PERIODIC DECIMAL FRACTIONS

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by

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CHAPTER I

INTRODUCTION

1.1. Introduction. Decimal fractions have intrigued mathematicians for many centuries. Famous mathematicians such as Fermat, Leibniz, Euler, Bernoulli, Gauss, and others who became very well-known in various branches of mathematics, shared a common interest in the "mystery" of the relationship between certain fractions and the corresponding lengths of the repeating cycle of digits in their decimal representations. More precisely, they were concerned with the inherent properties of a given fraction which seemingly forced restrictions, yet predictability, upon its decimal representation. From the middle of the seventeenth century to the present, mathematicians have struggled with ideas, theories, and conjectures concerning the notably characteristic patterns exhibited by periodic decimal fractions.

1.2. Statement of the problem. Number theory, one of the oldest branches of mathematics, thrives on the ideas of recurrence and predictability. Periodic decimal fractions attest to this fact. Thus, as in most areas of number theory, a need or desire is established to determine basic laws or properties which would enable one to reliably and accurately predict the length of the repeating cycle of digits corresponding to any given fraction. Consequently, throughout six months of extended research and preparation, the author has found that the decimal representations of rational numbers do indeed adhere to various laws or rules although a few "exceptions to the rule" do exist in some cases.

1.3. Definition and explanation of terms. The author prefers to use the name "periodic decimal fraction" to indicate a decimal that contains a repeating cycle of digits. Other names are commonly used to signify periodic decimal fractions. Repeating decimal, recurring decimal, and circulating decimal are the most frequent terms which are used synonymously with periodic decimal fraction.

The reader should be aware of the fact that only rational numbers are being considered throughout this paper. Furthermore, the theorems in this paper concern themselves only with rational numbers between 0 and 1 although the theorems and rules that are discussed readily apply to all rational numbers with very minor adjustments. Basically there are three distinct types:

1. Terminating decimal fractions (also called finite decimal fractions or exact decimal fractions). Terminating decimal fractions such as 1/4 = .25 and 1/10 = .1 consist of a finite number of digits to the right of the decimal point.

2. Purely periodic decimal fractions. A purely periodic decimal fraction begins its period or repeating cycle of digits with the digit directly to the right of the decimal point. The periodic or repeating sequence of digits (also referred to as the repetend) is denoted by a bar over the digits such as $1/3 = .\overline{3}$ and $1/7 = .\overline{142857}$. It is then understood that

this repeating cycle of digits will recur infinitely in the same sequential order.

3. Delayed periodic decimal fractions. This type of periodic decimal fraction behaves much as purely periodic decimal fractions except for the fact that the repetend does not begin directly to the right of the decimal point. The period is thus "delayed" by one or more digits and necessarily begins with some digit after the digit directly to the right of the decimal point. Examples of delayed periodic decimal fractions are $1/6 = .1\overline{6}$ and $1/28 = .03\overline{571428}$.

Two clarifications should be mentioned at this point. The first being the fact that every terminating decimal fraction *may* be written as a periodic decimal fraction by affixing a repeating sequence of zeros at the end of the decimal form. That is, a terminating decimal fraction such as $1/4 = .25$ may be written in the form $.25\overline{0}$. To prevent any confusion or ambiguities, this procedure will not be considered in which case a terminating decimal fraction consists of a finite sequence of digits.

Secondly, in similar fashion, many mathematicians allow an alternate representation for a terminating decimal fraction by subtracting 1 from the last digit of the decimal representation and affixing a repeating sequence of nines at the end. Such an instance would be .3 written as $.2\overline{9}$. Once again, the author prefers to consider only the terminating form of the decimal fraction.

Finally, the author finds it convenient to use the notation $a_1a_2...a_n$ to indicate an n-digit numeral where the subscripts give the order of the digits. In this notation, a succession of subscripted variables does not imply multiplication. Also, the reader should pay close attention to the position of the decimal point when this notation is used.

1.4. Brief history. Computations with decimal fractions first appeared in the early 1500's. But not until the late seventeenth and early eighteenth centuries were mathematicians beginning to focus their attention on the systematic characteristics of these numbers. It was then that mathematicians began investigating the properties of periodic decimal fractions. G.W. Leibniz, John Wallis, and J.H. Lambert were among the first to produce material relevant to periodic decimal fractions. At the close of the eighteenth century many other mathematicians joined in the investigation of the properties relating to rational numbers and their decimal expansions ". . . but not until Gauss were those indespensable tools of number theory created that were required for a systematic exploration (of decimal expansions)."[1] In 1801 Gauss proved an important theorem relating to the determination of the period for periodic decimal fractions.[2] The particular theorem is given in this paper as Theorem 4.1.

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[1]Oystein Ore, Number Theory and its History, (New York: McGraw-Hill Publishing Co., 1948), p. 315.

[2]L.E. Dickson, "Divisibility and Primality," History of the Theory of Numbers, I (New York: Chelsea Publishing Co., 1952), 159-179.

Widespread interest in the properties of periodic decimal fractions increased throughout the nineteenth century. During the middle of the nineteenth century mathematicians did an enormous amount of work relating to periodic decimal fractions. At this point in history most of the basic laws and properties applying to periodic decimal fractions had been established. Countless expositions were published and investigations with respect to all the properties of periodic decimal fractions were seemingly exhausted. Mathematicians became very specialized and technical in their methods and research regarding the characteristics of periodic decimal fractions. The world became well-acquainted with the fascinating properties of rational numbers and their decimal equivalents.[3]

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[3]Ibid.

CHAPTER II

TERMINATING VERSUS REPEATING

2.1. Terminating decimal fractions. Many questions arise upon the conversion of rational numbers of the form 1/q to their decimal representations. Possibly the most obvious and basic of these is: What definable property of the rational number 1/q determines whether the decimal expansion terminates or repeats?

Conversion of a rational number of the form 1/q to its decimal equivalent requires merely the division of 1.000... by q. Thus it may be noted that if q divides some integral power of 10 then the division process would be completed and a terminating decimal would be the resulting quotient. More precisely:

Theorem 2.1. 1/q has a terminating (finite) decimal fractional form if and only if $q = 2^m 5^n$ where m and n are non-negative integers.

Proof: First consider 1/q with a terminating decimal fractional form. There exists a positive integer t such that $1/q = .a_1 a_2 \ldots a_t$ where a_i is the i^{th} digit obtained in the quotient and a_t is the last non-zero digit obtained. Multiplying each side of this equation by 10^t gives $10^t/q = a_1 a_2 \ldots a_t.$, where $a_1 a_2 \ldots a_t.$ is an integer. Thus q divides 10^t. But 10^t has only the prime factors 2 and 5 which implies that $q = 2^m 5^n$

where m and n are non-negative integers.

Now assume $q = 2^m 5^n$ where m and n are non-negative integers. Suppose $m \geq n$. $1/q = 1/2^m 5^n$ and by multiplying both numerator and denominator of the right member by 5^{m-n} the result $1/q = 5^{m-n}/2^m 5^m = 5^{m-n}/10^m$ is obtained. Since $m - n$ is a non-negative integer then 5^{m-n} is a positive integer, say d, and hence $1/q = d/10^m$. Now since division by 10^m determines only the positioning of the decimal point (with respect to d) then $d/10^m$ is a finite decimal fraction. Now suppose $m < n$. As before, multiplying the numerator and denominator of the right member of the above equation by 2^{n-m}, in this case, gives $1/q = 2^{n-m}/2^n 5^n = 2^{n-m}/10^n$. Since $n - m$ is a positive integer then 2^{n-m} is a positive integer, say c. Hence $1/q = c/10^n$ and by the same reason as above, $c/10^n$ is a finite decimal fraction. Therefore 1/q has a terminating decimal fractional form.

Furthermore, the integers m and n in Theorem 2.1 are significant in that they determine exactly how many digits the quotient will contain.

Corollary to Theorem 2.1. Let 1/q be given where $q = 2^m 5^n$ and m and n are non-negative integers. The number, t, of digits to the right of the decimal point is equal to $\max\{m,n\}$.

Proof: Theorem 2.1 has established the fact that $1/q = 1/2^m 5^n = .a_1 a_2 ... a_t$ where $a_t \neq 0$. Also, $10^t/q = 10^t/2^m 5^n = 2^t 5^t/2^m 5^n = 2^{t-m} 5^{t-n} = a_1 a_2 ... a_t.$ where $a_1 a_2 ... a_t.$ is a positive integer. Since the product of 2^{t-m} and 5^{t-n} is an integer and

2 and 5 are relatively prime then 2^{t-m} and 5^{t-n} are positive integers. Hence $t - m \geq 0$, $t - n \geq 0$ which implies $t \geq m$, $t \geq n$. Suppose $t > m$ and $t > n$. Since t, m, and n are all non-negative integers then there exist positive integers h and k such that $m + h = t$ and $n + k = t$. Thus $h = t - m$ and $k = t - n$ where $h,k \geq 1$. By substitution, $2^h 5^k = 2\cdot 5(2^{h-1}5^{k-1}) = 10(2^{h-1}5^{k-1}) = a_1a_2...a_t.$. This implies that $a_1a_2...a_t.$ is divisible by 10 but this is impossible since $a_t \neq 0$. Hence $t = m$ or $t = n$. Thus t is greater than or equal to both m and n and from the discussion above, $t = m$ or $t = n$. If $t = m$ then $t = m \geq n$. Likewise, if $t = n$ then $t = n \geq m$. By this relationship one may conclude that t is equal to $\max\{m,n\}$.

2.2 Repeating decimal fractions. The conditions have now been established that must be met in order for the decimal representation of 1/q to terminate. If these conditions are not satisfied then the decimal representation of 1/q must be non-terminating, that is, a sequence of digits extending indefinitely to the right of the decimal point. More specifically, this unending sequence of digits has a distinct, predictable pattern as seen in the following theorem:

Theorem 2.2. If 1/q does not have a terminating decimal fractional form then it has a repeating (periodic) decimal fractional form.

Proof: Assume 1/q does not have a terminating decimal fractional form, that is, let $1/q = .a_1a_2a_3....$ By using the division algorithm for 1 divided by q, denote the successive remainders by $r_1, r_2, \ldots$ where the n^{th} remainder, r_n, is determined directly after the n^{th} digit of the quotient is obtained and $r_n < q$ for all n. One should note that the n^{th} remainder may always be obtained by $r_n = 10^n - q(a_1a_2\ldots a_n.)$ where $a_1a_2\ldots a_n.$ is the positive integer formed by the first n successive digits of the quotient. Also, define $r_0 = 1$. Furthermore, if $r_n = 0$ then $q(a_1a_2\ldots a_n.) = 10^n$ which implies that $.a_1a_2\ldots a_n$ is a terminating decimal fractional form for 1/q and this is a contradiction. Now since $r_n < q$ and r_n is a positive integer then there exactly q - 1 possible remainders. Thus after at most q - 1 steps of the division a remainder must be obtained, say r_h, that is either the same as one previously obtained, say r_k, or equal to 1 (in which case k = 0). Note that the next digit in the quotient, namely a_{h+1}, will necessarily be the exact digit that was acquired in the division after the remainder r_k was obtained. That is, $a_{h+1} = a_{k+1}$. In the same manner, the successive digits of the quotient $a_{h+2}, a_{h+3}, \ldots$ will correspond precisely to those directly after a_{k+1} until a remainder r_j is obtained that once again is equal to r_k. Therefore 1/q has a repeating (periodic) decimal fractional form.

Furthermore, it may be shown that if x is any integer such that $1 < x < q$ and $(x,q) = 1$ then 1/q having a periodic decimal fractional form is a sufficient condition for x/q to have a periodic decimal fractional form. Since $(x,q) = 1$ then the proof would be very similar to that of Theorem 2.2 with appropriate replacements. Hence, in statement form:

Corollary to Theorem 2.2. Let x be any integer such that $1 < x < q$ and $(x,q) = 1$. If $1/q$ has a periodic decimal fractional form then x/q has a periodic decimal fractional form.

By use of Theorem 2.2 together with the contrapositive of Theorem 2.1 a logical basis has been completed for the following important theorem:

Theorem 2.3. If q contains one or more prime factors different from 2 or 5 then $1/q$ has a periodic decimal fractional form.

The question "When does $1/q$ have a periodic decimal fractional form?" is now easily answered by the preceeding theorem. Thus sufficient conditions have been established to tell whether the decimal representation of $1/q$ is terminating or repeating.

CHAPTER III

GENERAL CHARACTERISTICS OF THE REPEATING CYCLE OF PERIODIC DECIMAL FRACTIONS

3.1. Purely periodic decimal fractions. Prime numbers play an extremely important role relative to the general characteristics of the repetend in periodic decimal fractions. As seen in the following two theorems and corollary, positioning of the repetend in the decimal representation of $1/q$ relies solely upon the specific prime factors of q.

Theorem 3.1. If q is a positive integer greater than 1 and contains no factors of 2 or 5 then the period for $1/q$ begins at the decimal point.

Proof: Since q contains no factors of 2 or 5 then the decimal representation of $1/q$ is a periodic decimal fraction by Theorem 2.3. Let $a_1, a_2, \ldots, a_n, \ldots$ be the successive digits of the quotient obtained in the division of 1 by q. Let the successive remainders in the division be denoted by $r_1, r_2, \ldots, r_n, \ldots$ where the n^{th} remainder is determined directly after the n^{th} digit of the quotient is obtained and $r_n < q$ for all n. The following relationships are established:

$$
\begin{aligned}
10 &= qa_1 + r_1 \\
10r_1 &= qa_2 + r_2 \\
&\vdots \\
10r_{k-1} &= qa_k + r_k \\
&\vdots \\
10r_{k+h-1} &= qa_{k+h} + r_{k+h}
\end{aligned}
$$

Note that if 1 occurs as a remainder, say after h steps of the division algorithm, before any remainder recurs then $a_{h+1} = a_1$ and the successive digits of the quotient following a_{h+1} will be exactly the same as those succeeding a_1. Hence the repeating cycle must begin with a_1. Suppose there exists a remainder that recurs before 1 occurs as a remainder. Denote the first remainder to recur in this manner by r_k where r_k recurs h steps later as r_{k+h} so the period must begin with the digit a_{k+1}.

By subtraction,

$$\begin{array}{rl} 10r_{k-1} &= qa_k + r_k \\ - \quad 10r_{k+h-1} &= qa_{k+h} + r_{k+h} \\ \hline \end{array}$$

$$10r_{k-1} - 10r_{k+h-1} = qa_k - qa_{k+h} \quad (\text{since } r_k = r_{k+h})$$

Applying the distributive law, $10(r_{k-1} - r_{k+h-1}) = q(a_k - a_{k+h})$. Since q divides the right member of the above equation then it must also divide the left member. But since q contains no factors of 2 or 5 then q and 10 are relatively prime and thus q must divide $(r_{k-1} - r_{k+h-1})$. But since both r_{k-1} and r_{k+h-1} are less than q then their difference is less than q. Hence $r_{k-1} - r_{k+h-1} = 0$ which implies $r_{k-1} = r_{k+h-1}$. This is contradictory to the fact that r_k is the first remainder to recur and therefore 1 occurs as a remainder before any remainder recurs. Thus the period for 1/q begins at the decimal point with a_1.

Corollary to Theorem 3.1. Let q be a positive integer greater than 1 that contains no factors of 2 or 5. If x is any positive integer such that $1 < x < q$ and $(x,q) = 1$ then the period for x/q begins at the decimal point and is the same length as the period for 1/q.

The proof of Corollary to Theorem 3.1 is very similar to that of Theorem 3.1 hence the author has not included a formal proof. Since $(x,q) = 1$ then one will also find that the periods for $1/q$ and x/q are of equal length although the specific digits in the periods are not necessarily the same. That is to say, the period for x/q depends solely on q.

3.2. Delayed periodic decimal fractions. As mentioned previously in Chapter I, some periodic decimal fractions have a delayed period. The following theorem gives sufficient conditions for this to occur.

Theorem 3.2. Let $1/q$ be given where $q = 2^m 5^n Q$ and Q is a positive integer greater than 1 that contains no factors of 2 or 5. Then the period of $1/q$ is delayed by $\max\{m,n\}$ digits. (i.e. the repeating cycle of digits begins $\max\{m+1,n+1\}$ places to the right of the decimal point.)

Proof: Consider $1/q$ where $q = 2^m 5^n Q$ and Q is a positive integer greater than 1 that contains no factors of 2 or 5. By Theorem 2.3, $1/q$ has a periodic decimal fractional form. Suppose $m \geq n$. Multiplying both numerator and denominator of $1/q$ by 5^{m-n} gives $5^{m-n}/q5^{m-n} = 5^{m-n}/2^m 5^n Q 5^{m-n} = 5^{m-n}/2^m 5^m Q = 5^{m-n}/10^m Q = (1/10^m)\cdot(5^{m-n}/Q)$. If $5^{m-n} < Q$ then $(5^{m-n},Q) = 1$ and by Corollary to Theorem 3.1 the period for $5^{m-n}/Q$ begins at the decimal point. Thus multiplication by $1/10^m$ simply moves the decimal point m places to the left and the period is delayed by m digits. Now if $5^{m-n} > Q$ then by Euclid's Theorem there exist positive integers a,b such that $5^{m-n} = aQ + b$ where

$b < Q$. Thus $5^{m-n}/Q = a + (b/Q)$ and in the same manner as above, $1/q = (1/10^m)\cdot(a + b/Q)$. But $a + (b/Q)$ is simply a mixed number with a fractional part consisting of b/Q. If $(b,Q) = 1$ then the period for b/Q begins at the decimal point and multiplication by $1/10^m$ moves the decimal point m places to the left. If b and Q contain like factors then the fraction b/Q may be reduced to b'/Q' where $(b',Q') = 1$ and the conclusion in the preceeding sentence also holds. Finally, if $n > m$ then multiplication by 2^{n-m} and a similar argument shows that the period is delayed, in this case, by n digits.

Thus, prime factorization of the denominator must inevitably be one's first consideration when attempting to predict the length of the repeating cycle of digits for any periodic decimal fraction.

3.3. Characteristics of the length of the period. One of the first observations that one may note when looking at a fraction of the form $1/q$ together with its periodic decimal representation is that the length of the period never exceeds q, the denominator of the fraction. As noted in the proof of Theorem 2.2, only $q - 1$ distinct remainders are possible in the division of 1 by q. This alone verifies the fact that indeed the length of the period cannot be greater than or equal to q. More specifically, consider $1/p$ where p is a prime other than 2 or 5. The length of the period has further restrictions.

Theorem 3.3. If p is a prime other than 2 or 5 then the length of the repeating cycle of digits in the decimal representation of $1/p$ is an exact divisor of $p - 1$.

Proof: As noted in the proof of Theorem 2.2, the n^{th} remainder in the division of 1.000... by p may be denoted by $r_n = 10^n - p(a_1a_2...a_n.)$. Thus each remainder, r_n, is the residue of 10^n. Also, when the remainder 1 appears then the period of the decimal fraction is completed. Let s denote the number of digits in the period of 1/p. Thus $r_s = 1$ and $1 = 10^s - p(a_1a_2...a_s.)$. This implies that $10^s \equiv 1 \pmod{p}$ where s is the smallest positive integral power of 10 for which the congruence is true. Suppose s does not divide p - 1. Thus $p - 1 = qs + r$ where q and r are positive integers, $r \neq 0$, $r < s$. Hence $10^{p-1} = 10^{qs+r} = 10^{qs} \cdot 10^r$ and thus the result $10^{p-1} \equiv 10^{qs} \cdot 10^r \pmod{p}$. Since $10^s \equiv 1 \pmod{p}$ then 10^{qs} is also congruent to 1 modulo p. Since p is prime then $(10,p) = 1$ and thus using Fermat's Theorem the important congruence $10^{p-1} \equiv 1 \pmod{p}$ is obtained. Thus by substitution into the above congruences, $1 \equiv 10^r \pmod{p}$. Hence $10^r \equiv 1 \pmod{p}$. But since $r > 0$ and $r < s$ then this is contradictory to the fact that s is the smallest positive integral power of 10 that is congruent to 1 modulo p. Thus $r = 0$ and $p - 1 = qs$. Therefore s divides p - 1.

In general, if P is a product of primes containing one or more factors different from 2 or 5, then the length of the repeating cycle of digits in the decimal representation of 1/P is an exact divisor of $\phi(P)$ where $\phi(P)$ is the Euler function defined as the number of positive integers less than P and relatively prime to P. A similar argument will verify this fact.

The special case when the period of 1/p does attain the maximum possible length, p - 1, is quite interesting in its own respects. If a maximum period occurs then p is a prime but the converse is not necessarily true. The decimal representation for 1/7 has a period of 7 - 1 = 6 digits while that of 1/11 is only 2 digits. The specific primes for which this is true have no particular order or sequence although, as the reader shall see in Theorem 3.5, necessary and sufficient conditions have been established for the period to consist of p - 1 digits. Table I at the conclusion of this paper gives the lengths of the periods for the reciprocals of the primes less than 500 and there the reader may observe the primes for which the period is of length p - 1.

Theorem 3.4. If the period of the decimal representation for 1/p contains p - 1 digits then p is a prime.

Proof: Let the period of 1/p contain p - 1 digits. By Theorem 2.2 there are exactly p - 1 distinct remainders possible in the division of 1 by p. But since the repeating cycle of 1/p contains p - 1 digits then one may conclude that every possible remainder was obtained. That is, every positive integer less than p was a remainder once and only once in the first p - 1 steps of the division. Suppose p is not prime and also $p \neq 1$. Thus p is a composite number and may be written as $p = b \cdot c$ where b and c are positive integers greater than 1 and $b, c < p$. Now p/b = c is a positive integer less than p, hence there exists some remainder, say r_k, in the first p - 1 steps of the division such that $r_k = p/b = c$. Hence by the relationship $r_{k+1} = 10r_k - pa_{k+1}$ established in Theorem 3.1, $r_{k+1} = 10c - pa_{k+1} = 10c - bca_{k+1} = c(10 - ba_{k+1})$. Thus c divides r_{k+1}. Assume c divides r_{k+i}. (i.e. $r_{k+i} = c \cdot d$ where d is a positive integer and $d < p$.) Now $r_{k+i+1} = 10r_{k+i} - pa_{k+i+1} = 10cd - pa_{k+i+1} = 10cd - bca_{k+i+1} = c(10d - ba_{k+i+1})$. Thus c divides r_{k+i+1}. Hence by the principle of finite induction, c divides every remainder after r_k and therefore $c = 1$. But this is a contradiction therefore p is a prime.

Theorem 3.5 establishes necessary and sufficient conditions for the period of 1/p to contain p - 1 digits. But before a formal statement and proof of Theorem 3.5 the following lemma is necessary.

Lemma to Theorem 3.5. Let p be any prime other than 2 or 5. s is the smallest positive integer for which $10^s \equiv 1 \pmod{p}$ if and only if s is the length of the period for 1/p.

Proof: First assume that s is the smallest positive integer such that $10^s \equiv 1 \pmod{p}$. Since $10^s \equiv 1 \pmod{p}$ then there exists a positive integer, a, such that $10^s - pa = 1$ and a is unique. In the division of 1 by p the fact that $r_s = 10^s - p(a_1a_2...a_s.)$ was established in Theorem 2.2. Since $r_s < p$ then $p(a_1a_2...a_s.)$ is necessarily the largest multiple of p less than 10^s. But pa must also be the largest multiple of p less than 10^s since $10^s - pa = 1$. Hence

$a = (a_1a_2...a_s.)$ and therefore $10^s - p(a_1a_2...a_s.) = 10^s - pa = 1 = r_s$. Since s was taken to be the smallest positive integer such that $10^s \equiv 1 \pmod p$ then r_s must be the first remainder to equal 1. Thus from the discussion in the proof of Theorem 3.1 the sequence of digits of the quotient beginning with a_{s+1} corresponds precisely to the sequence of digits beginning with a_1 thus a repeating cycle has been established in exactly s steps of the division process. Therefore since the period begins at the decimal point then s is the length of the period for 1/p.

Now assume that s is the length of the period for 1/p. From Theorem 3.1 the period for 1/p begins at the decimal point and also the remainder r_s is the first remainder to be equal to 1. Now $r_s = 10^s - p(a_1a_2...a_s.) = 1$ which implies that $10^s \equiv 1 \pmod p$. Suppose there exists a positive integer, say t, such that $0 < t < s$ and $10^t \equiv 1 \pmod p$. Hence there exists a positive integer b such that $10^t - pb = 1$ and b is unique. But in the division, $r_t = 10^t - p(a_1a_2...a_t.)$ where $(a_1a_2...a_t.)$ is necessarily the greatest integer for which $p(a_1a_2...a_t.)$ is less than 10^t since $r_t < p$. Hence $b = (a_1a_2...a_t.)$ and $r_t = 10^t - pb = 1$. But this is a contradiction since r_s is the first remainder to equal 1. Therefore s is the smallest positive integer such that $10^s \equiv 1 \pmod p$.

<u>Theorem 3.5</u>. Let p be any prime other than 2 or 5. The period for 1/p contains p - 1 digits if and only if 10 is a primitive root of p.

Proof: Assume the period for 1/p contains p - 1 digits. By Lemma to Theorem 3.5, p - 1 is the smallest positive integer such that $10^{p-1} \equiv 1 \pmod{p}$. But since p is prime then $\phi(p) = p - 1$ and thus $10^{\phi(p)} \equiv 1 \pmod{p}$ where $\phi(p)$ is the smallest integer for which the congruence is true. Thus by definition, 10 belongs to $\phi(p)$ modulo p. Therefore by definition of primitive root, 10 is a primitive root of p.

Now assume that 10 is a primitive root of p. By definition of primitive root, 10 belongs to $\phi(p)$ modulo p. That is, $\phi(p)$ is the smallest integral exponent such that the congruence $10^{\phi(p)} \equiv 1 \pmod{p}$ is true. But since p is prime then $\phi(p) = p - 1$. Hence $10^{p-1} \equiv 1 \pmod{p}$ and p - 1 is the smallest integer for which the congruence is true. Therefore by Lemma to Theorem 3.5, p - 1 is the length of the period for 1/p.

Finally, if the period of 1/p is of length p - 1 then another quite interesting fact may be noted. If x is any positive integer such that $1 < x < p$ then the period for x/p has exactly the same digits permuted cyclically. Proof is given below while a simple example would be the periods of fractions whose denominator is 7 which contain 6 digits. It is seen that $1/7 = .\overline{142857}$, $2/7 = .\overline{285714}$, $3/7 = \overline{428571}$, etc. do indeed have the same digits permuted cyclically.

The reader should be careful, however, to remember that the period must contain p - 1 digits as mentioned above. Various fractions such as $1/13 = .\overline{076923}$ and $3/13 = .\overline{230769}$ have the same digits permuted cyclically in their periods but this is not the case for <u>all</u> x such that $1 < x < p$. For instance, $5/13 = .\overline{384615}$.

<u>Theorem 3.6</u>. If p is a prime other than 2 or 5 whose period contains p - 1 digits then the period of x/p, where $1 < x < p$, has the same digits permuted cyclically.

<u>Proof</u>: Let p be a prime other than 2 or 5 and consider x/p where x is a positive integer such that $1 < x < p$ and the period of 1/p contains p - 1 digits. From Theorem 3.4, a necessary condition for the period of 1/p to contain p - 1 digits is that every positive integer less than p is a remainder once and only once in the first p - 1 steps of the division. Hence, x is equal to some remainder, say r_h, obtained in the first p - 1 steps of the division of 1 by p. One can easily see that the first digit in the decimal representation of x/p is exactly the same as the digit, a_{h+1}, of the quotient obtained in the division. In like manner, the successive digits in the decimal representation of x/p will correspond precisely to those immediately succeeding a_{h+1}. Hence after exactly p - 1 steps in the division of x by p a remainder equal to x will be obtained thus completing a repeating cycle the same length as that for 1/p with the same digits permuted cyclically.

CHAPTER IV

CALCULATION OF THE PERIOD LENGTH FOR PERIODIC DECIMAL FRACTIONS

4.1. Formulae for predicting the length of the period. Up to this point, very little mention has been made concerning the methods one would use to actually predict the length of the period for any rational number of the form 1/P where P is any positive integer except 1, 2, or 5. Lemma to Theorem 3.5, which was proven in Chapter III, gives the reader his first introduction to a reliable way to predict the period of 1/p where p is a prime other than 2 or 5. In general, if p is any prime number other than 2 or 5 then the theorem is also true for $1/p^n$.

Theorem 4.1. Let p be any prime other than 2 or 5. s is the smallest positive integer for which $10^s \equiv 1 \pmod{p^n}$ if and only if s is the length of the period for $1/p^n$.

Proof of Theorem 4.1 is very similar to that of the proof of Lemma to Theorem 3.5 therefore the author has not included a formal proof. Merely a substitution of p^n for p in the proof is sufficient.

The process of finding the smallest integral exponent, s, such that $10^s \equiv 1 \pmod{p^n}$, that is, the exponent to which 10 belongs modulo p^n, may not always be an easy task given any p^n. Since the period, s, is a divisor of $\phi(p^n)$ then all positive integers which are divisors of $p^{n-1}(p-1)$ may be considered as likely candidates for s. Thus the task of solving the congruence $10^s \equiv 1 \pmod{p^n}$ by a trial and error method may be cumbersome and quite time consuming.

Consequently the author gives the following theorem as an alternate method for finding the period of $1/p^n$ which, with very minor restrictions on the hypothesis, is much easier than the first method and as equally reliable.

Theorem 4.2. If the length of the period of the decimal representation of $1/p$ is s and $10^{p-1} \not\equiv 1 \pmod{p^2}$ then the period of $1/p^n$ is equal to sp^{n-1}.

Before giving a proof of the above theorem the author finds it necessary to mention two important facts.

First, the so called "restrictive" hypothesis seen in Theorem 4.2 is not as restrictive as one might suppose. To the author's knowledge, there exist only two prime numbers, 3 and 487, which satisfy the congruence $10^{p-1} \equiv 1 \pmod{p^2}$. Thus the fractions $1/3^n$ and $1/487^n$ are exceptions to the above theorem. Various articles make mention of material relative to this fact. In 1897, B. Reynolds discussed the two exceptions as follows:

> First, although $1/3 = .\overline{3}$ a period of one figure (in the decimal system), we see that $1/3^2 = .\overline{1}$ which is also a period of one figure. After this little irregularity, the general principle of the law is maintained, for $1/3^3$ has a 3-period, $1/3^4$ has a 9-period, and so on. This exception to the law is due to the fact that $3^2 = 10 - 1$, 10 being the radix. If n be any integer and $n^2 = r - 1$ (r the radix of notation) it is easily seen that $1/n = .\overline{n}$

and $1/n^2 = .\overline{1}$, so that $1/n$ and $1/n^2$ both have a period of 1 figure.[4]

An exception of quite a different type, discovered by Desmarest, is the number 487. While 1/487 repeats in 486 places, the repetend itself divides by 487, so that $1/487^2$ also repeats in 486 places. The present writer has verified these statements . . . .[5]

In 1878, J.W.L. Glaisher mentioned Desmarest's work in this area which was done around 1852 and verified Desmarest's statements concerning 3 and 487. He stated that:

> There seems no reason to suppose that there are not other solutions (to the congruence $10^{p-1} \equiv 1 \pmod{p^2}$), and that the congruences $10^{p-1} \equiv 1 \pmod{p^3}$, etc., may not have solutions. The next solution above 487 of the congruence $10^{p-1} \equiv 1 \pmod{p^2}$ may be a very high number, as is evident by merely considering the diminution of the chance of a number dividing exactly its own period . . . .[6]

Thus the only known exclusions to Theorem 4.2 are $1/3^n$ and $1/487^n$.

Secondly, the author uses an important theorem in the proofs of Theorems 4.2 and 4.3 which was given by T. Muir and stated as: "If s be the lowest solution of $N^s \equiv 1$, then any other solution is a multiple of s."[7]

The proof of Theorem 4.2 is now given.

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[4]B. Reynolds, "On the Frequency of Occurrence of the Digits in the Periods of Pure Circulates," Messenger of Mathematics, Vol. XXVII (1898), 184-185.

[5]Ibid., p. 185.

[6]J.W.L. Glaisher, "On Circulating Decimals," Cambridge Philosophical Society (Proceedings), Vol. III (October, 1878), 201.

[7]T. Muir, "Theorems on Congruences Bearing on the Question of the Number of Figures in the Periods of the Reciprocals of Integers," Messenger of Mathematics, Vol. IV (1875), 2.

Proof: Let w be the exponent to which 10 belongs modulo p^n. That is, $10^w \equiv 1 \pmod{p^n}$ where w is the smallest integral exponent for which the congruence is true. Since $10^s \equiv 1 \pmod{p}$ by Lemma to Theorem 3.5 then there exists a positive integer, a, such that $10^s = ap + 1$. Taking the p^{th} power of both sides of the above equation gives the result:

$$10^{sp} = (ap)^p + p(ap)^{p-1} + \frac{p(p-1)(ap)^{p-2}}{2!} + \ldots + pap + 1.$$

Note that every term of the right member of the above equation is integral and each contains a factor of p^2 except the last. Therefore, $10^{sp} \equiv 1 \pmod{p^2}$. In the same fashion, the congruence $10^{sp^2} \equiv 1 \pmod{p^3}$ may be obtained and so on until $10^{sp^{n-1}} \equiv 1 \pmod{p^n}$. But w is the smallest integral exponent such that $10^w \equiv 1 \pmod{p^n}$ thus sp^{n-1} must be a multiple of w by the theorem by T. Muir which was mentioned above. Also since $10^w \equiv 1 \pmod{p^n}$ then $10^w \equiv 1 \pmod{p}$ and w must be a multiple of s. Since p is prime then w must be of one of the following three forms: (1) $w = s$ (2) $w = sp^{n-1}$ (3) $w = sp^{n-t}$ where t is an integer such that $1 < t < n$. Suppose $w = sp^{n-t}$. Thus $10^{sp^{n-t}} \equiv 1 \pmod{p^n}$ and sp^{n-t} is the smallest integral exponent for which the congruence is true. Application of another theorem by T. Muir stated as, "If sp^{n-t} be a solution of the congruence $N^x \equiv 1 \pmod{p^n}$, p being any prime number, . . . then s is a solution of $N^x \equiv 1 \pmod{p^t}$."[8] gives the result, $10^s \equiv 1 \pmod{p^t}$. Thus since $10^s \equiv 1 \pmod{p^t}$ and $p - 1$ is a multiple of s then $10^{p-1} \equiv 1 \pmod{p^t}$. But $t > 1$

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[8]Ibid., p. 3.

hence a contradiction to the hypothesis.

Now suppose w = s. By substitution, $10^s \equiv 1 \pmod{p^n}$ which implies $10^{p-1} \equiv 1 \pmod{p^n}$ and again directly contradicts the hypothesis. Therefore $w = sp^{n-1}$ and the length of the period for $1/p^n$ is sp^{n-1}.

The periods for rational numbers of the form $1/3^n$ or $1/487^n$ do, however, have a predictable pattern. A pattern is actually established after what might be called a "delay" of one power of the denominator. In other words, the fractions and their squares have periods of equal length and then the pattern is the same as that observed in Theorem 4.2. That is to say, Theorem 4.2 may be modified, in these cases, to say that the period of $1/p^n$ is sp^{n-2} where $n \geq 2$. The reason for this so called "delay" is precisely the fact that $10^{p-1} \equiv 1 \pmod{p^2}$ which forces the periods of 1/p and $1/p^2$ to be of equal length as proven by the following theorem.

<u>Theorem 4.3</u>. Let p be any prime other than 2 or 5. 1/p and $1/p^2$ have periods of the same length if and only if $10^{p-1} \equiv 1 \pmod{p^2}$.

<u>Proof</u>: First assume that 1/p and $1/p^2$ have periods of the same length. Let s denote the length of the periods. From Lemma to Theorem 3.5, $10^s \equiv 1 \pmod{p}$. From Theorem 4.1, $10^s \equiv 1 \pmod{p^2}$. Theorem 3.3 established the fact that the length of the period for 1/p is an exact divisor of p - 1. Thus there exists a positive integer, n, such that sn = p - 1. Taking the n^{th} power of each side of the second congruence

gives $10^{sn} \equiv 1 \pmod{p^2}$. Hence by substitution, the result $10^{p-1} \equiv 1 \pmod{p^2}$ is obtained.

Now let s denote the length of the period for 1/p and assume $10^{p-1} \equiv 1 \pmod{p^2}$. Let w be the exponent to which 10 belongs modulo p^2. There exists a positive integer, a, such that $10^s = ap + 1$ since $10^s \equiv 1 \pmod{p}$. Taking the p^{th} power of both sides of the above equation gives:

$$10^{sp} = (ap)^p + p(ap)^{p-1} + \frac{p(p-1)(ap)^{p-2}}{2!} + \ldots + pap + 1.$$

Every term of the right member of the above equation is integral and each contains a factor of p^2 except the last and therefore $10^{sp} \equiv 1 \pmod{p^2}$. But w is the smallest exponent such that $10^w \equiv 1 \pmod{p^2}$ hence sp must be a multiple of w. Also since $10^w \equiv 1 \pmod{p}$ then w must be a multiple of s. But since p is prime then this is only possible if w = s or w = sp. Suppose w = sp. By hypothesis, $10^{p-1} \equiv 1 \pmod{p^2}$. By definition of w, $10^w \equiv 1 \pmod{p^2}$. Hence p - 1 is a multiple of w by T. Muir's theorem. Now sp = w is always greater than p - 1 since $s \geq 1$ and p is a prime, hence p - 1 cannot be a multiple of w. Thus a contradiction and w = s. Since w is the exponent to which 10 belongs modulo p^2 then by Theorem 4.1, w is the length of the period for $1/p^2$. Hence 1/p and $1/p^2$ have periods of the same length.

Thus the reader may conclude that the period of the decimal representation of all rational numbers of the form $1/p^n$ may be obtained by merely applying Theorem 4.2 or its modification discussed above.

More generally, the author now considers 1/P where P is any positive integer other than 1 that does not contain factors of 2 or 5. In the case of 1/p where p is a prime other than 2 or 5, Lemma to Theorem 3.5 established the fact that the length of the period is the exponent to which 10 belongs modulo p and this particular theorem was extended, in the form of Theorem 4.1, to also include $1/p^n$. A further extension may be made to include any composite number that does not contain factors of 2 or 5.

Theorem 4.4. Let P be any positive integer other than 1 that does not contain factors of 2 or 5. s is the smallest positive integer for which $10^s \equiv 1 \pmod{P}$ if and only if s is the length of the period for 1/P.

Once again, proof is similar to that of Lemma to Theorem 3.5 with appropriate replacements. Also, as before, an alternate method of determining the period follows directly.

Theorem 4.5. If $p_1, p_2, \ldots, p_n$ are distinct primes other than 2 or 5 and $1/p_1^a, 1/p_2^b, \ldots, 1/p_n^\alpha$ have periods of length $s_1p_1^{a-1}, s_2p_2^{b-1}, \ldots, s_np_n^{\alpha-1}$ where s_i is the length of the period for $1/p_i$ and no p_i satisfies the condition $10^{p_i-1} \equiv 1 \pmod{p_i^2}$ then the length of the period of $1/p_1^a p_2^b \cdots p_n^\alpha$ is the least common multiple of $s_1p_1^{a-1}, s_2p_2^{b-1}, \ldots, s_np_n^{\alpha-1}$.

Proof: Let $s_1p_1^{a-1}, \ldots, s_np_n^{\alpha-1}$ be the respective lengths of the periods of $1/p_1^a, \ldots, 1/p_n^\alpha$. From Theorem 4.1 the following relationships may be established:

$$10^{s_1 p_1^{a-1}} \equiv 1 \pmod{p_1^a}$$
$$10^{s_2 p_2^{b-1}} \equiv 1 \pmod{p_2^b}$$
$$\vdots \qquad\qquad \vdots$$
$$10^{s_n p_n^{\alpha-1}} \equiv 1 \pmod{p_n^\alpha}$$

where each exponent is the smallest integral exponent for which the congruence is true. Let s be the least common multiple of $s_1 p_1^{a-1}, \ldots, s_n p_n^{\alpha-1}$. Since s is a multiple of every $s_i p_i^{\beta-1}$ then $10^s \equiv 1 \pmod{p_i^\beta}$ for every i and this implies that $10^s \equiv 1 \pmod{p_1^a \cdots p_n^\alpha}$. Suppose there exists a positive integer $t < s$ such that $10^t \equiv 1 \pmod{p_1^a \cdots p_n^\alpha}$. This implies that $10^t \equiv 1 \pmod{p_i^\beta}$ for all i but this is impossible since s is the smallest such positive integer. Therefore s is the smallest integral exponent for which the congruence $10^s \equiv 1 \pmod{p_1^a \cdots p_n^\alpha}$ is true and by Theorem 4.4, s is the length of the period of $1/p_1^a \cdots p_n^\alpha$.

As in the discussion following Theorem 4.2, the author notes here that if the denominator of the fraction does contain a p_i such that $10^{p_i - 1} \equiv 1 \pmod{p_i^2}$, that is, 3 or 487, then Theorem 4.5 may be modified to include these primes by simply using $s_i p_i^{\beta-2}$ as the period for $1/p_i^\beta$ where $\beta \geq 2$.

Finally, if 1/P has a repeating decimal fractional form and does contain factors of 2 and/or 5, say $2^m 5^n$, then the period is simply delayed by max $\{m,n\}$ digits. That is to say, factors of 2 or 5 do not affect the length of the period and thus the length may be determined strictly from the formula given in Theorem 4.5.

CHAPTER V

SUMMARY AND CONCLUSION

5.1. Summary. Given any fraction of the form 1/P, a sufficient condition for the decimal representation to contain a repeating or periodic cycle of digits is that P contain one or more factors other than 2 or 5. If P contains factors of 2 and/or 5, say 2^m5^n, and also has other prime factors then the periodic cycle of digits does not start until m or n, whichever is larger, places to the right of the decimal point. But if P contains only factors of 2 and/or 5 then the decimal representation terminates and thus contains no repeating cycle of digits.

The length of the period for any fraction of the form 1/P is a divisor of P - 1 and more specifically, $\phi(P)$. Thus the length of the period may be as large as P - 1. If the length is P - 1 then P must be a prime, but not conversely. A sufficient condition for this to occur is that 10 be a primitive root of P.

There are two methods given to find the length of the period for any fraction of the form 1/P. First, the period length may be found by determining the exponent to which 10 belongs modulo P. If this process involves computations with very large numbers then the period may be determined by finding the lengths of the periods for the reciprocals of the powers of primes contained in P and computing the least common multiple of these numbers. And in the case when P does contain a power of a prime other than 2 or 5, say p^n, where $n > 1$, then the period for $1/p^n$ may be determined by the formula sp^{n-1} where s is the length of the period for 1/p unless p is 3 or 487 in which case sp^{n-2} is used.[9]

5.2. Examples. The following are examples of three rational fractions and an application of the method to find the periods of the decimal representations of the fractions:

Example 1. $1/77 = 1/(7\cdot 11)$

Since 1/7 has a period of 6 and 1/11 has a period of 2 then the period of 1/77 = l.c.m. 6;2 = 6.

Example 2. $1/17{,}199 = 1/(3^3\cdot 7^2\cdot 13)$

Since 3^3 has a period of $1\cdot 3^{3-2} = 3$, $1/7^2$ has a period of $6\cdot 7^{2-1} = 42$, and 1/13 has a period of 6 then the period of 1/17,199 = l.c.m. 3;42;6 = 42.

Example 3. $1/41{,}140 = 1/(2^2\cdot 5\cdot 11^2\cdot 17)$

Since $1/11^2$ has a period of $2\cdot 11^{2-1} = 22$ and 1/17 has a period of 16 then the period of 1/41,140 = l.c.m. 22;16 = 176. Also, the period is delayed by max $\{2,1\}$ = 2 digits.

5.3. Suggestions for further study. Throughout this thesis very little has been said concerning the specific digits of the period for repeating decimal fractions. Work has been done in this area and some quite interesting facts have been discovered. Investigations concerning the cyclic

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[9]For a clearer and more detailed explanation of these exceptions see Section 4.1, Chapter IV.

order of the digits of the period, odd and even period lengths, predictability of the occurrence of specific digits in the period, and properties of half-periods are a few of the related areas with which the author has become acquainted.

Another very closely related topic is that of periodic fractions in other base systems. Most of the same laws and properties apply in base systems different from base 10 but many interesting things occur when different bases are being considered. Prime bases give very interesting results.

All in all, the study of periodic decimal fractions is quite an experience in the unending investigation of the properties of numbers which we use every day. And to the interested reader, the author sincerely hopes that the information relayed by this thesis has been enriching in some manner.

TABLE I

PERIOD LENGTHS FOR THE RECIPROCALS OF THE PRIME NUMBERS LESS THAN 500

Fraction	Period length	Fraction	Period lengt
1/3	1	1/97*	96
1/7*	6	1/101	4
1/11	2	1/103	34
1/13	6	1/107	53
1/17*	16	1/109*	108
1/19*	18	1/113*	112
1/23*	22	1/127	42
1/29*	28	1/131*	130
1/31	15	1/137	8
1/37	3	1/139	46
1/41	5	1/149*	148
1/43	21	1/151	75
1/47*	46	1/157	78
1/53	13	1/163	81
1/59*	58	1/167*	166
1/61*	60	1/173	43
1/67	33	1/179*	178
1/71	35	1/181*	180
1/73	8	1/191	95
1/79	13	1/193*	192
1/83	41	1/197	98
1/89	44	1/199	99

TABLE I (Continued)

Fraction	Period length	Fraction	Period lengt
1/211	30	1/349	116
1/223*	222	1/353	32
1/227	113	1/359	179
1/229*	228	1/367*	366
1/233*	232	1/373	186
1/239	7	1/379*	378
1/241	30	1/383*	382
1/251	50	1/389*	388
1/257*	256	1/397	99
1/263*	262	1/401	200
1/269*	268	1/409	204
1/271	5	1/419*	418
1/277	69	1/421	140
1/281	28	1/431	215
1/283	141	1/433*	432
1/293	146	1/439	219
1/307	153	1/443	221
1/311	155	1/449	32
1/313*	312	1/457	152
1/317	79	1/461*	460
1/331	110	1/463	154
1/337*	336	1/467	233
1/347	173	1/479	239

TABLE I (Continued)

Fraction	Period length
1/487*	486
1/491*	490
1/499*	498

* indicates the primes p for which the length of the period is equal to p - 1.

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