A STUDY OF PROXIMITY SPACES

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CHAPTER I

INTRODUCTION

As early as 1908, Riesz [5] sketched the concepts of proximity spaces in his "theory of enchainment". However, his idea received no further development at that time.

In the early 1950's, Efremovič [1,2], a Russian mathematician, rediscovered the subject and gave the definition of a proximity space, which he called infinitesimal spaces in a series of his papers. Efremovič later found another way to generate proximity spaces by using the concept of proximity neighborhoods.

Smirnov [7] brought the concepts of filters and clusters into proximity theory in order to obtain the Smirnov compactification of a proximity space.

There are many research papers on proximity spaces published by modern mathematicians in the last ten years. The development of proximity spaces is growing rapidly.

This thesis presents the basic material about proximity spaces. The relationship between topological spaces and proximity spaces is investigated. A construction of the Smirnov compactification is presented.

Since clusters are used to construct the Smirnov compactification, and since the relationship between filters and clusters is very close, the author discussed filters in chapter III and discussed clusters in chapter IV.

There is an excellent list of publications on proximity spaces in the book Proximity Spaces [4], where the reader can find advanced material about proximity spaces.

CHAPTER II

ELEMENTARY PROPERTIES

1. THE DEFINITION AND SOME EXAMPLES

DEFINITION 1.1 A binary relation δ defined on the power set of X is called a proximity on X iff it satisfies the following axioms:

(A_1) A δ B implies B δ A

(A_2) (A ∪ B) δ C iff A δ C or B δ C

(A_3) A δ B implies A $\neq \emptyset$ and B $\neq \emptyset$

(A_4) A ∩ B $\neq \emptyset$ implies A δ B

(A_5) A $\not\delta$ B implies there exists a subset E of X such that A $\not\delta$ E and (X - E) $\not\delta$ B

The pair (X, δ) is called a proximity space.

DEFINITION 1.2 A proximity δ on X is separated if it satisfies

(A_6) x δ y implies x = y, and (X, δ) is called a separated proximity space.

Note that x δ y means $\{x\}$ δ $\{y\}$.

EXAMPLE 1. Let X = $\{a,b,c\}$ and define A δ B iff A ∩ B $\neq \emptyset$ for any subsets A and B of X. δ is a separated proximity. The proximity defined in this way is called a discrete proximity.

EXAMPLE 2. Let X be any non-empty set and define $A \,\delta\, B$ iff $A \neq \emptyset$ and $B \neq \emptyset$. δ is a proximity on X. If X contains two or more points, then δ is not a separated proximity. δ defined in this way is called the trivial proximity.

EXAMPLE 3. Let (X, d) be a pseudo-metric space. Define $A \,\delta\, B$ iff $d(A,B) = 0$, where $d(A,B) = \inf \{ d(x,y) : x \in A \text{ and } y \in B \}$.

If $A \not\delta B$, then $d(A,B) = r > 0$. Choosing $E = \{ x : d(x,B) \leq \frac{r}{2} \}$, then $d(A,E) \geq \frac{r}{2}$ and $d(X-E,B) \geq \frac{r}{2}$. It follows that $A \not\delta E$ and $(X-E) \not\delta B$. Hence δ satisfies axiom A_5. The rest of the axioms are clearly satisfied.

If (X,d) is a metric space, then $x \,\delta\, y$ implies $d(x\ y) = 0$ and hence $x = y$. Therefore (X, δ) is a separated proximity space.

A proximity is called a (pseudo-) metric proximity if it is derived from a (pseudo-) metric.

EXAMPLE 4. Consider a normal space (X,t). Define $A \,\delta\, B$ iff $\bar{A} \cap \bar{B} \neq \emptyset$. δ is a separated proximity on X.

The verification of all axioms except A_5 is straightforward. To prove A_5, let $A \not\delta B$. Then $\bar{A} \cap \bar{B} = \emptyset$, so that, since (X,t) is T_4, there exist disjoint open sets C and D such that $\bar{A} \subset C$ and $\bar{B} \subset D$. Hence X-C is closed and $\bar{A} \cap (X-C) = \emptyset$. This implies $A \not\delta (X-C)$. Since $C \cap D = \emptyset$, $C \subset (X-D)$. It follows that $\bar{C} \subset (X-D)$ since (X-D) is closed. Therefore $\bar{C} \cap \bar{B} = \emptyset$ and hence $C \not\delta B$. Let $E = X-C$. Then $A \not\delta B$ implies that there exists a subset E such that $A \not\delta E$ and $(X-E) \not\delta B$.

2. TOPOLOGY INDUCED BY A PROXIMITY.

A proximity on X always induces a topology on X.

DEFINITION 2.1 Let (X, δ) be a proximity space. A subset F of X is called closed iff $x \delta F$ implies $x \in F$.

LEMMA 2.2 (a) If $A \delta B$, $A \subset C$ and $B \subset D$, then $C \delta D$.

(b) If there is an x such that $A \delta x$ and $x \delta B$, then $A \delta B$.

PROOF. (a) If $A \delta B$, then $A \delta (B \cup D)$ by axioms A_1 and A_2. Since $B \subset D$, $A \delta D$. This implies $(A \cup C) \delta D$. Thus $C \delta D$ since $A \subset C$.

(b) Suppose $A \not\delta B$. By axiom A_5, there exists a subset E such that $A \not\delta E$ and $(X - E) \not\delta B$. x is either in E or in $X - E$. If $x \in E$, then $A \not\delta x$. For if $A \delta x$, then $A \delta E$ by part (a). If $x \in X - E$, then $x \not\delta B$. Therefore, if $A \delta x$ and $x \delta B$, then $A \delta B$.

THEOREM 2.3 The collection of the complements of all closed sets of (X, δ) forms a topology on X. This topology is denoted by $t(\delta)$.

PROOF. Since X and $\emptyset$ are closed in (X, δ), their complements $\emptyset$ and X are in $t(\delta)$. Let $\{F_i: i \in I\}$ be a collection of closed sets. If $x \delta \cap\{F_i: i \in I\}$, then $x \delta F_i$ for every $i \in I$ by lemma 2.2. Since F_i is closed, $x \in F_i$ for every $i \in I$. Hence $x \in \cap\{F_i: i \in I\}$ and $\cap\{F_i: i \in I\}$ is closed. Therefore, if $(X - F_i) \in t(\delta)$ for every $i \in I$, then $\cup\{X - F_i: i \in I\}$ the complement of $\cap\{F_i: i \in I\}$ belongs to $t(\delta)$. Finally if F_1 and F_2 are closed and $x \delta F_1 \cup F_2$, then $x \delta F_1$ or $x \delta F_2$. $x \in F_1$ or $x \in F_2$, since F_1 and F_2 are closed. This implies $x \in F_1 \cup F_2$. Thus

$F_1 \cup F_2$ is closed. Therefore, if $X - F_1 \in t(\delta)$ and $X - F_2 \in t(\delta)$, then $(X - F_1) \cap (X - F_2) = X - (F_1 \cup F_2) \in t(\delta)$. Hence $t(\delta)$ is a topology on X.

THEOREM 2.4 In a proximity space (X, δ), the set $\{x: x \delta A\}$ is the closure of A with respect to the topology $t(\delta)$.

PROOF. Let $A(\delta) = \{x: x \delta A\}$. If $x \in A(\delta)$, then $x \delta A$. By lemma 2.2, $x \delta \bar{A}$ since $A \subset \bar{A}$. Thus $x \in \bar{A}$. This shows that $A(\delta) \subset \bar{A}$. If $x \notin A(\delta)$, then $x \not\delta A$. By axiom A_5, there exists a subset E such that $x \not\delta E$ and $(X - E) \not\delta A$. Since there is no point of $X - E$ which is near A, $A(\delta) \subset E$. By lemma 2.2 and $x \not\delta E$ it follows that $x \not\delta A(\delta)$. Hence $A(\delta)$ is closed. Therefore, $\bar{A} \subset A(\delta)$, since $\bar{A}$ is the intersection of all closed sets containing A. Now $A(\delta) \subset \bar{A}$ and $\bar{A} \subset A(\delta)$ shows that $\bar{A} = A(\delta)$.

EXAMPLE 5. Let X be a non empty set. Define the proximity δ by $A \delta B$ iff $A \cap B \neq \emptyset$. This is the discrete proximity. Then $\bar{A} = \{x: x \delta A\} = \{x: \{x\} \cap A \neq \emptyset\} = \{x: x \in A\} = A$. Hence the topology $t(\delta)$ for X is the discrete topology.

EXAMPLE 6. Let (X, δ) be a proximity space and δ is defined by $A \delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$. Then the topology induced by this proximity is the trivial topology, since $\bar{A} = \{x: x \delta A\} = \emptyset$ if $A = \emptyset$ and $\bar{A} = X$ if $A \neq \emptyset$.

THEOREM 2.5 Let (X, δ) be a proximity space and let $O \subset X$. Then $O \in t(\delta)$ iff $x \not\delta (X - O)$ for every $x \in O$.

PROOF. If $O \in t(\delta)$, then $X - O$ is closed. Hence $x \notin X - O$ implies $x \not\delta X - O$, which shows that if $x \in O$, then $x \not\delta X - O$.

If for every $x \in O$, $x \not\delta X - O$, then $x \,\delta\, (X - O)$ implies $x \notin O$. This means that $x \,\delta\, (X - O)$ implies $x \in (X - O)$. Hence $X - O$ is closed. Thus $O \in t(\delta)$.

THEOREM 2.6 Let (X, δ) be a proximity space and let A and B be subsets of X such that $A \not\delta B$. Then (i) $\bar{B} \subset X - A$ (ii) $B \subset \text{Int}(X-A)$, where the closure and interior are taken with respect to $t(\delta)$.

PROOF. (i) If there exists some x such that $x \in \bar{B}$ and $x \in A$, then $x \,\delta\, B$ and $x \,\delta\, A$. By lemma 2.2, $A \,\delta\, B$. Hence if $x \in \bar{B}$, then $x \notin A$ since $A \not\delta B$. This means that $\bar{B} \subset X - A$.

(ii) If $x \in B$, then $x \,\delta\, B$. This implies $x \not\delta A$, for if $x \,\delta\, A$ then $A \,\delta\, B$ by lemma 2.2. Hence $x \notin \bar{A}$. Therefore $x \in X - \bar{A}$. Since $\text{Int}\,(X - A) = X - \bar{A}$, $x \in \text{Int}\,(X - A)$.

THEOREM 2.7 If A,B are subsets of (X, δ), then $A \,\delta\, B$ iff $\bar{A} \,\delta\, \bar{B}$, where the closure is taken with respect to $t(\delta)$.

PROOF. If $A \,\delta\, B$, then by lemma 2.2 $\bar{A} \,\delta\, \bar{B}$ since $A \subset \bar{A}$ and $B \subset \bar{B}$. If $A \not\delta B$, then there exists a subset E of X such that $A \not\delta E$ and $(X - E) \not\delta B$. Hence $\bar{B} \subset E$. This implies $A \not\delta \bar{B}$ for if $A \,\delta\, \bar{B}$, then by lemma 2.2 $A \,\delta\, E$ since $\bar{B} \subset E$. By applying lemma 2.2 again it follows that $\bar{A} \not\delta \bar{B}$.

Since a Kuratowski closure operator on X always introduces a topology for X. Hence if the operator $A \dashrightarrow \bar{A} = \{x: x \,\delta\, A\}$ defined on the power set of a proximity space (X, δ) is a Kuratowski closure operator, then the same topology as in theorem 2.4 can be introduced. The following theorem 2.9 will show that $A \dashrightarrow \overline{A}$ is a closure operator.

DEFINITION 2.8 Let X be a set and P (X) be the power set of X. The the operator C: $P(X) \dashrightarrow P(X)$ is a Kuratowski closure operator provided:

(i) $C(\emptyset) = \emptyset$

(ii) $A \subseteq C(A)$ for every $A \in P(X)$

(iii) $C(A \cup B) = C(A) \cup C(B)$ for any A,B belonging to P (X)

(iv) $C(C(A)) = C(A)$ for every $A \in P(X)$

THEOREM 2.9 Let (X, δ) be a proximity space and $A \subset X$. Define $\overline{A} = \{x \in X: x \delta A\}$. Then the operator $A \dashrightarrow \overline{A}$ is a Kuratowski closure operator on X.

PROOF. (i) Since there is no set which is near $\emptyset$, $\overline{\emptyset} = \{x \in X: x \delta \emptyset\} = \emptyset$.

(ii) If $x \in A$, then $x \delta A$. Hence $x \in \overline{A}$. This shows that $A \subseteq \overline{A}$.

(iii) Since $x \in \overline{(A \cup B)}$ iff $x \delta (A \cup B)$ iff $x \delta A$ or $x \delta B$ iff $x \in \overline{A}$ or $x \in \overline{B}$ iff $x \in \overline{A} \cup \overline{B}$, $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

(iv) If $x \in \overline{A}$, then $x \delta \overline{A}$ and hence $x \in \overline{(\overline{A})}$. Therefore $\overline{A} \subset \overline{(\overline{A})}$. If $x \notin \overline{A}$, then $x \not\delta A$. This implies that there exists a subset E such that $x \not\delta E$ and $(X - E) \not\delta A$. Now if $\overline{A}$ is not contained in E, then there exists an element t in $\overline{A}$ but t is not in E and hence $t \delta A$ and $t \in (X - E)$, contradicting $(X - E) \not\delta A$. Hence

$\bar{A} \subset E$. By lemma 2.2, $x \not\delta \bar{A}$ since $x \not\delta E$. This means that $x \notin \overline{(\bar{A})}$ and hence $\overline{(\bar{A})} \subset \bar{A}$.

DEFINITION 2.10 Let (X,t) be a topological space and δ a proximity on X such that $t = t(\delta)$. Then δ is said to be compatible with the topology t.

DEFINITION 2.11 A T_0 - space is a topological space in which, given any two distinct points x,y, there exists either a neighborhood N_x not containing y or a neighborhood N_y not containing x.

A T_1 - space is a topological space in which, given any two distinct points, each has a neighborhood which does not contain the other.

DEFINITION 2.12 A completely regular space is a topological space such that for each point x and neighborhood N of x, there is a continuous function with values in the interval [0,1] for which $f(x) = 1$ and $f(y) = 0$ if $y \notin N$.

DEFINITION 2.13 A Tychonoff space is a topological space which is a completely regular space and a T_1 - space.

DEFINITION 2.14 Given a completely regular space (X,t), the subsets A,B of X are functionally distinguishable iff there exists a continuous function f with values in the interval [0,1] such that $f(A) = 0$ and $f(B) = 1$.

THEOREM 2.15 If (X,t) is a completely regular space, then the proximity δ defined by $A \not\delta B$ iff A and B are functionally distinguishable, is compatible with t. If (X,t) is Tychonoff space, then δ is separated.

PROOF. It is first shown that δ is a proximity on X.

(i). Suppose $B \not\delta A$. Then B and A are functionally distinguishable. Hence there exists a continuous function $f: X \to [0,1]$ such that $f(B) = 0$ and $f(A) = 1$. Let $g = 1-f$, then g is continuous since f is continuous and $g(A) = 1 - f(A) = 0$, $g(B) = 1 - f(B) = 1$. This shows that A and B are functionally distinguishable. Hence $A \not\delta B$. Therefore $A \delta B$ implies $B \delta A$.

(ii). If $(A \cup B) \not\delta C$, then $(A \cup B)$ and C are functionally distinguishable and hence there exists a continuous function $f: X \to [0,1]$ such that $f(A \cup B) = 0$ and $f(C) = 1$. It follows that $f(A) = 0$, $f(B) = 0$, $f(C) = 1$. This implies $A \not\delta C$ and $B \not\delta C$.

If $A \not\delta C$ and $B \not\delta C$, then A, C are functionally distinguishable and B, C are functionally distinguishable. This implies that there exist continuous functions f_1 and f_2 such that $f_1(A) = 0$, $f_1(C) = 1$ and $f_2(B) = 0$, $f_2(C) = 1$. Let $f(x) = \text{g.l.b}\{f_1(x), f_2(x)\}$. Then $f(A \cup B) = 0$, $f(C) = 1$. f is continuous since f_1 and f_2 are continuous. Hence $(A \cup B) \not\delta C$. Therefore $(A \cup B) \delta C$ iff $A \delta C$ or $B \delta C$.

(iii). If $A = \emptyset$ or $B = \emptyset$, then A and B are functionally distinguishable and hence $A \not\delta B$ which shows $A \delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$.

(iv). Suppose $A \not\delta B$. Then there exists a continuous function $f: X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. It follows that $A \cap B = \emptyset$, for if $A \cap B \neq \emptyset$, then there exists a point $a \in A \cap B$

and $f(a) = 0$, $f(a) = 1$ which is impossible. Hence if $A \cap B \neq \emptyset$, then $A \,\delta\, B$.

(v). Let $A \not\delta B$. Then there exists a continuous function f from X to $[0,1]$ with $f(A) = 0$ and $f(B) = 1$. Let $E = \{ x \in X: \frac{1}{2} \leq f(x) \leq 1 \}$. (1). $A \not\delta E$ since there exists a continuous function g defined by $g(y) = 2y$ for $0 \leq y \leq \frac{1}{2}$ and $g(y) = 1$ for $\frac{1}{2} \leq y \leq 1$. The composite function gf is a continuous function such that $g(f(A)) = 0$ and $g(f(E)) = 1$. (2). $(X - E) \not\delta B$ since there exists a continuous function h such that $h(z) = 0$ for $0 \leq z \leq \frac{1}{2}$, $h(z) = 2z - 1$ for $\frac{1}{2} \leq z \leq 1$. The composite function hf is a continuous function from X to $[0,1]$ and $h(f(X - E)) = 0$ $h(f(B)) = 1$ where $X - E = \{x \in X; \ 0 \leq f(x) < \frac{1}{2}\}$.

It is now shown that δ is separated if (X,t) is Tychonoff. Since (X,t) is T_o - space, if $x \neq y$, then there exists a neighborhood N of y such that $x \notin N$. Since (X,t) is completely regular $\{x\}$ and N are functionally distinguishable. Hence $x \not\delta N$. By lemma 2.2, $x \not\delta y$. Therefore, if $x \,\delta\, y$, then $x = y$. This shows that is separated.

Finally, show that $t = t(\delta)$. Let $G \in t$ and $x \in G$. Then $x \notin X - G$, so that there exists a continuous function from X to $[0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$. Hence $x \not\delta X - G$. This shows that $G \in t(\delta)$, by theorem 2.5. Conversely, if $G \in t(\delta)$ and $x \in G$, then $x \not\delta X - G$ by theorem 2.5. Hence there exists a continuous function f from X to $[0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$. Then $f^{-1}([0,\frac{1}{2}))$ is an t open neighborhood of x in G, since f is continuous and $[0,\frac{1}{2})$ is open in $[0,1]$. Therefore $G \in t$.

DEFINITION 2.16 A T_4 - space is a topological space in which each pair of disjoint closed sets have disjoint neighborhoods. A normal space is a topological space that is T_4 and T_1.

The following is Urysohn's lemma which is stated without proof.

LEMMA 2.17 (Urysohn's lemma) Let X be a normal space, and let A and B be disjoint closed subsets of X. Then there exists a continuous real function f defined on X, all of whose values lie in the closed unit interval $[0,1]$, such that $f(A) = 0$ and $f(B) = 1$.

By the Urysohn's lemma, every normal space is completely regular and hence is a Tychonoff space.

THEOREM 2.18 Let (X,t) be a normal space. Then $\bar{A} \cap \bar{B} = \emptyset$ iff $\bar{A}$ and $\bar{B}$ are functionally distinguishable.

PROOF. By lemma 2.17 if $\bar{A} \cap \bar{B} = \emptyset$, then A and B are functionally distinguishable. If $\bar{A} \cap \bar{B} \neq \emptyset$, then there exists a point x such that $x \in \bar{A} \cap \bar{B}$. Since there exists no function f such that $f(x)$ has different values at one point x it follows that $\bar{A}$ and $\bar{B}$ are not funcitonally distinguishable.

THEOREM 2.19 Let (X,t) be a normal space. Then $A \,\delta\, B$ iff $\bar{A} \cap \bar{B} \neq \emptyset$ defines a compatible proximity.

PROOF. By theorem 2.18, $\bar{A} \cap \bar{B} = \emptyset$ iff $\bar{A}$ and $\bar{B}$ are functionally distinguishable. By the properties of a continuous function, $\bar{A}$ and $\bar{B}$ are functionally distinguishable iff A and B are functionally distinguishable. Hence, $A \not\delta B$ iff $\bar{A} \cap \bar{B} = \emptyset$ iff $\bar{A}$ and $\bar{B}$

are functionally distinguishable iff A and B are functionally distinguishable. Since every normal space is completely regular, by theorem 2.15, δ defines a compatible proximity.

THEOREM 2.20 If a completely regular space (X,t) has a compatible proximity δ defined by $A \delta B$ iff $\bar{A} \cap \bar{B} \neq \emptyset$, then (X,t) is T_4.

PROOF. Let P and Q be a pair of disjoint closed sets. Therefore $P \not\delta Q$, and there exists a subset E such that $P \not\delta E$ and $(X - E) \not\delta Q$. By theorem 2.6, $P \subset Int(X - E)$ and $Q \subset Int E$. Since $Int(E) \cap Int(X - E) = \emptyset$, (X,t) is T_4.

DEFINITION 2.21 If δ_1 and δ_2 are two proximities on a set X. Define $\delta_1 > \delta_2$ iff $A \delta_1 B$ implies $A \delta_2 B$. δ_1 is said to be finer than δ_2, or δ_2 is said to be coarser than δ_1.

The following theorem shows that a finer proximity induces a finer topology.

THEOREM 2.22 If δ_1 and δ_2 are two proximities defined on a set X, then $\delta_1 < \delta_2$ implies $t(\delta_1) \subset t(\delta_2)$.

PROOF. If $0 \in t(\delta_1)$, then by theorem 2.5, $x \not\delta_1 (X - 0)$ for every $x \in 0$. Since $\delta_1 < \delta_2$, $x \not\delta_2 (X - 0)$ for every $x \in 0$. Again by theorem 2.5, $0 \in t(\delta_2)$. Hence $t(\delta_1) \subset t(\delta_2)$.

THEOREM 2.23 Let t_1 and t_2 be two completely regular topologies on X and δ_1, δ_2 be the proximities on X defined by $A \delta_i B (i = 1,2)$ iff A and B are functionally distinguishable with respect to t_1 and t_2 respectively. Then $t_1 \subset t_2$ implies $\delta_1 < \delta_2$.

PROOF. If $A \not\delta_1 B$, then there exists a continuous function f from (X, t_1) to $[0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Since $t_1 \subset t_2$, f is also a continuous function from (X, t_2) to $[0,1]$ such that $f(A) = 0$ and $f(B) = 1$. This means that $A \not\delta_2 B$. By definition 2.21, $\delta_2 > \delta_1$.

3. PROXIMITY NEIGHBORHOOD

DEFINITION 3.1 A subset B of a proximity space (X, δ) is a δ- neighborhood of A if $A \not\delta X - B$. This is denoted by $A \ll B$.

THEOREM 3.2 Let (X, δ) be a proximity space, $\overline{A}$ and $\text{Int}(A)$ denote, respectively, the closure and interior of A in $t(\delta)$. Then

(i). $A \ll B$ implies $\overline{A} \ll B$, and

(ii). $A \ll B$ implies $A \ll \text{Int}(B)$.

PROOF (i). If $A \ll B$, then $A \not\delta X - B$. By theorem 2.7 and lemma 2.2, $\overline{A} \not\delta (X - B)$, which shows that $\overline{A} \ll B$.

(ii). $A \ll B$ implies $A \not\delta \overline{X - B}$. Since $\overline{X - B} = X - \text{Int}(B)$, $A \not\delta X - \text{Int}(B)$. Hence $A \ll \text{Int}(B)$.

LEMMA 3.3 Let (X, δ) be a proximity space. Then $A \not\delta B$ implies $A \subset X - B$.

PROOF. Suppose $A \not\subset X - B$. Then there exists at least one point a in A such that $a \notin X - B$. This means that $a \in B$. Hence $A \cap B \neq \emptyset$. It follows that $A \delta B$, which is impossible. Therefore $A \subset X - B$.

THEOREM 3.4 Axiom A_5 is equivalent to the statement: If $A \not\delta B$, then there exists subsets C and D such that $A \not\delta (X - C)$, $(X - D) \not\delta B$ and $C \not\delta D$.

PROOF. If A_5 holds, then $A \not\delta B$ implies there is a subset D such that $A \not\delta D$ and $(X - D) \not\delta B$. Since $A \not\delta D$, there exists a subset C such that $A \not\delta (X - C)$ and $C \not\delta D$. To prove the converse, let $E = X - C$. Then $A \not\delta E$. By lemma 3.3, $C \subset X - D$ since $C \not\delta D$. Hence $(X - E) = C$ and $C \not\delta B$, for if $C \, \delta \, B$, then $(X - D) \, \delta \, B$ by lemma 2.2, a contradiction. Therefore A_5 holds.

COROLLARY. In a proximity space (X, δ), $A \not\delta B$ implies that there exists subsets C and D such that $A \ll C$, $B \ll D$ and $C \not\delta D$.

If δ is separated, then the topology $t(\delta)$ is Hansdorff, since $x \neq y$ implies $x \not\delta y$ and there exist disjoint subsets C and D such that $\{x\} \ll C$ and $\{y\} \ll D$.

LEMMA 3.5 Let δ be a compatible proximity on a completely regular space (X,t). If A is compact and B is closed and $A \cap B = \emptyset$, then $A \not\delta B$.

PROOF. Since B is closed, $x \in B$ iff $x \, \delta \, B$. For each $a \in A$, $a \notin B$ since $A \cap B = \emptyset$. Hence $a \not\delta B$ for each a in A. By the corollary of theorem 3.4, there exists an open neighborhood N_a of a such that $N_a \not\delta B$. But $\{N_a : a \in A\}$ is an open cover of A, hence there is a finite subcover $\{N_{ai} : i = 1,2, \ldots, n\}$. Since $N_{ai} \not\delta B$ for each i, $\bigcup [N_{ai} : i = 1,2, \ldots, n] \not\delta B$. By lemma 2.2, $A \not\delta B$ since $A \subset \bigcup [N_{ai} : i = 1,2, \ldots, n]$.

THEOREM 3.6 Every compact topological space X which is completely regular (Tychonoff) has a unique compatible (separated) proximity, given by $A \, \delta \, B$ iff $\overline{A} \cap \overline{B} \neq \emptyset$.

PROOF. Let δ be any proximity and $\overline{A} \cap \overline{B} \neq \emptyset$. Then $\overline{A}\ \delta\ \overline{B}$. Since $\overline{A}\ \delta\ \overline{B}$ iff $A\ \delta\ B$ by theorem 2.7, $A\ \delta\ B$. Conversely, let δ be any proximity and $A\ \delta\ B$. Since $\overline{A}$ is a closed subset of a compact space X, $\overline{A}$ is compact. By lemma 3.5, $\overline{A} \cap \overline{B} \neq \emptyset$ since $\overline{B}$ is closed.

Now, if X is Tychonoff, then $\{x\}$ is closed. Hence if $x\ \delta\ y$, then $\{x\} \cap \{y\} \neq \emptyset$ and $x = y$.

THEOREM 3.7 In a proximity space (X, δ), the relation has the following properties.

(i). $X \ll X$.

(ii). $\emptyset \ll A$ for any subset A of X.

(iii). $A \ll B$ implies $A \subset B$.

(iv). $A \subset B$ implies $A \ll B$ iff δ is discrete.

(v). $A \subset B$, $B \ll C$ and $C \subset D$ imply $A \ll D$.

(vi). $A \ll B_i$ for $i = 1, 2, \ldots, n$ iff $A \ll \bigcap [B_i : i=1,2,\ldots n]$

(vii). $A \ll B$ implies $(X - B) \ll (X - A)$.

(viii). $A \ll B$ implies there is a C such that $A \ll C \ll B$.

(IX). If δ is separated, then $x \ll (X - Y)$ iff $x \neq y$.

(X). If $A_i \ll B_i$ for $i = 1,2,\ldots,n$, then

$\bigcap [A_i : i = 1,2,\ldots,n] \ll \bigcap [B_i : i = 1,2,\ldots,n]$ and

$\bigcup [A_i : i = 1,2,\ldots,n] \ll \bigcup [B_i : i = 1,2,\ldots,n]$

PROOF (i). Since $X \not\delta\ \emptyset = X - X$, $X \ll X$.

(ii). By axiom A_3, $\emptyset$ is not near to any subset of X. This means $\emptyset \not\delta\ (X - A)$ for any subset A of X. Hence $\emptyset \ll A$.

(iii). If $A \ll B$, then $A \not\delta\ (X - B)$. It follows that $A \cap (X - B) = \emptyset$, and hence $A \subset B$.

(iv). If δ is a discrete proximity, then $A \delta B$ iff $A \cap B \neq \emptyset$. Hence if $A \subset B$, then $A \cap (X - B) = \emptyset$. It follows that $A \not\delta X - B$ and hence $A \ll B$. Suppose $A \delta B$ and $A \cap B = \emptyset$. Then $A \subset X - B$ and $A \ll X - B$. By definition 3.1, $A \ll X - B$ implies $A \not\delta B$, which is a contradiction. Therefore $A \delta B$ implies $A \cap B \neq \emptyset$. By axiom A_4, $A \cap B \neq \emptyset$ implies $A \delta B$. Hense δ is the discrete proximity.

(v). By definition 3.1, $A \not\ll D$ implies $A \delta X - D$. Since $C \subset D$, $A \delta X - C$ by lemma 2.2. It follows $B \delta X - C$ since $A \subset B$. Hence $B \not\ll C$, a contradiction.

(vi). $A \ll B_i$ for $i = 1,2,\ldots,n$ iff $A \not\delta X - B_i$ iff $A \not\delta \bigcup [(X - B_i): i = 1,2,\ldots,n]$ by axiom A_2 iff $A \not\delta X - \bigcap [B_i: i = 1,2,\ldots,n]$ iff $A \ll \bigcap [B_i: i = 1,2,\ldots,n]$

(vii). If $A \ll B$, then $A \not\delta X - B$ and hence $(X - B) \not\delta A$. Since $A = X - (X - A)$, $(X - B) \not\delta X - (X - A)$. Therefore $(X - B) \ll (X - A)$.

(viii). If $A \ll B$, then $A \not\delta X - B$. There exists a subset C such that $A \not\delta (X - C)$ and $C \not\delta (X - B)$ which shows $A \ll C \ll B$.

(IX). If $x \neq y$, then $x \not\delta y$ and hence $x \ll (X - y)$. If $x \ll X - y$, then $x \not\delta y$. Hence $\{x\} \cap \{y\} = \emptyset$, which shows that $x \neq y$.

(X). Since $\bigcap [A_i: i = 1,2,\ldots,n] \subset A_i$, hence if $A_i \not\delta X - B_i$, then $\bigcap [A_i: i = 1,2,\ldots,n] \not\delta X - B_i$. Therefore $\bigcap [A_i: i = 1,2,\ldots,n] \ll B_i$. By property (vi), $\bigcap [A_i: i = 1,2,\ldots,n] \ll \bigcap [B_i: i = 1,2,\ldots,n]$.

Since $X - B_i \supset X - \bigcup [B_i: i = 1,2,\ldots,n]$, hence if $A_i \not\delta X - B_i$, then $A_i \not\delta X - \cup[B_i: i = 1,2,\ldots,n]$. It follows that $\cup[A_i: i = 1,2,\ldots,n] \not\delta X - \cup[B_i: i = 1,2,\ldots,n]$ and therefore $\cup[A_i: i = 1,2,\ldots,n] \ll \cup[B_i: i = 1,2,\ldots,n]$.

THEOREM 3.8 Let A be a subset of a proximity space (X, δ). Then $\bar{A} = \cap[B: A \ll B]$.

PROOF. By theorem 3.2, $A \ll B$ implies $\bar{A} \ll B$ and hence $\bar{A} \subset B$ by lemma 3.3. This shows $\bar{A} \subset \cap[B: A \ll B]$. If $x \notin \bar{A}$, then $x \not\delta \bar{A}$ for if $x \delta \bar{A}$, then $x \in \bar{\bar{A}} = \bar{A}$. By the corollary of theorem 3.4, $\bar{A}$ has a δ-neighborhood Bx and $x \notin Bx$. Hence $x \notin \cap [B: A \ll B]$ since Bx is also a δ-neighborhood of A.

4. PROXIMITY MAPPING.

Corresponding to the concept of continuous functions between topological spaces, there are proximity mappings between proximity spaces.

DEFINITION 4.1 Let (X, δ_1) and (Y, δ_2) be two proximity spaces. A function f from X into Y is said to be a proximity mapping or a proximally continuous mapping iff $A \delta_1 B$ implies $f(A) \delta_2 f(B)$.

EXAMPLE 7. Let (X, δ) be a proximity space and Y be a non-empty set. Define δ_t by $A \delta_t B$ iff $A \neq \emptyset$, $B \neq \emptyset$, the trivial proximity. Define δ_d by $A \delta_d B$ iff $A \cap B \neq \emptyset$, the discrete proximity.

(i). Any mapping f from (X, δ) to (Y, δ_t) is a proximity mapping.

Let A and B be subsets of X. $f(A) \not\delta_t f(B)$ iff $f(A) = \emptyset$ or $f(B) = \emptyset$. This implies $A = \emptyset$ or $B = \emptyset$ and hence $A \not\delta B$, which shows that $A \delta B$ implies $f(A) \delta_t f(B)$.

(ii). Any mapping g from (Y, δ_d) to (X, δ) is a proximity mapping.

Let A and B be subsets of Y. $A \delta_d B$ iff $A \cap B \neq \emptyset$. Therefore $g(A) \cap g(B) \neq \emptyset$ and hence $g(A) \delta g(B)$. Thus g is a proximity mapping.

(iii). The identity mapping I from (X, δ_t) to (X, δ_d) is not a proximity mapping, where X contains at least two points.

Let a,b be two distinct points of X. Then $a \delta_t b$ but $a \not\delta_d b$.

THEOREM 4.2 Let (Y, δ) be a proximity space. Let f be a function from X to (Y, δ). Define a relation ρ by $A \rho B$ iff $f(A) \delta f(B)$. Then ρ is a proximity on X.

PROOF (i). $A \rho B$ implies $f(A) \delta f(B)$ and hence $f(B) \delta f(A)$. Therefore $B \rho A$.

(ii). $(A \cup B) \rho C$ iff $f(A \cup B) \delta f(C)$ iff $(f(A) \cup f(B)) \delta f(C)$ iff $f(A) \delta f(C)$ or $f(B) \delta f(C)$ iff $A \rho C$ or $B \rho C$.

(iii). $A \rho B$ iff $f(A) \delta f(B)$. It follows $f(A) \neq \emptyset$ and $f(B) \neq \emptyset$ since δ is a proximity and hence $A \neq \emptyset$ and $B \neq \emptyset$.

(iv). If $A \cap B \neq \emptyset$, then $f(A \cap B) \neq \emptyset$ and hence $f(A) \cap f(B) \neq \emptyset$. It follows $f(A) \delta f(B)$ and therefore $A \rho B$.

(v). If $A \not\rho B$, then $f(A) \not\delta f(B)$ and hence there exists a subset E of Y such that $f(A) \not\delta E$ and $(Y - E) \not\delta f(B)$. Since $ff^{-1}(E) \subset E$, $f(A) \not\delta ff^{-1}(E)$ and hence $A \not\rho f^{-1}(E)$. Since $f^{-1}(Y - E) = X - f^{-1}(E)$, $f(X - f^{-1}(E)) = ff^{-1}(Y - E) \subset Y - E$. It follows that

$f(X - f^{-1}(E)) \not\delta f(B)$ and therefore $(X - f^{-1}(E)) \not\rho B$.

THEOREM 4.3 Let f be a one to one function from a proximity space (X, δ) onto a set Y. Define a relation ρ on Y such that $A \rho B$ iff $f^{-1}(A) \delta f^{-1}(B)$. Then ρ is a proximity on Y.

PROOF (i). $A \rho B$ iff $f^{-1}(A) \delta f^{-1}(B)$ which implies $f^{-1}(B) \delta f^{-1}(A)$ and hence $B \rho A$.

(ii). Since $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Hence $(A \cup B) \rho C$ iff $f^{-1}(A \cup B) \delta f^{-1}(C)$ iff $(f^{-1}(A) \cup f^{-1}(B)) \delta f^{-1}(C)$ iff $f^{-1}(A) \delta f^{-1}(C)$ or $f^{-1}(B) \delta f^{-1}(C)$ iff $A \rho C$ or $B \rho C$.

(iii). $A \rho B$ iff $f^{-1}(A) \delta f^{-1}(B)$ implies $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$. It follows that $A \neq \emptyset$ and $B \neq \emptyset$.

(iv). $f^{-1}(A) \not\delta f^{-1}(B)$ implies $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Therefore if $A \not\rho B$, then $A \cap B = \emptyset$.

(v). If $f^{-1}(A) \not\delta f^{-1}(B)$, then there exists a subset E of X such that $f^{-1}(A) \not\delta E$ and $(X - E) \not\delta f^{-1}(B)$. By lemma 3.3, $f^{-1}(A) \subset X - E$ and $f^{-1}(B) \subset E$. It follows that $A \subset f(X - E)$ and $B \subset f(E)$. Now, if $A \rho f(E)$, then $f^{-1}(A) \delta f^{-1}f(E)$ or $f^{-1}(A) \delta E$ since f is 1 - 1, contradicting $f^{-1}(A) \not\delta E$. Hence $A \not\rho f(E)$. Since $Y - f(E) = f(X - E)$, $f^{-1}(Y - f(E)) = X - E$ and hence $f^{-1}(Y - f(E))) \not\delta f^{-1}(B)$ which shows that $(Y - f(E)) \not\rho B$.

DEFINITION 4.4 Let X and Y be topological spaces and f be a mapping from X into Y. f is called a continuous mapping if $f^{-1}(G)$ is open in X whenever G is open in Y.

LEMMA 4.5 Let f be a mapping from one topological space X into another topological space Y. Then f is continuous iff $f^{-1}(F)$ is closed in X whenever F is closed in Y iff $f(\bar{A}) \subset \overline{f(A)}$ for any subset A of X.

PROOF (a). If f is continuous and F is closed in Y, then Y - F is open and $f^{-1}(Y - F) = X - f^{-1}(F)$ is open and hence $f^{-1}(F)$ is closed. Conversely, if G is open in Y, then Y - G is closed. Hence $f^{-1}(Y - G) = X - f^{-1}(G)$ is closed. This implies $f^{-1}(G)$ is open. By definition 4.4, f is continuous.

(b). If $f(\bar{A}) \subset \overline{f(A)}$ for every subset A of X and F is closed, then $f(\overline{f^{-1}(F)}) \subset \overline{f(f^{-1}(F))} \subset \bar{F} = F$. This means $\overline{f^{-1}(F)} \subset f^{-1}(F)$ which shows that $f^{-1}(F)$ is closed. On the other hand, if $f^{-1}(F)$ is closed whenever F is closed. then f is continuous by part (a). Let $y \in f(\bar{A})$. Then there exists an $x \in \bar{A}$ such that $y = f(x)$ and $Nx \cap A \neq \emptyset$ for every neighborhood Nx of x and hence $f(Nx \cap A) = f(Nx) \cap f(A) \neq \emptyset$ for any neighborhood Nx of x. Let Ny be a neighborhood of y. Then $f^{-1}(Ny)$ is a neighborhood of x since f is continuous. Hence $f(f^{-1}(Ny)) \cap f(A) \neq \emptyset$. Since $ff^{-1}(Ny) \subset Ny$, $Ny \cap f(A) \neq \emptyset$ which shows that $y \in \overline{f(A)}$. Therefore $f(\bar{A}) \subset \overline{f(A)}$.

THEOREM 4.6 A proximity mapping f from (X, δ_1) to (Y, δ_2) is continuous with respect to $t(\delta_1)$ and $t(\delta_2)$.

PROOF. Let $A \subset X$. Since f is a proximity mapping, $x \, \delta_1 A$ implies $f(x) \, \delta_2 f(A)$. Therefore, if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Hence $f(\bar{A}) \subset \overline{f(A)}$. By lemma 4.5, f is continuous with respect to $t(\delta_1)$ and $t(\delta_2)$.

DEFINITION 4.7 Two proximity space (X, δ_1) and (Y, δ_2) are called proximally isomorphic (or δ-homeomorphic) iff there exists a one - to - one mapping f from X onto Y such that both f and f^{-1} are proximity mappings. f is called a proximity isomorphism or δ-homeomorphism.

LEMMA 4.8 Let (X, δ) be a proximity space and let Y be a subset of X. For any subsets A, B of Y, define $A \delta_Y B$ iff $A \delta B$. Then δ_Y is a proximity on Y.

PROOF. The first four axioms of a proximity are easily verified. To prove the last axiom, let $A \not\delta_Y B$. Then $A \not\delta B$ and hence there exists a subset E^1 of X such that $A \not\delta E^1$ and $(X - E^1) \not\delta B$. If the intersection of Y and E^1 is empty, then Y is a subset of $X - E^1$. Since $(X - E^1) \not\delta B$, $Y \not\delta B$ contradicts B is a subset of Y. Hence the intersection of Y and E^1 is not empty. Set $E = Y \cap E^1$. Then $A \not\delta E$ since E is a subset of E^1 and $A \not\delta E^1$. $(Y - E) \not\delta B$ since $Y - E$ is a subset of $X - E^1$ and $X - E^1 \not\delta B$. Therefore if $A \not\delta_Y B$, then there exists a subset E of Y such that $A \not\delta E$ and $Y - E \not\delta B$ which shows that $A \not\delta_Y E$ and $(Y - E) \not\delta_Y B$.

DEFINITION 4.9 The proximity δ_Y defined in the previous lemma is called the induced (or subspace) proximity on Y and $t(\delta_Y)$ is the subspace topology induced on Y by $t(\delta)$.

CHAPTER III

FILTERS

DEFINITION 1. Let X be a non-empty set. A filter $\mathcal{F}$ on X is a non-empty collection of subsets of X such that

(1). $\phi \notin \mathcal{F}$

(2). $A \in \mathcal{F}$, $B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$.

(3). $A \in \mathcal{F}$ and $A \subset B$ imply $B \in \mathcal{F}$.

EXAMPLE 1. Let X be a non-empty set. Then $\{ X \}$ is a filter on X.

EXAMPLE 2. Let (X, t) be a topological space and $x \in X$. The collection $N(x) = \{ B: B$ is a neighborhood of $x \}$ is a filter on X called the neighborhood filter of x.

EXAMPLE 3. Let $\{ x_n \}$ be a sequence in a topological space (X,t). Define $F_k = \{ x_n: n \geq k \}$ for k a natural number. Then the collection of subsets of X defined by $\mathcal{F} = \{ F \subset X: F \supset F_k$ for some $k \}$ is a filter, called the filter generated by the sequence.

DEFINITION 2. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters on a given set X. Define $\mathcal{F}_1 \leq \mathcal{F}_2$ iff $\mathcal{F}_1 \subset \mathcal{F}_2$.

DEFINITION 3. A filter $\mathcal{U}$ on X is a ultrafilter if $\mathcal{U} \leq \mathcal{U}_1$, a filter on X, then $\mathcal{U} = \mathcal{U}_1$.

ZORN'S LEMMA. If P is a non-empty partially ordered set in which every chain has an upper bound, then P possesses a maximal element.

THEOREM 4. For any filter $\mathcal{V}$ on X, there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{V} \leq \mathcal{U}$.

PROOF. Let $\alpha(\mathcal{V})$ be the set of all filters on X which contains $\mathcal{V}$. Define a partial order on $\alpha(\mathcal{V})$ by definition 2. Every chain C in $\alpha(\mathcal{V})$ has an upper bound in $\alpha(\mathcal{V})$. This upper bound is the union of all elements of the chain C. To show that $\mathcal{V} = \bigcup [\mathcal{V}_i : \mathcal{V}_i \in C]$ is a filter, it is enough to note that (i). $\emptyset \notin \mathcal{V}$ since $0 \notin \mathcal{V}_i$. (ii). If $A \in \mathcal{V}$ and $A \subset B$, then A is in $\mathcal{V}_i$ for some $\mathcal{V}_i$ in C. Since $\mathcal{V}_i$ is a filter, $B \in \mathcal{V}_i$. Hence $B \in \mathcal{V}$. (iii). If $A \in \mathcal{V}$, $B \in \mathcal{V}$, then $A \in \mathcal{V}_i$, $B \in \mathcal{V}_i$ for some $\mathcal{V}_i$, $\mathcal{V}_j$ in C. If $\mathcal{V}_i \leq \mathcal{V}_j$, then $A \in \mathcal{V}_j$ and $A \cap B \in \mathcal{V}_j$ since $\mathcal{V}_j$ is a filter. If $\mathcal{V}_j \leq \mathcal{V}_i$, then $A \cap B \in \mathcal{V}_i$. It follows that $A \cap B$ is in $\mathcal{V}$. By Zorn's lemma, $\alpha(\mathcal{V})$ has a maximal element. This maximal element is an ultrafilter which contains $\mathcal{V}$.

THEOREM 5. A filter $\mathcal{U}$ is an ultrafilter on X iff $A \cup B \in \mathcal{U}$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

PROOF. Suppose $A \cup B \in \mathcal{U}$ and $A \notin \mathcal{U}$, $B \notin \mathcal{U}$. Let $\mathcal{V}$ be the set of all subsets Y of X such that $Y \cup A \in \mathcal{U}$. Then $\mathcal{V}$ is a filter by the following argument.

(1). $\emptyset \notin \mathcal{V}$ since $\emptyset \cup A = A \notin \mathcal{U}$ and $\mathcal{V} \neq \emptyset$ since $B \in \mathcal{V}$.

(2). If Y_1, Y_2 are in $\mathcal{V}$, then $Y_1 \cup A \in \mathcal{U}$ and $Y_2 \cup A \in \mathcal{U}$. $(Y_1 \cup A) \cap (Y_2 \cup A) = (Y_1 \cap Y_2) \cup A \in \mathcal{U}$ since $\mathcal{U}$ is a filter. Hence $Y_1 \cap Y_2 \in \mathcal{V}$.

(3). If $Y_1 \in \mathcal{V}$ and $Y_1 \subset Y_2$, then $Y_1 \cup A \subset Y_2 \cup A$ and $Y_2 \cup A \in \mathcal{U}$. Hence $Y_2 \in \mathcal{V}$. $\mathcal{U} \lesssim \mathcal{V}$ for if $Y \in \mathcal{U}$, then $Y \subset Y \cup A \in \mathcal{U}$ and hence $Y \in \mathcal{V}$. $\mathcal{U} \neq \mathcal{V}$ since $B \in \mathcal{V}$ but $B \notin \mathcal{U}$. Hence $\mathcal{U}$ is not an ultrafilter.

Suppose $A \cup B \in \mathcal{U}$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$ and $\mathcal{U}$ is not an ultrafilter. By theorem 4, there exists an ultrafilter $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{V}$. Choose an A such that $A \notin \mathcal{U}$ but $A \in \mathcal{V}$. $(X - A) \in \mathcal{U}$ since $(X - A) \cup A = X \in \mathcal{U}$. It follows that $X - A \in \mathcal{V}$ since $\mathcal{U} \subset \mathcal{V}$. $(X - A) \cap A \in \mathcal{V}$ since $X - A$ and A are in $\mathcal{V}$. Hence $\emptyset \in \mathcal{V}$, but this is impossible.

COROLLARY. If $\mathcal{U}$ is an ultrafilter on X, then for any subset A of X either A is in $\mathcal{U}$ or its complement is in $\mathcal{U}$.

COROLLARY. If $\bigcup [A_i:\ i = 1,2,\ldots,n]$ is in an ultrafilter $\mathcal{U}$, then at least one A_i is in $\mathcal{U}$.

EXAMPLE 4. Let a be a fixed point of X. The collection $\mathcal{U}$ of all subsets of X which contains a is an ultrafilter, called a fixed ultrafilter.

(1). $\mathcal{U}$ is not empty and $\emptyset$ is not in $\mathcal{U}$ since $a \notin \emptyset$.

(2). If $A \in \mathcal{U}$, $B \in \mathcal{U}$, then $a \in A \cap B$. It follows that $A \cap B \in \mathcal{U}$.

(3). $A \in \mathcal{U}$, $A \subset B$ imply $B \in \mathcal{U}$.

(4). If $A \cup B \in \mathcal{U}$, then $a \in A \cup B$. It follows that $a \in A$ or $a \in B$. Hence $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

EXAMPLE 5. Let N denote the set of natural numbers. Set $F_k = \{n: n \geq k\}$. Define $\mathcal{F} = \{F \subset N: F \geq F_k$ for some k in $N\}$. Then

(1). $\mathcal{F}$ is a filter generated by the sequence $\{n\}_1^{\infty}$

(2). By theorem 4, there exists an ultrafilter containing $\mathcal{F}$. It is clear that such an ultrafilter is not a fixed ultrafilter.

DEFINITION 6. A non-empty collection β of subsets of X is called a filter base iff

(1). $\emptyset \notin \beta$

(2). If $B_1, B_2 \in \beta$, then there exists $B \in \beta$ such that $B \subset B_1 \cap B_2$.

THEOREM 7. If β is a filter base on X, then the collection $\mathcal{F}(\beta)$ consisting of all sets A such that $A \supset B$ for some B in β is a filter.

PROOF. (1). $\emptyset \notin \mathcal{F}(\beta)$ since $\emptyset \notin \beta$. (2). $\mathcal{F}(\beta) \neq \emptyset$ since β is not empty. (3). If $A \in \mathcal{F}(\beta)$ and $A \subset C$, then there exists $B \in \beta$ with $B \subset A$. Hence $B \subset C$ and $C \in \mathcal{F}(\beta)$. (4). If A_1 and A_2 are in $\mathcal{F}(\beta)$, then there exists B_1 and B_2 in β such that $A_1 \supset B_1$ and $A_2 \supset B_2$. Hence $A_1 \cap A_2 \supset B_1 \cap B_2$.

Since β is a filter base, there exists B in β such that $B_1 \cap B_2 \supset B$. Therefore $A_1 \cap A_2 \supset B_1 \cap B_2 \supset B$ and hence $A_1 \cap A_2 \in \mathcal{F}(\beta)$.

$F(\beta)$ is called the filter generated by β and β is called a base of the filter $F(\beta)$. β is called an ultrafilter base if $F(\beta)$ is an ultrafilter.

EXAMPLE 6. Let R be the set of all real numbers. Let A be the close interval [0,1] and $\beta = \{A\}$. Then

(1). β is a filter base since $\beta \neq \emptyset$, $\emptyset \notin \beta$ and $A \subset (A \cap A)$.

(2). The collection $F(\beta) = \{F \subset R: F \supset [0,1]\}$ is the filter generated by β. By theorem 5, $F(\beta)$ is not an ultrafilter since $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1] \in F(\beta)$ but $[0, \frac{1}{2}] \notin F(\beta)$ and $[\frac{1}{2}, 1] \notin F(\beta)$. Hence β is not anultrafilter base.

EXAMPLE 7. Let R denote the set of real numbers. Set $\beta = \{(a,b): 1 \in (a,b),\ a,\ b \in R\}$, the set of all open intervals containing 1. Then

(1). β is a filter base since $\beta \neq \emptyset$, $\emptyset \notin \beta$ and the intersection of two open sets containing 1 is an open interval containg 1.

(2). β is not a filter since $(0,2) \in \beta$ and $(0,2) \subset [0,2]$ but $[0,2] \notin \beta$.

(3). The collection $F(\beta) = \{F \subset R : F \supset (a, b)$ and $1 \in (a, b)\}$ is the filter generated by β.

(4). $F(\beta)$ is not an ultrafilter since $(0,1] \cup [1,2) = (0,2) \in F(\beta)$ but $(0,1] \notin F(\beta)$ and $[1,2) \notin F(\beta)$.

EXAMPLE 8. Let R be the set of real numbers. Set $\beta = \{\{1\}\}$

Then

(1). β is a filter base but β is not a filter.

(2). The collection $F(\beta) = \{ F \subset R: F \supset \{1\} \}$ is the filter generated by β .

(3). $F(\beta)$ is a fixed ultrafilter.

EXAMPLE 9. Let N denote the set of natural numbers. Set $S_n = \{ n, n+1, n+2, \ldots \}$. Let $\beta = \{ S_n: n = 1,2,\ldots \}$. Then

(1). β is a filter base.

(2). The collection $F(\beta) = \{ F \subset N: F \supset S_n \text{ for some } S_n \text{ in } \beta \}$ is a filter.

(3). By theorem 5, $F(\beta)$ is not an ultrafilter since the union of the set E, of all even numbers, and the set O, of all odd numbers, in in $F(\beta)$ but neither E nor O is an element of $F(\beta)$.

THEOREM 8. A filter F is an ultrafilter on X iff $A \cap F \neq \emptyset$ for all F in F implies that A belongs to F.

PROOF. Let F be an ultrafilter and $A \subset X$ such that $A \cap F \neq \emptyset$ for all F in F. Then the collection consisting of all finite intersections of elements of $F \cup \{A\}$ is a filter base and hence determines a filter F' such that $F' \supset F$. Since F is an ultrafilter, $A \in F' = F$.

If F is not an ultrafilter, then there exists a filter F' such that $F' \supset F$ and $F' \neq F$. Hence there exists a set A such that $A \in F'$ and $A \notin F$ and $A \cap F \neq \emptyset$ for all F in F'. It follows that $A \cap F \neq \emptyset$ for all F in F since $F' \supset F$. Therefore if F is not an ultrafilter, then there exists an A such that $A \notin F$ and $A \cap F \neq \emptyset$ for all F in F.

THEOREM 9. Let $\mathcal{F}$ be a filter on X and f a function from X to Y. Then the set of all f(A), A $\in \mathcal{F}$ is a filter base on Y.

PROOF. $\emptyset \notin f(A)$ since $A \neq \emptyset$. For any A, B in $\mathcal{F}$, $f(A) \cap f(B) \supset f(A \cap B)$. Hence $\{ f(A): A \in \mathcal{F} \}$ is a filter base on Y.

The set $\{ f(A): A \in \mathcal{F} \}$ is denoted by $f(\mathcal{F})$. Since $f(\mathcal{F})$ is a filter base on Y, by theorem 7, the collection $\mathcal{E} = \{ E \subset Y: E \supset f(A)$ for some f(A) in $f(\mathcal{F}) \}$ is a filter on Y generated by the filter base $f(\mathcal{F})$ on Y.

EXAMPLE 10. Consider example 9. Let f: $N \longrightarrow N$ defined by $f(n) = n+2$. Set $\mathcal{F} = \mathcal{F}(\beta)$. Then $f(\mathcal{F}) = \{ F' \subset N: 1 \notin F', 2 \notin F'$ and $F' \supset S_{n+2}$ for some n in N $\}$. By theorem 9, $f(\mathcal{F})$ is a filter base on N. $f(\mathcal{F})$ is not a filter since for any F' in $f(\mathcal{F})$, F' is contained in S_1 but $S_1 \notin f(\mathcal{F})$.

DEFINITION 10. Let (X,t) be a topological space and let $\mathcal{F}$ be a filter on X.

(a). The limit set of $\mathcal{F}$ is $\lim \mathcal{F} = \{ x: N_x \in \mathcal{F}$ for each neighborhood N_x of x $\}$. The element x is said to be a limit point of $\mathcal{F}$ or $\mathcal{F}$ is said to converge to x. This is denoted by $\mathcal{F} \longrightarrow x$.

(b). The adherent set of $\mathcal{F}$ is adh $\mathcal{F} = \{ x: N_x \cap F \neq \emptyset$ for each F in $\mathcal{F}$ and for every neighborhood N_x of x $\}$. The element x is said to be an adherent point of $\mathcal{F}$.

EXAMPLE 11. Let $(R, |\ |)$ be a metric space with the usual topology. Define $\mathcal{F} = \{ F \subset R: 1 \in F \}$. Then $\mathcal{F}$ is a filter on R and $\lim \mathcal{F} = \{1\}$, adh $\mathcal{F} = \{ 1 \}$.

THEOREM 11. Let (X,t) be a topological space and let $\mathcal{F}$ be a filter on X. Then adh $\mathcal{F} = \bigcap \{\overline{F}: F \in \mathcal{F}\}$.

PROOF. If $x \in \bigcap \{\overline{F}: F \in \mathcal{F}\}$, then $x \in \overline{F}$ for every F in $\mathcal{F}$. Since x is a closure point of F. $N_x \cap F \neq \emptyset$ for every neighborhood N_x of x and for every F in $\mathcal{F}$. Hence $x \in$ adh $\mathcal{F}$.

If $x \notin \bigcap \{\overline{F}: F \in \mathcal{F}\}$, then there exists a F in $\mathcal{F}$ such that $x \notin \overline{F}$. Hence there exist at least one neighborhood N_x of x such that $N_x \cap F = \emptyset$. This implies $x \notin$ adh $\mathcal{F}$.

THEOREM 12. If $\mathcal{F}$ is a filter on a topological space (X,t) and $x \in \lim \mathcal{F}$, then $x \in$ adh $\mathcal{F}$. This means that $\lim \mathcal{F} \subset$ adh $\mathcal{F}$.

PROOF. If x is a limit of $\mathcal{F}$, then every neighborhood N_x of x is contained in $\mathcal{F}$. Since $\mathcal{F}$ is a filter, $N_x \cap F \neq \emptyset$ for every F in $\mathcal{F}$. Hence x is a closure point of F for every F in $\mathcal{F}$. By theorem 11, x is an adherent of $\mathcal{F}$.

THEOREM 13. If $\mathcal{U}$ is an ultrafilter on a topological space (X,t) and if y is an adherent point of $\mathcal{U}$, then y is a limit point of $\mathcal{U}$. This means that adh $\mathcal{U} = \lim \mathcal{U}$, for $\mathcal{U}$ an ultrafilter on X.

PROOF. If $y \in$ adh $\mathcal{U}$, then for every neighborhood N_y of y, $N_y \cap F \neq \emptyset$ for any F in $\mathcal{U}$. By theorem 8, $N_y \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter. Hence y is a limit point of $\mathcal{U}$.

THEOREM 14. If (X, t) is T_2 - space, then a filter $\mathcal{F}$ has at most one limit point.

PROOF. Suppose x and y belong to $\lim \mathcal{F}$ and $x \neq y$. By the definition of a limit point, $N_x \in \mathcal{F}$, $N_y \in \mathcal{F}$ for any neighborhoods

N_x of x and N_y of y. Since $\mathcal{F}$ is a filter, $N_x \cap N_y \neq \emptyset$. Hence there are no disjoint neighborhoods for x and y. But this is impossible since X is T_2.

THEOREM 15. Let (X,d) and (Y, ρ) be metrice spaces and let f be a function from (X, d) to (Y, ρ), Then f is continuous iff $x_n \longrightarrow x$ implies $f(x_n) \longrightarrow f(x)$.

PROOF. If f is continuous, then f is continuous at each point x of X. Let $\{x_n\}_1^\infty$ be a sequence in X such that $x_n \longrightarrow x$. Then for each open sphere $S_\epsilon(f(x))$, there exists an open sphere $S_\delta(x)$ such that $f(S_\delta(x)) \subset S_\epsilon(f(x))$. Since $x_n \longrightarrow x$, there exists a natural number N such that $x_n \in S_\delta(x)$ for each $n > N$. Hence $f(x_n) \in S_\epsilon(f(x))$ for each $n > N$ since $f(S_\delta(x)) \subset S_\epsilon(f(x))$. This means that $f(x_n) \longrightarrow f(x)$.

Suppose f is not continuous at some points x of X. Then there exists an open sphere $S_\epsilon(f(x))$ such that $f(S_\delta(x)) \not\subset S_\epsilon(f(x))$ for each $\delta > 0$. Thus there exists $x_n \in S_{\frac{1}{n}}(x)$ but $f(x_n) \notin S_\epsilon(f(x))$ for each natural number n. Hence there exists a sequence $\{x_n\}$ such that $x_n \longrightarrow x$ but $f(x_n) \not\longrightarrow f(x)$.

THEOREM 16. Let (X,t) and (Y, s) be topological spaces and let f be a function from (X,t) to (Y,s). If f is continuous, then $x_n \longrightarrow x$ implies $f(x_n) \longrightarrow f(x)$.

PROOF. Let $\{x_n\}_1^\infty$ be a sequence in X such that $x_n \longrightarrow x$. If f is continuous, then for each neighborhood N_y of f(x), there exists a neighborhood N_x of x such that $f(N_x) \subset N_y$. Since $x_n \longrightarrow x$, there exists a natural number N such that $x_n \in N_x$ for each $n > N$. Hence $f(x_n) \in N_y$ for each $n > N$ since $f(N_x) \subset N_y$. This means that $f(x_n) \longrightarrow f(x)$.

The converse of this theorem does not hold as the following counterexample shows.

EXAMPLE 12. Let R denote the set of real numbers with the cocountable topology t for R. That is, $t = \{ O \subset R:\ R - O$ is countable$\} \cup R$. Let d be the discrete topology for R. Let $f(x) = x$ be the identity mapping from (R,t) to (R,d). Let $\{a_n\}$ be a sequence in (R,t) such that $a_n \longrightarrow a$. Then

(i) $a_n \longrightarrow a$ iff there exists a natural number n_o such that $a_n = a$ for every $n \geq n_o$. To show this statement, it suffices to show that if there exists NO such n_o, then let $F = \{ a_n:\ a_n \neq a \}$ and hence $R - F$ is a neighborhood of a and $a_n \notin R - F$. Hence $a_n \not\longrightarrow a$.

(ii). For every $n \geq n_o$, $f(a_n) = a_n = a$. Hence $f(a_n) \longrightarrow a$ in (R,d) if $a_n \longrightarrow a$ in (R, t).

(iii). Let $a \in R$. Then $\{ a \}$ is open in (R,d) but $\{ a \} = f^{-1}(\{a\})$ is not open in (R,t). Therefore, f is not continuous.

THEOREM 17. Let (X, t) and (Y, s) be topological spaces. Let $\mathcal{F}$ be any filter on X. Then f is continuous iff $\mathcal{F} \longrightarrow x$ implies that $\mathcal{F}^*$, the filter generated by $f(\mathcal{F})$ converges to $f(x)$.

PROOF. Suppose f is continuous and let $x \in X$. Let $N_{f(x)}$ be a neighborhood of $f(x)$. Then $f^{-1}(N_{f(x)})$ is a neighborhood of x. If $\mathcal{F} \longrightarrow x$, then $f^{-1}(N_{f(x)}) \in \mathcal{F}$. Hence $f(f^{-1}(N_{f(x)})) \in \mathcal{F}^*$ and since $N_{f(x)} \supset f(f^{-1}(N_{f(x)}))$ it follows that $N_{f(x)} \in \mathcal{F}^*$. Hence

$\mathcal{F}^* \longrightarrow f(x)$.

If f is not continuous at x, then there exists a neighborhood $N_{f(x)}$ of $f(x)$ such that each neighborhood N_x of x is not contained in $f^{-1}(N_{f(x)})$. Hence $N_{f(x)}$ does not belong to the filter $\mathcal{F}^*$ generated by the filter base $\{ f(N_x)$: N_x is a neighborhood of $x \}$. Therefore $\mathcal{F}^* \not\longrightarrow f(x)$.

The following theorem is well - known and is state here without proof.

THEOREM 18. A topological space is compact iff every collection of closed sets with the finite intersection property has a non-empty intersection.

THEOREM 19. Let (X, t) be a topological space. X is compact iff every filter on X has a non-empty adherence.

PROOF. Let $\mathcal{F}$ be a filter on X. By the definition of a filter, the collection of closed set $\{ \overline{F}: F \in \mathcal{F} \}$ has the finite intersection property. By theorem 11, adh $\mathcal{F} = \bigcap \{ \overline{F}: F \in \mathcal{F} \} \neq \emptyset$. Therefore (X, t) is compact iff every filter on X has a non-empty adherence by applying theorem 18.

THEOREM 20. A topological space (X, t) is compact iff every ultrafilter converges.

PROOF. If (X, t) is compact, then every ultrafilter $\mathcal{U}$ has a non-empty adherence by theorem 19. By theorem 13, adh $\mathcal{U} \subset \lim \mathcal{U}$. Hence $\lim \mathcal{U} \neq \emptyset$. Therefore $\mathcal{U}$ converges.

To prove the converse, let $\mathcal{F}$ be a filter on X. By theorem 4, there exists an ultrafilter $\mathcal{U}$ on X with $\mathcal{F} \subset \mathcal{U}$. By hypothesis, $\emptyset \neq \lim \mathcal{U} = \operatorname{adh} \mathcal{U}$. Since $\mathcal{F} \subset \mathcal{U}$, it follows that adh $\mathcal{F} \supset$ adh $\mathcal{U} \neq \emptyset$. By theorem 19. X is compact.

CHAPTER IV

CLUSTERS

It is easy to see that a collection $\mathcal{U}$ of subsets of a non-empty set X is an ultrafilter iff the following conditions are satisfied:

(i). If A and B belong to $\mathcal{U}$, then $A \cap B \neq \emptyset$.

(ii). If $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}$, then $A \in \mathcal{U}$.

(iii). If $(A \cup B) \in \mathcal{U}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

The definition of a cluster in a proximity space can be motivated from these three conditions by replacing non-empty intersection with nearness. Clusters are extremely useful in the study of proximity spaces.

DEFINITION 1. Let (X, δ) be a proximity space. A cluster is a collection of subsets of X such that

(i). If A and B belong to σ, then $A \delta B$.

(ii). If $A \delta C$ for every C in σ, then A is in σ.

(iii). If $(A \cup B) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$.

EXAMPLE 1. Let (X, δ) be a proximity space. Let a be a point of X. Then the collection $\sigma_a = \{ A \subset X: A \delta a \}$ is a cluster. (i). If A and B are in σ, then $A \delta a$ and $B \delta a$. It follows $A \delta B$. (ii). If $A \delta C$ for every C in σ, then $A \delta a$ since $\{ a \}$ is in σ. Hence $A \in \sigma$. (iii). If $(A \cup B) \in \sigma$, then $(A \cup B) \delta a$.

By the definition of proximity, $A \delta a$ or $B \delta a$. Hence $A \in \sigma$ or $B \in \sigma$. σ_a is called a point cluster.

EXAMPLE 2. Define a proximity δ on X by $A \delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$. Then the collection $\sigma = \{ A \subset X: A \neq \emptyset \}$ is a cluster. (i). If A_1, A_2 belong to σ, then $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. By the definition of δ, $A_1 \delta A_2$. (ii). If $A \delta C$ for every C in σ, then $A \neq \emptyset$ and hence $A \in \sigma$. (iii). If $(A \cup B) \in \sigma$, then $A \cup B \neq \emptyset$. It follows $A \neq \emptyset$ or $B \neq \emptyset$. Thus $A \in \sigma$ or $B \in \sigma$.

The cluster in example 1 is a filter on X. However, the cluster in example 2 is not a filter if X contains more than one point.

LEMMA 2. Let (X, δ) be a proximity space and let σ be a cluster in (X, δ). Then

(a). For any subset E of X, either $E \in \sigma$ or $(X - E) \in \sigma$.

(b). If $A \in \sigma$ and $A \subset B$, then $B \in \sigma$.

(c). $A \in \sigma$ iff $\bar{A} \in \sigma$.

(d). If $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x$ a point cluster.

PROOF (a). Since $A \delta X$ for any subset A of X, $X \in \sigma$. Therefore, $E \in \sigma$ or $(X - E) \in \sigma$ since $E \cup (X - E) = X \in \sigma$.

(b). If $A \in \sigma$ and $A \subset B$, then $A \delta C$ for every C in σ and $B \delta C$ for every C in σ by lemma 2.2 in chapter II. Hence $B \in \sigma$.

(c). If $A \in \sigma$, then $\bar{A} \in \sigma$ since $A \subset \bar{A}$ by part (b).

If $A \notin \sigma$, then $A \not\delta C$ for some C in σ. By theorem 2.7 in chapter II, $\bar{A} \not\delta \bar{C}$. Therefore $\bar{A} \not\delta C$ and hence $\bar{A} \notin \sigma$.

(d). If $A \in \sigma$, then $A \delta x$ since $\{x\} \in \sigma$. Hence $A \in \sigma_x$. If $A \notin \sigma$, then $A \not\delta C$ for some C in σ. Since $\{x\}$ and C are in σ, $x \delta C$. Suppose $A \delta x$. Then $A \delta C$ a contradiction. Therefore, $A \not\delta x$ and hence $A \notin \sigma_x$. Since $\sigma \subset \sigma_x$ and $\sigma_x \subset \sigma$, $\sigma = \sigma_x$.

LEMMA 3. If σ_1, σ_2 are two clusters in (X, δ) and $\sigma_1 \subset \sigma_2$, then $\sigma_1 = \sigma_2$.

PROOF. Let $A \notin \sigma_1$. Then $A \not\delta C$ for some C in σ_1. Since $\sigma_1 \subset \sigma_2$, $A \not\delta C$ for some C in σ_2 which shows that $A \notin \sigma_2$. Hence $\sigma_2 \subset \sigma_1$.

LEMMA 4. Let P be a collection of subsets of X such that $\emptyset \notin P$ and $(A \cup B) \in P$ iff $A \in P$ or $B \in P$. If $P \in P$, then there exists an ultrafilter F such that $P \in F$ and $F \subset P$.

PROOF. By Zorn's lemma, there exists a maximal collection F of subsets of X such that F contains P and if $\{A_i: i = 1,2,\ldots n\} \subset F$, then $\bigcap \{A_i: i = 1,2,\ldots,n\} \in P$.

One must show that F is a filter.

(i). $\emptyset \notin F$ since $\emptyset \notin P$.

(ii). If A and B belong to F, then $A \cap B \in P$. Since F is maximal, $A \cap B \in F$.

(iii). If $A \in F$ and $A \subset C$, then $C \in P$ and hence $C \in F$ since F is maximal.

Now suppose F is not an ultrafilter. Then there exists a subset E of X such that neither E nor X - E belongs to $\mathcal{F}$. Hence there exist A_1, A_2 in $\mathcal{F}$ such that $A_1 \cap E \notin \mathcal{P}$ and $A_2 \cap (X - E) \notin \mathcal{P}$. Let $A = A_1 \cap A_2$. Then $A \in \mathcal{P}$ and $A \cap E \notin \mathcal{P}$ and $A \cap (X - E) \notin \mathcal{P}$. This is impossible since $A = (A \cap E) \cup (A \cap (X - E))$. Therefore $\mathcal{F}$ is an ultrafilter.

THEOREM 5. Let σ be a collection of subsets of a proximity space (X, δ). Then σ is a cluster iff there exists an ultrafilter $\mathcal{F}$ on X such that $\sigma = \{A \subset X: A \delta B$ for every B in $\mathcal{F}\}$.

PROOF. Let $\mathcal{F}$ be an ultrafilter and $\sigma = \{A \subset X: A \delta B$ for every B in $\mathcal{F}\}$. Then (i). If A_1 and A_2 belong to σ, then $A_1 \delta B$ and $A_2 \delta B$ for every B in $\mathcal{F}$. Since for any subset C in X, either C or X - C is in $\mathcal{F}$. Hence A_1 and A_2 are near to C or X - C for every subset C. Suppose $A_1 \not\delta A_2$. Then there exists a subset E of X such that $A_1 \not\delta E$ and $(X - E) \not\delta A_2$, a contradiction. Therefore $A_1 \delta A_2$. (ii). If $A \delta C$ for every C in σ, then $A \delta B$ for every B in $\mathcal{F}$ since $\mathcal{F} \subset \sigma$. Hence $A \in \sigma$. (iii). Suppose $A \notin \sigma$ and $C \notin \sigma$. Then there exist B_1 and B_2 in $\mathcal{F}$ such that $A \not\delta B_1$ and $C \not\delta B_2$. By lemma 2.2 of chapter II, $A \not\delta (B_1 \cap B_2)$ and $C \not\delta (B_1 \cap B_2)$. Thus $(A \cup C) \not\delta (B_1 \cap B_2)$. Since $B_1 \cap B_2 \in \mathcal{F}$, it follows that $(A \cup C) \notin \sigma$. Hence if $(A \cup C) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$. Therefore σ is a cluster.

Conversely let σ be a cluster and let P be an element of σ. Let $\mathcal{P} = \sigma$, then by lemma 4 there exists an ultrafilter $\mathcal{F} \subset \sigma$ such that $P \in \mathcal{F}$. If $\sigma' = \{A \subset X: A \delta B$ for every $B \in \mathcal{F}\}$, then $\sigma \subset \sigma'$. Applying lemma 3, $\sigma = \sigma'$.

A cluster σ is said to be determined by an ultrafilter $\mathcal{F}$ iff $\sigma = \{ A \subset X: A \,\delta\, B$ for each B in $\mathcal{F} \}$.

LEMMA 6. Let σ be a cluster in (X, δ) determined by an ultrafilter $\mathcal{F}$. Then, σ is a point cluster σ_x iff $\mathcal{F}$ converges to x.

PROOF. If $\sigma = \sigma_x$, then $\{ x \} \in \sigma$. Since σ is determined by $\mathcal{F}$, $x \,\delta\, A$ for every A in $\mathcal{F}$. Hence $x \in \bar{A}$ for every A in $\mathcal{F}$. Therefore $x \in \bigcap \{ \bar{A}: A \in \mathcal{F} \}$ = adh $\mathcal{F}$ = lim $\mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. This means that $\mathcal{F}$ converges to x.

If $\mathcal{F}$ converges to x, then $x \in$ lim $\mathcal{F}$ = adh $\mathcal{F}$ = $\bigcap \{ \bar{A}: A \in \mathcal{F} \}$. Hence $x \,\delta\, A$ for every A in $\mathcal{F}$. Thus $\{ x \} \in \sigma$ since σ is determined by $\mathcal{F}$. By lemma 2, $\sigma = \sigma_x$.

THEOREM 7. A proximity space (X, δ) is compact iff every cluster in (X, δ) is a point cluster.

PROOF. By theorem 20 of chapter III, (X, δ) is compact iff every ultrafilter $\mathcal{F}$ converges. By lemma 6, $\mathcal{F}$ converges iff the cluster it determines is a point cluster.

THEOREM 8. Let (X, δ) be a proximity space. If $A \,\delta\, B$, then there exists a cluster σ containing A and B.

PROOF. Set $\mathcal{P} = \{ C \subset X: C \,\delta\, B \}$. Then $\mathcal{P} \neq \emptyset$ since $B \in \mathcal{P}$. $\emptyset \notin \mathcal{P}$ since $\emptyset \,\bar{\delta}\, B$. Since $A \,\delta\, B$, $A \in \mathcal{P}$. If $(E \cup F) \in \mathcal{P}$, then $(E \cup F) \,\delta\, B$, and hence $E \,\delta\, B$ or $F \,\delta\, B$. It follows that $E \in \mathcal{P}$ or $F \in \mathcal{P}$. By lemma 4, there exists an ultrafilter $\mathcal{F}$ such that $A \in \mathcal{F} \subset \mathcal{P}$. Hence the cluster $\sigma = \{ S \subset X: S \,\delta\, F$ for every F in $\mathcal{F} \}$ is determined by $\mathcal{F}$. $A \in \sigma$ since $A \in \mathcal{F}$. $B \in \sigma$ since $\mathcal{F} \subset \mathcal{P}$.

THEOREM 9. Let $\mathcal{F}$ be an ultrafilter in Y and $X \subset Y$. Then $\mathcal{F}_X = \{ F \cap X: F \in \mathcal{F} \}$, the trace of $\mathcal{F}$ on X, is an ultrafilter in X iff $X \in \mathcal{F}$.

PROOF. If $\mathcal{F}_X$ is an ultrafilter in X, then $F \cap X \neq \emptyset$ for every F in $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $X \in \mathcal{F}$ by theorem 8 of chapter III.

If $X \in \mathcal{F}$, then $X \cap X = X \in \mathcal{F}_X$. Hence $\mathcal{F}_X \neq \emptyset$ and $\emptyset \notin \mathcal{F}_X$. If $F_1 \cap X \in \mathcal{F}_X$ and $F_2 \cap X \in \mathcal{F}_X$, then $(F_1 \cap X) \cap (F_2 \cap X) = (F_1 \cap F_2) \cap X \in \mathcal{F}_X$ since $F_1 \cap F_2 \in \mathcal{F}$. If $F_1 \cap X \in \mathcal{F}_X$ and $F_1 \cap X \subset F' \subset X$, then $F' \in \mathcal{F}$ since F_1 and X are in $\mathcal{F}$. Hence $F' = F' \cap X \in \mathcal{F}_X$. Therefore $\mathcal{F}_X$ is a filter in X.

Since $(F_1 \cap X) \cup (F_2 \cap X) = (F_1 \cup F_2) \cap X \in \mathcal{F}_X$ iff $F_1 \cup F_2 \in \mathcal{F}$. It follows $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Hence $F_1 \cap X \in \mathcal{F}_X$ or $F_2 \cap X \in \mathcal{F}_X$. By theorem 5 of chapter III, $\mathcal{F}_X$ is an ultrafilter in X.

THEOREM 10. Let (Y, δ) be a proximity space and σ a cluster in Y. Let $X \subset Y$ and $X \in \sigma$. Then the cluster $\sigma' = \{ A \subset X: A \in \sigma \}$ is the only cluster in (X, δ_X) contained in σ.

PROOF. Since σ is a cluster in a proximity space (Y, δ) and $X \in \sigma$, σ is determined by an ultrafilter $\mathcal{F}$ containing X, as in theorem 5. Then $\mathcal{F}_X = \{ F \cap X: F \in \mathcal{F} \}$, the trace of $\mathcal{F}$ on X is an ultrafilter in X by theorem 9. Hence $\mathcal{F}_X$ generates a cluster σ' in X. If $A \in \sigma'$, then $A \, \delta \, (F \cap X)$ for every F in $\mathcal{F}$. It follows that $A \, \delta \, F$ for every F in $\mathcal{F}$. Hence $A \in \sigma$. Thus $\sigma' \subset \sigma$ and hence $\sigma' = \{ A \subset X: A \in \sigma \}$. Suppose there is another cluster σ_1' in (X, δ_X) contained in σ, then $\sigma_1' \subset \sigma'$. By lemma 3, $\sigma_1' = \sigma'$.

THEOREM 11. Let f be a proximity mapping from (X, δ_1) to (Y, δ_2). Then for each cluster σ_1 in X, there corresponds a cluster σ_2 in Y such that $\sigma_2 = \{A \subset Y: A\ \delta_2 f(B)$ for every B in $\sigma_1\}$.

PROOF. Let σ_1 be a cluster in X. Then σ_1 is determined by an ultrafilter $\mathcal{F}$ in X. $f(\mathcal{F})$ is a filter base by theorem 9 of chapter III. $f(\mathcal{F})$ generates a filter and hence there exists an ultrafilter $\mathcal{F}^*$ containing $f(\mathcal{F})$ and $\mathcal{F}^*$ generates a cluster σ_2 in Y. If $A\ \delta_2 f(B)$ for every B in σ_1, then $A\ \delta_2 f(F)$ for every F in $\mathcal{F}$ since $\mathcal{F} \subset \sigma_1$. Hence $A \in \sigma_2$. Since $f(B) \in \sigma_2$ for each B in σ_1, $f(\sigma_1) \subset \sigma_2$. $B \in \sigma_1$ implies $B\ \delta_1 F$ for every F in $\mathcal{F}$. $f(B)\ \delta_2 f(F)$ since f is a proximity mapping. Hence $f(B) \in \sigma_2$. Therefore if $A \in \sigma_2$, then $A\ \delta_2 f(B)$ for every B in σ_1.

EXAMPLE 3. Let N denote the set of natural numbers. Define a proximity δ by $A\ \delta\ B$ iff $A \cap B \neq \emptyset$. Let $a \in N$. Then

(a). The collection $\mathcal{F} = \{F \subset N: a \in F\}$ is an ultrafilter in N.

(b). The collection $\sigma_a = \{A \subset N: A\ \delta\ a\}$ is a cluster.

(c). The cluster $\sigma = \{B \subset N: B\ \delta\ F$ for every F in $\mathcal{F}\}$ is determined by $\mathcal{F}$. $\sigma \subset \sigma_a$ since $B \in \sigma$ implies $B\ \delta\ F$ for every F in $\mathcal{F}$ and hence $B\ \delta\ a$. By lemma 3, $\sigma = \sigma_a$.

(d). (N, δ) is separated since if $x \delta y$, then $\{x\} \cap \{y\} \neq \emptyset$ and hence $x = y$. If $\sigma_a = \sigma_b$, then $a \delta B$. This implies $a = b$. Therefore each cluster in (N, δ) can contain at most one singleton set.

(e). Since $\bar{A} = \{x: x \delta A\} = \{x: \{x\} \cap A \neq \emptyset\} = \{x: x \in A\} = A$, the topology induced by δ is the discrete topology. Hence (N, δ) is not compact. By theorem 7, there exists a cluster which is not a point cluster.

EXAMPLE 4. Let N denote the set of natural numbers. Define the proximity δ by $A \delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$. Then

(a). The collection $\sigma = \{A \subset N: A \neq \emptyset\}$ is a cluster. is the only one cluster in (N, δ) since if there exists another cluster σ' then $\sigma' \subset \sigma$ and hence $\sigma' = \sigma$.

(b). Every point cluster σ_a is equal to the cluster σ.

(c). (N, δ) is not separated since $\{2\} \delta \{3\}$ but $2 \neq 3$.

(d). Every cluster in (N, δ), there is only one, is a point cluster. Thus by theorem 7, (N, δ) is compact.

CHAPTER V

SMIRNOV COMPACTIFICATION

This chapter is an attempt to investigate a way to construct the Smirnov compactification of a separated proximity space. All the proximity spaces considered in this chapter are separated.

The following notation is used throughout this chapter. Let (X, δ) be a separated proximity space.

$\mathcal{X}$: the set of all clusters in X.

$\mathcal{A}$: the set of clusters in X which contain a subset A of X.

f: A mapping from X to $\mathcal{X}$ defined by $f(x) = \sigma_x$ the point cluster determined by the point x.

δ^*: A proximity on $\mathcal{X}$, as defined in lemma 2 of this chapter.

Using clusters, the following results will be proved later.

(i) f(X) is dense in $\mathcal{X}$.

(ii) X is proximally isomorphic to f(X).

(iii) $(\mathcal{X}, \delta^*)$ is compact. Therefore (X, δ) is embeded in a compact proximity space $(\mathcal{X}, \delta^*)$. $\mathcal{X}$ is a compactification of X called the Smirnov compactification of X.

DEFINITION 1. Let $\mathcal{P}$ be a subset of $\mathcal{X}$. Then a subset A of X absorbs $\mathcal{P}$ iff for every σ in $\mathcal{P}$, σ contains A. That is $\mathcal{P} \subset \mathcal{A}$.

LEMMA 2. Let δ^* be the binary relation on $\mathcal{X}$ defined by

$\mathcal{P}\ \delta^* \mathcal{Q}$ iff A absorbs $\mathcal{P}$ and B absorbs $\mathcal{Q}$ implies $A\ \delta\ B$. Then δ^* is a separated proximity on X and hence (X, δ^*) is a separated proximity space.

PROOF (i). The symmetry of δ^* follows directly from the definition of δ^* and the symmetry of δ .

(ii). $(\mathcal{P} \cup \mathcal{Q})\ \delta^* \mathcal{R}$ iff $\mathcal{P}\ \delta^*\ \mathcal{R}$ or $\mathcal{Q}\ \delta^* \mathcal{R}$. Suppose $\mathcal{Q}\ \delta^* \mathcal{R}$, D absorbs $(\mathcal{P} \cup \mathcal{Q})$, C absorbs $\mathcal{R}$. Then $\mathcal{P} \cup \mathcal{Q} \subset \mathcal{D}$, where $\mathcal{D} = \{\sigma \in X: D \in \sigma\}$. This implies $\mathcal{Q} \subset \mathcal{D}$ and hence D absorbs $\mathcal{Q}$. Since $\mathcal{Q}\ \delta^* \mathcal{R}$, $D\ \delta\ C$. Hence $(\mathcal{P} \cup \mathcal{Q})\ \delta^* \mathcal{R}$. Conversely, suppose $(\mathcal{P} \cup \mathcal{Q})\ \delta^* \mathcal{R}$ and $\mathcal{P}\ \not\delta^* \mathcal{R}$. Let B absorbs $\mathcal{Q}$ and C absorbs $\mathcal{R}$. Then there exist sets A, D absorbing $\mathcal{P}$ and $\mathcal{R}$ respectively such that $A \not\delta D$ and hence there exists a subset E of X such that $A \not\delta E$ and $(X - E) \not\delta D$. Since D absorbs $\mathcal{R}$ and $(X - E) \not\delta D$, X - E belongs to no cluster in $\mathcal{R}$, for if not, $(X - E)\ \delta\ D$ by the definition of a cluster. Since $C - E \subset X - E$, C - E belongs to no cluster in $\mathcal{R}$, for if not, X - E will belong to one cluster in $\mathcal{R}$. But $(C - E) \cup (C \cap E) = C$ absorbs $\mathcal{R}$ and C - E belongs to no cluster in $\mathcal{R}$. Hence $C \cap E$ absorbs $\mathcal{R}$. Now $A \cup B$ absorbs $\mathcal{P} \cup \mathcal{Q}$ and $C \cap E$ absorbs $\mathcal{R}$, which shows that $(A \cup B)\ \delta\ (C \cap E)$. Since $A \not\delta E$ and $C \cap E$ is contained in E, $A \not\delta (C \cap E)$ by lemma 2.2 in chapter II. It follows that $B\ \delta\ (C \cap E)$ and hence $B\ \delta\ C$ since $C \cap E$ is contained in C. Therefore, if $(\mathcal{P} \cup \mathcal{Q})\ \delta^* \mathcal{R}$ and $\mathcal{P}\ \not\delta^* \mathcal{R}$, then $\mathcal{Q}\ \delta^* \mathcal{R}$.

(iii). If $\mathcal{P}\ \delta^* \mathcal{Q}$, then $A\ \delta\ B$. It follows that $A \neq \emptyset$ and $B \neq \emptyset$ and hence $\mathcal{P} \neq \emptyset$ and $\mathcal{Q} \neq \emptyset$.

(iv). If $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, then A absorbs $\mathcal{P} \cap \mathcal{Q}$ and B absorbs $\mathcal{P} \cap \mathcal{Q}$. Thus $A\ \delta\ B$ and $\mathcal{P}\ \delta^* \mathcal{Q}$.

(v). If $\mathcal{P} \not\delta^* \mathcal{Q}$, then $A \not\delta B$ which implies that there exists a subset E such that $A \not\delta E$ and $(X - E) \not\delta B$. Since B absorbs $\mathcal{Q}$ adn $X - E \not\delta B$, $X - E$ belongs to no cluster in $\mathcal{Q}$. It follows that E absorbs $\mathcal{Q}$. Let $\mathcal{R} = \mathcal{E} = \{ \sigma \in x: E \in \sigma \}$. Then $\mathcal{P} \not\delta^* \mathcal{R}$ since A absorbs $\mathcal{P}$, E absorbs $\mathcal{E}$ and $A \not\delta E$. Since E belongs to no cluster in $X - \mathcal{R}$, $X - E$ absorbs $X - \mathcal{R}$. Therefore, $(X - E) \not\delta B$ implies $(X - \mathcal{R}) \not\delta^* \mathcal{Q}$.

(vi). Since every set in σ_1 absorbs $\{ \sigma_1 \}$ and every set in σ_2 absorbs $\{ \sigma_2 \}$, hence $\sigma_1 \, \delta^* \, \sigma_2$ implies that every set in σ_1 is near to every set in σ_2 and hence $\sigma_1 = \sigma_2$. Therefore δ^* is separated.

LEMMA 3. Let (X, δ) be a proximity space and $f: (X, \delta) \dashrightarrow (\mathcal{X}, \delta^*)$ defined by $f(x) = \sigma_x$. (σ_x is the point cluster containing x; thus $\sigma_x = \{ A \subset X: x \, \delta \, A \}$.) Then

(i). f is a one - to - one mapping, and

(ii). $f(A) \subset \mathcal{A}$ for each $A \subset X$.

PROOF (i). Since δ is separated, $\sigma_x = \sigma_y$ implies that $x = y$. Therefore $f(x) = f(y)$ implies that $x = y$ and f is one-to-one.

(ii). Let $\sigma_a \in f(A)$. Since $\{ a \} \subset A$, $A \in \sigma_a$. Hence $\sigma_a \in \mathcal{A}$, and hence $f(A) \subset \mathcal{A}$.

LEMMA 4. A absorbs $f(B)$ iff $B \subset \bar{A}$, where $\bar{A}$ is the $t(\delta)$ closure of A.

PROOF. Suppose A absorbs $f(B)$. Let $b \in B$. Then $\sigma_b \in \mathcal{A}$ and $A \in \sigma_b$. Therefore $b \, \delta \, A$ and hence $b \in \bar{A}$. Conversely, if $B \subset \bar{A}$, then for every b in B, $b \in \bar{A}$ and hence $b \, \delta \, A$. It follows that $A \in \sigma_b$ which shows that $A \in \sigma_b$ for any σ_b in $f(B)$. Therefore A absorbs $f(B)$.

LEMMA 5. Let $\mathcal{Q}$ be any subset of $\mathcal{X}$. Then $\mathcal{Q}\ \delta^* f(A)$ iff C absorbs $\mathcal{Q}$ implies $C\ \delta\ A$.

PROOF. Let $\mathcal{Q}\ \delta^* f(A)$ and C absorbs $\mathcal{Q}$. Since A belongs to every cluster in $f(A)$, A absorbs $f(A)$. By the definition of δ^*, $C\ \delta\ A$. If C absorbs $\mathcal{Q}$ implies $C\ \delta\ A$ and D absorbs $f(A)$, then $A \subset \bar{D}$ by lemma 4. Thus $C\ \delta\ \bar{D}$ and $\bar{C}\ \delta\ \bar{D}$. Therefore, $C\ \delta\ D$ by theorem 2.7 in chapter II.

LEMMA 6. $f(X)$ is dense in $\mathcal{X}$ with respect to the topology $t(\delta^*)$.

PROOF. In lemma 5, let $\mathcal{Q}$ be the singleton $\{\sigma\}$. Then $\{\sigma\}^* f(A)$ iff $C \in \sigma$ implies $C\ \delta\ A$ iff $A \in \sigma$. Hence $\mathcal{A}$ is the $t(\delta^*)$ closure of $f(A)$. Since X belongs to each cluster in $\mathcal{X}$, $\mathcal{X}$ is the $t(\delta^*)$ closure of $f(X)$. Therefore $f(X)$ is dense in $\mathcal{X}$.

LEMMA 7. (X, δ) is proximally isomorphic to $f(X)$ with the subspace proximity $\delta^*_{f(X)}$.

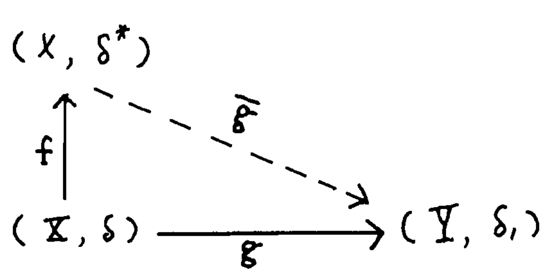
PROOF. Let C absorbs $f(A)$ and D absorbs $f(B)$. If $f(A)\ \delta^* f(B)$, then $C\ \delta\ D$. Since A absorbs $f(A)$ and B absorbs $f(B)$, it follows that $A\ \delta\ B$. Conversely, suppose that $A\ \delta\ B$. Let C absorbs $f(A)$ and D absorbs $f(B)$. Then $A \subset \bar{C}$ and $B \subset \bar{D}$ by lemma 4 and $\bar{C}\ \delta\ \bar{D}$ by lemma 2.2 in chapter II. Therefore $C\ \delta\ D$ by theorem 2.7 in chapter II. By the definition of δ^*, $f(A)\ \delta^* f(B)$. Therefore, (X, δ) is proximally isomorphic to $(f(X), \delta^*_{f(X)})$.

LEMMA 8. (X, δ^*) is compact.

PROOF. By theorem 7 in chapter IV, (X, δ^*) is compact iff every cluster in X is a point cluster. Hence it suffices to show that any cluster in X is a point cluster. Let σ be an arbitrary cluster in X. Since $\overline{f(X)} = X \in \sigma$, $\overline{f(X)}\ \delta\ B$ for every B in σ and hence $\overline{f(X)}\ \delta\ \overline{B}$. But this implies $f(X)\ \delta\ B$ for every B in σ. It follows that $f(X) \in \sigma$. Applying theorem 10 in chapter IV, there exists a unique cluster in $(f(X), \delta^*_{f(X)})$ contained in σ, namely $\sigma' = \{A \subset f(X) :\ A \in \sigma\}$. By lemma 7, (X, δ) is proximally isomorphic to $(f(X), \delta^*_{f(X)})$. Hence there exists a cluster σ'' in X such that $\sigma' = \{f(A):\ A \in \sigma''\}$. From the proof of lemma 6, $\{\sigma''\}\ \delta^* f(A)$ iff $A \in \sigma''$. Hence $\{\sigma''\}\ \delta^* C$ for every $C \in \sigma'$. It follows $\{\sigma''\} \in \sigma' \subset \sigma$. Therefore there exists $\sigma'' \in X$ and $\{\sigma''\} \in \sigma$ which shows that σ is a point cluster.

LEMMA 9. If g is a δ-homeomorphism of (X, δ) onto a dense subset of a compact proximity space (Y, δ_1), then g can be extended to a δ-homeomorphism $\overline{g}$ of (X, δ^*) onto (Y, δ_1).

PROOF. Consider the following diagram.



From the hypothesis and theorem 11 in chapter IV, for every cluster σ in X, there corresponds a cluster σ' in Y such that $\sigma' = \{A' \subset Y: A' \ \delta_1 g(A)$ for every A in $\sigma\}$. Since Y is compact, σ' is a point cluster. By theorem 10 of chapter IV, every point in Y determines a unique cluster in X. Hence the clusters in X are in a one-to-one correspondence with the points of Y. Thus an extension $\bar{g}$ of g exists and $\bar{g}$ is a bijection from $\mathcal{X}$ to Y.

In order to show that $\bar{g}$ is a δ- homeomorphism one must show that $P \ \delta^* Q$ iff $\bar{g}(P) \ \delta_1 \ \bar{g}(Q)$. Let P and Q be subsets of $\mathcal{X}$ and $P \ \delta^* Q$. Since $(\mathcal{X}, \delta^*)$ is a compact Hansdorff space, $(\mathcal{X}, \delta^*)$ is normal. Hence if $P \ \delta^* Q$, then $\bar{P} \cap \bar{Q} \neq \emptyset$. It follows that there exists a $\sigma \in \mathcal{X}$ such that $\sigma \in \bar{P}$ and $\sigma \in \bar{Q}$. Hence $\sigma \ \delta^* P$ and $\sigma \ \delta^* Q$. Let $y = \bar{g}(\sigma)$. By axiom A_5 and the definition of a cluster it follows that $\{y\} \ \delta_1 \ \bar{g}(P)$ and $\{y\} \ \delta_1 \ \bar{g}(Q)$. Hence $\bar{g}(P) \ \delta_1 \ \bar{g}(Q)$.

Conversely, if $\bar{g}(P) \ \delta_1 \ \bar{g}(Q)$, then there exists a $y \in \overline{\bar{g}(P)} \cap \overline{\bar{g}(Q)}$ since Y ic compact. Let $\sigma = g^{-1}(y)$. Since X is proximally homeomorphism to a dense subset of Y, X can be considered as a subspace of Y. Therefore, if $A \in \sigma$ and B absorbs P, then $A \ \delta \ \bar{g}(P)$ and $\bar{g}(P) \subset \bar{B}$. It follows that $A \ \delta \ B$ and hence $\{\sigma\} \ \delta^* P$. Similarly, if $A \in \sigma$ and C absorbs Q, then $A \ \delta \ C$ and hence $\{\sigma\} \ \delta^* Q$. Therefore $P \ \delta^* Q$.

The main theorem of this chapter follows as a result of the above lemmas.

THEOREM 10. Every separated proximity space (X, δ) is a dense subspace of a unique (up to a δ - homeomorphism) compact Hausdorff space $\mathcal{X}$. Since $\mathcal{X}$ has a unique compatible separated proximity, subsets A and B of X are near iff their closures in $\mathcal{X}$ have a non-empty intersection. $\mathcal{X}$ is called the Smirnov compactification of X.

THEOREM 11. Let g be a proximity mapping of (X, δ_1) onto (Y, δ_2). Then g can be extended uniquely to a proximity mapping $\bar{g}$ which maps the compactification of X onto the compactification of Y.

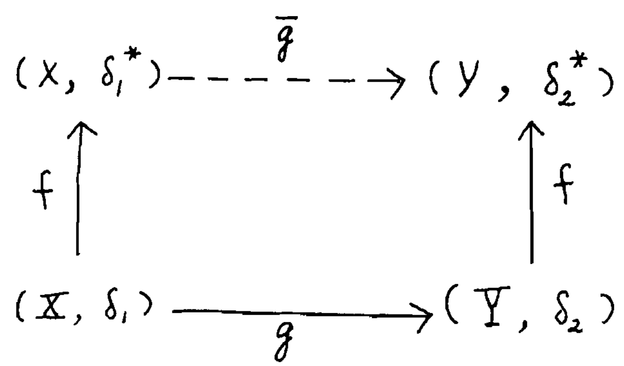
PROOF. In chapter IV, theorem 11 shows that if σ_1 is a cluster in X, then there corresponds a cluster σ_2 in Y such that $\sigma_2 = \{ P \subset Y: P\ \delta_2 g(c)$ for every c in $\sigma_1 \}$. Define $\bar{g}(\sigma_1) = \sigma_2$. Then g is a mapping from $\mathcal{X}$ to $\mathcal{Y}$ and $\bar{g}$ maps the point cluster σ_x to the point cluster $\sigma_{g(x)}$. Hence $\bar{g}$ is an extension of g.

The following proof is to show that $\bar{g}$ is a proximity mapping. Let $\mathcal{P}\ \delta_1^* \mathcal{Q}$. Suppose A absorbs $\bar{g}(\mathcal{P})$ and B absorbs $\bar{g}(\mathcal{Q})$. If $A\ \delta_2 B$, then there exist subsets C and D of Y such that $A\ \not\delta_2 (Y - C)$, $(Y - D)\ \not\delta_2 B$ and $C\ \not\delta_2 D$ by theorem 3.4 in chapter II. Since A absorbs $\bar{g}(\mathcal{P})$, (Y - C) belongs to no cluster in $\bar{g}(\mathcal{P})$. It follows that $g^{-1} (Y - C) = X - g^{-1}(c)$ belongs to no cluster in $\mathcal{P}$. Since $g^{-1}(c) \cup (X - g^{-1}(c)) = X$ belongs to every cluster in $\mathcal{P}$, $g^{-1}(c)$ belongs to every cluster in $\mathcal{P}$. This means that $g^{-1}(c)$ absorbs $\mathcal{P}$. Similarly $g^{-1}(D)$ absorbs $\mathcal{Q}$. Hence $g^{-1}(c)\ \delta_1\ g^{-1}(D)$ since $\mathcal{P}\ \delta_1^* \mathcal{Q}$. Since g is a proximity mapping, it follows that $C\ \delta_2 D$, but this is a contradiciton. Therefore $A \, \delta_2 B$.

Since $f(Y)$ is dense in $\mathcal{Y}$ and $f(Y) \subset \bar{g}(\mathcal{X}) \subset \mathcal{Y}$ and $\bar{g}(\mathcal{X})$ is compact in $\mathcal{Y}$, it follows that $\bar{g}(\mathcal{X}) = \mathcal{Y}$. Thus $\bar{g}$ maps $\mathcal{X}$ onto $\mathcal{Y}$.

The uniqueness of $\bar{g}$ is proved as follows. Suppose there exists another extension $\bar{g}'$ of g mapping $\mathcal{X}$ onto $\mathcal{Y}$ and $\bar{g}' \neq \bar{g}$. Then there is a $\sigma \in \mathcal{X}$ such that $\bar{g}(\sigma) \neq \bar{g}'(\sigma)$. Since $\mathcal{Y}$ is Hansdorff, $\bar{g}$ and $\bar{g}'$ are continuous, there exists a neighborhood E of σ such that $\bar{g}(E) \cap \bar{g}'(E) = \emptyset$. Now $f(X)$ is dense in $\mathcal{X}$ by lemma 6. Hense $E \cap f(X) \neq \emptyset$. Let $\sigma_x \in E \cap f(X)$. then $\bar{g}(\sigma_x) \neq \bar{g}'(\sigma_x)$. Hence $\bar{g}$ and $\bar{g}'$ are different on X and that contradicts the fact that $\bar{g}$ and $\bar{g}'$ are extensions of g. Therefore $\bar{g} = \bar{g}'$.

The following diagram shows the relations among the four proximity spaces.



Since every compact Hansdorff space is normal, $\mathcal{X}$ is a compact Hansdorff space and thus $\mathcal{X}$ is a normal space. The following theorem is an analogue of Urysohn's lemma for normal spaces.

THEOREM 12. Let (X, δ) be a separated proximity space. If $A \not\delta B$, then there exists a proximity mapping g: $X \dashrightarrow [0,1]$ such that $g(A) = 0$ and $g(B) = 1$.

PROOF. If $A \not\delta B$, then $\bar{A} \cap \bar{B} = \emptyset$ in X. Since X is a compact Hansdorff space, X is normal. By Urysohn's lemma, there exists a continuous mapping $\bar{g}$: $X \dashrightarrow [0,1]$ such that $\bar{g}(\bar{A}) = 0$ and $\bar{g}(\bar{B}) = 1$. Since X is compact and $\bar{g}$ is continuous, $\bar{g}$ is a proximity mapping. Let g be the restriction of $\bar{g}$, then g is the required mapping.

DEFINITION 13. A proximal (or δ -) extension of a proximity space (X, δ) is a separated proximity space (Y, δ') such that $\bar{X} = Y$ and $\delta = \delta'_X$. A proximity space is maximal (or absolutely closed) iff it has no proper δ - extension.

THEOREM 14. A separated proximity space (X, δ) is maximal iff every cluster in X is a point cluster.

PROOF. If X is not maximal, then there exists a proper δ - extension Y and hence $Y - X \neq \emptyset$. Let $a \in Y - X$. Then there exists a unique cluster σ in X such that $\sigma = \{A \subset X: A \in \sigma_a\}$ where σ_a is a point cluster determined by a point a of Y - X. Since Y is separated, σ is not a point cluster in X.

If there exists a cluster in X which is not a point cluster, then the proximal extension of X given in lemma 6 is proper and hence X is not maximal.

A proximity space (X, δ) is compact iff every cluster in the space is a point cluster. Hence the following corollary is obvious.

COROLLARY. A separated proximity space is maximal iff it is compact.

DEFINITION 15. A separated proximity space (X, δ) is equinormal iff $\bar{A} \cap \bar{B} = \emptyset$ implies $A \not\delta B$.

LEMMA 16. Every equinormal proximity space is normal.

PROOF. Let (X, δ) be a equinormal proximity space. Then for any disjoint closed subsets $\bar{A}$ and $\bar{B}$ in X implies that $A \not\delta B$ and hence $\bar{A} \not\delta \bar{B}$. By theorem 12, there exists a proximity mapping $g: X \to [0,1]$ such that $g(\bar{A}) = 0$ and $g(\bar{B}) = 1$. Let $O_1 = [0,\frac{1}{2})$, $O_2 = (\frac{1}{2},1]$. Then O_1, O_2 are open in $[0,1]$. Since g is continuous with respect to $t(\delta)$ by theorem 4.6 in chapter II. $g^{-1}(O_1)$ and $g^{-1}(O_2)$ are open in X and $g^{-1}(O_1) \cap g^{-1}(O_2) = \emptyset$, $\bar{A} \subset g^{-1}(O_1)$ and $\bar{B} \subset g^{-1}(O_2)$ shows that X is normal.

The converse of lemma 16 is not true. For example, let X be the real line with the usual topology. δ is defined by $A \delta B$ iff $D(A,B) = \inf \{ |a - b| : a \in A, b \in B \} = 0$. Set $A = \{n : n \in N\}$ and $B = \{ n - \frac{1}{n} : n \in N \}$. Then $\bar{A} \cap \bar{B} = \emptyset$ but $A \delta B$.

THEOREM 17. A normal separated proximity space (X, δ) is equinormal iff every real-valued continuous function on X is a proximity mapping.

PROOF. Let R denote the set of real numbers and let δ_1 be any proximity on R compatible with the usual topology. If (X, δ) is equinormal, then $\bar{A} \cap \bar{B} = \emptyset$ implies $A \not\delta B$. Let f be a real-valued continuous function defined on X. $A \delta B$ implies $\bar{A} \cap \bar{B} \neq 0$. Hence $f(\bar{A}) \cap f(\bar{B}) \neq \emptyset$ which shows that $f(\bar{A}) \ \delta_1 f(\bar{B})$. Since f is continuous, $f(\bar{A}) \subset \overline{f(A)}$ and $f(\bar{B}) \subset \overline{f(B)}$. It follows that $\overline{f(A)} \ \delta_1 \ \overline{f(B)}$ and hence $f(A) \ \delta_1 f(B)$. Therefore, f is a proximity mapping.

Conversely, suppose any real-valued continuous function f defined on the normal proximity space X is a proximity mapping. If $\bar{A} \cap \bar{B} = \emptyset$, then by Urysohn's lemma, there exists a continuous function $f\colon X \longrightarrow [0,1]$ such that $f(\bar{A}) = 0$ and $f(\bar{B}) = 1$. Hence $f(\bar{A}) \not\delta_1 f(\bar{B})$. Since f is a proximity mapping, $\bar{A} \not\delta \bar{B}$ and hence $A \not\delta B$. Therefore $\bar{A} \cap \bar{B} = \emptyset$ implies $A \not\delta B$ which shows that (X, δ) is equinormal.

CHAPTER VI

SUMMARY AND SUGGESTIONS FOR FURTHER STUDY

This paper covered the fundamentals of proximity spaces, basic definitions and basic theorems. There are many additional topics that could be considered. For example: (i). The development of the concept of proximity structures in a uniform space. (ii). In theorem 2.15 in chapter II, it was shown that every completely regular space (X,t) has a compatible proximity. It can be shown that if (X, δ) is a proximity space, then t(δ) is completely regular. Hence a compatible proximity can be introduced on a topological space if and only if it is a completely regular topological space. One can study generalized proximity structures that can be introduced in any topological space.

The books and paper written by Naimpally [4], Thron [8] and Tukey [9] are useful in the further study of proximity spaces.

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