^2}{x_1^2} = 0,$$

$$(jB' - eC')^2 - (CB' - BC')^2 x_1^2 - (kC' + hB')^2 = 0,$$

Expanding this equation results in:

$$j^2B'^2 - 2jeC'B' + e^2C'^2 -$$
$$(C^2B'^2 - 2BCC'B' + B^2C'^2)x_1^2 -$$
$$(k^2C'^2 + 2khC'B' + h^2B'^2) = 0,$$

$$j^2B'^2 - 2jeC'B' + e^2C'^2 -$$
$$C^2x_1^2B'^2 + 2BCx_1^2C'B' - B^2x_1^2C'^2 -$$
$$h^2B'^2 - 2khC'B' - k^2C'^2 = 0 \qquad \text{OR}$$

$$(j^2 - C^2x_1^2 - h^2)B'^2 + 2(-je + BCx_1^2 - kh)C'B' + (e^2 - B^2x_1^2 - k^2)C'^2 = 0$$

Let $p = j^2 - C^2x_1^2 - h^2$

$m = -je + BCx_1^2 - kh$

$n = e^2 - B^2x_1^2 - k^2$, and the equation is:

$$pB'^2 + 2mC'B' + nC'^2 = 0$$

If $p \neq 0$, this equation can be solved for B' in terms of C' using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Proper substitution yields:

$$B' = \frac{-2mC' \pm \sqrt{4m^2C'^2 - 4pnC'^2}}{2p}$$

$$B' = \frac{-2mC' \pm 2C'\sqrt{m^2 - pn}}{2p}$$

$$B' = \left(\frac{-m \pm \sqrt{m^2 - pn}}{p}\right)C'$$

$$B' = \left(\frac{-m + \sqrt{m^2 - pn}}{p}\right)C'$$

OR

$$B' = \left(\frac{-m - \sqrt{m^2 - pn}}{p}\right)C' \quad .$$

There are now, two solutions of B' in terms of C'. Since $A' = \frac{C'z_1 - B'y_1}{x_1}$, there are, also, two solutions of A' in terms of C';

$$A' = \frac{C'z_1 - \left(\frac{-m + \sqrt{m^2 - pn}}{p}\right)C'y_1}{x_1} \quad ,$$

$$A' = \left(\frac{pz_1 + my_1 - y_1\sqrt{m^2 - pn}}{px_1}\right)C' \quad ,$$

OR

$$A' = \frac{C'z_1 - \left(\frac{-m - \sqrt{m^2 - pn}}{p}\right)C'y_1}{x_1}$$

$$A' = \left(\frac{pz_1 + my_1 + y_1\sqrt{m^2 - pn}}{px_1}\right)C'$$

In terms of C', two ordered triples, (A', B', C'), have been determined:

$$\left(\left(\frac{pz_1 + my_1 - y_1\sqrt{m^2 - pn}}{px_1}\right)C', \quad \left(\frac{-m + \sqrt{m^2 - pn}}{px_1}\right)C', \quad C'\right)$$

$$\left(\left(\frac{pz_1 + my_1 + y_1\sqrt{m^2 - pn}}{px_1}\right)C', \quad \left(\frac{-m - \sqrt{m^2 - pn}}{px_1}\right)C', \quad C'\right)$$

Let $C' = px_1$, and the two vectors become:

$$((pz_1 + my_1 - y_1\sqrt{m^2 - pn}), \quad (-m + \sqrt{m^2 - pn}), \quad px_1) \text{ AND}$$
$$((pz_1 + my_1 + y_1\sqrt{m^2 - pn}), \quad (-m - \sqrt{m^2 - pn}), \quad px_1) \ .$$

These two vectors determine two distinct lines in the space of the sphere of radius ri.

$$(pz_1 + my_1 - y_1\sqrt{m^2 - pn})x + (-m + \sqrt{m^2 - pn})y = (px_1)z \text{ , and}$$
$$(pz_1 + my_1 + y_1\sqrt{m^2 - pn})x + (-m - \sqrt{m^2 - pn})y = (px_1)z \ .$$

The two lines contain the point (x_1, y_1, z_1) and do not intersect $Ax + By = Cz$.

If $p = 0$ and $n \neq 0$, the equation becomes $2mC'B' + nC' = 0$. Solving for C produces:

$$C' = 0 \qquad \text{or} \qquad C' = \frac{-2m}{n} B'$$

$$A' = \frac{-y_1}{x_1} B' \qquad \text{or} \qquad A' = \left(\frac{-2mz_1 - ny_1}{nx_1}\right)B'$$

Let $B' = x_1$ or let $B' = nx_1$

producing vectors

$$(-y_1, x_1, 0) \qquad \text{and} \qquad ((-2mz_1 - ny_1), nx_1, -2mx_1)$$

or lines

$$(-y_1)x + (x_1)y = 0 \quad \text{and} \quad (-2mz_1 - ny_1)x + (nx_1)y = (-2mx_1)z$$

If $P = 0$, and $n = 0$, the equation is $2mC'B' = 0$. Solving this equation gives:

$$C' = 0 \qquad \text{or} \qquad B' = 0$$

$$A' = \frac{-y_1}{x_1}B' \qquad \text{or} \qquad A' = \frac{z_1}{x_1}C'$$

$$\text{let} \quad B' = x_1 \qquad \text{or} \quad \text{let} \quad C' = x_1$$

two vectors result

$$(-y_1, x_1, 0) \qquad \text{and} \qquad (z_1, 0, x_1)$$

determining

$$(-y_1)x + (x_1)y = 0 \qquad \text{and} \qquad (z_1)x = (x_1)c$$

In each of the parts of Case I, two lines have been determined that contain (x_1, y_1, z_1) and do not intersect $Ax + By = Cz$

<u>Case II</u>: Let $x_1 = 0$ and $y_1 \neq 0$ and $A'x_1 + B'y_1 = C'z$

becomes

$$B'y_1 = C'z$$

$$B' = \frac{C'z_1}{y_1}$$

Substituting for B' in $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$ gives:

$$\left(\frac{AC'z_1}{y_1} - A'B\right)^2 - \left(\frac{CC'z_1}{y_1} - C'B\right)^2 - (AC' - A'C)^2 = 0$$

$$\left(\frac{Az_1C' - By_1A'}{y_1}\right)^2 - \left(\frac{Cz_1C' - By_1C'}{y_1}\right)^2 - (AC' - CA')^2 = 0$$

Let $d = Az_1$, $e = By_1$, $f = Cz_1$ and

$$\frac{(dC' - eA')^2}{{y_1}^2} - \frac{(f - e)^2 C'^2}{{y_1}^2} - (AC' - CA')^2 = 0$$

Let $f - e = j$

$$(dC' - eA')^2 - j^2C'^2 - {y_1}^2(AC' - CA')^2 = 0$$

$$e^2A'^2 - 2deC'A' + d^2C'^2 - j^2C'^2 -$$
$$(C^2y_1^2A'^2 - 2ACy_1^2C'A' + A^2y_1^2C'^2) = 0$$

Let $p = e^2 - C^2y_1^2$

$m = de - ACy_1^2$

$n = d^2 - j^2 - A^2y_1^2$

$$pA'^2 - 2mC'A' + nC'^2 = 0$$

If $p \neq 0$, then

$$A' = \left(\frac{2m \pm \sqrt{4m^2 - 4pn}}{2p}\right) C'$$

and

$$A' = \left(\frac{m + \sqrt{m^2 - pn}}{p}\right)C' \qquad \text{or} \qquad A' = \left(\frac{m - \sqrt{m^2 - pn}}{p}\right) C'$$

Let $C' = py_1$

and vectors

$((my_1 + y_1\sqrt{m^2 - pn}), pz_1, py_1)$ and

$((my_1 - y_1\sqrt{m^2 - pn}), pz_1, py_1)$ are determined producing

$$(my_1 + y_1\sqrt{m^2 - pn})x + (pz_1)y = (py_1)z$$
$$(my_1 - y_1\sqrt{m^2 - pn})x + (pz_1)y = (py_1)z$$

If $p = 0$ and $n \neq 0$, the equation becomes $-2mC'A' + nC'^2$

$C' = 0$ or $C' = \frac{2m}{n}A'$

$B' = 0$ or $B' = \frac{2mz_1}{ny_1}A'$

A' = any real number

Let $A' = s \neq 0$ Let $A' = ny_1$

producing vectors

$(s, 0, 0)$ and $(ny_1, 2mz_1, 2my_1)$

and lines

$sx = 0$ and $(ny_1)x + (2mz_1)y = (2my_1)z$

If $p = 0$ and $n = 0$, then

$-2mC'A'$ has solutions

$$C' = 0 \qquad \text{or} \qquad A' = 0$$

$$B' = 0 \qquad \text{or} \qquad B' = \frac{z_1}{y_1} C'$$

$$A' = s \neq 0 \qquad\qquad C' = y_1$$

giving vectors

$$(s, 0, 0) \qquad \text{and} \qquad (0, z_1, y_1)$$

or lines

$$sx = 0 \qquad \text{and} \qquad (z_1)y = (y_1)z$$

In each part of Case II, two lines have been determined that contain (x_1, y_1, z_1) and do not intersect $Ax + By = Cz$.

<u>Case III</u>: Let $x_1 = y_1 = 0$ and

$A'x_1 + B'y_1 = C'z_1$ becomes $0 = C'z_1$

Since $z_1 > 0$, $C' = 0$

Substitution for C' in $(AB' - A'B)^2 - (CB' - C'B)^2 - (AC' - A'C)^2 = 0$ gives:

$$(AB' - A'B)^2 - (CB')^2 - (A'C)^2 = 0$$

$$A^2B'^2 - 2ABA'B' + B^2A'^2 - C^2B'^2 - C^2A'^2 = 0$$

$$(A^2 - C^2)B'^2 - 2ABA'B' + (B^2 - C^2)A'^2 = 0$$

Let $p = A^2 - C^2$

$m = AB$

$n = B^2 - C^2$ and

$$pB'^2 - 2mA'B' + nA'^2 = 0$$

If $p \neq 0$ then

$$B' = \left(\frac{2m \pm \sqrt{4m^2 - 4pn}}{2p}\right) A' \quad \text{and}$$

$$B' = \left(\frac{m + \sqrt{m^2 - pn}}{p}\right)A' \qquad \text{or} \qquad B' = \left(\frac{m - \sqrt{m^2 - pn}}{p}\right)A'$$

$$C' = 0$$

Let $A' = p$

Determining vectors

$(p, (m + \sqrt{m^2 - pn}), 0)$ and $(p, (m - \sqrt{m^2 - pn}), 0)$

or lines

$(p)x + (m + \sqrt{m^2 - pn})y = 0$ and $(p)x + (m - \sqrt{m^2 - pn})y = 0$

If $p = 0$ and $n \neq 0$, then

$-2mA'B' + nA'^2 = 0$ has solutions

$A' = 0$ or $A' = \frac{2m}{n} B'$

Let $B' = s \neq 0$ $\qquad$ $B' = n$

and vectors

$(0, s, 0)$ and $(2m, n, 0)$

determine lines

$sy = 0$ and $(2m)x + (n)y = 0$

If $p = 0$ and $n = 0$, the equation $-2mA'B' = 0$ has solutions:

$A' = 0$ or $B' = 0$

Let $B' = s \neq 0$ or $A' = s \neq 0$

determining vectors

$(0, s, 0)$ and $(s, 0, 0)$

and lines

$sy = 0$ and $sx = 0$

In each of the three parts of Case III, two distinct lines have been determined that contain (x_1, y_1, z_1) and do not intersect $Ax + By = Cz$.

By Case I, II, and III in the space of the sphere of radius ri, through a given point not on a given line there is more than one line that does not intersect the given line.

The values of p, m, and n were:

Case I: $p = (Ax_1 + By_1)^2 - (x_1^2 + y_1^2)C^2$

$m = Bz_1(-Ax_1 - By_1) + Cy_1(Cz_1 - Ax_1) + CBx_1^2$

$n = B^2(z_1^2 - x_1^2) - (Ax_1 - Cz_1)^2$

Case II: $p = (B^2 - C^2)y_1^2$

$m = Ay_1(Bz_1 - Cy_1)$

$n = A^2(z_1^2 - y_1^2) - Cz_1 + By_1$

Case III: $p = A^2 - C^2$

$m = AB$

$n = B^2 - C^2$