QUASI-UNIFORM STRUCTURES

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CHAPTER I

QUASI-UNIFORM STRUCTURES

INTRODUCTION

A quasi-uniform structure is a natural generalization of a uniform structure . In this chapter it is shown that every topological space admits a quasi-uniform structure . In general , a topological space will admit more than one compatible quasi-uniform structure . As with uniform structure , it is possible to study the concepts of completeness and totally boundedness and a notion of uniform or quasi-uniform continuity and other related concepts which can not be studied in a topological space .

DEFINITION 1.1 Let X be a non-empty set . A quasi-uniform structure $\mathcal{U}$ for the set X is a non-empty collection , $\mathcal{U}$, of subsets of $X \times X$ satisfying :

(1). $\Delta = \{ (x,x) : x \in X \} \subset U$ for each $U \in \mathcal{U}$,

(2). U_1 and $U_2 \in \mathcal{U}$ implies that $U_1 \cap U_2 \in \mathcal{U}$,

(3). $U_1 \in \mathcal{U}$ and $U \supset U_1$ implies that $U \in \mathcal{U}$,

(4). For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V \circ V \subset U$.

($U \circ V = \{ (x,y)$: there exists $z \in X$ with $(x,z) \in U$ and $(z,y) \in V \}$)

Then the pair (X , $\mathcal{U}$) is called a quasi-uniform space .

DEFINITION 1.2 Let $\mathcal{U}$ be a quasi-uniform structure for a set X satisfying :

(5). For each U in $\mathcal{U}$, then $U^{-1} = \{ (x,y):(y,x) \in U \} \in \mathcal{U}$, and $\mathcal{U}$ is called a uniform structure for X . The pair (X , $\mathcal{U}$) is called a uniform space .

DEFINITION 1.3 If $A \subset X \times X$, then A is symmetric if and only if $A^{-1} = A$, A is anti-symmetric if and only if $A \cap A^{-1} \subset \Delta$.

THEOREM 1.1 Let (X , $\mathcal{U}$) be a uniform space , then for each U in $\mathcal{U}$ there exists a symmetric V in $\mathcal{U}$ such that $V \circ V \subset U$.

PROOF. For each U in $\mathcal{U}$ there exists a W in $\mathcal{U}$ such that $W \circ W \subset U$. Since $W^{-1} \in \mathcal{U}$, it follows that $V = W \cap W^{-1} \in \mathcal{U}$. Now V is symmetric and $V \circ V \subset U$.

DEFINITION 1.4 Let $\mathcal{U}$ be a quasi-uniform structure for X , and let x be a point in X . Then $U[x] = \{ y:(x,y) \in U \}$ for $U \in \mathcal{U}$, and $\mathcal{U}[x] = \{ U[x] : U \in \mathcal{U} \}$.

THEOREM 1.2 Let $\mathcal{U}$ be a quasi-uniform structure for the set X . Set

$t_{\mathcal{U}} = \{ O \subset X$: if $x \in O$ then there exists $U \in \mathcal{U}$ such that $x \in U[x] \subset O \}$. Then $t_{\mathcal{U}}$ is a topology for X .

PROOF. Suppose for each $\alpha \in \Lambda$, $O_\alpha \in t_{\mathcal{U}}$. If $x \in \bigcup \{ O_\alpha : \alpha \in \Lambda \}$, then $x \in O_\alpha$ for some α in Λ . There exists a $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha[x] \subset O_\alpha \subset \bigcup \{ O_\alpha : \alpha \in \Lambda \}$. Therefore $\bigcup \{ O_\alpha : \alpha \in \Lambda \} \in t_{\mathcal{U}}$. Clearly $\phi \in t_{\mathcal{U}}$.

Suppose O_1 , $O_2 \in t_{\mathcal{U}}$. If $x \in O_1 \cap O_2$, then $x \in O_1$ and $x \in O_2$. There exist U_1 , $U_2 \in \mathcal{U}$ such that $x \in U_1[x] \subset O_1$, $x \in U_2[x] \subset O_2$. However , $x \in (U_1 \cap U_2)[x] = U_1[x] \cap U_2[x] \subset O_1 \cap O_2$. Hence $O_1 \cap O_2 \in t_{\mathcal{U}}$. Clearly X belongs to $t_{\mathcal{U}}$. Therefore $t_{\mathcal{U}}$ is a topology for the set X .

THEOREM 1.3 Let $\mathcal{U}$ be a quasi-uniform structure for the set X , then the collection $\mathcal{N} = \{ U[x] : U \in \mathcal{U} , x \in X \}$ is a neighborhood system for the topology $t_{\mathcal{U}}$.

PROOF. For each $x \in X$, $\mathcal{U}[x]$ forms a neighborhood system of x which satisfy four axioms as follows :

(N-1). For each x in X and for each U in $\mathcal{U}$, $x \in U[x]$, since $\Delta \subset U$, for each U in $\mathcal{U}$.

(N-2). $U[x]$, $V[x] \in \mathcal{N}$ implies that $U[x] \cap V[x] = (U \cap V)[x] \in \mathcal{N}$, since $U \cap V$ is in $\mathcal{U}$.

(N-3). Suppose that $U[x] \in \mathcal{N}$ and $U[x] \subset A$. Set $V = U \cup A \times A$, then $U \subset V \in \mathcal{U}$, and $V[x] = A$. Hence $A \in \mathcal{N}$.

(N-4). Let $U[x] \in \mathcal{N}$, then there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$. Thus $V \subset V \circ V \subset U$. Let $t \in V[x]$ and p be an arbitary point of $V[t]$. Then $(x,t) \in V$ and $(x,p) \in V \circ V \subset U$. This implies that $V[t] \subset U[x] \in \mathcal{N}$ by (N-3) . Hence for every $U[x] \in \mathcal{N}$, there exists $V[x]$ such that $U[x]$ is a neighborhood of each point of $V[x]$. Therefore $\mathcal{N}$ is a neighborhood system . Clearly , the topology generated by the neighborhood system $\mathcal{N}$ is $t_{\mathcal{U}}$. Hence $\mathcal{N}$ is a neighborhood system for the topology $t_{\mathcal{U}}$.

DEFINITION 1.5 Let (x , t) be a topological space and $\mathcal{U}$ be a quasi-uniform structure on X . Then $\mathcal{U}$ is said to be compatible with the topology t if $t = t_{\mathcal{U}}$.

The following are some examples of quasi-uniform spaces.

EXAMPLE 1.1 Let X be a non-empty set and $\mathcal{U} = \{ U : \Delta \subset U \subset X \times X \}$. Then $\mathcal{U}$ is a quasi-uniform structure and $t_{\mathcal{U}}$ is the discrete topology for X .

EXAMPLE 1.2 Let X be a non-empty set , and let $\mathcal{U} = \{ X \times X \}$ then (X , $\mathcal{U}$) is a quasi-uniform space . $t_{\mathcal{U}}$ is the trivial topology for X .

EXAMPLE 1.3 Let R denote the set of real numbers and let $r > 0$. Set $D_r = \{ (x,y) : |x - y| < r \}$. Then $\mathcal{U} = \{ U : D_r \subset U \subset R \times R , r > 0 \}$ is an uniform structure for R , and $t_{\mathcal{U}}$ is the usual topology for R . This follows since $D_r[x] = (x - r, x + r)$.

EXAMPLE 1.4 Let (X , d) be a metric space , and let $S_\epsilon = \{ (x,y) \in X \times X : d(x,y) < \epsilon \}$. Then the collection $\mathcal{U} = \{ U : S_\epsilon \subset U \subset X \times X , \epsilon > 0 \}$ is an uniform structure . $t_{\mathcal{U}}$ is the original topology for X .

EXAMPLE 1.5 Let X be a non-empty set linearly ordered by $<$ and let $W = \{ (x,y) : x \leq y \}$. $\mathcal{U} = \{ U : W \subset U \subset X \times X \}$ is a quasi-uniform structure for the set X , and $t_{\mathcal{U}}$ is generated by the family of all intervals of the form $[a , \infty) = \{ x : x \geq a \}$ for any $a \in X$.

EXAMPLE 1.6 Let X be a non-empty set linearly ordered by $<$. Let $W_a = \{ (x,y) : x = y \text{ or } a < x < y \}$ for $a \in X$. Then $\mathcal{U} = \{ U : W_a \subset U \subset X \times X , a \in X \}$ is a quasi-uniform structure for the set X . $t_{\mathcal{U}}$ is the discrete topology for the set X .

EXAMPLE 1.7 Let X be a non-empty set linearly ordered by $<$. For some fixed points a , b of X , define $V_{a,b} = \{ (x,y) : x = y \text{ or } a \leq x \leq b \}$ then $\mathcal{U} = \{ U : V_{a,b} \subset U \subset X \times X , a \leq b , a , b \in X \}$ is a quasi-uniform structure for the set X , and $t_{\mathcal{U}}$ is a discrete topology for the set X . In fact , for each $x \in X$ there exists $a , b \in X$ such that $x < a < b$ and $x = V_{a,b}[x]$.

EXAMPLE 1.8 Let X be a non-empty set linearly ordered by $<$, and let a and b be fixed points of X . Define $T_{a,b} = \{ (x,y): a \leq x \leq y \leq b \} \cup \Delta$. Then $\mathcal{U} = \{ U: T_{a,b} \subset U \subset X \times X , a \leq b , a , b \in X \}$ is a quasi-uniform structure for the set X , and $t_{\mathcal{U}}$ is a discrete topology for the set X . This follows by the fact that for each $x \in X$ there exists $a , b \in X$ such that $x < a < b$ and $x = T_{a,b}[x]$.

EXAMPLE 1.9 Let X be a non-empty set linearly ordered by $<$, and let $a \in X$ be given . If $H_a = \{ (x,y): x = y \text{ or } a \leq x \}$, then $\mathcal{U} = \{ U: H_a \subset U \subset X \times X, a \in X \}$ is a quasi-uniform structure for the set X and $t_{\mathcal{U}}$ is a discrete topology for the set X . In fact for each $x \in X$, there exists a point a in X such that $x < a$ and $x = H_a[x]$.

EXAMPLE 1.10 Let X be a non-empty set linearly ordered by $<$, and let $a \in X$ be given . If $L_a = \{ (x,y): x \leq a \leq y \} \cup \Delta$, then $\mathcal{U} = \{ U: L_a \subset U \subset X \times X , a \in X \}$ is a quasi-uniform structure for the set X and $t_{\mathcal{U}}$ is a discrete topology for X since for each $x \in X$, $x = L_a[x]$ by choosing $a \in X$ such that $x < a$.

EXAMPLE 1.11 Let R denote the set of real numbers with the usual order $<$ and let $\epsilon > 0$ be given . Set $W_\epsilon = \{ (x,y): y < x + \epsilon \}$. Then $\mathcal{U} = \{ U: W_\epsilon \subset U \subset X \times X , \epsilon > 0 \}$ is a quasi-uniform structure for R , and $t_{\mathcal{U}}$ is the left-hand topology for X generated by the base consisting of all intervals of the form $(-\infty, a) = \{ x: x < a \}$ for any real number a .

EXAMPLE 1.12 Let N denote the set of natural numbers and $U_n = \{ (x,y) \in N \times N : x = y \text{ or } x \geq n \}$, then $\mathcal{U} = \{ U : U_n \subset U \subset X \times X , n \in N \}$ is a quasi-uniform structure for the set N , and $t_{\mathcal{U}}$ is the discrete topology since $U_{n+1}[n] = \{ n \}$ for each $n \in N$.

DEFINITION 1.6 Let $\mathcal{U}$ be a quasi-uniform structure on X . The conjugate $\mathcal{U}^{-1}$ of $\mathcal{U}$ is the collection of subsets of $X \times X$ defined by $\mathcal{U}^{-1} = \{ U^{-1} : U \in \mathcal{U} \}$.

THEOREM 1.4 The conjugate of a quasi-uniform structure is a quasi-uniform structure .

PROOF. Let (X , $\mathcal{U}$) be a quasi-uniform space , and let $\mathcal{U}^{-1}$ be the conjugate of $\mathcal{U}$. $\Delta \subset U^{-1}$ for every $U^{-1} \in \mathcal{U}^{-1}$, since $\Delta = \Delta^{-1} \subset U^{-1}$. Let U^{-1} , $V^{-1} \in \mathcal{U}^{-1}$ then $U^{-1} \cap V^{-1} = (U \cap V)^{-1}$. Hence $U^{-1} \cap V^{-1} \in \mathcal{U}^{-1}$. Let $U^{-1} \in \mathcal{U}^{-1}$ and $U^{-1} \subset A \subset X \times X$, then $U \subset A^{-1}$ and $A^{-1} \in \mathcal{U}$. Hence $A \in \mathcal{U}^{-1}$. Let $U^{-1} \in \mathcal{U}^{-1}$, then $U \in \mathcal{U}$. There exists a V in $\mathcal{U}$ such that $V \circ V \subset U$. This implies that $V^{-1} \circ V^{-1} = (V \circ V)^{-1} \subset U^{-1}$. Thus $\mathcal{U}^{-1}$ is a quasi-uniform structure for the set X .

DEFINITION 1.7 Let $f : X \rightarrow Y$, and $g : S \rightarrow T$, then the function $f \times g : X \times S \rightarrow Y \times T$ is defined by $(f \times g)(x,y) = (f(x) , f(y))$ for every $(x,y) \in X \times S$. In particular , if f = g and Y = T , and then denoted $f \times f$ by f_2 .

DEFINITION 1.8 Let $\mathcal{B}$ be a non-empty collection of subsets of $X \times X$. $\mathcal{B}$ is a base for a quasi-uniform structure on X if and only if

(1). $\Delta \subset B$ for every $B \in \mathcal{B}$,

(2). If B_1 , $B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$,

(3). For each $B \in \mathcal{B}$ there exists a $B' \in \mathcal{B}$ such that $B' \circ B' \subset B$.

THEOREM 1.5 Let $f : X \rightarrow Y$, and let $\mathcal{U}$ be a quasi-uniform structure on Y , then the collection $f_2^{-1}(\mathcal{U}) = \{ f_2^{-1}(U) : U \in \mathcal{U} \}$ is a quasi-uniform base on X .

PROOF. Since $\Delta_Y \subset U$ for each $U \in \mathcal{U}$, it follows that $\Delta_X \subset f_2^{-1}(U)$. Let $f_2^{-1}(U)$, $f_2^{-1}(V) \in f_2^{-1}(\mathcal{U})$. Then $f_2^{-1}(U) \cap f_2^{-1}(V) = f_2^{-1}(U \cap V) \in f_2^{-1}(\mathcal{U})$. Let $f_2^{-1}(U) \in f_2^{-1}(\mathcal{U})$, then there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$. Hence $f_2^{-1}(V) \circ f_2^{-1}(V) \subset f_2^{-1}(U)$. Thus $f_2^{-1}(\mathcal{U})$ is a quasi-uniform base on X .

DEFINITION 1.9 Let (X , $\mathcal{U}$) and (Y , $\mathcal{V}$) be quasi-uniform spaces . A function $f : X \rightarrow Y$ is said to be quasi-uniformly continuous if and only if for every V in $\mathcal{V}$, $f_2^{-1}(V)$ in $\mathcal{U}$.

THEOREM 1.6 Let (X , $\mathcal{U}$) and (Y , $\mathcal{V}$) be quasi-uniform spaces , and let $f : X \rightarrow Y$. Then the following statements are equivalment .

(1). f is quasi-uniformly continuous .

(2). For each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $(x,y) \in U$

implies that $(f(x) , f(y)) \in V$.

(3). $\mathcal{U}$ is finer than $f_2^{-1}(\mathcal{V})$.

PROOF. (1) $\Rightarrow$ (2). For each $V \in \mathcal{V}$, $f_2^{-1}(V) \in \mathcal{U}$. Let $U = f_2^{-1}(V)$, then $f_2(U) = f_2(f_2^{-1}(V)) \subset V$.

(2) $\Rightarrow$ (3). For each $f_2^{-1}(V)$ in $f_2^{-1}(\mathcal{V})$ there exists a $U \in \mathcal{U}$ such that $U \subset f_2^{-1}(V)$. Hence $f_2^{-1}(V) \in \mathcal{U}$.

(3) $\Rightarrow$ (1). Since $\mathcal{U}$ is finer than $f_2^{-1}(\mathcal{V})$, then for every $V \in \mathcal{V}$, $f_2^{-1}(V) \in f_2^{-1}(\mathcal{V})$ and therefore $f_2^{-1}(V) \in \mathcal{U}$.

THEOREM 1.7 Every quasi-uniformly continuous function is continuous .

PROOF. Let $f : (X , \mathcal{U}) \rightarrow (Y , \mathcal{V})$ be a quasi-uniformly continuous function . For each $V \in \mathcal{V}$, $V[f(x)]$ is a neighborhood of $f(x)$. Then there exists a $U \in \mathcal{U}$ such that $f_2(U) \subset V$. Hence $f(U[x]) = (f_2(U))[f(x)] \subset V[f(x)]$. Therefore f is a continuous function .

Let R denote the set of real numbers . For each $r > 0$, set $D_r = \{ (x,y) \in R \times R : |x - y| < r \}$. Then $\mathcal{D} = \{ D_r : r > 0 \}$ forms a quasi-uniform base for R . Let $\mathcal{U}$ be the quasi-uniform structure on R which is generated by $\mathcal{D}$. If $f : (R , \mathcal{U}) \rightarrow (R , \mathcal{U})$ is a quasi-uniformly continuous function , then , equivalently , for each U in $\mathcal{U}$ there exists $U' \in \mathcal{U}$ such that $f_2(U') \subset U$. In other words , for each $U \in \mathcal{U}$ there exists a D_δ with $\delta > 0$ such that $f_2(D_\delta) \subset U$. That is , for each base element D_ϵ of $\mathfrak{D}$ there exists another base element D_δ such that $f_2(D_\delta) \subset D_\epsilon$. Or equivalently , for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $(x,y) \in D_\delta$ implies that $(f(x), f(y)) \in D_\epsilon$. Hence , for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. Thus f is a uniformly continuous function on the reals R .

CHAPTER II

PERVIN QUASI-UNIFORM STRUCTURE

DEFINITION 2.1 Let $\mathcal{S}$ be a non-empty collection of subsets of $X \times X$. $\mathcal{S}$ is called a subbase for a quasi-uniform structure on X if and only if

(1). $\Delta \subset S$ for each S in $\mathcal{S}$,

(2). For each S in $\mathcal{S}$, there exists a T in $\mathcal{S}$ such that $T \circ T \subset S$.

DEFINITION 2.2 Let (X , t) be a topological space . Set $\mathcal{S} = \{ O \times O \cup (X - O) \times X : O \in t \}$. Let $\mathcal{P}$ denoted the family of subsets of $X \times X$ which are supersets of finite intersections of members of $\mathcal{S}$. $\mathcal{P}$ is called the Pervin quasi-uniform structure for the topological space (X , t) . The following theorem justifies the above terminology .

THEOREM 2.1 Let (X , t) be a topological space , $\mathcal{P}$ as defined in definition 2.2 , then

(a). $\mathcal{P}$ is a quasi-uniform structure ,

(b). $\mathcal{t}_{\mathcal{P}} = t$.

PROOF. (a). Let $S = O \times O \cup (X - O) \times X \in \mathcal{S}$. Clearly $\Delta \subset S$. Suppose $(x,y) \in S$ and $(y,z) \in S$. If $x \in O$ then $y \in O$ and $z \in O$. This implies that $(x,z) \in O \times O \subset S$. If $x \in X - O$ then $(x,z) \in (X - O) \times X \subset S$. Hence $S \circ S \subset S$. Therefore the collection $\mathcal{S}$ forms a quasi-uniform subbase which generates the quasi-uniform structure $\mathcal{P}$.

(b). Let $x \in O \in t$. Then $S = O \times O \cup (X - O) \times X$ belongs to $\mathcal{P}$ and $x \in S[x] \subset O$. Therefore $t \leq t_{\mathcal{P}}$. Clearly , $t_{\mathcal{P}} \leq t$ and hence $t = t_{\mathcal{P}}$.

EXAMPLE 2.1 Let t be the usual topology for the set of real numbers R . Then the Pervin quasi-uniform structure $\mathcal{P}$ for R is generated by the subbase $\mathcal{S} = \{ (a,b) \times (a,b) \cup ((-\infty,a] \cup [b,\infty)) \times X : (a,b) \in t \}$.

Let $O = (a,b)$, then figure 2.1 illustrates the subbasic element $S = O \times O \cup (X - O) \times X$.

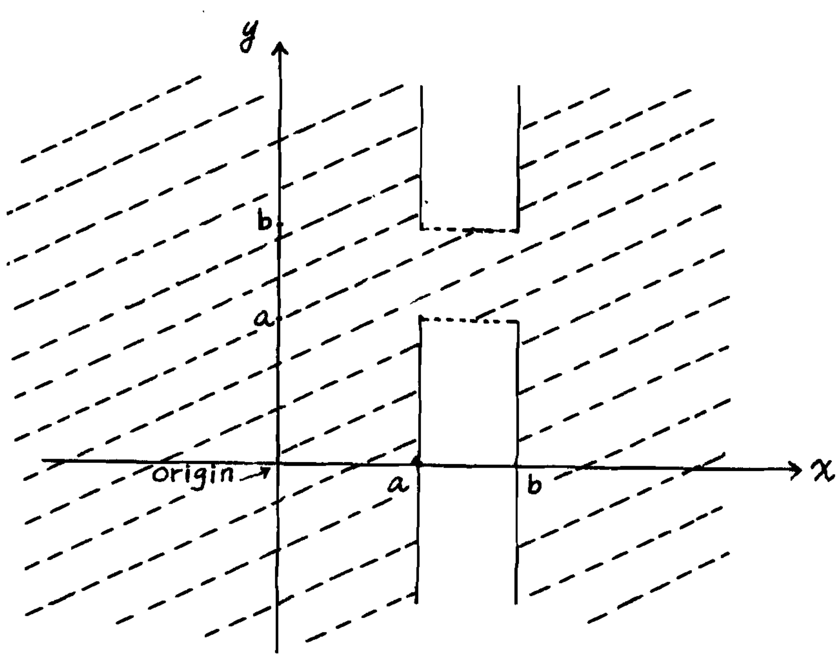
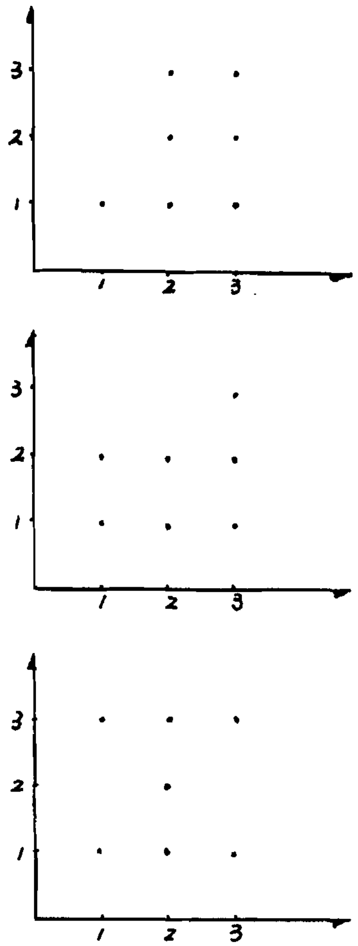


FIGURE 2.1

EXAMPLE 2.2 Let $X = \{1,2,3\}$, $t = \{\Phi, X, \{1\}, \{1,2\}, \{1,3\}\}$. Then (X, t) is a topological space . The Pervin quasi-uniform structure $\mathcal{P}$ for (X, t) is generated by the subbase $\mathcal{S} = \{\{1\}\times\{1\} \cup \{2,3\}\times\{1,2,3\}, \{1,2\}\times\{1,2\}\cup\{3\}\times\{1,2,3\}, \{1,3\}\times\{1,3\} \cup \{2\}\times\{1,2,3\}\}$. These subbasic elements are illustrated in figure 2.2 .



$S_1 = \{1\}\times\{1\} \cup \{2,3\}\times\{1,2,3\} \in \mathcal{S}$

$S_2 = \{1,2\}\times\{1,2\} \cup \{3\}\times\{1,2,3\} \in \mathcal{S}$

$S_3 = \{1,3\}\times\{1,3\} \cup \{2\}\times\{1,2,3\} \in \mathcal{S}$

FIGURE 2.2

CHAPTER III

SEPARATION AXIOMS

DEFINITION 3.1 A topological space (X , t) is a R_0-space if and only if for every open set 0 in t , containing x in X , it follows that $\bar{x} \subset 0$.

THEOREM 3.1 Every subspace of a R_0-space is a R_0-space .

PROOF. Let (Y , s) be a subspace of (X , t) , and let 0' be an open set in Y containing y with $0' = Y \cap 0$, where 0 is open in (X , t) . Then $y \in 0$ and $Cl_X\{y\} \subset 0$, since X is a R_0-space . Now $Cl_Y\{y\} = Y \cap Cl_X\{y\} \subset Y \cap 0 = 0'$ and Y is a R_0-space .

THEOREM 3.2 Let (X , t) be a topological space . Then the following three statements are equivalent .

(1). (X , t) is a R_0-space .

(2). For any closed subset A , and a point x not in A there exists a neighborhood of A not containing x .

(3). If $x \neq y$, then either $\bar{x} = \bar{y}$ or $\bar{x} \cap \bar{y} = \phi$.

PROOF. (1) $\Longrightarrow$ (2). Let A be closed and $x \in X - A$. Since (X , t) is a R_0-space , it follows that $\bar{x} \subset X - A$. Thus $X - \bar{x}$ is a neighborhood of A which does not contain the point x .

(2) $\Longrightarrow$ (1). Let 0 be an open set containing $x \in X$. Let $y \in \bar{x}$, then every neighborhood of y contains x . Suppose $y \notin 0$, then $y \in X - 0$ and , by (2) , there exists a neighborhood containing $X - 0$, but not containing x . This is impossible . Therefore $y \in 0$ and hence $\bar{x} \subset 0$.

(1) $\Longrightarrow$ (3). Suppose that $x \neq y$ and $\bar{x} \neq \bar{y}$, then it may be assumed that there exists an $a \in \bar{x}$ and $a \notin \bar{y}$. Now $x \notin \bar{y}$ for otherwise $x \in \bar{x} \subset \bar{y}$ and $a \in \bar{x} \subset \bar{y}$ contradicts the fact that $a \notin \bar{y}$. Since $x \notin \bar{y}$ then $x \in X - \bar{y}$. By the R_0 hypothesis , it follows that $\bar{x} \subset X - \bar{y}$. Therefore $\bar{x} \cap \bar{y} = \phi$.

(3) $\Longrightarrow$ (1). Let 0 be an open set and $x \in 0$. Let $y \in \bar{x}$ and $y \neq x$, then by (3) , either $\bar{x} = \bar{y}$, or $\bar{x} \cap \bar{y} = \phi$. But the second case is impossible , since $y \in \bar{x} \cap \bar{y}$. Thus $\bar{x} = \bar{y}$ and $x \in \bar{y}$ and therefore every open set 0 containing x must contain y . Hence $\bar{x} \subset 0$.

THEOREM 3.3 Let (X , t) be a topological space . Then the following statements are equivalent .

(1). (X , t) is a R_0-space .

(2). (X , t) has a compatible quasi-uniform structure $\mathcal{U}$ such that for each x in X and for each U in $\mathcal{U}$ there exists a symmetric $V \in \mathcal{U}$ with $V[x] \subset U[x]$.

(3). (X , t) has a compatible quasi-uniform structure $\mathcal{U}$ such that the collection $\{ V[x] : V \text{ symmetric and } V \in \mathcal{U} \}$ forms a local base at the point $x \in X$.

(4). (X , t) has a compatible quasi-uniform structure $\mathcal{U}$ such that for each x in X and for each U in $\mathcal{U}$ there exists a V in $\mathcal{U}$ such that $V^{-1}[x] \subset U[x]$.

(5). (X , t) has a compatible quasi-uniform structure $\mathcal{U}$ such that $t = t_{\mathcal{U}} \subset t_{\mathcal{U}^{-1}}$.

PROOF. $(1) \Longrightarrow (2)$. Let $\mathcal{P}$ be the Pervin quasi-uniform structure for the set X . For each x in X , and each $U \in \mathcal{U}$, there exists an open set $O \in t$ with $x \in O \subset U[x]$.

Define $S_O = O \times O \cup (X - O) \times X$,

$S_{\bar{x}^c} = \bar{x}^c \times \bar{x}^c \cup \bar{x} \times X$,

$V = O \times O \cup \bar{x}^c \times \bar{x}^c$.

Set $A = O \times O$,

$B = (X - O) \times X$,

$C = \bar{x}^c \times \bar{x}^c$,

$D = \bar{x} \times X$.

Then $S_O \cap S_{\bar{x}^c} = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$,

where $A \cap C = (O - x) \times (O - x) \subset O \times O \subset V$,

$A \cap D = \bar{x} \times O \subset O \times O \subset V$,

$B \cap C = (X - O) \times \bar{x}^c \subset \bar{x}^c \times \bar{x}^c \subset V$,

$B \cap D = \phi$.

Hence $S_O \cap S_{\bar{x}^c} \subset V$ and $V \in \mathcal{U}$. Furthermore , V is symmetric and $O = V[x] \subset U[x]$.

(2)$\Longleftrightarrow$(3). For each $x \in X$, the collection $\{ V[x] : V$ symmetric and $V \in \mathcal{U} \}$ is a local base at x if and only if for each $x \in X$ and each $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$ with $V[x] \subset U[x]$.

(3)$\Longrightarrow$(4). For each $x \in X$ and each $U \in \mathcal{U}$ there exists a symmetric $V \in \mathcal{U}$ with $V[x] \subset U[x]$. Since V is symmetric it follows that $V^{-1}[x] = V[x] \subset U[x]$.

(4)$\Longrightarrow$(5). Let $\mathcal{U}$ be a compatible quasi-uniform structure for the set X which satisfies condition (4) , then $t = t_{\mathcal{U}}$. For each open set $0 \in t_{\mathcal{U}}$, there exists a U in $\mathcal{U}$ with $x \in U[x] \subset 0$, and by (4) there exists a V in $\mathcal{U}$ with $V^{-1}[x] \subset U[x]$. Therefore $x \in V^{-1}[x] \subset U[x] \subset 0$. This implies that 0 belongs to $t_{\mathcal{U}^{-1}}$. Hence $t = t_{\mathcal{U}} \subset t_{\mathcal{U}^{-1}}$.

(5)$\Longrightarrow$(4). For every x in X and each U in $\mathcal{U}$, $x \in U[x]$. However , U[x] is a neighborhood of the point x , and there exists an open set 0 in $t_{\mathcal{U}}$ such that $x \in 0 \subset U[x]$. But $0 \in t_{\mathcal{U}^{-1}}$, since $t_{\mathcal{U}} \subset t_{\mathcal{U}^{-1}}$. Therefore there exists a $V^{-1} \in \mathcal{U}^{-1}$ such that $x \in V^{-1}[x] \subset 0 \subset U[x]$.

(4)$\Longrightarrow$(1). The point y belongs to $\bar{x}$ if and only if $U[y] \cap \{x\} \neq \phi$ for each $U \in \mathcal{U}$, or equivalently , $x \in U[y]$ for every U in $\mathcal{U}$. That is , $y \in U^{-1}[x]$ for each U in $\mathcal{U}$. Hence $\bar{x} = \cap \{ U^{-1}[x] : U \in \mathcal{U} \} \subset \cap \{ U[x] : U \in \mathcal{U}$ by condition (4) . This implies that $\bar{x}$ is contained in every neighborhood of x . Thus for each open set $0 \in t = t_{\mathcal{U}}$ containing the point $x \in X$ there exists $U \in \mathcal{U}$ with $x \in U[x] \subset 0$. Since $\bar{x} \subset U[x]$ for each U in $\mathcal{U}$ it follows that $\bar{x} \subset U[x] \subset 0$. Hence (X , t) is a R_0-space .

THEOREM 3.4 A quasi-uniform space $(X , \mathcal{U})$ is R_0 if and only if $\cap\{ U:U \in \mathcal{U}\}$ is symmetric .

PROOF. $(X , \mathcal{U})$ is a R_0-space , if and only if for each $x \in X$, $\bar{x} = \cap\{ U^{-1}[x]:U \in \mathcal{U}\} \subset \cap\{U[x]:U \in \mathcal{U}\}$ by theorem 3.3 . Now $\cap\{ U^{-1}:U \in \mathcal{U} \} \subset \cap\{U:U \in \mathcal{U}\}$ and this is equivalent to the statement that $\cap\{ U:U \in \mathcal{U}\}$ is symmetric .

COROLLARY $(X , \mathcal{U})$ is a R_0-space if and only if $\bar{x} = \cap\{ U[x]:U \in \mathcal{U}\}$.

The proof follows immediately from theorem 3.4 .

THEOREM 3.5 A quasi-uniform space $(X , \mathcal{U})$ is T_0 if and only if $\cap\{ U:U \in \mathcal{U}\}$ is anti-symmetric .

PROOF. Suppose $(X , \mathcal{U})$ is T_0-space . If x , y are two distinct points in X , then there exists an open set $0 \in t_{\mathcal{U}}$ which contains one of them but not the other . Suppose that $x \in 0$, $y \notin 0$, then there exists a U in $\mathcal{U}$ such that $x \in U[x] \subset 0$ and $y \notin U[x]$. Hence $x \neq y$ implies that there exists a U in $\mathcal{U}$ such that either $y \notin U[x]$ or $x \notin U[y]$. In other words , for each U in $\mathcal{U}$, $y \in U[x]$ and $x \in U[y]$ implies that $x = y$. Therefore $\cap\{ U:U \in \mathcal{U}\}$ is anti-symmetric .

If collection $\cap\{ U:U \in \mathcal{U}\}$ is anti-symmetric , then for any two distinct points x and y in X , there exists a U in $\mathcal{U}$ with either $(x,y) \notin U$ or $(y,x) \notin U$. Hence either $y \notin U[x]$ or $x \notin U[y]$. Therefore X must be a T_0-space .

THEOREM 3.6 A quasi-uniform space (X , $\mathcal{U}$) is T_1 if and only if $\Delta = \bigcap \{ U : U \in \mathcal{U} \}$.

PROOF. Suppose (X , $\mathcal{U}$) is a T_1-space . Clearly , $\Delta \subset \bigcap \{ U : U \in \mathcal{U} \}$. Suppose $x \neq y$, then there exists a U in $\mathcal{U}$ with $y \notin U[x]$. Therefore $(x,y) \notin \bigcap \{ U : U \in \mathcal{U} \}$. Hence $\bigcap \{ U : U \in \mathcal{U} \} \subset \Delta$ and therefore $\Delta = \bigcap \{ U : U \in \mathcal{U} \}$. The other part of the proof is natural and omitted .

THEOREM 3.7 A quasi-uniform space (X , $\mathcal{U}$) is T_1 if and only if it is T_0 and R_0 .

PROOF. Let the quasi-uniform space (X , $\mathcal{U}$) be T_1 , then for every open set O in $t_{\mathcal{U}}$ containing x in X , it follows that $\overline{x} = \{x\} \subset O$. Hence (X , $\mathcal{U}$) is a R_0-space . Clearly , every T_1-space is a T_0-space .

Suppose that (X , $\mathcal{U}$) is a T_0 and R_0-space , then , by theorems 3.4 and 3.5 , it follows that $S = \bigcap \{ U : U \in \mathcal{U} \}$ is both symmetric and anti-symmetric . Thus $S = S^{-1}$ and $S \cap S^{-1} = \Delta$. Hence $S = S \cap S^{-1} = \Delta$ and (X , $\mathcal{U}$) is a T_1-space by theorem 3.6 .

THEOREM 3.8 Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniform structures for a set X . If $M \subset X \times X$, then $\overline{M} = \bigcap \{ U \circ M \circ V^{-1} : U \in \mathcal{U} , V \in \mathcal{V} \}$.

PROOF. The ordered pair $(x,y) \in \overline{M}$ if and only if for each U in $\mathcal{U}$ and for each V in $\mathcal{V}$, $U[x] \times V[y] \cap M \neq \phi$. Equivalently , for each U in $\mathcal{U}$ and for each V in $\mathcal{V}$, there exists a point $(a,b) \in M$ such that $a \in U[x]$,

$b \in V[y]$. This is true , if and only if $(x,y) \in U \circ M \circ V^{-1}$ for each U in $\mathcal{U}$ and for each V in $\mathcal{V}$.

THEOREM 3.9 A quasi-uniform space (X , $\mathcal{U}$) is T_2 if and only if $\Delta = \bigcap \{ U \circ U^{-1} : U \in \mathcal{U} \}$.

PROOF. A topological space is T_2 if and only if Δ is closed in $X \times X$, that is , $\Delta = \bar{\Delta}$. By theorem 3.8 , $\bar{\Delta} = \bigcap \{ U \circ \Delta \circ U^{-1} : U \in \mathcal{U} \}$, but $U \circ \Delta \circ U^{-1} = U \circ U^{-1}$ for each $U \in \mathcal{U}$. Thus (X , $\mathcal{U}$) is T_2 if and only if $\Delta = \bar{\Delta} = \bigcap \{ U \circ U^{-1} : U \in \mathcal{U} \}$.

DEFINITION 3.2 A topological space (X , t) is a R_1-space if and only if $x \neq y$ implies that $\bar{x}$ and $\bar{y}$ have disjoint neighborhoods .

THEOREM 3.10 Every subspace of a R_1-space is a R_1-space .

PROOF. Suppose (Y , s) is a subspace of (X , t) , let y_1 , y_2 be points in Y , then $Cl_Y\{y_1\} = Y \cap Cl_X\{y_1\}$ and $Cl_Y\{y_2\} = Y \cap Cl_X\{y_2\}$. If $Cl_Y\{y_1\} \neq Cl_Y\{y_2\}$, then $Cl_X\{y_1\} \neq Cl_X\{y_2\}$. Since (X , t) is a R_1-space , then there exist two disjoint neighborhoods N_{y_1} , N_{y_2} containing $Cl_X\{y_1\}$, $Cl_X\{y_2\}$ respectively . That is,there exist open sets O_1 and O_2 such that $Cl_X\{y_1\} \subset O_1 \subset N_{y_1}$, and $Cl_X\{y_2\} \subset O_2 \subset N_{y_2}$. This implies that $Cl_Y\{y_1\} = Y \cap Cl_X\{y_1\} \subset Y \cap O_1 \subset Y \cap N_{y_1}$ and $Cl_Y\{y_2\} = Y \cap Cl_X\{y_2\} \subset Y \cap O_2 \subset Y \cap N_{y_2}$. Now , $(Y \cap N_{y_1}) \cap (Y \cap N_{y_2}) = Y \cap (N_{y_1} \cap N_{y_2}) = \phi$.

Hence $Cl_Y\{y_1\}$, $Cl_Y\{y_2\}$ have disjoint neighborhoods $Y \cap N_{y_1}$ and $Y \cap N_{y_2}$. Therefore the subspace (Y , s) is a R_1-space .

THEOREM 3.11 A R_1-space (X , t) is a R_0-space .

PROOF. Let 0 be any open set in t which contains $x \in X$, and let $y \in X - 0$, then $\bar{y} \subset X - 0$ and $x \notin \bar{y}$. Hence $\bar{x} \neq \bar{y}$. Since (X , t) is a R_1-space , there exist disjoint neighborhoods $N_{\bar{x}}$, $N_{\bar{y}}$ such that $\bar{x} \subset N_{\bar{x}}$, $\bar{y} \subset N_{\bar{y}}$. Therefore $y \notin \bar{x}$ and $\bar{x} \subset 0$. Hence (X , t) is a R_0-space .

THEOREM 3.12 The following three statements are equivalent .

(1). (X , t) is a R_1-space .

(2). For any points x , y in X , $\bar{x} \neq \bar{y}$ implies that x and y have disjoint neighborhoods .

(3). $\nabla = \{ (x,y) : \bar{x} = \bar{y} \} = \bar{\Delta}$.

PROOF. (1) $\Longrightarrow$ (2). For any points x , y in X , if $\bar{x} \neq \bar{y}$, then there exist disjoint neighborhoods $N_{\bar{x}}$ and $N_{\bar{y}}$ of $\bar{x}$ and $\bar{y}$, respectively . Since $x \in \bar{x}$ and $y \in \bar{y}$, there exist disjoint neighborhoods of x and y .

(2) $\Longrightarrow$ (3). For any open set 0_x containing x , and for any point $z \in X - 0$, then $\bar{z} \subset X - 0$. Hence $x \notin \bar{z}$ and x , z have disjoint neighborhoods N_x and N_z respectively . This implies that $z \notin \bar{x}$ and $\bar{x} \subset 0$.

Let $(x,y) \in \nabla$, then $\bar{x} = \bar{y}$. Let 0_x , 0_y be arbitary open sets of x and y , respectively . Then $(x,x) \in \bar{x} \times \bar{y} \subset O_x \times O_y$ and $O_x \times O_y \cap \Delta \neq \phi$. This implies that $(x,y) \in \bar{\Delta}$ and $\nabla \subset \bar{\Delta}$. On the other hand , let $(x,y) \in \bar{\Delta}$, then $O_x \times O_y \cap \Delta \neq \phi$ for any open sets O_x and O_y of x and y , respectively . That is to say , $O_x \cap O_y \neq \phi$ for any open sets O_x and O_y of x and y , respectively . Therefore , by (2) , $\bar{x} = \bar{y}$ and hence $(x,y) \in \nabla$.

(3) $\Longrightarrow$ (1). For any point x , y in X , if $\bar{x} \neq \bar{y}$, then $(x,y) \notin \nabla$. That is (x,y) in $X - \nabla = X - \bar{\Delta}$ which is open in the product topology of $X \times X$. Hence there exist open sets O_x , O_y of x and y respectively with $(x,y) \in O_x \times O_y \subset X - \nabla = X - \bar{\Delta}$. Therefore $O_x \cap O_y \cap \Delta = \phi$ and $O_x \cap O_y = \phi$. Hence (X , t) is a R_1-space .

THEOREM 3.13 A quasi-uniform space (X , $\mathcal{U}$) is R_1 if and only if $\nabla = \cap \{ U \circ U^{-1} : U \in \mathcal{U} \}$.

PROOF. By theorem 3.12 , (X , $\mathcal{U}$) is R_1 if and only if $\nabla = \bar{\Delta}$. $\bar{\Delta} = \cap \{ U \circ \Delta \circ U^{-1} : U \in \mathcal{U} \} = \cap \{ U \circ U^{-1} : U \in \mathcal{U} \}$ by theorem 3.7 . Thus (X , $\mathcal{U}$) is R_1 if and only if $\nabla = \cap \{ U \circ U^{-1} : U \in \mathcal{U} \}$.

THEOREM 3.14 A quasi-uniform (X , $\mathcal{U}$) is T_2 if and only if it is T_1 and R_1 .

PROOF. It is well-known fact that every T_2-space is also a T_1-space . Let (X , $\mathcal{U}$) be a T_2-space , then $\bar{x} \neq \bar{y}$ implies that $x \neq y$ since $\bar{x} = x$, and $\bar{y} = y$. Since (X , $\mathcal{U}$) is T_2 , then there exist disjoint neighborhoods N_x and N_y of x , y respectively . Hence x , y have disjoint neighborhoods N_x and N_y . Therefore (X , $\mathcal{U}$) is a R_1-space .

Let (X , $\mathcal{U}$) be a T_1 and R_1-space . Clearly , $\Delta \subset \nabla$. Let $(x,y) \in \nabla$, then $\bar{x} = \bar{y}$ and , since X is T_1 , x = y . Therefore , $\Delta = \nabla$ and $\nabla = \bar{\Delta}$, since X is R_1 by theorem 3.12 . Thus $\Delta = \bigcap \{ U \circ U^{-1} : U \in \mathcal{U} \}$ and by theorem 3.9 (X , $\mathcal{U}$) is a T_2-space .

CHAPTER IV

COMPLETENESS AND COMPACTNESS

DEFINITION 4.1 Let X be a non-empty set , then a non-empty family $\mathcal{F}$ of subsets of X is a filter on X if and only if

(1). $\phi \notin \mathcal{F}$,

(2). F_1 , $F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$,

(3). $F_1 \in \mathcal{F}$ and $F_1 \subset F \Rightarrow F \in \mathcal{F}$.

DEFINITION 4.2 A collection $\mathcal{B}$ of subsets of the set X is said to be a base for a filter $\mathcal{F}$ on X if and only if $\mathcal{F} = \{ E : B \subset E$ for some $B \in \mathcal{B} \}$.

DEFINITION 4.3 A collection $\mathcal{S}$ of subsets of the set X is said to be a subbase for a filter $\mathcal{F}$ on X if and only if the collection of all finite intersection members of $\mathcal{S}$ is a base for the filter $\mathcal{F}$.

EXAMPLE 4.1 Let (X , t) be a topological space and let x be a fixed point of X , then the set $S_x = \{ N_x : x \in O_x \subset N_x$, for some $O_x \in t \}$ is a filter on X . This is called the neighborhood filter of the point x .

EXAMPLE 4.2 Let X be a non-empty set , and let x be a fixed point of

X , then the collection $S_x = \{ N : x \in N \subset X \}$ is a filter on X .

EXAMPLE 4.3 Let X be a non-empty set and let A be a non-empty subset of X . Then the collection $S_A = \{ N : A \subset N \subset X \}$ is a filter on X .

EXAMPLE 4.4 Let (X , t) be a topological space . The collection $\mathcal{F}$ of all neighborhoods of an arbitary non-empty subset A of X is a filter , called the neighborhood filter of A .

EXAMPLE 4.5 Let X be an infinite set , then the set $\mathcal{F} = \{ F : X - F$ is finite in $X \}$ is a filter on X .

EXAMPLE 4.6 If $X \neq \phi$, then $\mathcal{F} = \{ X \}$ is a filter on X .

THEOREM 4.1 Let $\mathcal{U}$ be a collection of subsets of X , then there exists a filter $\mathcal{F}$ on X which contains $\mathcal{U}$ if and only if $\mathcal{U}$ has the finite intersection property .

PROOF. The proof of this theorem is immediately from definitions 4.1 and 4.3 .

THEOREM 4.2 Let f be a function from X onto Y , and $\mathcal{F}$ be a filter on Y , then $f^{-1}(\mathcal{F}) = \{ f^{-1}(F) : F \in \mathcal{F} \}$ is a filter on X .

PROOF. $\phi \notin f^{-1}(\mathcal{F})$, since $\phi \notin \mathcal{F}$ and f is an onto function . Let

$f^{-1}(F_1)$, $f^{-1}(F_2) \in f^{-1}(\mathcal{F})$. Then $f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2)$ which is in $f^{-1}(\mathcal{F})$, since $F_1 \cap F_2 \in \mathcal{F}$. If $f^{-1}(F) \in f^{-1}(\mathcal{F})$ and $f^{-1}(F) \subset A$ then $F \subset f(A) \in \mathcal{F}$. Therefore $A \in f^{-1}(\mathcal{F})$ and $f^{-1}(\mathcal{F})$ is a filter on X .

THEOREM 4.3 Let f be a function from X onto Y , and let $\mathcal{F}$ be a filter on X , then $f(\mathcal{F}) = \{ f(F) : F \in \mathcal{F} \}$ is a filter on Y .

PROOF. $\phi \notin f(\mathcal{F})$, otherwise $\phi = f(F)$ for some $F \in \mathcal{F}$ which is impossible . Let $f(F_1)$, $f(F_2) \in f(\mathcal{F})$. Then $f(F_1 \cap F_2) \subset f(F_1) \cap f(F_2)$. This implies that $F_1 \cap F_2 \subset f^{-1}(f(F_1 \cap F_2)) \subset f^{-1}(f(F_1) \cap f(F_2)) \in \mathcal{F}$. Since f is onto , $f(F_1) \cap f(F_2) = f(f^{-1}(f(F_1) \cap f(F_2)))$. Hence , $f(F_1) \cap f(F_2) \in f(\mathcal{F})$. If $f(F) \in f(\mathcal{F})$ and $f(F) \subset A$, then $F \subset f^{-1}(A) \in \mathcal{F}$, and $f^{-1}(A) \in \mathcal{F}$. Since f is onto , $A = f(f^{-1}(A))$ and hence $A \in f(\mathcal{F})$.

COROLLARY. Let f be a function from X onto Y , and let $\mathcal{B}$ be a filter base on X , then $f(\mathcal{B}) = \{ f(B) : B \in \mathcal{B} \}$ is a filter base on Y .

DEFINITION 4.4 An ultrafilter $\mathcal{F}$ on a set X is a filter on X which is maximal in the collection of all filters partially ordered by inclusion ; that is to say , a filter which is not properly contained in any other filter .

EXAMPLE 4.7 Let X be a non-empty set and let a be fixed point of X , then the collection $\mathcal{F} = \{ F : a \in F \subset X \}$ is an ultrafilter . This follows since if $\mathcal{S}$ is a filter with $\mathfrak{F} \subset \mathcal{S}$ and $\mathfrak{F} \neq \mathcal{S}$, then there exists a $S \in \mathcal{S}$, $S \notin \mathfrak{F}$. This implies that $a \notin S$. However $\{a\} \in \mathfrak{F} \subset \mathcal{S}$. Thus $\{a\} \cap S = \phi \in \mathcal{S}$ which is a contradiction .

ZORN'S LEMMA. Let X be a non-empty partially ordered set such that every linearly ordered subset has an upper bound , then X contains a maximal element .

THEOREM 4.4 If $\mathfrak{F}$ is any filter on a set X , then there exists an ultrafilter finer than $\mathfrak{F}$.

PROOF. Let $\mathfrak{F}$ be a filter on a set X , and let $\mathcal{A}$ be the collection of all filters containing $\mathfrak{F}$. Then $\mathcal{A}$ is non-empty set , since $\mathfrak{F} \in \mathcal{A}$, and is partially ordered by inclusion . Let L be a linearly ordered subset of $\mathcal{A}$, then for any pair $\mathfrak{F}_1$, $\mathfrak{F}_2 \in L$, it follows that either $\mathfrak{F}_1 \subset \mathfrak{F}_2$ or $\mathfrak{F}_2 \subset \mathfrak{F}_1$. Let H be a set defined by $H = \{ E : E \in \mathfrak{F} \in L \}$, then H is a filter containing every filter in L . This is true because it satisfies the following three properties .

(1). $\phi \notin H$, since no filter $\mathfrak{F}$ in L contains ϕ .

(2). Let E_1 , $E_2 \in H$, then there exist filters $\mathfrak{F}_1$, $\mathfrak{F}_2$ in L such that $E_1 \in \mathfrak{F}_1$, $E_2 \in \mathfrak{F}_2$. Since L is a linearly ordered set , then either $\mathfrak{F}_1 \subset \mathfrak{F}_2$ or $\mathfrak{F}_2 \subset \mathfrak{F}_1$ and hence either $E_1 \in \mathfrak{F}_2$, $E_1 \cap E_2 \in \mathfrak{F}_2$ or $E_2 \in \mathfrak{F}_1$, $E_1 \cap E_2 \in \mathfrak{F}_1$. In both cases $E_1 \cap E_2 \in H$ for every pair E_1 , $E_2 \in H$.

(3). Let $E_1 \subset E$, and $E_1 \in H$, then there exists a filter $\mathcal{F}_1 \in L$ with $E_1 \in \mathcal{F}_1$. Since $E_1 \subset E$, hence $E \in \mathcal{F}_1 \in L$ and hence $E \in H$.

Thus H is a filter which is finer than any other filters in L . Therefore $\mathcal{G}$ is a non-empty partially ordered set such that every linearly ordered subset L has an upper bound H , then $\mathcal{G}$ contains a maximal element by Zorn's Lemma . This maximal element is by definition an ultrafilter finer than $\mathcal{F}$.

THEOREM 4.5 Let X be a non-empty set and $\mathcal{F}$ be a filter on X . The following three statements are equivalent .

(1). $\mathcal{F}$ is an ultrafilter on X .

(2). If $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(3). If $E \subset X$, then either $E \in \mathcal{F}$ or $X - E \in \mathcal{F}$.

PROOF. (1) $\Longrightarrow$ (2). Let $\mathcal{F}$ be an ultrafilter on X . Suppose condition (2) is not true , then there exist subsets A , B in X , such that $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$ and $A \cup B \in \mathcal{F}$. Let $\mathcal{G}$ be defined as follows ,

$$\mathcal{G} = \{ E \subset X : A \cup E \in \mathcal{F} \} .$$

Then $\mathcal{G}$ is a filter on X . This is true because it satisfies the following three properties .

(a). $\phi \notin \mathcal{G}$ since $B \in \mathcal{G}$.

(b). If E_1 and E_2 belong to $\mathcal{G}$, then $A \cup E_1 \in \mathcal{F}$ and $A \cup E_2 \in \mathcal{F}$. Now $(A \cup E_1) \cap (A \cup E_2) \in \mathcal{F}$, therefore $A \cup (E_1 \cap E_2) \in \mathcal{F}$ and

$E_1 \cap E_2 \in \mathcal{G}$.

(c). If $E_1 \in \mathcal{G}$ and $E_1 \subset E$, then $A \cup E_1 \subset A \cup E$. Hence $A \cup E \in \mathcal{F}$ and $E \in \mathcal{G}$.

Since $B \in \mathcal{G}$ and $B \notin \mathcal{F}$, it follows that $\mathcal{G}$ is a filter strictly finer than $\mathcal{F}$. This contradicts the fact that $\mathcal{F}$ is an ultrafilter . Hence for every pairs of subsets A , B in X , with $A \cup B$ in an ultrafilter $\mathcal{F}$ it follows that either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(2) $\Rightarrow$ (3). If $E \subset X$, then $E \cup (X - E) = X \in \mathcal{F}$. By condition (2) , it follows that either $E \in \mathcal{F}$ or $X - E \in \mathcal{F}$.

(3) $\Rightarrow$ (1). By theorem 4.4 , $\mathcal{F}$ is contained in an ultrafilter $\mathcal{F}'$. For each subset E in $\mathcal{F}'$, $X - E \notin \mathcal{F}'$. Hence $X - E \notin \mathcal{F}$ and by condition (3) it follows that $E \in \mathcal{F}$. This implies that $\mathcal{F}' \subset \mathcal{F}$. Since $\mathcal{F}'$ is an ultrafilter , then $\mathcal{F}' = \mathcal{F}$. Therefore $\mathcal{F}$ is an ultrafilter .

DEFINITION 4.5 Let (X , t) be a topological space , and let $\mathcal{F}$ be a filter on X . A point $x \in X$ is called a limit point of $\mathcal{F}$, denoted by $x \in \lim \mathcal{F}$, if and only if every neighborhood N_x of x belongs to $\mathcal{F}$.

DEFINITION 4.6 Let (X , t) be a topological space , and let $\mathcal{F}$ be a filter on X . A point $x \in X$ is an adherence point of the filter $\mathcal{F}$, denoted by $x \in \text{adh}\, \mathcal{F}$, if and only if for every $F \in \mathcal{F}$ and for every neighborhood N_x of x , $N_x \cap F \neq \phi$.

THEOREM 4.6 Let $\mathcal{F}$ be a filter on (X , t) , then x is an adherence point if and only if $x \in \cap \{ \bar{F} : F \in \mathcal{F} \}$.

PROOF. Let x be an adherence point of $\mathcal{F}$, then for every $F \in \mathcal{F}$ and for every neighborhood N_x of x , it follows that $N_x \cap F \neq \phi$. That is to say , for every $F \in \mathcal{F}$, $x \in \bar{F}$. Hence $x \in \cap \{ \bar{F} : F \in \mathcal{F} \}$.

On the other hand , if $x \in \cap \{ \bar{F} : F \in \mathcal{F} \}$, then $x \in \bar{F}$ for every $F \in \mathcal{F}$. That is for every $F \in \mathcal{F}$, and for every neighborhood N_x of x , $N_x \cap F \neq \phi$. Hence x is an adherence point of $\mathcal{F}$.

DEFINITION 4.7 Let $\{ x_n \}_1^{\infty}$ be a given sequence , and set $F_K = \{ x_i : i \geq k \}$. Then the collection $\{ F_K : k = 1 , 2 \cdots \}$ is a filter base . The generated filter $\mathcal{F}$ will be called the natural filter generated by the given sequence .

Let $\{ x_n \}_1^{\infty}$ be a sequence of real numbers and let $\mathcal{F}$ be the natural filter generated by the sequence . The point x is a limit point of $\mathcal{F}$ if and only if every neighborhood N_x of x belongs to $\mathcal{F}$ and this is true provided every open set of the form $(x - \epsilon , x + \epsilon)$ containing x belongs to $\mathcal{F}$. Or equivalently , for every $\epsilon > 0$ there exists F in $\mathcal{F}$ containing x such that $F \subset (x - \epsilon , x + \epsilon)$, that is for every $\epsilon > 0$ there exists an integer $k > 0$ such that $F_K \subset F \subset (x - \epsilon , x + \epsilon)$ where $F_K = \{ x_K , x_{K+1} , \cdots \}$. Hence for every $\epsilon > 0$ there exists an integer $k > 0$ such that for $n > k$, $x_n \in (x - \epsilon , x + \epsilon)$. Thus $x \in \lim \mathcal{F}$ if and only if $\lim_{n \to \infty} x_n = x$.

The point x is an adherence point of a filter $\mathcal{F}$ on X , if and only if for every neighborhood N_x of x and for every $F \in \mathcal{F}$, $N_x \cap F \neq \phi$, or equivalently if every open set of the form $(x - \epsilon , x + \epsilon)$ intersets every member F in $\mathcal{F}$. That is , for every $\epsilon > 0$ and for any integer $k > 0$, then $(x - \epsilon , x + \epsilon) \cap F_K \neq \phi$. And this is equivalent to the statement that for every $\epsilon > 0$, and for every integer $k > 0$ there exists a $n > k$ such that $x_n \in (x - \epsilon , x + \epsilon)$. Hence $x \in \text{adh } \mathcal{F}$, if and only if x is a cluster point of the sequence .

In examples 4.8 through 4.15 , sequences are considered in the set of real numbers with the usual topology .

EXAMPLE 4.8 The sequence $\{ n \}_1^\infty$ generates a filter $\mathcal{F}$ with $\lim \mathcal{F} = \phi$ and $\text{adh } \mathcal{F} = \phi$.

EXAMPLE 4.9 The sequence $\{ 1/n \}_1^\infty$ generates a filter $\mathcal{F}$ with $\lim \mathcal{F} = \{ 0 \}$ and $\text{adh } \mathcal{F} = \{ 0 \}$.

EXAMPLE 4.10 The sequence $\{ (-1)^n \}_1^\infty$ generates a filter $\mathcal{F}$, with $\lim \mathcal{F} = \phi$ and $\text{adh } \mathcal{F} = \{ -1, 1 \}$.

EXAMPLE 4.11 Let $\{ 2+n/n+5 \}_1^\infty$ be a given sequence which generates a filter $\mathcal{F}$, then $\lim \mathcal{F} = \{ 1 \}$ and $\text{adh } \mathcal{F} = \{ 1 \}$.

EXAMPLE 4.12 Let $\{ (-1)^n (n+1/n) \}_1^\infty$ be a sequence which generates a filter $\mathcal{F}$, then lim $\mathcal{F} = \Phi$ and adh $\mathcal{F} = \{-1,1\}$.

EXAMPLE 4.13 Let $\{(-1)^n(1/n)\}_1^\infty$ be a sequence which generates a filter $\mathcal{F}$, then lim $\mathcal{F} = \{0\}$, and adh $\mathcal{F} = \{0\}$.

EXAMPLE 4.14 Let $\{x_n\}_1^\infty$ be defined by

$$x_n = \begin{cases} n+1 & \text{when } n \text{ is even ,} \\ 1/n & \text{when } n \text{ is odd .} \end{cases}$$

Let $\mathcal{F}$ denote the natural filter generated by this sequence . Then lim $\mathcal{F} = \Phi$ and adh $\mathcal{F} = \{0\}$.

EXAMPLE 4.15 The sequence $\{1 , 1/2 , 1 , 1/3 \cdots\}$ generates a filter $\mathcal{F}$ with lim $\mathcal{F} = \Phi$ and adh $\mathcal{F} = \{1,0\}$.

THEOREM 4.7 Let (X , t) be a topological space , and let $\mathcal{F}$ be a filter on X , then every limit point of $\mathcal{F}$ is an adherence point of $\mathcal{F}$.

PROOF. Let $x \in \lim \mathcal{F}$, then every neighborhood N_x of x belongs to $\mathcal{F}$. Therefore $N_x \cap F \neq \Phi$ for every neighborhood N_x of x and for every F in $\mathcal{F}$. Hence $x \in \text{adh } \mathcal{F}$.

THEOREM 4.8 Let (X , t) be a topological space , and let $\mathcal{F}$ be a filter on X , then $x \in \text{adh } \mathcal{F}$ if and only if $x \in \lim \mathcal{F}'$ for some $\mathcal{F}'$ finer than $\mathcal{F}$.

PROOF. Let $x \in \text{adh } \mathcal{F}$, then , by definition , for every neighborhood N_x of x and for every $F \in \mathcal{F}$, $N_x \cap F \neq \phi$. Let $\mathcal{B} = \{ N_x \cap F : N_x$ is a neighborhood of x , $F \in \mathcal{F} \}$. Then $\mathcal{B}$ is a base for a filter $\mathcal{F}'$ and $x \in \lim \mathcal{F}'$.

On the other hand , let $\mathcal{F}'$ be a filter finer than $\mathcal{F}$ and let $x \in \lim \mathcal{F}'$. Then , by theorem 4.7 , $x \in \text{adh } \mathcal{F}'$. Hence $x \in \text{adh } \mathcal{F}$.

THEOREM 4.9 Let (X , t) be a topological space . If $A \subset X$ is closed then $\text{adh } \mathcal{F} \subset A$ for every filter containing A .

PROOF. Let $\mathcal{F}$ be a filter containing the set A . Then $\text{adh } \mathcal{F} \equiv \cap \{ \bar{F} : F \in \mathcal{F} \} \subset \bar{A} = A$.

THEOREM 4.10 Let (X , t) be a topological space . Then (X , t) is T_2 if and only if every filter has at most one limit point .

PROOF. Let $\mathcal{F}$ be any filter on a T_2-space (X , t) , and let x , y are distinct limit points of $\mathcal{F}$. Then there are two disjoint open sets O_1 and O_2 containing x and y respectively . Since x , y are limit points of $\mathcal{F}$, then O_1 , O_2 are members of $\mathcal{F}$. This implies $\phi = O_1 \cap O_2 \in \mathcal{F}$ which is impossible . Hence the filter $\mathcal{F}$ has at most one limit point .

Suppose the condition holds and (X , t) is not T_2 . Then there exist two points x , y in X such that any open sets O_x , O_y in t containing x and y respectively , have a non-empty intersection .

Therefore , the collection $\mathcal{B} = \{ O_x \cap O_y : x \in O_x \in t \, , y \in O_y \in t \}$ has the finite intersection property and hence generate a filter $\mathcal{F}$ on X . However , the filter $\mathcal{F}$ has two distinct limit points x and y which contradicts the hypothesis . Therefore (X , t) must be a T_2-space .

THEOREM 4.11 Let (X , t) be a topological space , then the following statements are equivalent .

(1). (X , t) is compact .

(2). Every non-empty collection of closed sets with the finite intersection property has a non-empty intersection .

(3). Every filter has a non-empty adherence .

(4). Every ultrafilter has a non-empty limit .

PROOF. (1) $\Leftrightarrow$ (2). (X , t) is compact if and only if every open cover has a finite subcover . That is , every collection of closed sets with an empty intersection has a finite subcollection with an empty intersection . Or equivalently , every non-empty collection of closed sets with the finite intersection property has a non-empty intersection .

(2) $\Rightarrow$ (3). Suppose condition (2) is true , then $\phi \neq \cap \{ \overline{F} : F \in \mathcal{F} \}$ $=$ adh $\mathcal{F}$, where $\mathcal{F}$ is arbitary filter on X .

(3) $\Rightarrow$ (4). Suppose condition (3) is true . Then every ultrafilter $\mathcal{F}$ on X has a non-empty adherence . Therefore by theorem 4.8 every ultrafilter has a non-empty limit .

(4) $\Rightarrow$ (2). Let $\mathcal{C}$ be a non-empty collection of closed subsets of

X with the finite intersection property . Then $\mathcal{C}$ generates a filter $\mathcal{F}$ on X which is contained in an ultrafilter $\mathcal{F}'$. By condition (4) , it follows that adh $\mathcal{F} \neq \phi$. Now adh $\mathcal{F} \equiv \cap\{\bar{F}:F \in \mathcal{F}\} \subset \cap\{A:A \in \mathcal{C}\}$. Hence $\cap\{A:A \in \mathcal{C}\} \neq \phi$.

DEFINITION 4.8 Let (X , $\mathcal{U}$) be a quasi-uniform space , (X , $\mathcal{U}$) is totally bounded if and only if for each U in $\mathcal{U}$ there exist finite number of subsets A_1 , A_2 , $\cdots$, A_n such that

(1). $\cup\{A_i:1 \leq i \leq n\} = X$,

(2). $A_i \times A_i \subset U$, for each $1 \leq i \leq n$.

DEFINITION 4.9 Let (X , $\mathcal{U}$) be a quasi-uniform space , (X , $\mathcal{U}$) is pre-compact if and only if for each U in $\mathcal{U}$ there exists a finite set $A = \{x_1, x_2, x_3, \cdots, x_n\} \subset X$ such that $U[A] = X$.

THEOREM 4.12 If a quasi-uniform space (X , $\mathcal{U}$) is totally bounded then it is pre-compact .

PROOF. Let (X , $\mathcal{U}$) be a totally bounded quasi-uniform space . Let $U \in \mathcal{U}$. Then there exist a finite number of subsets A_1 , A_2 , $\cdots$, A_n such that $\cup\{A_i:1 \leq i \leq n\} = X$ and $A_i \times A_i \subset U$ for each $1 \leq i \leq n$. Let $x_i \in A$, $1 \leq i \leq n$, then $U[x_i] = A_i$ for each i and $\cup\{U[x_i]:1 \leq i \leq n\} \supset \cup\{A_i:1 \leq i \leq n\} \supset X$. Therefore (X , $\mathcal{U}$) is pre-compact .

THEOREM 4.13 In a uniform space , totally boundedness and pre-compactness are equivalent .

PROOF. Let (X , $\mathcal{U}$) be a pre-compact uniform space . For each U in $\mathcal{U}$ there exists a symmetric V in $\mathcal{U}$ such that $V \circ V \subset U$. Since (X , $\mathcal{U}$) is pre-compact , there exists a finite set $A = \{ x_1 , x_2 , \cdots , x_n \}$ such that $\cup \{ V[x_i] : 1 \leq i \leq n \} = X$. Let $A_i = V[x_i]$, $1 \leq i \leq n$. For each ordered pair (y,z) in $A_i \times A_i$, $1 \leq i \leq n$, $(x_i , y) \in V$ and $(x_i , z) \in V$. Then $(y,z) \in V^{-1} \circ V = V \circ V \subset U$. Hence $A_i \times A_i \subset U$ for each U in $\mathcal{U}$ and $1 \leq i \leq n$. Since $\cup \{ V[x_i] : 1 \leq i \leq n \} = X$, it follows that $\cup \{ A_i : 1 \leq i \leq n \} = X$. Hence every pre-compact uniform space is a totally bounded uniform space . The proof is now completed by theorem 4.12 .

THEOREM 4.14 Every topological space has a compatible totally bounded quasi-uniform structure .

PROOF. Let (X , t) be a topological space and let $\mathcal{P}$ be the compatible Pervin quasi-uniform structure . The Pervin quasi-uniform structure has a totally bounded subbase and hence it is totally bounded .

THEOREM 4.15 The inverse image of a totally bounded quasi-uniform structure is totally bounded .

The proof of this theorem is natural and omitted .

THEOREM 4.16 A quasi-uniform space (X , $\mathcal{U}$) is totally bounded if and only if (X , $\mathcal{U}^{-1}$) is totally bounded .

PROOF. (X , $\mathcal{U}$) is totally bounded if and only if for each U in $\mathcal{U}$ there exist A_1 , A_2 , $\cdots$, A_n such that $A_i \times A_i \subset U$ for each $1 \leq i \leq n$ and $\cup \{ A_i : 1 \leq i \leq n \} = X$. Or equivalently , for each $U^{-1} \in \mathcal{U}^{-1}$ there exists A_1 , A_2 , $\cdots$, A_n such that $A_i \times A_i = (A_i \times A_i)^{-1} \subset U^{-1}$ for $1 \leq i \leq n$ and $\cup \{ A_i : 1 \leq i \leq n \} = X$. Hence (X , $\mathcal{U}$) is totally bounded if and only if (X , $\mathcal{U}^{-1}$) is totally bounded .

DEFINITION 4.10 Let (X , $\mathcal{U}$) be a quasi-uniform space . A filter $\mathcal{F}$ on (X , $\mathcal{U}$) is said to be a Cauchy filter if and only if for each U in $\mathcal{U}$ there exists a point x in X such that $U[x] \in \mathcal{F}$.

DEFINITION 4.11 A filter base $\mathcal{B}$ is said to be a Cauchy filter base if and only if the generated filter $\mathcal{F}$ is Cauchy .

THEOREM 4.17 Let (X , $\mathcal{U}$) be a quasi-uniform space . Then (X , $\mathcal{U}$) is pre-compact if and only if every ultrafilter $\mathcal{F}$ on X is a Cauchy filter .

PROOF. Suppose (X , $\mathcal{U}$) is pre-compact , then there exists a finite set $A = \{ x_1 , x_2 , \cdots , x_n \}$ such that $U[A] = \cup \{ U[x_i] : 1 \leq i \leq n \} = X$. Let $\mathcal{F}$ be an ultrafilter on X , then there exists a $x_K \in A$ with $U[x_K] \in \mathcal{F}$. Hence $\mathcal{F}$ is a Cauchy filter .

Let every ultrafilter $\mathcal{F}$ on X be Cauchy . Suppose that X is not pre-compact , then there exists a U in $\mathcal{U}$, such that for any finite subset A of X , X - U[A] $\neq \phi$. Hence the collection $\mathcal{S}$ = { X - U[A]: A is a finite subset of X } has the finite intersection property . Now $\mathcal{S}$ is contained in an ultrafilter $\mathcal{F}$. Since every ultrafilter $\mathcal{F}$ is Cauchy , for each U $\in \mathcal{U}$ there exists a point z $\in$ X such that U[z] $\in \mathcal{F}$. But X - U[z] $\in \mathcal{F}$. This implies that ϕ = U[z] $\cap$ (X - U[z]) $\in \mathcal{F}$ which is impossible . Hence (X , $\mathcal{U}$) must be pre-compact .

THEOREM 4.18 Let $\mathcal{B}$ be a filter base for (X , $\mathcal{U}$) . $\mathcal{B}$ is a Cauchy filter base if and only if for each U in $\mathcal{U}$ there exists a point x in X such that B $\subset$ U[x] for some B $\in \mathcal{B}$.

PROOF. Let $\mathcal{B}$ be a Cauchy filter base , then $\mathcal{B}$ generates a Cauchy filter $\mathcal{F}$. That is to say , for each U in $\mathcal{U}$ there exists a point x in X such that U[x] $\in \mathcal{F}$. Since $\mathcal{B}$ is a filter base for $\mathcal{F}$, it follows that B $\subset$ U[x] for some B $\in \mathcal{B}$.

If for each U in $\mathcal{U}$ there exists a point x in X such that B $\subset$ U[x] for some B $\in \mathcal{B}$, then for the filter $\mathcal{F}$ generated by $\mathcal{B}$, U[x] $\in \mathcal{F}$ since B $\in \mathcal{F}$. Hence the filter base $\mathcal{B}$ is a Cauchy filter base .

THEOREM 4.19 Let $\mathcal{F}$ be a filter on a quasi-uniform space (X , $\mathcal{U}$) . If for each U $\in \mathcal{U}$, there exists F $\in \mathcal{F}$ such that F $\times$ F $\subset$ U , then $\mathcal{F}$ is a Cauchy filter .

PROOF. Suppose the given condition is true. Then for each U in $\mathcal{U}$, there exists $F \in \mathcal{F}$ with $F \times F \subset U$. Let $x \in F$, then $F = (F \times F)[x] \subset U[x]$ and hence $U[x] \in \mathcal{F}$. Therefore $\mathcal{F}$ is a Cauchy filter.

THEOREM 4.20 Every convergent filter is Cauchy.

PROOF. Let $\mathcal{F}$ be a filter on a quasi-uniform space $(X , \mathcal{U})$, and let $x \in \lim \mathcal{F}$. Then for each U in $\mathcal{U}$, $U[x] \in \mathcal{F}$. Hence $\mathcal{F}$ is Cauchy.

EXAMPLE 4.16 Let R denote the set of real numbers with the usual order. Let $W = \{ (x,y) \in R \times R : x \leq y \}$ then $\{ W \}$ forms a quasi-uniform base. Let $\{ 1/n \}_1^{\infty}$ be a sequence and $\mathcal{F}$ the natural filter generated by this sequence. Then $\mathcal{F}$ is convergent and hence $\mathcal{F}$ is a Cauchy filter. However, there does not exist a F_K with $k > 0$ such that $F_K \times F_K \subset W$. Therefore the converse of theorem 4.19 is not always true.

THEOREM 4.21 Let F be a filter on a uniform space $(X , \mathcal{U})$. $\mathcal{F}$ is Cauchy if and only if for each U in $\mathcal{U}$, there exists an element F of $\mathcal{F}$ such that $F \times F \subset U$.

PROOF. Let $\mathcal{F}$ be a Cauchy filter on a uniform space. For each U in $\mathcal{U}$ there exists a symmetric V in $\mathcal{U}$ such that $V \circ V \subset U$. Since $\mathcal{F}$ is Cauchy, there exists a point x in X such that $V[x] \in \mathcal{F}$. Set $F = V[x]$. Let $(y,z) \in F \times F = V[x] \times V[x]$. Then $(x,y) \in V$ and $(x,z) \in V$. And thus

$(y,z) \in V^{-1} \circ V = V \circ V \subset U$. Hence $F \times F \subset V \circ V \subset U$ for each U in $\mathcal{U}$. The result now follows by theorem 4.19 .

THEOREM 4.22 A filter finer than a Cauchy filter is a Cauchy filter .

PROOF. Let $\mathcal{F}$ be a Cauchy filter on (X , $\mathcal{U}$) and $\mathcal{F} \leq \mathcal{F}'$. For each U in $\mathcal{U}$ there exists a point $x \in X$ such that $U[x]$ since $\mathcal{F}$ is Cauchy . Since $\mathcal{F}'$ is finer than $\mathcal{F}$, it follows that $U[x] \in \mathcal{F}'$. Hence $\mathcal{F}'$ is a Cauchy filter .

THEOREM 4.23 If $\mathcal{F}$ is a Cauchy filter on (X , $\mathcal{U}$) and $\mathcal{U}'$ is coarser than $\mathcal{U}$ then $\mathcal{F}$ is a Cauchy filter on (X , $\mathcal{U}'$) .

PROOF. Let $\mathcal{U}'$ be a quasi-uniform structure coarser than the quasi-uniform structure $\mathcal{U}$ on X , and let $\mathcal{F}$ be a Cauchy filter on (X , $\mathcal{U}$) . Let U in $\mathcal{U}'$ then U in $\mathcal{U}$ and there exists a point x in X , such that $U[x] \in \mathcal{F}$. Hence $\mathcal{F}$ is Cauchy on (X , $\mathcal{U}'$) .

THEOREM 4.24 Let f be a function of X onto Y , and let $\mathcal{F}$ be a Cauchy filter on (Y , $\mathcal{U}$) . Then the filter $f^{-1}(\mathcal{F}) = \{ f^{-1}(F) : F \in \mathcal{F} \}$ is a Cauchy filter on (X , $f_2^{-1}(\mathcal{U})$) .

PROOF. $f^{-1}(\mathcal{F})$ is a filter on (X , $f_2^{-1}(\mathcal{U})$) by theorem 4.2 . Let V belongs to $f_2^{-1}(\mathcal{U})$ then $V = f_2^{-1}(U)$ for some U in $\mathcal{U}$. Since $\mathcal{F}$ is Cauchy on (Y , $\mathcal{U}$) , there exists a point y in Y such that $U[y] \in \mathcal{F}$. Now

$f^{-1}(\ U[f(x)]\ ) \in f^{-1}(\mathcal{F})$ where $y = f(x)$ and $V[x] = (f_2^{-1}(U))[x] =$ $f^{-1}(\ U[f(x)]\ ) \in f^{-1}(\mathcal{F})$. Therefore $f^{-1}(\mathcal{F})$ is a Cauchy filter on $(\ X\ ,\ f_2^{-1}(\mathcal{U})\ )$.

The next example shows that the image of a Cauchy filter need not be a Cauchy filter .

EXAMPLE 4.17 Let $D_r = \{\ (x,y) \in R^+ \times R^+ : |x - y| < r\}$ and let $\mathcal{U}$ be a quasi-uniform structure on R^+ , the set of positive real numbers , generated by the quasi-uniform base $\{\ D_r : r > 0\ \}$. Let f be a function of R^+ into R^+ defined by $f(x) = 1/x$ for every $x \in R^+$. Let $\{\ 1/n\}_1^\infty$ be a given sequence in the domain of f , then the filter $\mathcal{F}$ generated by the sequence $\{\ 1/n\ \}_1^\infty$ is Cauchy . But the filter $f(\mathcal{F})$ generated by the sequence $\{\ n\ \}_1^\infty$ is not a Cauchy filter .

THEOREM 4.25 Let f be a quasi-uniformly continuous function of $(\ X\ ,\ \mathcal{U}\ )$ onto $(\ Y\ ,\ \mathcal{V}\ )$ and let $\mathcal{F}$ be a Cauchy filter on X , then $f(\mathcal{F})$ is a Cauchy filter .

PROOF. $f(\mathcal{F})$ is a filter on $(\ Y\ ,\ \mathcal{V}\ )$ by theorem 4.3 . For each V in $\mathcal{V}$ there exists an entourage U in $\mathcal{U}$ such that $f_2(U) \subset V$, since f is quasi-uniformly continuous function . Since $\mathcal{F}$ is Cauchy on X , there exists a point $x \in X$ such that $U[x] \in \mathcal{F}$. Hence $f(U[x]) \in f(\mathcal{F})$. Now $f(U[x]) \subset f_2(U)[f(x)] \in f(\mathcal{F})$, since $f(\mathcal{F})$ is a filter on Y . Hence for each V in $\mathcal{V}$ there exists $f(x) \in Y$ such that $V[f(x)] \in f(\mathcal{F})$, and $f(\mathcal{F})$ is a Cauchy filter .

DEFINITION 4.12 A quasi-uniform space (X , $\mathcal{U}$) is complete if and only if every Cauchy filter has non-empty adherence .

DEFINITION 4.13 A quasi-uniform space (X , $\mathcal{U}$) is strongly complete if and only if every Cauchy filter has non-empty limit .

THEOREM 4.26 A strongly complete quasi-uniform space (X , $\mathcal{U}$) is complete .

PROOF. This follows since every limit point of a filter $\mathcal{F}$ is an adherence point by theorem 4.7 .

THEOREM 4.27 In a uniform space , completeness and strong completeness are equivalent .

PROOF. A strongly complete uniform space is always complete by theorem 4.26 . Let $\mathcal{F}$ be a Cauchy filter in the uniform space (X , $\mathcal{U}$) and let $x \in \text{adh}\,\mathcal{F}$. For each U in $\mathcal{U}$ there exists a symmetric V in $\mathcal{U}$ such that $V \circ V \subset U$. Since $\mathcal{F}$ is a Cauchy filter on a uniform space (X , $\mathcal{U}$) there exists a $F \in \mathcal{F}$ such that $F \times F \subset V$. Since $x \in \text{adh}\,\mathcal{F}$, there exists a point $y \in V[x] \cap F \neq \phi$. That is $(x,y) \in V$ and $y \in F$. Let z be any point in F , then $(y,z) \in F \times F \subset V$. This implies that $(x,z) \in V \circ V \subset U$ and $z \in U[x]$.

That is $F \subset U[x]$ and hence $U[x] \in \mathcal{F}$. Therefore , in a uniform space , every Cauchy filter converges to its adherence point . Hence every complete uniform space is strongly complete .

THEOREM 4.28 Completeness and strong completeness are invariant under quasi-uniformly continuous function .

PROOF. Let f be a quasi-uniformly continuous function from a quasi-uniform space (X , $\mathcal{U}$) onto a quasi-uniform space (Y , $\mathcal{V}$) .

Let (X , $\mathcal{U}$) be a complete space . Suppose (Y , $\mathcal{V}$) is not complete . Then there is a Cauchy filter $\mathcal{F}$ on (Y , $\mathcal{V}$) such that adh $\mathcal{F} = \phi$. For every $x \in X$, then $y = f(x) \notin$ adh $\mathcal{F}$. That is to say , there exists a V_o in $\mathcal{V}$ and a F_o in $\mathcal{F}$ such that $V_o[y] \cap F_o = \phi$. Now that $f_2^{-1}(V_o)[x] \cap f^{-1}(F_o) = f^{-1}(V_o[y]) \cap f^{-1}(F_o) = f^{-1}(V_o[y] \cap F_o) = \phi$, where $f_2^{-1}(V_o)$ is in $\mathcal{U}$ and $f^{-1}(F_o)$ is in the Cauchy filter $f^{-1}(\mathcal{F})$. This implies that $x \notin$ adh $f^{-1}(\mathcal{F})$ for every $x \in X$, or equivalently adh $f^{-1}(\mathcal{F}) = \phi$ which is impossible . Hence (Y , $\mathcal{V}$) must be a complete quasi-uniform space .

Let (X , $\mathcal{U}$) be strongly complete . Suppose (Y , $\mathcal{V}$) is not strongly complete . Then there is a Cauchy filter $\mathcal{F}$ on (Y , $\mathcal{V}$) such that lim $\mathcal{F} = \phi$. For every $x \in X$, then $y = f(x) \notin$ lim $\mathcal{F}$, or equivalently , there is a V_o in $\mathcal{V}$ such that $V_o[y] \notin \mathcal{F}$ or $f^{-1}(V_o[y]) \notin f^{-1}(\mathcal{F})$. Now that $f_2^{-1}(V_o)[x] = f^{-1}(V_o[y])$. Hence $f_2^{-1}(V_o)[x] \notin f^{-1}(\mathcal{F})$ for every $x \in X$ and then the Cauchy filter $f^{-1}(\mathcal{F})$ has empty limit which is impossible . Therefore (Y , $\mathcal{V}$) is strongly complete .

THEOREM 4.29 Every closed subset A of a complete quasi-uniform space is complete .

PROOF. Let A be a closed subspace of a complete quasi-uniform space . Let $\mathcal{F}$ be a Cauchy filter on A , then $\mathcal{F}$ is a collection of subsets of X which has the finite intersection property . Let $\mathcal{F}'$ be the Cauchy filter on X generated by $\mathcal{F}$. Since (X , $\mathcal{U}$) is complete , it follows that there exists a $x \in \text{adh } \mathcal{F}' \equiv \cap \{ Cl_X F : F \in \mathcal{F}' \}$. Now , $x \in \bar{A}$. Since A is closed , then $x \in A$. However , $Cl_A F = A \cap Cl_X F$. Hence , $x \in \cap \{ Cl_A F : F \in \mathcal{F} \}$. Therefore A is complete .

THEOREM 4.30 Every closed subset A of a strongly complete quasi-uniform space (X , $\mathcal{U}$) is strongly complete .

PROOF. Let $\mathcal{F}$ be a Cauchy filter on a closed subspace A , then $\mathcal{F}$ is a collection of subsets of X with the finite intersection property and hence generates a Cauchy filter $\mathcal{F}'$ on X . Since (X , $\mathcal{U}$) is strongly complete , then there exists a point $x \in X$ such that $U[x] \in \mathcal{F}'$ for each $U \in \mathcal{U}$. Hence each neighborhood of x is contained in $\mathcal{F}'$ and therefore meets A . Thus $x \in \bar{A} = A$. Therefore A is strongly complete .

THEOREM 4.31 Let (X , $\mathcal{U}$) be a T_2 , uniform space and let A be complete subspace of (X , $\mathcal{U}$) , then A is closed .

PROOF. Let x belongs to $\bar{A}$. Then for each U in $\mathcal{U}$, it follows that $U[x] \cap A \neq \phi$. Let $\mathcal{B} = \{ U[x] \cap A : U \in \mathcal{U} \}$. Then $\mathcal{B}$ has the finite intersection property and hence generates a filter $\mathcal{F}$ on A . Since (X , $\mathcal{U}$) is a uniform space , then for each U in $\mathcal{U}$ there is a symmetric V in $\mathcal{U}$ such that $V \circ V \subset U$. This implies that $(V[x] \cap A) \times (V[x] \cap A) \subset U \cap (A \times A)$ and hence $\mathcal{F}$ is a Cauchy filter on A . Then there exists a point $x' \in \lim \mathcal{F}$ and $x' \in A$. However $\mathcal{F}$ is a Cauchy filter base in X and generates a Cauchy filter $\mathcal{F}'$ on X . Furthermore $x' \in \lim \mathcal{F}'$ and $x' = x$ by theorem 4.10 . Therefore $\bar{A} \subset A$ and A is closed .

THEOREM 4.32 Every compact quasi-uniform space (X , $\mathcal{U}$) is strongly complete .

PROOF. Let (X , $\mathcal{U}$) be a compact quasi-uniform space . Suppose (X , $\mathcal{U}$) is not strongly complete . That is to say , there exists a Cauchy filter $\mathcal{F}$ on (X , $\mathcal{U}$) such that for every point x in (X , $\mathcal{U}$) there exists a U_x in $\mathcal{U}$ with $U_x[x] \notin \mathcal{F}$. Let V_x be in $\mathcal{U}$ such that $V_x \circ V_x \subset U_x$. Since X is compact , then there exists a finite set $A = \{ x_1 , x_2 , \cdots , x_n \}$ such that $X = \cup \{ V_{x_i}[x_i] : V_{x_i} \in \mathcal{U}$ and $V_{x_i} \circ V_{x_i} \subset U_{x_i} , 1 \leq i \leq n \}$. Set $V = \cap \{ V_{x_i} : 1 \leq i \leq n \} \in \mathcal{U}$, then there exists a point $a \in X$ such that $V[a] = \cap \{ V_{x_i}[a] : 1 \leq i \leq n \} \in \mathcal{F}$, since $\mathcal{F}$ is Cauchy . For each point y in V[a] , then $(a,y) \in V = \cap \{ V_{x_i} : 1 \leq i \leq n \}$. However $a \in V_{x_i}[x_i]$ for some i , hence $(a,y) \in V_{x_i}$ for this i . That is $(x_i,y) \in V_{x_i} \circ V_{x_i} \subset U_{x_i}$. Hence $V[a] \subset U_{x_i}[x_i] \in \mathcal{F}$. This contradicts our assumption , hence (X , $\mathcal{U}$) must be strongly complete .

THEOREM 4.33 A quasi-uniform space (X , $\mathcal{U}$) is compact if and only if it is complete and pre-compact .

PROOF. Let (X , $\mathcal{U}$) be a compact quasi-uniform space . Then by theorem 4.11 every ultrafilter is Cauchy . Hence by theorem 4.17 (X , $\mathcal{U}$) is pre-compact .

Since (X , $\mathcal{U}$) is compact , then by theorem 4.11 every Cauchy filter has a non-empty adherence . Hence (X , $\mathcal{U}$) is complete .

Suppose (X , $\mathcal{U}$) is a complete and pre-compact quasi-uniform space . Then every ultrafilter $\mathcal{F}$ on (X , $\mathcal{U}$) is Cauchy by theorem 4.17 . Since (X , $\mathcal{U}$) is complete , then adh $\mathcal{F} \neq \Phi$. However $\mathcal{F}$ is an ultrafilter , then adh $\mathcal{F}$ = lim $\mathcal{F} \neq \Phi$. Therefore every ultrafilter $\mathcal{F}$ converges . Hence (X , $\mathcal{U}$) is compact by theorem 4.11 .

COROLLARY. A quasi-uniform space (X , $\mathcal{U}$) is compact if and only if it is strongly complete and pre-compact .

The proof follows immediately from theorem 4.32 and theorem 4.33 .

BIBLIOGRAPHY

1. Bourbaki, N. General Topology, chapter I and II. Hermann, Paris, 1961.

2. J.W. Carlson and T.L. Hicks. "Some Properties of Quasi-Uniform Structures," Archiv der Mathematik, (24), 82-86 (1973).

3. G.F. Simmons. Introduction to Topology and Modern Analysis. New York: McGraw-Hill, 1963.

4. M.G. Murdeshwar and S.A. Naimpally. Quasi-Uniform Topological Spaces. Groningen, 1966.