LINEAR PROGRAMMING AND APPLICATIONS

A Thesis

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Chapter I

LINEAR PROGRAMMING AND SIMPLEX METHOD

1. Introduction

As early as 1826, the French mathematician Fourier formulated linear programming problems for use in mechanics and probability theory and suggested a method of solution.

In 1939 the Russian mathematician Kantorovich formulated production problems as linear programming problems, emphasized their importance and suggested a method of solution in his book, Mathematical Methods of Organizing and Planning of Production.

The basic method to solve linear programming problems was developed by G.B. Dantzig in 1947 and is called the simplex method. It is a mathematical technique, though a straight forward economic interpretation can be given to it.

In 1949 T.C. Koopmans collected the papers, which were presented by economists, mathematicians, and statisticians who joined the Cowles Commission conference on linear programming at the University of Chicago, in the book, Activity Analysis of Production and Allocation. From that time, linear programming has had wide application in business, industry, and government.

The general linear programming problem can be described as follows. Given a set of m inequalities or equations in r variables, find non-negative values of these variables which satisfy the constraints and maximize or minimize some linear function of these variables.

2. Graphical Examples.

Linear programming problems which involve only two variables can be solved graphically. The procedure will be illustrated by several examples.

EXAMPLE 1-1

$$\begin{aligned} 3x_1 + x_2 &\le 6 \\ x_1 + 4x_2 &\le 8 \qquad\qquad (1\text{-}1) \\ x_1,\ x_2 &\ge 0 \\ \max z &= 2x_1 + 4x_2 \end{aligned}$$

The object is to find the set of points (x_1, x_2) which satisfy the first three inequalities. This set of points is called the feasible solution.

A point in the feasible solution for which z is maximized is an optimal solution to the problem.

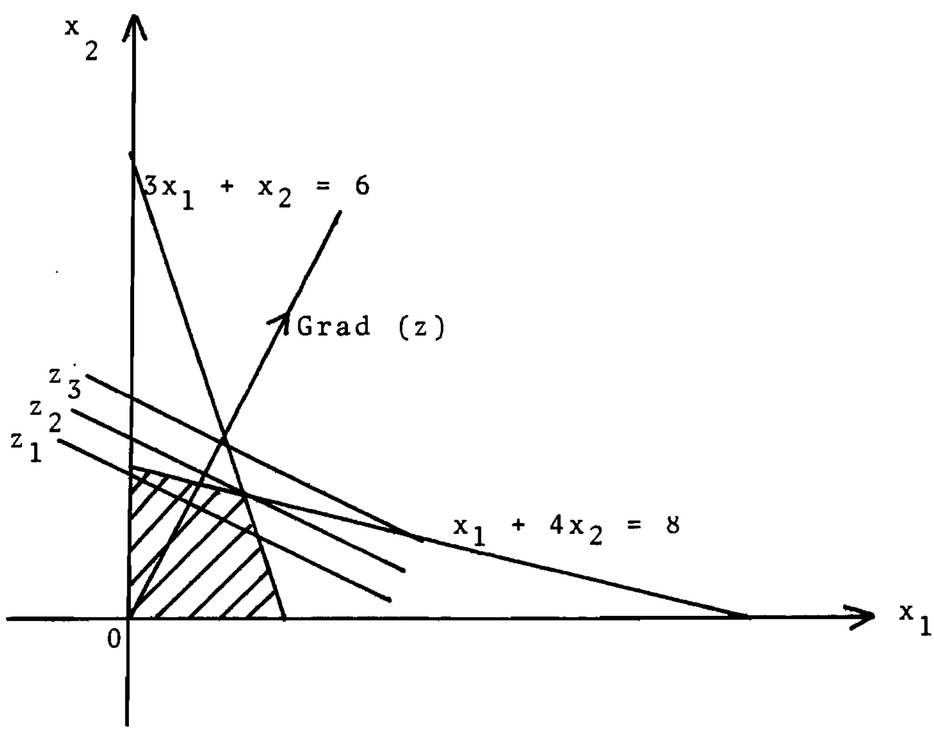
The feasible solution of (1-1) is shown in the shaded region of Figure 1-1. Consider

$$\text{Grad } (z) = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}\right) = \left(\frac{\partial(2x_1 + 4x_2)}{\partial x_1}, \frac{\partial(2x_1 + 4x_2)}{\partial x_2}\right)$$

$$= (2,\ 4)$$

which is a vector that indicates the direction to move in order to increase z most rapidly. Consider a set of lines, say $S = \{z_1, z_2, z_3, \ldots\}$, each of which is perpendicular to the Grad (z). There is a $z_2 \in S$ which has a maximum value in the area of the feasible solutions. Then the intersection point $(\bar{x}_1, \bar{x}_2)$ of the z_2 and the area of the feasible solutions is an optimal solution. From figure 1-1 it is clear that $(\bar{x}_1, \bar{x}_2)$ satisfies $3\bar{x}_1 + \bar{x}_2 = 6$ and $\bar{x}_1 + 4\bar{x}_2 = 8$. Therefore, $\bar{x}_1 = \frac{16}{11}$, $\bar{x}_2 = \frac{18}{11}$, and max $z = 2\bar{x}_1 + 4\bar{x}_2 = 9\frac{5}{11}$.

FIGURE 1-1

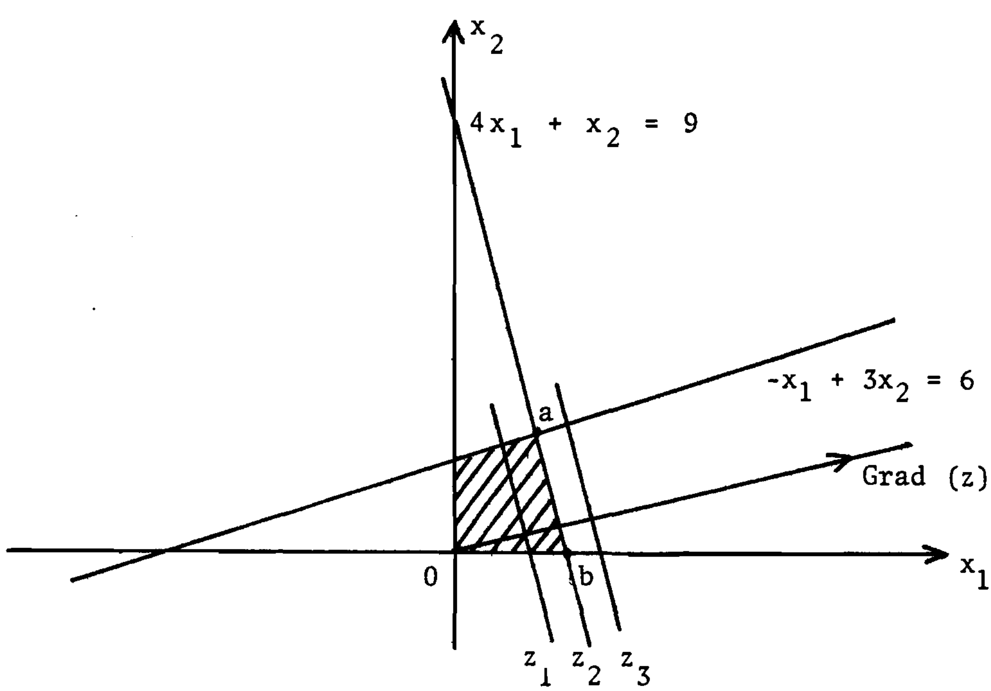


EXAMPLE 1-2

$$\begin{aligned} -x_1 + 3x_2 &\leq 6 \\ 4x_1 + x_2 &\leq 9 \\ x_1,\ x_2 &\geq 0 \\ \max \quad z &= 8x_1 + 2x_2 \end{aligned} \tag{1-2}$$

In this example the Grad (z) = (8, 2) is perpendicular to the line $4x_1 + x_2 = 9$. The line z_2 coincides with the line $4x_1 + x_2 = 9$. Therefore, any point on the edge ab in figure 1-2 is an optimal solution. So the optimal solution of (1-2) is not unique.

FIGURE 1-2



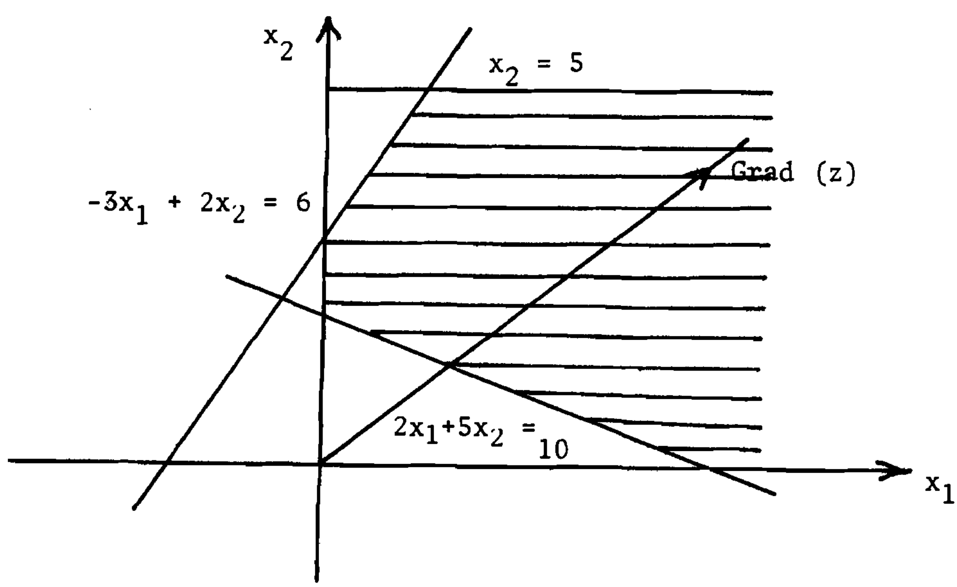
EXAMPLE 1-3

$$\begin{aligned} 2x_1 + 5x_2 &\geq 10 \\ -3x_1 + 2x_2 &\leq 6 \\ x_2 &\leq 5 \\ x_1,\ x_2 &\geq 0 \\ \max z = 5x_1 &+ 4x_2 \end{aligned} \tag{1-3}$$

The geometric interpretation of (1-3) is given in figure (1-3).

The four constraints of this example form an unbounded area representing the feasible solution. The value of z increases in the direction of the Grad (z) = (5, 4). A maximum value of z in the area of the feasible solutions cannot be found. Therefore the solution is unbounded.

FIGURE 1-3

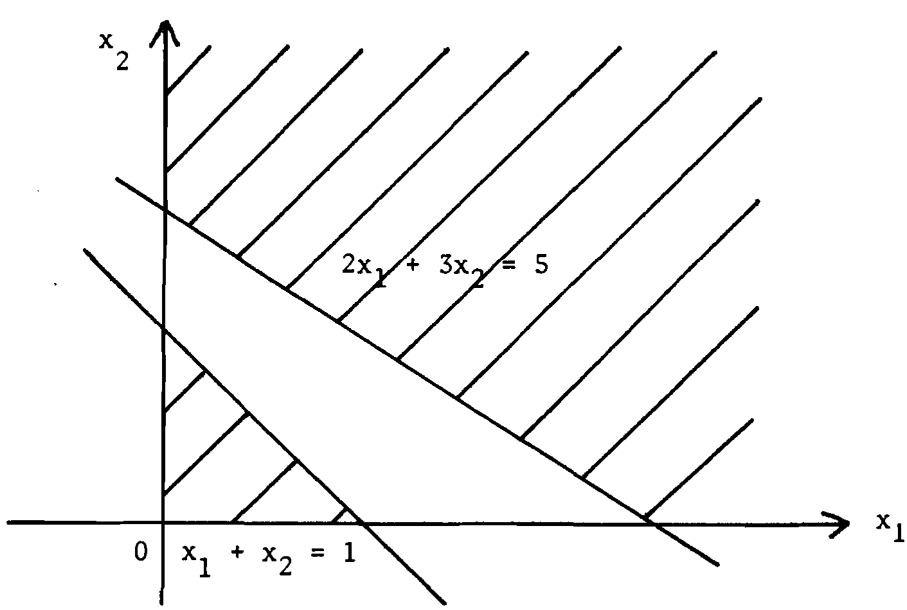


EXAMPLE 1-4.

$$\begin{aligned} x_1 + x_2 &\le 1 \\ 2x_1 + 3x_2 &\le 5 \qquad (1\text{-}4) \\ x_1,\ x_2 &\ge 0 \\ \max z &= x_1 + 2x_2 \end{aligned}$$

This example is expressed graphically in figure 1-4. The area representing the solution of the first inequality does not meet the area representing the solution of the second inequality. Therefore, there is no feasible solution.

FIGURE 1-4



3. Numerical Examples.

Consider the inequality

$$\sum_{i=1}^{n} \alpha_i x_i \leq b. \tag{1-5}$$

A variable $x' \geq 0$ can be added to the inequality such that

$$\sum_{i=1}^{n} \alpha_i x_i + x' = b. \tag{1-6}$$

x' is called a slack variable. Consider the inequality

$$\sum_{j=1}^{m} \alpha_j x_j \geq b. \tag{1-7}$$

A variable $x'' \geq 0$ can be supplied to the inequality such that

$$\sum_{j=1}^{m} \alpha_j x_j - x'' = b. \tag{1-8-a}$$

x'' is called a surplus variable. In this case an artificial variable $x''' \geq 0$ must also be added and then

$$\sum_{j=1}^{m} \alpha_j x_j - x'' + x''' = b \tag{1-8-b}$$

In general, a linear programming problem can be written as follows.

$$\sum_{j=1}^{n} a_{ij} x_j \begin{pmatrix} \geq \\ = \\ \leq \end{pmatrix} b_i, \ i=1, \ldots, m \tag{1-9}$$

$$x_1, x_2, \ldots, x_n \geq 0.$$

$$\max z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \tag{1-10}$$

For each constraint in (1-9) there exists one and only one of the signs $\leq$, $=$, $\geq$. Function (1-10) is called the objective function.

Provide (1-9) with the slack variables and the surplus variables. The following system of linear equations is obtained.

$$\sum_{j=1}^{n} a_{ij} x_j (\pm) x_{n+i} = b_i, \; i = 1, \ldots, m \qquad (1\text{-}11)$$

For each equation in (1-11) there exists one and only one of the signs + or - depending on whether a slack variable or a surplus variable was added to the inequality. In each case where a surplus variable was added it is necessary to also add an artificial variable. Consider example 1-1.

$$\begin{aligned} 3x_1 + x_2 &\leq 6 \\ x_1 + 4x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \\ \max z &= 2x_1 + 4x_2 \; . \end{aligned}$$

It can be changed to

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 6 \\ x_1 + 4x_2 + x_4 &= 8 \\ x_1, x_2 &\geq 0 \\ \max z &= 2x_1 + 4x_2 + 0 \cdot x_3 + 0 \cdot x_4 \end{aligned} \qquad (1\text{-}12)$$

where x_3 and x_4 are slack variables.

Suppose a system of the equations (1-13), (1-14) is obtained by providing the slack variables variables for a linear programming problem. The modification necessary to handle surplus variables will be discussed subsequently.

$$
\begin{array}{l}
a_{11}x_1 + \dots + a_{1n}x_n + a_{1n+1}x_{n+1} + \dots + a_{1n+m}x_{n+m} = b_1 \\
a_{21}x_1 + \dots + a_{2n}x_n + a_{2n+1}x_{n+1} + \dots + a_{2n+m}x_{n+m} = b_2 \\
\quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
a_{m1}x_1 + \dots + a_{mn}x_n + a_{mn+1}x_{n+1} + \dots + a_{mn+m}x_{n+m} = b_n
\end{array}
\tag{1-13}
$$

where $a_{in+j} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$$
\max z = c_1x_1 + \dots + c_nx_n + c_{n+1}x_{n+1} + \dots + c_{n+m}x
\tag{1-14}
$$

where $c_{n+1} = \dots = c_{n+m} = 0$

Then set up the first tableau.

TABLEAU 1-1

V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	c_1 $\mathbf{a}_1$	$\dots$ $\dots$	c_r $\mathbf{a}_r$	$\dots$ $\dots$	c_{n+m} $\mathbf{a}_{n+m}$
$n+1$	c_{n+1}	$x_{B1} = b_1$	a_{11}	$\dots$	a_{1r}	$\dots$	a_{1n+m}
$\cdot$							
$\cdot$							
$n+k$	c_{n+k}	$x_{Bk} = b_k$	a_{k1}	$\dots$	a_{kr}	$\dots$	a_{kn+m}
$\cdot$							
$\cdot$							
$n+m$	c_{n+m}	$x_{Bm} = bm$	a_{m1}	$\dots$	a_{mr}	$\dots$	a_{mn+m}
		z'	$z_1 - c_1$		$z_r - c_r$		$z_{n+m} - c_{n+m}$

$$\text{where} \quad z' = \not{c}_B \cdot \not{x}_B + c_{n+1} \cdot b_1 + \ldots + c_{n+m} \cdot b_m \tag{1-15}$$

$$z_j = \not{c}_B \cdot \not{a}_j = c_{n+1} \cdot a_{1j} + \ldots + c_{n+m} a_{mj}, \quad j = 1, \ldots, n+m \tag{1-16}$$

NOTE 1: In tableau 1-1 all components of the third column must be non-negative. If some components of the third column are negative, then some artificial variables must be supplied.

If all $z_j - c_j \geq 0$ in tableau 1-1, then the problem is completed. Consider the first negative value $z_r - c_r$ in the last row of tableau 1-1 or consider the minimum negative value $z_r - c_r$ in the last row of tableau 1-1. That negative value determines the variable x_r and the corresponding c_r that will enter the new tableau. The kth row which satisfies

$$\frac{x_{Bk}}{a_{kr}} = \min_{a_{kr} > 0} \left\{ \frac{x_{Bi}}{a_{ir}} \right\}, \tag{1-17}$$

will leave tableau 1-1. a_{kr} is called a pivot element. Tableau 1-2 is constructed in the following manner.

$$\begin{aligned}
\bar{a}_{kj} &= \frac{a_{kj}}{a_{kr}} \\
\bar{a}_{ij} &= a_{ij} - a_{ir} \frac{a_{kj}}{a_{kr}} \quad \text{for } i \neq k \\
\bar{x}_{Br} &= \frac{x_{Br}}{a_{kr}} \\
\bar{x}_{Bi} &= x_{Bi} - a_{ir} \frac{x_{Bk}}{a_{kr}} \quad \text{for } i \neq k
\end{aligned} \tag{1-18}$$

$j = 1, \ldots, n+m,$

and $z'' = c_{n+1} \cdot x_{B1} + \ldots + c_r x_{Bk} + \ldots + c_{n+m} x_{Bm}$

$$z''_j = c_{n+1} a_{ij} + \ldots + c_r a_{kj} + \ldots + c_{n+m} a_{mj} \qquad (1\text{-}19)$$

$j = 1, \ldots, n+m$

TABLEAU 1-2

			c_1		c_r		c_{n+m}
V_B	$\cancel{c}_B$	$\cancel{x}_B$	$\cancel{a}_1$		$\cancel{a}_r$		$\cancel{a}_{n+m}$
$n+1$	c_{n+1}	$\bar{x}_{B1}$	$\bar{a}_{11}$	...	$\bar{a}_{1r}$	...	$\bar{a}_{1n+m}$
.							
.							
r	c_r	$\bar{x}_{Bk}$	$\bar{a}_{k1}$	...	$\bar{a}_{kr}$	...	$\bar{a}_{kn+m}$
.							
.							
$n+m$	c_{n+m}	$\bar{x}_{Bm}$	$\bar{a}_{m1}$	...	$\bar{a}_{m2}$	...	$\bar{a}_{mn+m}$
		z''	$z''_1 - c_1$		$z''_1 - c_r$		$z''_{n+m} - c_r$

If all $z_j - c_j$ are non-negative, then the problem is completed. If not, continue the process until all $z_j - c_j$ are non-negative. This method is called the simplex method.

The simplex method is used to solve example 1-1.

$$3x_1 + x_2 \leq 6$$
$$x_1 + 4x_2 \leq 8$$
$$x_1, x_2 \geq 0$$
$$\max z = 2x_1 + 4x_2 \quad .$$

Two slack variables x_3 and x_4 are added to the constraints and to the objective function. Thus,

$$3x_1 + x_2 + x_3 + 0 \cdot x_4 = 6$$

$$x_1 + 4x_2 + 0 \cdot x_3 + x_4 = 8$$

$$x_1,\ x_2,\ x_3,\ x_4 \geqslant 0$$

$$\max z = 2x_1 + 4x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

Next, set up the first tableau.

TABLEAU 1-3

			2	4	0	0
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
3	0	6	3	1	1	0
4	0	8	1	4	0	1
		0	-2	-4	0	0

where $z' = 0 \cdot 6 + 0 \cdot 8 = 0$

$$z_1 - c_1 = 0 \cdot 3 + 0 \cdot 1 - 2 = -2$$

$$z_2 - c_2 = 0 \cdot 1 + 0 \cdot 4 - 4 = -4$$

$$z_3 - c_3 = 0 \cdot 1 + 0 \cdot 0 - 0 = 0$$

$$z_4 - c_4 = 0 \cdot 0 + 0 \cdot 1 - 0 = 0$$

In the last row of tableau 1-3, -2 is the first negative number. Since the corresponding column of -2 contains the numbers 3 and 1 such that

$$\frac{x_{B1}}{a_{11}} = \frac{6}{3} \leq \frac{8}{1} = \frac{x_{B2}}{a_{21}},$$

3 is the pivot element. Therefore, the first row leaves tableau 1-3 and the variable x_1 and the corresponding

coefficient $c_1 = 2$ enter the tableau. Then the formulae (1-18) and (1-19) are applied to form the next tableau.

TABLEAU 1-4

			2	4	0	0
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
1	2	2	1	1/3	1/3	0
4	0	6	0	$3\frac{2}{3}$	$-\frac{1}{3}$	1
		4	0	$-3\frac{1}{3}$	$\frac{2}{3}$	0

Using the same method, the following tableau is obtained.

TABLEAU 1-5

			2	4	0	0
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
1	2	$\frac{16}{11}$	1	0	$\frac{20}{33}$	$-\frac{1}{11}$
2	4	$\frac{18}{11}$	0	1	$-\frac{1}{11}$	$\frac{3}{11}$
		$9\frac{5}{11}$	0	0	$\frac{28}{33}$	$\frac{10}{11}$

All elements of the last row in tableau 1-5 are non-negative. Hence the optimal solution obtained is:

$$x_1 = \frac{16}{11}$$

$$x_2 = \frac{18}{11}$$

$$\max z = 9\frac{5}{11}$$

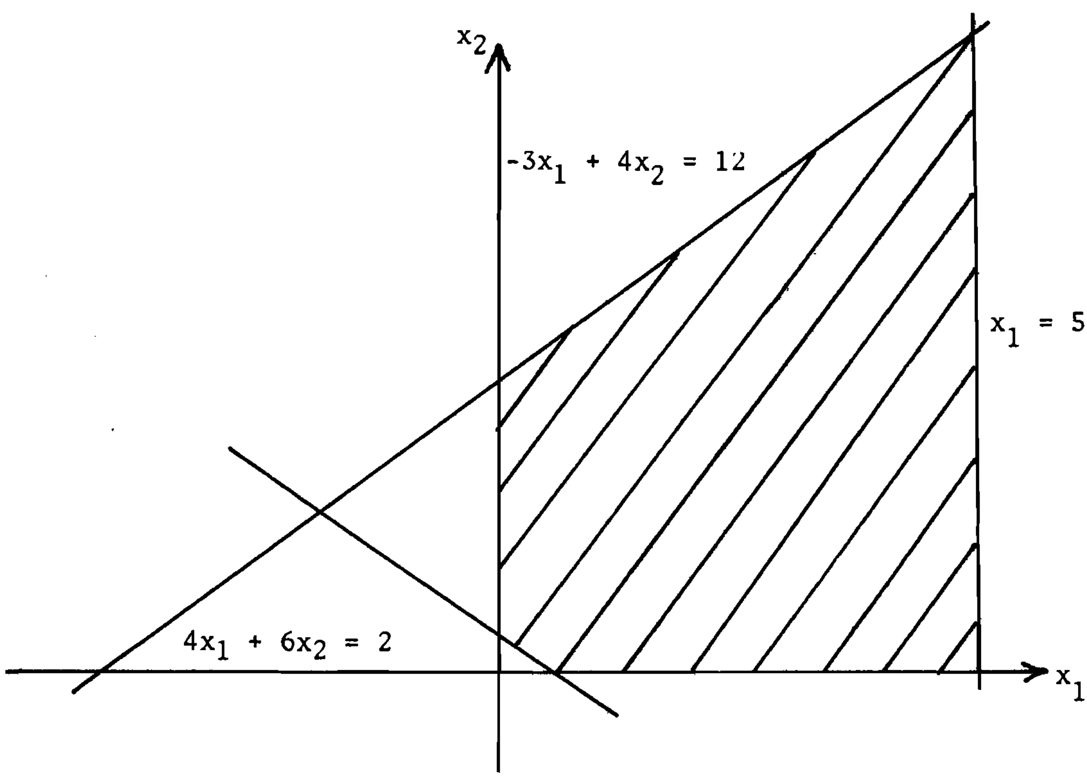
Consider another example.

EXAMPLE 1-5

$$\begin{aligned} 4x_1 + 6x_2 &\geq 2 \\ -3x_1 + 4x_2 &\leq 12 \\ x_1 &\leq 5 \qquad\qquad (1\text{-}20) \\ x_1,\ x_2 &\geq 0 \\ \max z = 5x_1 + x_2 & \end{aligned}$$

The feasible solutions of example 1-5 are shown in the shaded part of figure 1-5.

FIGURE 1-5



By providing a surplus variable x_3 and two slack variables x_4, x_5 (1-20) can be converted into a system of equations. Thus,

$$\begin{aligned} 4x_1 + 6x_2 - x_3 &= 2 \\ -3x_1 + 4x_2 + x_4 &= 12 \\ x_1 + x_5 &= 5 \\ x_1, x_2 &\geqslant 0 \end{aligned} \tag{1-21}$$

$$\max z = 5x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$$

Set up the first tableau letting $x_1 = x_2 = 0$. Thus, the components of the third column are $x_3 = -2$, $x_4 = 12$, $x_5 = 5$. This is impossible since each variable must be non-negative. Therefore, an artificial variable must be supplied. Convert (1-21) into (1-22).

$$\begin{aligned} 4x_1 + 6x_2 - x_3 \qquad\qquad + x_6 &= 2 \\ -3x_1 + 4x_2 \qquad + x_4 \qquad\qquad &= 12 \\ x_1 \qquad\qquad\qquad + x_5 \qquad &= 5 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geqslant 0 \end{aligned} \tag{1-22}$$

$$\max z = 5x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 - Mx_6$$

where M is considered to be an arbitrary large positive number.

Then set up the first tableau. This is shown in tableau 1-6.

The same method has been used to construct the remaining tableaux, which are presented in tableau 1-7 through tableau 1-9.

TABLEAU 1-6

			5	1	0	0	0	-M
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$
6	-M	2	4	6	-1	0	0	1
4	0	12	-3	4	0	1	0	0
5	0	5	1	0	0	0	1	0
		-	-	-	M	0	0	0

TABLEAU 1-7

			5	1	0	0	0	-M
V_B	$\not{c}_B$	x_B	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$
1	5	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$
4	0	$12\frac{3}{2}$	0	$8\frac{1}{2}$	$-\frac{3}{4}$	1	0	$\frac{3}{4}$
5	0	$4\frac{1}{2}$	0	$-\frac{3}{2}$	$\frac{1}{4}$	0	1	$-\frac{1}{4}$
		$\frac{5}{2}$	0	$6\frac{1}{2}$	$-\frac{5}{4}$	0	0	+

TABLEAU 1-8

			5	1	0	0	0	-M
V_B	$\not{c}_B$	x_B	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$
1	5	5	1	0	0	0	1	0
4	0	27	0	4	0	1	$-\frac{3}{4}$	$\frac{9}{16}$
3	0	18	0	-6	1	0	4	-1
		25	0	-1	0	0	5	+

TABLEAU 1-9

			5	1	0	0	0	-M
V_B	$\not c_B$	$\not x_B$	$\not a_1$	$\not a_2$	$\not a_3$	$\not a_4$	$\not a_5$	$\not a_6$
1	5	5	1	0	0	0	1	0
2	1	$\frac{27}{4}$	0	1	0	$\frac{1}{4}$	$-\frac{9}{16}$	$\frac{9}{64}$
3	0	$58\frac{1}{2}$	0	0	1	$\frac{3}{2}$	$\frac{5}{8}$	$-\frac{5}{32}$
		$31\frac{3}{4}$	0	0	0	$\frac{1}{4}$	$4\frac{7}{16}$	+

All elements of the last row in tableau 1-9 are non-negative. The optimal solution is:

$$x_1 = 5$$
$$x_2 = 27/4$$

$$\max z = 31\tfrac{3}{4}\ .$$

Consider example 1-2,

$$-x_1 + 3x_2 \leq 6$$
$$4x_1 + x_2 \leq 9$$
$$x_1,\ x_2 \geq 0$$
$$\max z = 8x_1 + 2x_2$$

Supply two slack variables x_3 and x_4. Then,

$$-x_1 + 3x_2 + x_3 = 6$$
$$4x_1 + x_2 + x_4 = 9$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0$$

$$\max z = 8x_1 + 2x_2 + 0\cdot x_3 + 0\cdot x_4$$

The following tableaux are set up by the simplex method.

TABLEAU 1-10

			8	2	0	0
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$
3	0	6	-1	3	1	0
4	0	9	4	1	0	1
		0	-8	-2	0	0

TABLEAU 1-11

			8	2	0	0
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$
3	0	$8\frac{1}{4}$	0	$3\frac{1}{4}$	1	$\frac{1}{4}$
1	8	$\frac{9}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$
		18	0	0	0	2

All elements of the last row in tableau 1-11 are non-negative. The problem is solved. An optimal solution is:

$$x_1 = 9/4$$

$$x_2 = 0$$

$$\max z = 18$$

Note that there are three 0's in the last row of the two-row tableau, tableau 1-11. In the previous examples, there has always been m 0's in the last row of the last m-row tableau. In tableau 1-11 let the variable x_2

enter the tableau and remove the first row from the tableau. By doing so, tableau 1-12 is obtained.

TABLEAU 1-12

			8	2	0	0
V_B	c_B	x_B	a_1	a_2	a_3	a_4
2	2	$\frac{33}{13}$	0	1	$\frac{4}{13}$	$\frac{1}{13}$
1	8	$\frac{21}{13}$	1	0	$-\frac{1}{13}$	$\frac{3}{13}$
		18	0	0	0	2

All elements of the last row in tableau 1-12 are also non-negative. Then another optimal solution is:

$$x_1 = \frac{21}{13}$$

$$x_2 = \frac{33}{13}$$

$$\max z = 18 .$$

The maximum value of z is unchanged. Therefore, in this example the optimal solution is not unique.

Consider example 1-3.

$$2x_1 + 5x_2 \geq 10$$

$$-3x_1 + 2x_2 \leq 6$$

$$x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

$$\max z = 5x_1 + 4x_2$$

Provide two slack variables x_4 and x_5, one surplus variable

x_3, and one artificial variable x_6, to obtain the following system of equations.

$$
\begin{aligned}
2x_1 + 5x_2 - x_3 \qquad\qquad\qquad + x_6 &= 10 \\
-3x_1 + 2x_2 \qquad + x_4 \qquad\qquad &= 6 \\
x_2 \qquad\qquad + x_5 \qquad &= 5
\end{aligned}
$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0$$

$$\max z = 5x_1 + 4x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 - Mx_6$$

Use the simplex method to form the following tableau.

TABLEAU 1-13

			5	4	0	0	0	-M
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5	a_6
6	-M	10	2	5	-1	0	0	1
4	0	6	-3	2	0	1	0	0
5	0	5	0	1	0	0	1	0
		-10M	-	-	M	0	0	0

TABLEAU 1-14

			5	4	0	0	0	-M
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5	a_6
1	5	5	1	$\frac{5}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
4	0	21	0	$\frac{19}{2}$	$-\frac{3}{2}$	1	0	$\frac{3}{2}$
5	0	5	0	1	0	0	1	0
		25	0	$\frac{17}{2}$	$-\frac{5}{2}$	0	0	+

In tableau 1-14, $-\frac{5}{2}$ is the only negative number of the last row. There is no positive number in the corresponding column a_3. In this case, the solution is unbounded.

Consider example 1-4.

$$x_1 + x_2 \leq 1$$
$$2x_1 + 3x_2 \geq 5$$
$$x_1,\ x_2 \geq 0$$
$$\max z = x_1 + 2x_2$$

Provide the slack variable x_3, the surplus variable x_4, and the artificial variable x_5, to obtain the following system of equations.

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &= 1 \\ 2x_1 + 3x_2 \qquad - x_4 + x_5 &= 5 \end{aligned}$$
$$x_1,\ x_2 \geq 0$$
$$\max z = x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4 - Mx_5$$

Use the simplex method to form the following tableaux.

TABLEAU 1-15

			1	2	0	0	-M
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5
3	0	1	1	1	1	0	0
5	-M	5	2	3	0	-1	1
		-5M	-	-	0	M	0

TABLEAU 1-16

			1	2	0	0	-M
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5
1	1	1	1	1	1	0	0
5	-M	3	0	1	-2	-1	1
		-3M	0	-	+	M	0

TABLEAU 1-17

			1	2	0	0	-M
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5
2	2	1	1	1	1	0	0
5	-M	2	-1	0	-3	-1	1
		-	+	0	+	M	0

All elements of the last row in tableau 1-17 are non-negative. But the value of the artificial variable is not zero in the final solution. Thus, in this case, the problem has no feasible solution.

The following observations have been made in the previous examples.

CASE A. All elements of the last row in the final tableau are non-negative.

(1) No feasible solution: One or more of the artificial variables remains in the final solution at a nonzero level.

(2) Unique optimal solution: Same number of zero's in the last row as the number of variables.

(3) Not unique optimal solution: More zero's in the last row than the number of variables.

CASE B. An element of the last row in the final tableau is negative.

(4) Unbounded solution: There is no positive element in the corresponding column.

4. Justification of the Simplex Method

After adding the slack variables, the surplus variables, and the artificial variables, a system of inequalities in a linear programming problem can be converted into a system of equations.

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
a_{m1}x_1 & + & a_{m2}x_n & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
\qquad (1\text{-}23)
$$

where $x_1, x_2, \ldots, x_n \geq 0$

and $\max z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$.

let

$$
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdot & \cdot & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

and $\mathbf{c} = [c_1, c_2, \ldots , c_n]$

Then (1-23) can be written as:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq 0 \\ \max z &= \mathbf{c}\,\mathbf{x} . \end{aligned} \tag{1-24}$$

Consider a basis $B = \{\mathbf{t}_1, \mathbf{t}_2, \ldots , \mathbf{t}_m\}$ in this system, and construct the following tableau.

TABLEAU 1-18*

			c_1	...	c_j	...	c_n
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	...	$\mathbf{a}_j$	...	$\mathbf{a}_n$
	c_{B1}	x_{B1}	y_{11}	...	y_{1j}	...	y_{1n}
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
	c_{Bi}	x_{Bi}	y_{i1}	...	y_{i2}	...	y_{in}
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
	c_{Bm}	x_{Bm}	y_{m1}	...	y_{mj}	...	y_{mn}
		z_0	. . .		$z_j - c_j$		

*Tableau 1-1 is the special case of tableau 1-18. The basis of tableau 1-1 is $\{e_1, e_2, \ldots, e_m\}$ where e_i is a unit vector containing m components.

In tableau 1-18 y_{ij}, x_{Bi}, and c_{Bi} are scalars and

$$\sum_{i=1}^{m} y_{ij}\mathbf{b}_i = \;_j \qquad (1\text{-}25)$$

$$\sum_{i=1}^{m} x_{Bi}\mathbf{b}_i = b \qquad (1\text{-}26)$$

$$z_0 = \sum_{i=1}^{m} c_{Bi}x_{Bi} \qquad (1\text{-}27)$$

$$z_j = \sum_{i=1}^{m} c_{Bi}y_{ij},\ j = 1, \ldots, n \qquad (1\text{-}28)$$

The $\mathbf{x}_B$ provides the basic solution. The z_0 is the value of the objective function at $\mathbf{x}_B$ and sometimes is expressed by $z_0(\mathbf{x}_B)$.

In the simplex method one vector is changed at a time in the basis and the change is required to be such that:

(1) The new basic solution is feasible.

(2) The objective function does not decrease as the vector is changed.

The first objective is accomplished by the choice of the vector to leave the basis, while the second is accomplished by the choice of the vector to enter the basis.

Suppose $\mathbf{a}_k$ is to enter the basis. From

$$\mathbf{a}_k = \sum_{i=1}^{m} y_{ik}\mathbf{b}_i .$$

In order to maintain a basis, choose an r such that $y_{rk} \neq 0$. Then

$$\mathbf{b}_r = \frac{1}{y_{rk}} \mathbf{a}_k - \sum_{i \neq r} \frac{y_{ik}}{y_{rk}} \mathbf{b}_i \tag{1-29}$$

From (1-26),

$$\mathbf{b} = \sum x_{Bi} \mathbf{b}_i = x_{Br} \mathbf{b}_r + \sum_{i \neq r} x_{Bi} \mathbf{b}_i$$

$$= x_{Br} \left(\frac{1}{y_{rk}} \mathbf{a}_k - \sum_{i \neq r} \frac{y_{ik}}{y_{rk}} \mathbf{b}_i \right) + \sum_{i \neq r} x_{Bi} \mathbf{b}_i$$

$$= \frac{x_{Br}}{y_{rk}} \mathbf{a}_k - \sum_{i \neq r} \frac{y_{ik}}{y_{rk}} x_{Br} \mathbf{b}_i + \sum_{i \neq r} x_{Bi} \mathbf{b}_i$$

Hence,

$$\mathbf{b} = \frac{x_{Br}}{y_{rk}} \mathbf{a}_k + \sum_{i \neq r} \left(x_{Bi} - x_{Br} \frac{y_{ik}}{y_k} \right) \mathbf{b}_i \ . \tag{1-30}$$

This provides that a new basic solution is feasible if

(1) $\dfrac{x_{Br}}{y_{rk}} \geq 0$

and

(2) $x_{Bi} - x_{Br} \cdot \dfrac{y_{ik}}{y_{rk}} \geq 0$, for all $i \neq r$.

Since $x_{Bi} \geq 0$ for all $i = 1, \ldots, m$, the condition (1) can be considered by $y_{rk} > 0$ unless $x_{Br} = 0$. Since $y_{rk} > 0$, condition (2) is satisfied for those i's such that

$y_{ik} \leq 0$. Hence, the object is to choose the r so that

$$x_{Bi} - x_{Br} \frac{y_{ik}}{y_{rk}} \geq 0$$

or

$$x_{Bi} \geq x_{Br} \cdot \frac{y_{ik}}{y_{rk}}$$

for all $i \neq r$ such that $y_{ik} > 0$,

or

$$\frac{x_{Bi}}{y_{ik}} \geq \frac{x_{Br}}{y_{rk}} \qquad (1\text{-}31)$$

This r is chosen by satisfying

$$\frac{x_{Br}}{y_{rk}} = \min_{y_{ik} > 0} \left\{ \frac{x_{Bi}}{y_{ik}} \right\} \quad . \qquad (1\text{-}32)$$

This proves that if $\not{a}_k$ is chosen to enter the basis, then the vector r is chosen to leave the basis, where r satisfies the formula (1-32).

In (1-31) if a new basic solution $\not{x}_B = (\bar{x}_{B1}, \ldots, \bar{x}_{Bm})$ is obtained, then

$$\bar{x}_{Br} = \frac{x_{Br}}{y_{rk}}$$

$$\bar{x}_{Bi} = x_{Bi} - x_{Br} \frac{y_{ik}}{y_{rk}} \qquad i \neq r \; . \qquad (1\text{-}33)$$

The vector $\not{a}_k$ replaces the vector $\not{b}_r$ in the basis. Therefore, a new value of z, say $z(\not{x}_B)$, is

$$z(\bar{\mathbf{a}}_B) = \frac{x_{Br}}{y_{rk}} c_k + \sum_{i \neq r} \left(x_{Bi} - x_{Br} \frac{y_{ik}}{y_{rk}}\right) c_{Bi}$$

$$= \frac{x_{Br}}{y_{rk}} c_k + \sum_{i \neq r} x_{Bi} c_{Bi} - \sum_{i \neq r} \frac{x_{Br}}{y_{rk}} y_{ik} c_{Bi}$$

$$= \frac{x_{Br}}{y_{rk}} c_k + \sum_{i=1}^{m} x_{Bi} c_{Bi} - x_{Br} c_{Br} - \sum_{i=1}^{m} \frac{x_{Br}}{y_{rk}} y_{ik} c_{Bi}$$

$$+ \frac{x_{Br}}{y_{rk}} y_{rk} c_{Br}$$

$$= z(\mathbf{a}_B) + \frac{x_{Br}}{y_{rk}} (c_k - z_k) \tag{1-34}$$

If $z_j - c_j \geq 0$ for all $j=1, \ldots, n$, then the solution is optimal. This will be later proven in theorem 1-1. Since $\frac{x_{Br}}{y_{rk}} \geq 0$, take the first negative $z_k - c_k$ and insert that vector into the basis. This assures that the new value of z does not decrease.

From (1-25) and (1-29), then

$$\sum_{i \neq r} y_{ij} \mathbf{a}_i + y_{rj} \left(\frac{1}{y_{rk}} \mathbf{a}_k - \sum_{i \neq r} \frac{y_{ik}}{y_{rk}} \mathbf{a}_i \right) = \mathbf{a}_j$$

or $$\frac{y_{rj}}{y_{rk}} \mathbf{a}_k + \sum_{i \neq r} \left(y_{ij} - y_{rj} \frac{y_{ik}}{y_{rk}} \right) \mathbf{a}_i = \mathbf{a}_j .$$

It is known that the vector $\not{a}_k$ replaces the vector $\not{\ell}_r$ in the basis. Hence, the new y_{ij}'s, say $\bar{y}_{ij}$'s, which are needed in the new tableau are:

$$\bar{y}_{rj} = \frac{y_{rj}}{y_{rk}}$$

$$\bar{y}_{ij} = y_{ij} - y_{rj}\frac{y_{ik}}{y_{rk}} \quad i \neq r \tag{1-35}$$

where y_{rk} is the pivot element. A new tableau is formed by (1-33) and (1-35). The procedure for the transformation of tableaus is:

(1) Divide the r-th row (x_{Br} and all y_{rj}) by y_{rk} to get a new r-th row.

(2) For $i \neq r$, suutract y_{ik} times the new r-th row from the i-th row to get the new i-th row.

Continue this process until all $z_i - c_i$ are non-negative.

THEOREM 1-1 If for some feasible basic solution $\not{x}_B$ all $z_j - c_j \geq 0$, then $\not{x}_B$ is an optimal solution.

PROOF. Suppose $\not{x}'$ is any feasible solution,

then $\sum_{j=1}^{n} \not{a}_j x_j' = \not{b}$.

Since $\not{a}_j = \sum_{i=1}^{m} y_{ij} \not{\ell}_i$,

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_{ij} \not{\ell}_i \right) x_j' = \not{b},$$

or $$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} y_{ij}x_j'\right)\mathbf{a}_i = \mathbf{b}.$$

But $\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i = \mathbf{b}$ is known, and this representation of $\mathbf{b}$ in terms of $\mathbf{a}_i$ is unique. Therefore,

$$x_{Bi} = \sum_{j=1}^{n} y_{ij}x_j'$$

Since

$$z(\mathbf{x}_B) = \sum_{i=1}^{m} c_i x_{Bi} \quad ,$$

$$z(\mathbf{x}_B) = \sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} y_{ij}x_j'\right)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} c_i y_{ij}\right) x_j'$$

$$= \sum_{j=1}^{n} z_j x_j' \quad . \tag{1-36}$$

However, since $z_j > c_j$ and $x_j' \geq 0$ for all j, then

$$z(\mathbf{x}_B) \geq \sum_{j=1}^{n} c_j x_j' = z(\mathbf{x}').$$

This implies that $z(\mathbf{x}_B)$ is maximum. Therefore, $\mathbf{x}_B$ is an optimal solution.

THEOREM 1-2 Suppose for some basis B, there exists that $z_k - c_k < 0$ and $y_{ik} \leq 0$, for i=1, ... , m. Then the objective function is unbounded above by the constraints.

PROOF. Let T be any number such that $T > z(\mathbf{x}_B)$, where $\mathbf{x}_B$ is any basic solution. It is known that

$$\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i = \mathbf{b} .$$

Suppose $\theta > 0$, thus

$$\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i - \theta\mathbf{a}_k + \theta\mathbf{a}_k = \mathbf{b} . \qquad (1\text{-}37)$$

From (1-25),

$$\mathbf{a}_k = \sum_{i=1}^{m} y_{ik}\mathbf{a}_i .$$

Hence,

$$\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i - \theta \sum_{i=1}^{m} y_{ik}\mathbf{a}_i + \theta\mathbf{a}_k = \mathbf{b}$$

or

$$\sum_{i=1}^{m} (x_{Bi} - \theta y_{ik})\mathbf{a}_i + \theta\mathbf{a}_k = \mathbf{b}. \qquad (1\text{-}38)$$

Since $y_{ik} < 0$, $i=1, \ldots, m$, and $\theta \geq 0$,

$$x_{Bi} - \theta y_{ik} \geq 0.$$

Hence, a new feasible solution $\mathbf{x}(\theta)$ is given by the equation (1-38). Then

$$\begin{aligned} z(\mathbf{x}(\theta)) &= \sum_{i=1}^{m} (x_{Bi} - \theta y_{ik})c_{Bi} + \theta c_k \\ &= \sum_{i=1}^{m} x_{Bi}c_{Bi} - \theta \left(\sum_{i=1}^{m} y_{ik}c_{Bi} - c_k \right) \\ &= z(\mathbf{x}_B) - \theta (z_k - c_k) \end{aligned}$$

Let $\theta = \dfrac{T - z(\mathbf{x}_B)}{-(z_k - c_k)}$, where $T - z(\mathbf{x}_B) > 0$ and $-(z_k - c_k) > 0$, then $z(\mathbf{k}(\theta)) = T$. Therefore, the objective function is unbounded above.

In the simplex method, the most important concept is to change the basis to obtain a new feasible solution such that the value of z is not decreased.

The following remarks deal with a technique used in the simplex method.

Consider the system of linear equations.

$$\begin{aligned}
&(1)\quad 3x_1 - 4x_2 + 2x_3 + 3x_4 - x_5 + 2x_6 = 7\\
&(2)\quad x_1 - 3x_2 + 6x_3 + 5x_4 - 2x_5 = 1\\
&(3)\quad -2x_1 + x_2 - 4x_3 - 3x_5 + 4x_6 = 6\ .
\end{aligned}$$

One solution to this system is

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 1,\ x_5 = 2,\ x_6 = 3$.

Suppose, however, a solution involving x_3, x_5, and x_6 is needed. Divide equation (1) by 2, then

$$\begin{aligned}
&(1)\quad \tfrac{3}{2}x_1 - 2x_2 + x_3 + \tfrac{3}{2}x_4 - \tfrac{1}{2}x_5 + x_6 = \tfrac{7}{2}\\
&(2)\quad x_1 - 3x_2 + 6x_3 + 5x_4 - 2x_5 = 1\\
&(3)\quad -2x_1 + x_2 - 4x_3 - 3x_5 + 4x_6 = 6
\end{aligned}$$

Add -6 times equation (1) to equation (2) and add 4 times equation (1) to equation (3).

$$\begin{aligned}
&(1)\quad \tfrac{3}{2}x_1 - 2x_2 + x_3 + \tfrac{3}{2}x_4 - \tfrac{1}{2}x_5 + x_6 = \tfrac{7}{2}\\
&(2)\quad -8x_1 + 9x_2 - 4x_4 + x_5 - 6x_6 = -20\\
&(3)\quad 4x_1 - 7x_2 + 6x_4 - 5x_5 + 8x_6 = 20
\end{aligned}$$

Now, a solution is

$x_1 = 0,\ x_2 = 0,\ x_3 = -\frac{7}{38},\ x_4 = 0,\ x_5 = -\frac{20}{19},\ x_6 = \frac{60}{19}.$

Then suppose a solution involving x_2, x_3, and x_6 is needed. Divide equation (2) by 9.

$$\begin{aligned}
&(1)\ \tfrac{3}{2}x_1 - 2x_2 + x_3 + \tfrac{3}{2}x_4 - \tfrac{1}{2}x_5 + x_6 = \tfrac{7}{2}\\
&(2) -\tfrac{8}{9}x_1 + x_2 - \tfrac{4}{9}x_4 + \tfrac{1}{9}x_5 - \tfrac{6}{9}x_6 = -\tfrac{20}{9}\\
&(3)\ 4x_1 - 7x_2 + 6x_4 - 5x_5 + 8x_6 = 20
\end{aligned}$$

Add 2 times equation (2) to equation (1) and add 7 times equation (2) to equation (3).

$$\begin{aligned}
&(1) -\tfrac{5}{18}x_1 + x_3 + \tfrac{11}{18}x_4 - \tfrac{5}{18}x_5 - \tfrac{3}{9}x_6 = -\tfrac{17}{18}\\
&(2) -\tfrac{8}{9}x_1 + x_2 - \tfrac{4}{9}x_4 + \tfrac{1}{9}x_5 - \tfrac{6}{9}x_6 = -\tfrac{20}{9}\\
&(3) -\tfrac{20}{9}x_1 + \tfrac{26}{9}x_4 - \tfrac{38}{9}x_5 + \tfrac{30}{9}x_6 = \tfrac{40}{9}
\end{aligned}$$

Hence, the solution involving x_2, x_3 and x_6 is $x_1 = 0$, $x_2 = -\frac{4}{3}$, $x_3 = -\frac{1}{2}$, $x_4 = 0$, $x_5 = 0$, $x_6 = \frac{4}{3}$. Now suppose a solution involving x_1, x_2, and x_3 is needed. Divide equation (3) by $-\frac{20}{9}$.

$$\begin{aligned}
&(1) -\tfrac{5}{18}x_1 + x_3 + \tfrac{11}{18}x_4 - \tfrac{5}{18}x_5 - \tfrac{3}{9}x_6 = -\tfrac{17}{18}\\
&(2) -\tfrac{8}{9}x_1 + x_2 - \tfrac{4}{9}x_4 + \tfrac{1}{9}x_5 - \tfrac{6}{9}x_6 = -\tfrac{20}{9}\\
&(3)\ x_1 - \tfrac{13}{10}x_4 + \tfrac{19}{10}x_5 - \tfrac{3}{2}x_6 = -2
\end{aligned}$$

Add $\frac{5}{18}$ times equation (3) to equation (1) and add $\frac{8}{9}$ times equation (3) to equation (2).

$$
\begin{array}{llrrrrl}
(1) & & x_3 & + \frac{1}{4}x_4 & + \frac{1}{4}x_5 & - \frac{3}{4}x_6 & = -\frac{3}{2} \\
 & x_2 & & - \frac{8}{5}x_4 & + \frac{9}{5}x_5 & - 2x_6 & = -4 \\
x_1 & & & - \frac{13}{10}x_4 & + \frac{19}{10}x_5 & - \frac{3}{2}x_6 & = -2
\end{array}
$$

Therefore, the solution involving x_1, x_2, and x_3 is:
$x_1 = -2$, $x_2 = -4$, $x_3 = -\frac{3}{2}$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$

If a linear programming problem is required to minimize the value of z, the objective function, then either one of two following methods can be used to solve the problem.

(1) Find the value of max $(-z)$ by the simplex method. Then times -1 to $\max(-z)$ so that $-\max(-z) = \min z$.

(2) Choose the first positive $z_k - c_k$ in the tableau and insert that vector into the basis. Let the vector r leave the basis, where r satisfies the formula (1-33). Then make a new tableau. Iterate the same way until all $z_j - c_j \leq 0$.

Method (2) is easy to verify using formula (1-36) in the proof of theorem 1-1. Since all $z_j - c_j \leq 0$, $z(\mathbf{x}_B) \leq z(\mathbf{x}')$, where $\mathbf{x}_B$ is an optimal solution and $\mathbf{x}'$ is any feasible solution.

Consider an example.

EXAMPLE 1-7

$$x_1 + 3x_2 \geq 3$$

$$x_1 + x_2 \geq 2$$
$$x_1,\ x_2 \geq 0$$
$$\min z = 1.5x_1 + 2.5x_2$$

METHOD 1.

Consider $\max(-z) = -1.5x_1 - 2.5x_2$. Then form a system of equations such that

$$x_1 + 3x_2 - x_3 + x_5 = 3$$
$$x_1 + x_2 - x_4 + x_6 = 2$$
$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0.$$
$$\max(-z) = -1.5x_1 - 2.5x_2 + 0\cdot x_3 + 0\cdot x_4 - M\cdot x_5 - M\cdot x_6$$

where M is an arbitrary large positive number.

The following tableaux are formed by the use of the simplex method.

TABLEAU 1-19

			-1.5	-2.5	0	0	-M	-M
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
5	-M	3	1	3	-1	0	1	0
6	-M	2	1	1	0	-1	0	1
		-	-	-	+	0	0	0

TABLEAU 1-20

			-1.5	-2.5	0	0	-M	-M
V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
5	-M	1	0	2	-1	1	1	-1
1	-1.5	2	1	1	0	-1	0	1
		-	0	-	+	-	0	+

TABLEAU 1-21

			-1.5	-2.5	0	0	-M	-M
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
2	-2.5	0.5	0	1	-0.5	0.5	0.5	-0.5
1	-1.5	1.5	1	0	0.5	-1.5	-0.5	1.5
		-3.5	0	0	0.5	1	+	+

Therefore, an optimal solution is

$$x_1 = 1.5,\ x_2 = 0.5$$

$$\min z = -(\max(-z)) = -(-3.5) = 3.5$$

METHOD 2.

Form a system of equations such that,

$$\begin{aligned} x_1 + 3x_2 - x_3 \quad\quad + x_5 \quad\quad &= 3 \\ x_1 + x_2 \quad\quad - x_4 \quad\quad + x_6 &= 2 \end{aligned}$$

where $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

and $\min z = 1.5x_1 + 2.5x_2 + 0\cdot x_3 + 0\cdot x_4 + Mx_5 + Mx_6$.

where M is an arbitrary large positive number.

Then the following tableaux are obtained by the second method.

TABLEAU 1-22

			1.5	2.5	0	0	M	M
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
5	M	3	1	3	-1	0	1	0
6	M	2	1	1	0	-1	0	1
		+	+	+	-	-	0	0

TABLEAU 1-23

			1.5	2.5	0	0	M	M
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
5	M	1	0	2	-1	1	1	-1
1	1.5	2	1	1	0	-1	0	1
		+	0	+	-	+	0	-

TABLEAU 1-24

			1.5	2.5	0	0	M	M
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
2	2.5	0.5	0	1	-0.5	0.5	0.5	-0.5
1	1.5	1.5	1	0	0.5	-1.5	-0.5	1.5
		3.5	0	0	-0.5	-1	-	-

All elements of the last row in tableau 1-24 are non-positive. Therefore, an optimal solution is:

$$x_1 = 1.5, \; x_2 = 0.5$$

and $\min z = 3.5$.

Chapter II

SOME APPLICATIONS OF LINEAR PROGRAMMING

Linear programming can be applied to almost any industrial operation. Oil refinery operations is a large field of application. Companies in this field spend large amounts of money to formulate accurate models as well as solve and implement them. Other well known applications are in cattle-feed mixing, the steel industry, the paper industry and the dairy industry. Since linear programming is concerned with the basic problem of allocation of resources to various uses, it is applicable to almost any economic activity. The cases in which it is not of much use are those in which the problem is so trivial that the solution is obvious or cases in which the model is complicated by constraints which do not fit into the linear programming model. Other methods have been and are being developed to cope with these problems.

Consider some simple practical linear programming problems which are solved by the use of the simplex method.

1. Problem 1

At the refinery of a petroleum company two grades of gasoline are produced: high test and regular. To produce each grade of gasoline, a fixed proportion of straight gasoline, octane, and additives is needed. A gallon of high test requires 30 percent straight gasoline, 50 percent octane, and 20 percent additives; a gallon of regular requires 60 percent straight gasoline, 30 percent octane, and 10 percent additives. It is known that the petroleum company receives a profit of 6 and 5 cents for each gallon on the high test and regular gasoline respectively. If the supplies of straight gasoline, octane, and additives are restricted to 6,000,000, 4,000,000, and 2,000,000 gallons respectively, how much of each grade should be produced in a given time period, to maximize profits and make the best use of resources?

Suppose x_1 gallons of high test and x_2 gallons of regular should be produced. Then the following constraints are formed:

$$0.3x_1 + 0.6x_2 \leq 6{,}000{,}000$$

$$0.5x_1 + 0.3x_2 \leq 4{,}000{,}000$$

$$0.2x_1 + 0.1x_2 \leq 2{,}000{,}000 \ .$$

$$\max z = 6x_1 + 5x_2$$

After adding slack variables to the above inequalities, the following system of linear equations is obtained.

$$\begin{aligned} 0.3x_1 + 0.6x_2 + x_3 \qquad\qquad &= 6{,}000{,}000 \\ 0.5x_1 + 0.3x_2 \qquad + x_4 \qquad &= 4{,}000{,}000 \\ 0.2x_1 + 0.1x_2 \qquad\qquad + x_5 &= 2{,}000{,}000 \end{aligned}$$

$$\max z = 6x_1 + 5x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5$$

Use the simplex method to form the following tableaux.

TABLEAU 2-1

V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	6	5	0	0	0
			$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$
3	0	6,000,000	0.3	0.6	1	0	0
4	0	4,000,000	0.5	0.3	0	1	0
5	0	2,000,000	0.2	0.1	0	0	1
		0	-6	-5	0	0	0

TABLEAU 2-2

V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	6	5	0	0	0
			$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$
3	0	3,600,000	0	0.42	1	-0.6	0
1	6	8,000,000	1	0.6	0	2	0
5	0	4,000,000	0	-0.02	0	-0.4	1
		48,000,000	0	-1.4	0	12	0

TABLEAU 2-3

V_B	$\mathbf{c}_B$	$\mathbf{x}_B$	6	5	0	0	0
			$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$
2	5	$\frac{60{,}000{,}000}{7}$	0	1	$\frac{50}{21}$	$-\frac{10}{7}$	0
1	6	$\frac{20{,}000{,}000}{7}$	1	0	$-\frac{10}{7}$	$\frac{20}{7}$	0
5	0	$\frac{40{,}000{,}000}{7}$	0	0	$\frac{1}{21}$	$-\frac{3}{7}$	1
		60,000,000	0	0	10	10	0

All elements of the last row in tableau 2-3 are non-negative. The problem is completed. The optimal solution is:

$$x_1 = \frac{20,000,000}{7} = 2,857,142\ \frac{6}{7}$$

$$x_2 = \frac{60,000,000}{7} = 8,571,428\ \frac{4}{7}$$

$$\max z = 60,000,000$$

Therefore, 2,857,142 $\frac{6}{7}$ gallons of high test and 8,571,428 $\frac{4}{7}$ gallons of regular are produced to obtain the maximal profits of 600,000 dollars.

2. Problem 2

A car plant can produce both automobiles and trucks. It has four departments: metal stamping, engine assembly, automobile assembly, and truck assembly. The net revenue for an automobile is \$300 and for a truck is \$250. The following data is known:

Department	Capacity h/year	Automobile requirements h/unit	Truck requirements h/unit
Metal stamping	110,000	4	$2\frac{6}{7}$
Engine Assembly	70,400	2	3
Automobile Assembly	66,000	3	-
Truck Assembly	41,250	-	$2\frac{1}{2}$

(a) Find the optimal production program for this plant using the simplex method.

(b) Interpret every element in the final tableau.

Suppose there are x_1 automobiles and x_2 trucks to be produced. Then

$$4x_1 + 2\tfrac{6}{7}x_2 \leq 110{,}000$$

$$2x_1 + 3x_2 \leq 70{,}400$$

$$3x_1 \leq 66{,}000$$

$$2\tfrac{1}{2} \leq 41{,}250$$

and an objective function is

$$\max z = 300x_1 + 250x_2$$

Add the slack variables x_3, x_4, x_5, and x_6 to the above inequalities. The following system of equations is formed:

$$4x_1 + 2\tfrac{6}{7}x_2 + x_3 = 110{,}000$$

$$2x_1 + 3x_2 + x_4 = 70{,}400$$

$$3x_1 + x_5 = 66{,}000$$

$$2\tfrac{1}{2}x_2 + x_6 = 41{,}250$$

$$\max z = 300x_1 + 250x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 .$$

Then form the following tableaux using the simplex method.

TABLEAU 2-4

			300	250	0	0	0	0
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
3	0	110,000	4	$2\frac{6}{7}$	1	0	0	0
4	0	70,400	2	3	0	1	0	0
5	0	66,000	3	0	0	0	1	0
6	0	41,250	0	$2\frac{1}{2}$	0	0	0	1
		0	-300	-250	0	0	0	0

TABLEAU 2-5

			300	250	0	0	0	0
V_B	$\mathbf{c}_B$	$\mathbf{b}_B$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
3	0	22,000	0	$2\frac{6}{7}$	1	0	$-\frac{4}{3}$	0
4	0	26,400	0	3	0	1	$-\frac{2}{3}$	0
1	300	22,000	1	0	0	0	$\frac{1}{3}$	0
6	0	41,250	0	$2\frac{1}{2}$	0	0	0	1
		6,600,000	0	-250	0	0	100	0

TABLEAU 2-6

			300	250	0	0	0	0
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$
2	250	7,700	0	1	$\frac{7}{20}$	0	$-\frac{7}{15}$	0
4	0	3,300	0	0	$-\frac{21}{20}$	1	$\frac{11}{15}$	0
1	300	22,000	1	0	0	0	$\frac{1}{3}$	0
6	0	22,000	0	0	$-\frac{7}{8}$	0	$\frac{7}{6}$	1
		8,525,000	0	0	$\frac{175}{2}$	0	$-\frac{50}{3}$	0

TABLEAU 2-7

			300	250	0	0	0	0
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$
2	250	9,800	0	1	$-\frac{7}{22}$	$\frac{7}{11}$	0	0
5	0	4,500	0	0	$-\frac{63}{44}$	$\frac{15}{11}$	1	0
1	300	20,500	1	0	$\frac{21}{44}$	$-\frac{5}{11}$	0	0
6	0	16,750	0	0	$\frac{70}{88}$	$-\frac{35}{22}$	0	1
		8,600,000	0	0	$\frac{700}{11}$	$\frac{250}{11}$	0	0

In tableau 2-7 all elements of the last row are non-negative. The problem is done. The optimal solution is:

$$x_1 = 20{,}500,\ x_2 = 9{,}800$$

and

$$\max z = 8{,}600{,}000$$

Therefore, the optimal production program is to produce 20,500 automobiles and 9,800 trucks to make the maximum profits of $8,600,000 per year. There are $x_5 = 4{,}500$ and $x_6 = 16{,}750$ in the final tableau. This means that there are 4,500 hours in the automobile assembly and 16,750 hours in the truck assembly which have not been used each year. Hence, it is enough to spend 66,000 - 4,500 = 61,500 hours in the automobile assembly and 41,250 - 16,750 = 24,500 hours in the truck assembly each year.

3. Problem 3

A company has three warehouses, denoted by w_1, w_2, and w_3, containing 8,000, 5,000, and 3,000 units of its products, respectively. In the next month, 2,000; 1,000; 3,000; 4,500 units must be shipped to four retail outlets denoted by 0_1, 0_2, 0_3, and 0_4. The unit cost of shipment from any warehouse to any retail outlet is contained in the following matrix. Find the minimum cost shipping schedule.

	0_1	0_2	0_3	0_4
w_1	10	8	16	3
w_2	19	25	18	7
w_3	20	17	20	5

Let a_{ij} be the amount sent from the warehouse w_i to the retail outlet 0_j, where i = 1, 2, 3, and j = 1, 2, 3, 4. Set up the restraints for each warehouse and for each outlet.

In warehouse 1:

$$a_{11} + a_{12} + a_{13} + a_{14} \leq 8{,}000.$$

In warehouse 2:

$$a_{21} + a_{22} + a_{23} + a_{24} \leq 5{,}000.$$

In warehouse 3:

$$a_{31} + a_{32} + a_{33} + a_{34} \leq 3{,}000.$$

In outlet 1:

$$a_{11} + a_{21} + a_{31} = 2{,}000.$$

In outlet 2:

$$a_{12} + a_{22} + a_{32} = 1{,}000.$$

In outlet 3:

$$a_{13} + a_{23} + a_{33} = 3{,}000.$$

In outlet 4:

$$a_{14} + a_{24} + a_{34} = 4{,}500.$$

The objective function is to minimize the cost of shipment

$$\min z = 10\cdot a_{11} + 8\cdot a_{12} + 16\cdot a_{13} + 3\cdot a_{14} + 19\cdot a_{21}$$
$$+ 25\cdot a_{22} + 18\cdot a_{23} + 7\cdot a_{24} + 20\cdot a_{31} + 17\cdot a_{32}$$
$$+ 20\cdot a_{33} + 5\cdot a_{34} .$$

Provide the slack variables a_{10}, a_{20}, a_{30} and the artificial variables a_{01}, a_{02}, a_{03}, a_{04} to the above restraints. Then

$$a_{11} + a_{12} + a_{13} + a_{14} + a_{10} = 8{,}000$$

$$a_{21} + a_{22} + a_{23} + a_{24} + a_{20} = 5{,}000$$

$$a_{31} + a_{32} + a_{33} + a_{34} + a_{30} = 3{,}000$$

$$a_{11} + a_{21} + a_{31} \qquad + a_{01} = 2{,}000$$

$$a_{12} + a_{22} + a_{32} \qquad + a_{02} = 1{,}000$$

$$a_{13} + a_{23} + a_{33} \qquad + a_{03} = 3{,}000$$

$$a_{14} + a_{24} + a_{34} \qquad + a_{04} = 4{,}500$$

Since min z = -max(-z), consider max (-z) as follows:

$$\begin{aligned} \max(-z) = {} & -10a_{11} - 8a_{12} - 16a_{13} - 3a_{14} - 19a_{21} \\ & -25a_{22} - 18a_{23} - 7a_{24} - 20a_{31} - 17a_{32} \\ & -20a_{33} - 5a_{34} + 0 \cdot a_{10} + 0 \cdot a_{20} + 0 \cdot a_{30} \\ & - Ma_{01} - Ma_{02} - Ma_{03} - Ma_{04}, \end{aligned}$$

where M is an arbitrary large positive number.

This problem is too complicated to compute by hand, using the simplex method. A FORTRAN program for the simplex method is listed in appendix I. Using this program, the result is:

x(4) = 0.45000000 E 04

x(14) = 0.25000000 E 04

x(15) = 0.30000000 E 04

$x(1) = 0.20000000$ E 04

$x(2) = 0.10000000$ E 04

$x(3) = 0.50000000$ E 03

$x(7) = 0.25000000$ E 04

max $(-z) = -0.94500000$ E 05, where

$x(1) = a_{11}$, $x(2) = a_{12}$, $x(3) = a_{13}$, $x(4) = a_{14}$

$x(7) = a_{23}$, $x(14) = a_{20}$, $x(15) = a_{30}$

Therefore, the optimal solution is:

$a_{11} = 2{,}000$, $a_{12} = 1{,}000$, $a_{13} = 500$, $a_{14} = 4{,}500$,
$a_{23} = 2{,}500$

and min $z = -\max(-z) = 94{,}500$.

This means that 2,000, 1,000, 500, and 4,500 units are shipped from warehouse 1 to outlets 0_1, 0_2, 0_3, and 0_4 respectively, and 2,500 units are shipped from warehouse 2 to outlet 3 having a minimum cost of 94,500.

Chapter III

CLASSIC TRANSPORTATION PROBLEMS

1. Transportation Problem Tableau and Initial Feasible Solutions

A product is available in known quantities g_i at each i of m origins. It is required that given quantities d_j of the product be shipped to each j of n destinations. The cost of shipping a unit of the product from origin i to destination j is c_{ij}. Determining the shipping schedule which minimizes the total cost of shipment is a transportation problem.

Assume that

$$\sum_{i=1}^{m} g_i = \sum_{j=1}^{n} d_j \quad .$$

Let a_{ij} be the quantity of the product sent from origin i to destination j. Set up

$$\sum_{j=1}^{n} a_{ij} = g_i \quad g_i > 0 \quad i = 1, \ldots, m \tag{3-1}$$

$$\sum_{i=1}^{m} a_{ij} = d_j \quad d_j > 0 \quad j = 1, \ldots, n \tag{3-2}$$

and minimize

$$z = \sum_{i,j} c_{ij} a_{ij} \quad . \tag{3-3}$$

Consider the transportation problem tableaus.

TABLEAU 3-1

c_{11}	c_{12}	$\cdots$	c_{1j}	$\cdots$	c_{1n}
c_{21}	c_{22}	$\cdots$	c_{2j}	$\cdots$	c_{2n}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
c_{i1}	c_{i2}	$\cdots$	c_{ij}	$\cdots$	c_{in}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
c_{m1}	c_{m2}	$\cdots$	c_{mj}	$\cdots$	c_{mn}

TABLEAU 3-2

	D_1	D_2		D_j		D_n	g_i
G_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}	g_1
G_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}	g_2
	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
G_i	a_{i1}	a_{i2}	$\cdots$	a_{ij}	$\cdots$	a_{in}	g_i
	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
G_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	g_m
d_j	d_1	d_2	$\cdots$	d_j	$\cdots$	d_n	$\sum g_i = \sum d_j$

Tableau 3-1 is a cost tableau, and tableau 3-2 is an activity tableau.

EXAMPLE 3-1: A transportation problem has

$$g_1 = 30 \qquad d_1 = 20$$
$$g_2 = 50 \qquad d_2 = 40$$
$$g_3 = 75 \qquad d_3 = 30$$
$$g_4 = 20 \qquad d_4 = 10 \qquad (3\text{-}4)$$
$$d_5 = 50$$
$$d_6 = 25$$

and a cost tableau shown in tableau 3-3.

TABLEAU 3-3

1	2	1	4	5	2
3	3	2	1	4	3
4	2	5	9	6	2
3	1	7	3	4	6

The first step of solving the transportation problem is to find an initial feasible solution. There are five methods used to find an initial feasible solution.

(1) NORTHWEST CORNER RULE.

In tableau 3-2, let $a_{11} = \min(g_1, d_1)$.

(a) If $a_{11} = g_1$, then set $a_{21} = \min(g_2, d_1 - g_1)$, and if $a_{21} = g_2$, then set $a_{31} = \min(g_3, d_1 - g_1 - g_2)$ and continue in this manner. When $a_{11} + \ldots + a_{k1} = d_1$ then set $a_{k2} = \min(g_k - a_{k1}, d_2)$.

(b) If $a_{11} = d_1$, set $a_{12} = \min(g_1 - d_1, d_2)$. If $a_{12} = d_2$, set $a_{13} = \min(g_1 - d_1 - d_2, d_3)$.

Continue in this way, until $a_{11} + \ldots + a_{1h} = g$, then set $a_{2h} = \min(g_2, d_h - a_{1h})$. Repeat the process for the resulting tableau. Then there is an initial feasible solution which contains m + n - 1 basic variables. All values of the other variables are zero

This method is used to find an initial feasible solution of Example 3-1. It is shown in tableau 3-4.

If the i-th row constraint and the j-th column constraint are satisfied simultaneously, then set $a_{i+1,j+1} = \min(g_{i+1}, d_{j+1})$ and put $a_{i,j+1}$ or $a_{i+1,j} = 0$ in the tableau.

TABLEAU 3-4

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20	10					30
G_2		30	20				50
G_3			10	10	50	5	75
G_4						20	20
d_j	20	40	30	10	50	25	175

(2) COLUMN MINIMA.

Choose the minimum cost in column 1. If it is not unique, select any one of the minima. Suppose it is in row k. Then let $a_{k1} = \min(g_k, d_1)$. If $a_{k1} = d_1$ then remove column 1 from the tableau and consider column 2. If $a_{k1} = g_k$ then remove row k from the tableau, and consider the next lowest cost in column 1, say in row h. Set $a_{h1} = \min(g_h, b_1 - g_k)$ and continue in this way until the requirement of destination 1 is satisfied. Then remove column 1 from the tableau and repeat the same procedure for column 2. Continue until the requirement of column n is satisfied.

If the i-th row constraint and the j-th column constraint are satisfied simultaneously, remove only the i-th row and consider the next lowest cost of the j-th column. Assume it occurs in row P. Then set $a_{pj} = 0$ in the tableau, remove column j and consider column j+1. The column minima method is used to find an initial feasible solution of the same example. It is shown in tableau 3-5.

TABLEAU 3-5

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20		10				30
G_2			20	10	20		50
G_3		20			30	25	75
G_4		20					20
d_j	20	40	30	10	50	25	175

(3) ROW MINIMA.

The procedure of this method is the same as the procedure of column minima method except that where the column minima method moves horizontally from column 1 to column n, the row minima method moves vertically from row 1 to row m.

The row minima method is applied to find an initial feasible solution of the same problem. It is shown in tableau 3-6.

TABLEAU 3-6

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20		10				30
G_2		20	20	10			50
G_3		20			30	25	75
G_4					20		20
d_j	20	40	30	10	50	25	175

(4) MATRIX MINIMA.

Choose the smallest cost in the whole tableau. Say it occurs in (i, j). Then set $a_{ij} = \min(g_i, d_j)$. Remove row i from the tableau if $a_{ij} = g_i$ and subtract g_i from d_j. Remove column j from the tableau if $a_{ij} = d_j$ and subtract d_j from g_i. Repeat the process for the resulting tableau. If a row and a column constraint are satisfied simultaneously, remove either the column or the row but not both. If the minimum is not unique choose the i+j that is the smallest.

This method yields the initial feasible solution for the same example. It is shown in the following tableau.

TABLEAU 3-7

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20		10				30
G_2			20	10	20		50
G_3		20			30	25	75
G_4		20					20
d_j	20	40	30	10	50	25	175

(5) VOGEL'S METHOD.

For each row and each column, find the difference of the lowest cost and the next lowest cost. There are m + n numbers. Choose the largest of the m + n numbers. Suppose it is in row i. Select the lowest cost c_{ij} in row i. Then set $a_{ij} = \min(g_i, d_j)$. Remove row i or column j depending on which constraint is satisfied. Repeat the process to get an initial feasible solution. If the largest difference of m + n numbers is not unique, select any one maximum. If a row constraint and a column constraint are satisfied simultaneously, remove either the row or the column, but not both.

The Vogel method is used to find an initial feasible solution for Example 3-1. This initial feasible solution is shown in tableau 3-8.

TABLEAU 3-8

	D_1	D_2	D_3	D_4	D_5	D_6	g_i	
G_1	20		10				30	1
G_2			10	10	30		50	1
G_3		40	10			25	75	2
G_4					20		20	2
d_j	20	40	30	10	50	25	175	
	2	1	1	2	1	1		

2. Stepping Stone Method.

Select the northwest corner rule to find an initial feasible solution of example 3-1. Denote the cell of the basic variable with an asterisk (*). This is shown in the following tableau.

TABLEAU 3-9

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*					30
G_2		30*	20*				50
G_3			10*	10*	50*	5*	75
G_4						20*	20
d_j	20	40	30	10	50	25	175

Fill the blank cell (i, j) with the value of $z_{ij} - c_{ij}$.

Let $z_{ij} - c_{ij} = c_{ik} - c_{\ k} + c_{\ m} - \ldots - c_{rs} + c_{rj} - c_{ij}$ (3-5)

where c_{ik}, $c_{\ k}$, $c_{\ m}$, ... , c_{rs}, c_{rj} are costs of the basic variables a_{ik}, a_{hk}, a_{hm}, ... , a_{rs}, a_{rj}. The technique used to choose these basic variables is illustrated as follows. In tableau 3-3 there are 9 basic costs.

$$c_{11} = 1 \qquad c_{12} = 10 \qquad c_{22} = 3$$
$$c_{23} = 2 \qquad c_{33} = 5 \qquad c_{34} = 9$$
$$c_{35} = 6 \qquad c_{36} = 2 \qquad c_{46} = 6$$

By (3-5),

$$z_{21} - c_{21} = c_{22} - c_{12} + c_{11} - c_{21} = 3 - 2 + 1 - 3 = -1,$$

$$z_{31} - c_{31} = c_{33} - c_{23} + c_{22} - c_{12} + c_{11} - c_{31}$$
$$= 5 - 2 + 3 - 2 + 1 - 4 = 1 .$$

Similarly, the other values can be found and tableau 3-9 completed. The new tableau is shown in tableau 3-10.

TABLEAU 3-10

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	0	1	-3	-4	30
G_2	-1	30*	20*	5	-1	-4	50
G_3	1	4	10*	10*	50*	5*	75
G_5	6	9	2	10	6	20*	20
d_j	20	40	30	10	50	25	175

If all $z_{ij} - c_{ij} \leq 0$, then an optimal solution is obtained.

Not all $z_{ij} - c_{ij} \leq 0$ in tableau 3-10. Consider the largest one, in this case $z_{44} - c_{44} = 10$ where

$$z_{44} - c_{44} = c_{46} - c_{36} + c_{34} - c_{44} . \qquad (3-5)$$

The coefficients of c_{46} and c_{34} are positive in (3-5). By the simplex method the variable $\min(a_{46}, a_{34}) = a_{34}$ is determined to leave the simplex method tableau. Hence, replace a_{34} by a_{44}. Now a_{44} is a basic variable in the new transportation problem tableau.

The new feasible solution is obtained in the following way.

$$\bar{a}_{44} = a_{34} = 10$$

$$\bar{a}_{46} = a_{46} - a_{34} = 20 - 10$$

$$\bar{a}_{36} = a_{36} + a_{34} = 5 + 10 = 15 .$$

$\bar{a}_{46}$ and $\bar{a}_{36}$ are obtained by the formula (3-6).

$$\bar{a}_{ij} = \begin{cases} a_{ij} - a_{34}, & \text{if the coefficient of } c_{ij} \text{ in (3-5) is } +1 \\ a_{ij} + a_{34}, & \text{if the coefficient of } c_{ij} \text{ in (3-5) is } -1. \end{cases}$$

All other basic variables will not change their values if their costs did not occur in (3-5).

A new feasible solution is shown in the following tableau.

TABLEAU 3-11

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*					30
G_2		30*	20*				50
G_3			10*		50*	15*	75
G_4				10*		10*	20
d_j	20	40	30	10	50	25	175

By referring to tableau 3-3, compare the value of z from tableau 3-9 with the value of z from tableau 3-11.

In tableau 3-9

$$z = 1\cdot20 + 2\cdot10 + 3\cdot30 + 2\cdot20 + 5\cdot10 + 9\cdot10 + 6\cdot50 + 2\cdot5 + 6\cdot20 = 740$$

In tableau 3-11

$$z = 1\cdot20 + 2\cdot10 + 3\cdot30 + 2\cdot20 + 5\cdot10 + 3\cdot10 + 6\cdot50 + 2\cdot15 + 6\cdot10 = 640$$

The value of z for the new feasible solution is smaller than the value of z for the initial feasible solution.

Use the same method to complete tableau 3-11. This yields tableau 3-12.

TABLEAU 3-12

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	0	-9	-3	-4	30
G_2	-1	30*	20*	-5	-1	-4	50
G_3	1	4	10*	-10	50*	15*	75
G_4	6	9	2	10*	6	10*	20
d_j	20	40	30	10	50	25	175

Not all $z_{ij} - c_{ij} \leq 0$ in tableau 3-12. Continue using the same method for each subsequent tableau until the optimal solution is reached. These tableaus are shown in tableau 3-13 through tableau 3-18.

TABLEAU 3-13

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	0	0	-3	-4	30
G_2	-1	20*	30*	4	-1	-4	50
G_3	1	4	0*	-1	50*	25*	75
G_4	-3	10*	-7	10*	-3	-9	20
d_j	20	40	30	10	50	25	175

TABLEAU 3-14

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	0	0	1	0	30
G_2	-1	20*	30*	4	3	0	50
G_3	-3	0*	-4	-5	50*	25*	75
G_4	-3	10*	-7	10*	1	-5	20
d_j	20	40	30	10	50	25	175

TABLEAU 3-15

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	0	-4	1	0	30
G_2	-1	10*	30*	10*	3	0	50
G_3	-3	0*	-4	-9	50*	25*	75
G_4	-3	20*	-7	-4	1	-5	20
d_j	20	40	30	10	50	25	175

TABLEAU 3-16

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	10*	3	-1	1	0	30
G_2	-4	-3	30*	10*	10*	-3	50
G_3	-3	10*	-1	-6	40*	25*	75
G_4	-3	20*	-4	1	1	-5	20
d_j	20	40	30	10	50	25	175

TABLEAU 3-17

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	-3	10*	-4	-2	-3	30
G_2	-1	-3	20*	10*	20*	-3	50
G_3	0	20*	-1	-6	30*	25*	75
G_4	0	20*	-4	-1	1	-5	20
d_j	20	40	30	10	50	25	175

TABLEAU 3-18

	D_1	D_2	D_3	D_4	D_5	D_6	g_i
G_1	20*	-3	10*	-4	-2	-3	30
G_2	-1	-3	20*	10*	20*	-3	50
G_3	0	40*	-1	-6	10*	25*	75
G_4	-1	-1	-5	-2	20*	-6	20
d_j	20	40	30	10	50	25	175

In tableau 3-18 all $z_{ij} - c_{ij} \leq 0$. Therefore, an optimal solution is

$$
\begin{array}{lll}
a_{11} = 20 & a_{13} = 10 & a_{23} = 20 \\
a_{24} = 10 & a_{25} = 20 & a_{32} = 40 \\
a_{35} = 10 & a_{36} = 25 & a_{45} = 20
\end{array}
$$

All other $a_{ij} - 0$. Then

$$
\begin{aligned}
\min z = {} & 1\cdot 20 + 1\cdot 10 + 2\cdot 20 + 1\cdot 10 + 4\cdot 20 + 2\cdot 40 + 6\cdot 10 \\
& + 2\cdot 25 + 4\cdot 20 = 430.
\end{aligned}
$$

3. Justification of the Stepping Stone Method.

Consider the transportation problem

$$
\begin{array}{lllllll}
a_{11} + \cdots + a_{1n} & & & & & = & g_1 \\
 & a_{21} + \cdots + a_{2n} & & & & = & g_2 \\
 & & \ddots & & & & \vdots \\
 & & & a_{m1} + \cdots + a_{mn} & & = & g_m \\
a_{11} & + \; a_{21} & + \cdots & + \; a_{m1} & & = & d_1 \\
\ddots & & & & & & \vdots \\
\quad a_{1n} & + \; a_{2n} & + \cdots & + \; a_{mn} & & = & d_n
\end{array}
\tag{3-6}
$$

or in matrix form

$$
P\not{a} = \not{b} \tag{3-7}
$$

where P is a matrix with $m + n$ rows and mn columns. Denote the origin rows of P by $\emptyset_i$, $i = 1, \ldots, m$, and the destination rows by γ_j, $j = 1, \ldots, n$. Then

$$\sum_{i=1}^{m} \emptyset_i - \sum_{j=1}^{n} \not{N}_j = \emptyset \qquad (3\text{-}8)$$

Hence, the rank of P is less than m + n.

Consider a matrix C which is formed from P by first deleting the last row and then taking columns n, 2n, ... , mn, 1, 2, ... , n-1. Hence

$$C = \underbrace{\left|\begin{array}{cccccccccccc}
1 & 0 & \cdot & \cdot\cdot & \cdot & 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\
0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1
\end{array}\right|}_{m \;\; | \;\; n-1} \Bigg\} \; m+n-1$$

Since C is a square matrix of order m+n-1, r(P) = m+n-1. Hence, there exist a set of m+n-1 constraints in (3-6) which are linearly independent. Therefore, an optimal solution of a transportation problem never need have more than m+n-1 of the a_{ij} with a nonzero value.

From (3-8) the remaining row vector is a linear combination of the set of linearly independent row vectors.

Thus, choose any m+n-1 row vectors which are linearly independent.

THEOREM 3-1 Any determinant of KxK submatrix obtained from P by crossing out m+n-K rows and mn-K columns have the value ± 1 or 0, where $1 \leq K \leq m+n-1$.

PROOF. Suppose that P_K is such a matrix.

If P_K contains one or more columns of zeros, then $|P_K| = 0$.

If each column of P_K contains two of the number 1, then one must be in an origin row and the other one must be in a destination row. Then the sum of the origin rows minus the sum of the destination rows equals 0. Hence $|P_K| = 0$.

If every column of P_K contains one or two 1's, and at least one column contains a single 1 then

$$|P_K| = \pm |P_{K-1}|$$

where P_{K-1} is a (K-1) x (K-1) submatrix. Either $|P_{K-1}| = 0$ or $|P_{K-1}| = \pm |P_{K-2}|$. However, every $|P_1| = 0$ or 1, so $|P_K| = 0$ or ± 1.

THEOREM 3-2 Let $P\not{x} = b$ represent the transportation problem. From the simplex method

$$\sum_{cd} y^{ij}_{cd} \not{f}_{cd} = \not{p}_{ij} \tag{3-9}$$

where $\not{f}_{cd}$ is a basis vector. Then every $y^{ij}_{cd} = \pm 1$ or 0.

PROOF. In (3-9) $R\not{y}_{ij} = \not{p}_{ij}$, where R is a matrix formed from m+n-1 linearly independent columns of P. From the above discussion, any one row of R can be crossed out and the new matrix S will be nonsingular. Suppose the i-th row is crossed out, then all components of $\bar{p}_{ij}$ are zeros except the j+m-1th component, where $\bar{p}_{ij}$ is a column vector formed from p_{ij} by deleting the i-th component. Hence, $S\bar{p}_{ij} = e_{m+j-1}$, where e_{m+j-1} is a unit vector containing m+n-1 components. Then

$$\bar{p}_{ij} = S^{-1}e_{m+j-1} \, .$$

$S^{-1}e_{m+j-1}$ is the (j+m-1)th column of S^{-1} and each component of this column is a cofactor of an element in S divided by $|S|$. A cofactor of an element in S and $|S|$ are determinants of the submatrix obtained from P by deleting certain rows and columns. Therefore, from theorem 3-1, they are either ± 1 or 0 but $|S| \neq 0$. Thus, every $y^{ij}_{cd} = 0$ or ± 1.

Equation (1-33) gives the method for moving from one simplex method tableau to the next simplex tableau. Therefore,

$$\bar{a}_{kh} = \frac{a^B_{rs}}{y^{kh}_{rs}} \tag{3-10a}$$

and

$$\bar{a}^B_{ij} = a^B_{ij} - a^B_{rs}\frac{y^{kh}_{ij}}{y^{kh}_{rs}} \quad \text{where } ij \neq rs \, .$$

Now, y^{kh}_{rs} must equal 1, thus

$$\bar{a}_{kh} = a^B_{rs}$$

and (3-10b)

$$\bar{a}^B_{ij} = a^B_{ij} \pm a^B_{rs} \quad \text{for all } ij \neq rs$$

In (3-6), every column vector $\not{p}_{ij}$ in P has the following form:

$$\not{p}_{ij} = \not{e}_i + \not{e}_{m+j} , \tag{3-11}$$

where the $\not{e}_k$ are the unit vectors containing m+n components. The basis $\{\not{\ell}_{cd}\}$ contains m+n-1 vectors. Each $\not{\ell}_{cd}$ is a column vector of the matrix P. Using (3-9), omit the terms for which $y_{ij} = 0$. Then,

$$\not{p}_{ij} = \Sigma(\pm)\not{\ell}_{cd} \tag{3-12}$$

since $\not{p}_{ij}$ and $\not{\ell}_{cd}$ have the form (3-11). Hence,

$$\not{p}_{ij} = \not{\ell}_{ip} - \not{\ell}_{qp} + \not{\ell}_{qr} - \dots - \not{\ell}_{vu} + \not{\ell}_{vj} \tag{3-13}$$

where $\not{p}_{ij}$ is the column ij of P and $\not{\ell}_{st}$ is the column st of P. Therefore, the value of y^{ij}_{cd} depends on (3-13). From (1-28),

$$z_{ij} - c_{ij} = \sum_{cd} y^{ij}_{cd}\, c^B_{cd} - c_{ij} \tag{3-14}$$

where c^B_{cd} is the cost of the basic variable a_{cd}. For $\not{p}_{ij}$ in (3-13), (3-14) can be written as

$$z_{ij} - c_{ij} = c^B_{ip} - c^B_{qp} + c^B_{qr} - \dots - c^B_{vu} + c^B_{vj} - c^B_{ij} . \tag{3-15}$$

This justifies equation (3-5). By the same reason and from (3-10b), the new basic variables are

$$\bar{a}_{kh} = a^B_{rs} \tag{3-16}$$

and

$$\bar{a}^B_{ef} = a^B_{ef} \qquad (3\text{-}17)$$

if c^B_{ef} does not occur in the equation (3-15) representing $z_{kh} - c_{kh}$. The new basic variable

$$\bar{a}^B_{ef} = a^B_{ef} - a^B_{rs} \qquad (3\text{-}18)$$

if the coefficient of c^B_{ef} in $z_{kh} - c_{kh}$ is 1, and

$$\bar{a}^B_{ef} = a^B_{ef} + a^B_{rs} \qquad (3\text{-}19)$$

if the coefficient of c^B_{ef} in $z_{kh} - c_{kh}$ is -1.

These equations determine the new tableau.

4. Inequalities in the Constraints of a Transportation Problem

If, in the transportation problem, more units are available at the origins than are required at the destinations, then it has the following form:

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij} \leq g_i \qquad i=1, \dots, m \\ &\sum_{i=1}^{m} a_{ij} = d_j \qquad j=1, \dots, n \end{aligned} \qquad (3\text{-}20)$$

$a_{ij} \geq 0$, for all i, j.

$$\min z = \sum_{i,j} c_{ij} a_{ij} .$$

The inequalities can be converted to equalities by the addition of m slack variables.

Thus,

$$\sum_{j=1}^{n} a_{ij} + a_{in+1} = g_i \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} a_{ij} = d_j \qquad j = 1, \ldots, n \qquad (3\text{-}21)$$

$$\sum_{i=1}^{m} a_{in+1} = \sum_{i=1}^{m} g_i - \sum_{j=1}^{n} d_j = d_{n+1}$$

where a_{in+1}, $i = 1, \ldots, m$ are the slack variables and

$$\min z = \sum_{i,j} c_{ij}a_{ij} .$$

There are m+n+1 constraints and mn+m variables in (3-21). Hence, there is an optimal solution which never need have more than m+n of the a_{ij} different from zero.

Consider problem 3 in the last chapter. The cost tableau is presented in tableau 3-19.

TABLEAU 3-19

10	8	16	3	0
19	25	18	7	0
20	17	20	5	0

The northwest corner rule yields an initial feasible solution for this problem. Then the stepping stone method is used to set up the following tableaux.

TABLEAU 3-20

	D_1	D_2	D_3	D_4	S	g_i
G_1	2000*	1000*	3000*	2000*	-4	8000
G_2	-5	-13	2	2500*	2500*	5000
G_3	-6	-5	0	2	3000*	3000
d_j	2000	1000	3000	4500	5500	16000

The column S is the column of the slack variables.

TABLEAU 3-21

G_1	2000*	1000*	500*	4500*	-2	8000
G_2	-7	-15	2500*	-2	2500*	5000
G_3	-8	-7	-2	0	3000*	3000
d_j	2000	1000	3000	4500	5500	16000

All $z_{ij} - c_{ij} \leq 0$ in tableau 3-21. Therefore an optimal solution is:

$$a_{11} = 2000$$

$$a_{12} = 1000$$

$$a_{13} = 500$$

$$a_{14} = 4500$$

$$a_{23} = 2500$$

$$a_{25} = 2500$$

$$a_{35} = 3000$$

and $\min z = 10 \cdot 2000 + 8 \cdot 1000 + 16 \cdot 500 + 3 \cdot 4500 + 18 \cdot 2500$

$$+ 0 \cdot 2500 + 0 \cdot 3000$$

$$= 94500.$$

This is the same as (2-2).

Chapter IV

CURVE FITTING BY LINEAR PROGRAMMING

Linear programming can be applied to curve fitting problems.

Given x_i, y_i, $i = 1, \ldots, n$, (4-1)

fit this data to the equation:

$$y = c_1\emptyset_1(x) + c_2\emptyset_2(x) + \ldots + c_k\emptyset_k(x), \tag{4-2}$$

where the $\emptyset_i(x)$ are known functions without constants to be determined. The object is to find values of c_j, $j = 1, \ldots, k$ so that the data in (4-1) "best" fits the curve in (4-2). Consider two methods whereby the problem can be solved by the use of linear programming.

1. Method 1.

The first method is an attempt to minimize the sum of the absolute values of the deviations. Thus, a system of equations are formed as follows:

$$\sum_{j=1}^{k} c_j\emptyset_j(x_i) + S_i - T_i = y_i \text{ , } i = 1, \ldots, n \tag{4-3}$$

where S_i and T_i are the positive slack variables, the positive surplus variables or the artificial variables. In the linear programming problem the objective function is:

$$\min z = \sum_{i=1}^{n} S_i + T_i \tag{4-4}$$

Consider an example.

EXAMPLE 4-1

Given the set of points which are shown in (4-5), try to fit the curve $y = ax + b$ to the set of points, under the assumption that a and b are positive.

x	y
-1	1.1
0	4.1
1	6.8
2	9.9

(4-5)

Set up and solve the linear programming problem that determines a and b so that the sum of the absolute values of the deviations is a minimum. By (4-3), there is a set of constraints such that

$$a\cdot(-1) + b + S_1 - T_1 = 1.1$$
$$a\cdot 0 + b + S_2 - T_2 = 4.1$$
$$a\cdot 1 + b + S_3 - T_3 = 6.8$$
$$a\cdot 2 + b + S_4 - T_4 = 9.9$$

The objective function is:

$$\min z = S_1 + S_2 + S_3 + S_4 + T_1 + T_2 + T_3 + T_4$$

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, and x_{10} be a, b, S_1, S_2, S_3, S_4, T_1, T_2, T_3, and T_4 respectively. Use the simplex method to form the following tableaux.

TABLEAU 4-1

			0	0	1	1	1	1	1	1	1	1
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$	$\not{a}_7$	$\not{a}_8$	$\not{a}_9$	$\not{a}_{10}$
3	1	1.1	-1	1	1	0	0	0	-1	0	0	0
4	1	4.1	0	1	0	1	0	0	0	-1	0	0
5	1	6.8	1	1	0	0	1	0	0	0	-1	0
6	1	9.9	2	1	0	0	0	1	0	0	0	-1
		21.9	2	4	0	0	0	0	-2	-2	-2	-2

TABLEAU 4-2

			0	0	1	1	1	1	1	1	1	1
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$	$\not{a}_7$	$\not{a}_8$	$\not{a}_9$	$\not{a}_{10}$
3	1	6.05	0	1.5	1	0	0	0.5	-1	0	0	-0.5
4	1	4.1	0	1	0	1	0	0	0	-1	0	0
5	1	1.85	0	0.5	0	0	1	-0.5	0	0	-1	0.5
1	0	4.95	1	0.5	0	0	0	0.5	0	0	0	-0.5
		12.0	0	3	0	0	0	-1	-2	-2	-2	-1

TABLEAU 4-3

			0	0	1	1	1	1	1	1	1	1
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
3	1	0.5	0	0	1	0	-3	2	-1	0	3	-2
4	1	0.4	0	0	0	1	-2	1	0	-1	2	-1
2	0	3.7	0	1	0	0	2	-1	0	0	-2	1
1	0	3.1	1	0	0	0	-1	1	0	0	1	-1
		0.9	0	0	0	0	-6	2	-2	-2	4	-4

TABLEAU 4-4

			0	0	1	1	1	1	1	1	1	1
V_B	c_B	x_B	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
6	1	0.25	0	0	0.5	0	-1.5	1	-0.5	0	1.5	-1
4	1	0.15	0	0	-0.5	1	-0.5	0	0.5	-1	0.5	0
2	0	3.95	0	1	0.5	0	0.5	0	-0.5	0	-0.5	0
1	0	2.85	1	0	-0.5	0	0.5	0	0.5	0	-0.5	0
		0.4	0	0	-1	0	-3	0	-1	-2	1	-2

TABLEAU 4-5

			0	0	1	1	1	1	1	1	1	1
V_B	$\not{c}_B$	$\not{x}_B$	$\not{a}_1$	$\not{a}_2$	$\not{a}_3$	$\not{a}_4$	$\not{a}_5$	$\not{a}_6$	$\not{a}_7$	$\not{a}_8$	$\not{a}_9$	$\not{a}_{10}$
9	1	$\frac{1}{6}$	0	0	$\frac{1}{3}$	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	$-\frac{2}{3}$
4	1	$\frac{1}{15}$	0	0	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	-1	0	$\frac{1}{3}$
2	0	$4\frac{1}{30}$	0	1	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$
1	0	$2\frac{14}{15}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$-\frac{1}{3}$
		$\frac{7}{30}$	0	0	$-1\frac{1}{3}$	0	-2	$-\frac{2}{3}$	$-\frac{2}{3}$	-2	0	$-1\frac{1}{3}$

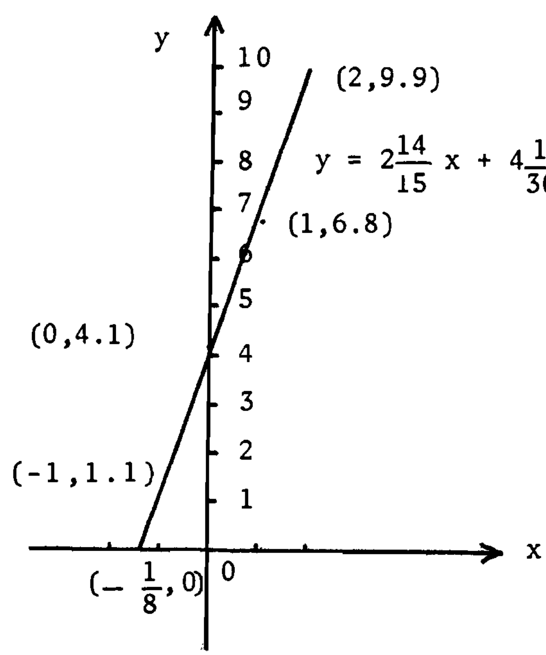
All $z_j - c_j$ in tableau 4-5 are non-positive. An optimal solution is: $a = 2\frac{14}{15}$, $b = 4\frac{1}{30}$

$$S_2 = \frac{1}{15}, \quad T_3 = \frac{1}{6},$$

and $\min z = \frac{7}{30}$.

Therefore, the curve is $y = 2\frac{14}{15}x + 4\frac{1}{30}$. This result is expressed graphically in figure 4-1.

FIGURE 4-1



2. Method 2.

The second method is to minimize the maximum absolute value of the deviation. Consider

$$r = \max_i \left| y_i - \sum_{j=1}^{k} c_j \phi_j(x_i) \right| \tag{4-6}$$

Then the following equations must be satisfied:

$$y_i - \sum_{j=1}^{k} c_j \phi_j(x_i) \leq r \tag{4-7}$$

and

$$-y_i + \sum_{j=1}^{k} c_j \phi_j(x_i) \leq r \tag{4-8}$$

$i = 1, \ldots, n.$

Therefore, the linear programming problem is to minimize r and have the following constraints:

$$-\sum_{j=1}^{k} c_j \phi_j(x_i) - r \leq -y_i \tag{4-9}$$

$$\sum_{j=1}^{k} c_j \phi_j(x_i) - r \leq y_i \tag{4-10}$$

$i=1, \ldots, n,$

where c_j, $j=1, \ldots, k$, and r are variables.

This method is used to solve example 4-1. First, set up the following constraints:

$$(-a)\cdot(-1) - b - r \leq -1.1$$

$$a\cdot(-1) + b - r \leq 1.1$$

$$-b - r \leq -4.1$$

$$b - r \leq 4.1$$

$$(-a)\cdot 1 - b - r \leq -6.8$$

$$a\cdot 1 + b - r \leq 6.8$$

$$(-a)\cdot 2 - b - r \leq -9.9$$

$$a\cdot 2 + b - r \leq 9.9$$

and an objective function

$$\min z = r \ .$$

Then add the slack variables, the surplus variables, and the artificial variables to the above inequalities. Thus,

$$\begin{aligned}
a\cdot(-1) + b + r - x_4 + x_{12} &= 1.1 \\
a\cdot(-1) + b - r + x_5 &= 1.1 \\
b + r - x_6 + x_{13} &= 4.1 \\
b - r + x_7 &= 4.1 \\
a\cdot 1 + b + r - x_8 + x_{14} &= 6.8 \\
a\cdot 1 + b - r + x_9 &= 6.8 \\
a\cdot 2 + b + r - x_{10} + x_{15} &= 9.9 \\
a\cdot 2 + b - r + x_{11} &= 9.9
\end{aligned}$$

and $\min z = 0\cdot a + 0\cdot b + r + 0\cdot x_4 + 0\cdot x_5 + 0\cdot x_6 + 0\cdot x_7$

$$+ 0\cdot x_8 + 0\cdot x_9 + 0\cdot x_{10} + 0\cdot x_{11} + Mx_{12} + Mx_{13}$$

$$+ Mx_{14} + Mx_{15}$$

where M is an arbitrary large positive number.

It is too complicated to compute this problem by hand. The FORTRAN program which is listed in appendix I, with necessary format changes, is used to solve this linear programming problem by the computer. The following result is obtained.

Thus, max $(-z) = -0.99999840\ E\ -01$

$x(4) = 0.99999480\ E\ -01$

$x(3) = 0.99999840\ E\ -01$

$x(6) = 0.19999980\ E\ 00$

$x(5) = 0.10000060\ E\ 00$

$x(9) = 0.19999970\ E\ 00$

$x(2) = 0.40000000\ E\ 01$

$x(10) = 0.19999970\ E\ 00$

$x(1) = 0.28999980\ E\ 01$

Hence,

$$r = \min(z) = -\max(-z) = 0.09999984 \approx 0.1$$

and

$$a = x(1) \approx 2.9$$

$$b = x(2) \approx 4$$

Therefore, the curve is $y \approx 2.9x + 4$.

Chapter V

SUMMARY

In this paper the basic linear programming problems have been discussed graphically and numerically. The simplex method has been presented and justified, and some practical applications of linear programming discussed. A special method, the transportation method, of linear programming has been discussed and justified. It was shown that linear programming is also applicable to curve fitting problems.

BIBLIOGRAPHY

1. Antle, Charles. Notes on Linear Programming. Penn State University.

2. Dantzig, G.B. Linear Programming and Extensions. Princeton, N.J.: Princeton University Press, 1963.

3. Hadley, George. Linear Programming. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1962.

4. Loomba, N. Paul. Linear Programming. McGraw-Hill Book Co., 1964.

5. Moore, B.C. Notes on Linear Programming. Texas A&M University.

6. Van de Panne, C. Linear Programming and Relative Techniques. Amsterdam, The Netherlands: North-Holland Publishing Co.; New York: American Elsevier Publishing Co., Inc., 1971.

APPENDIX I

The following is FORTRAN IV program to solve a linear programming problem using the Simplex method.

```
C  USING THE SIMPLEX METHØD TØ SØLVE THE
      LINEAR PRØGRAMMING PRØBLEM
      DIMENSIØN A(25, 25), C(20), R(20)
      READ (5, 1) N, M
    1 FØRMAT (2I5)
      M3 = M + 3
      N1 = N + 1
C  READ TABLEAU
      READ (5, 2) ((A(I,J), J = 1, M3), I = 1, N)
    2 FØRMAT (F6.0, 2F8.0, 8F5.0/11F5.0)
      READ (5, 3) (C(I), I = 1, M)
    3 FØRMAT (10F7.0/9F7.0)
C  CØMPUTE THE LAST RØW
   30 DØ 6 J = 3, M3
      A(N1, J) = 0.
      DØ 4 I = 1, N
    4 A(N1, J) = A(N1, J) + A(I, 2) * A(I, J)
      IF (J-3) 100, 6, 5
    5 K = J-3
      A(N1, J) = A(N1, J) - C(K)
    6 CØNTINUE
C  DETERMINE THE CØLUMN THAT ENTERS
      J=4
```

```
   18 IF (A(N1, J)) 7, 8, 8
    7 L = J
      GØ TØ 11
    8 J = J + 1
      IF (J-M3) 9, 9, 10
    9 GØ TØ 18
   10 GØ TØ 90
C  DETERMINE THE RØW THAT LEAVES
   11 DØ 14 I = 1, N
      IF (A(I, L)) 12, 12, 13
   12 R(I) = -1.
      GØ TØ 14
   13 R(I) = A(I, 3)/A(I, L)
   14 CØNTINUE
      K = -1
      Z = 1.E + 20
      DØ 17 I = 1, N
      IF (R(I)) 17, 15, 15
   15 IF (R(I) - Z) 16, 16, 17
   16 X = R(I)
      K = I
   17 CØNTINUE
      IF (K.EQ. -1) GØ TØ 80
      KK = L - 3
C  NEW TABLEAU
      D = L - 3
```

```
      A(K, 1) = D
      A(K, 2) = C(KK)
      S = A(K, L)
      DØ 21 J = 3, M3
   21 A(K, J) = A(K, J)/S
      DØ 23 I = 1, N
      IF (I.EQ. K) GØ TØ 23
      T = A(I, L)
      DØ 22 J = 3, M3
   22 A(I, J) = A(I, J) - T * A(K, J)
   23 CØNTINUE
      GØ TO 30
C  UNBØUNDED SØLUTIØN
   80 WRITE (6, 85)
   85 FØRMAT (19H UNBØUNDED SØLUTIØN)
      GØ TØ 100
C  WRITE SØLUTIØN
   90 WRITE (6, 91) A(N1, 3)
   91 FØRMAT (18H ØPTIMAL VALUE IS , E20.8)
      DØ 92 I = 1, N
      II = A(I, 1)
   92 WRITE (6, 93) II, A(I, 3)
   93 FØRMAT (10X, 3H X(, I3, 5H ) =  , E20.8)
  100 STØP
      END
```