COMPACTIFICATIONS

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CHAPTER I

INTRODUCTION

There are many topological spaces that are not compact, just as there are metric spaces that are not complete. The question arises, is there a process to compactify a topological space corresponding to the completion process of a metric space? And, if a process does exist, what relationships exist between the original space and the new compact space? Topologists have produced a large quantity of material in answering these questions.

1. THE PROBLEM

The purpose of this study is to develope the methods of compactifying a topological space and to show what relationships hold between the original space and the new compact space. Proofs concerning topological spaces are, at times, made simpler if the topological space in question is compact. Thus, if the space in question can be compactified and if several important relationships hold between the new compact space and the original space, then the topologist has a useful tool. Compactifying different types of topological spaces produces examples and counterexamples of various topological spaces.

Since there has been a very extensive development of the problem under consideration, this paper can only develope a portion of the solution. The areas covered in this paper are such that the reader with an introduction to topology should be able to follow with little difficulty.

2. BRIEF HISTORY

Compactifications are in part dense embeddings, and as such are sometimes referred to as extensions of spaces. In 1913 Caratheodory stimulated the development of extensions by his work on "prime ends." But it was not until 1924 (Thron [14, 131]) that Tietze, Urysohn, and Alexandroff used the concepts of and formed the one-point compactification. In 1930 Tychonoff showed that completely regular spaces could be extended to compact T_2-spaces, for which he used the product space of closed intervals [11, 144]. Then in 1937, independently and using different approaches, E. Čech and M.H. Stone obtained what is now known as the Stone-Čech compactification. Other types of compactifications have been developed since then, see for example Thron [14, 131] or Gillman and Jerison [5, 269-70] .

3. DEFINITIONS OF TERMS USED

The reader is assumed to have a basic knowledge of topology and to be familiar with the basic terminology.

Definitions of terms fundamental to the entire paper are presented here, other definitions can be found in the chapters where they first appear. The reader is referred to Simmons [12] for definitions not listed in this paper.

DEFINITION 3.1 A class $\{O_i\}$ of open sets of the topological space $(X,\mathcal{T})$ is said to be an open cover of X if and only if X is contained in the union of $\{O_i\}$.

DEFINITION 3.2 If $\{O_j\}$ is a subclass of the open cover $\{O_i\}$ such that $\{O_j\}$ is an open cover itself, then $\{O_j\}$ is called a subcover.

DEFINITION 3.3 $(X,\mathcal{T})$ is called a compact topological space if every open cover of X contains a finite subcover.

DEFINITION 3.4 If $(Y,\mathcal{T}')$ is a compact topological space such that $(X,\mathcal{T})$ is homeomorphic to a dense subspace of Y, then $(Y,\mathcal{T}')$ is called a compactification of $(X,\mathcal{T})$.

After it has been established that $(X,\mathcal{T})$ is homeomorphic to a subspace of $(Y,\mathcal{T}')$, X will be fully identified with its homeomorphic image. For example, instead of saying $h(X)$ is open in $(Y,\mathcal{T}')$ it will be stated that X is open in $(Y,\mathcal{T}')$.

CHAPTER II

ONE-POINT COMPACTIFICATION

1. INTRODUCTION

In this chapter the concept of a one-point compactification is developed. The main purpose of this chapter is to introduce the structure of a one-point compactification and to note some of the relationships between a topological space and a one-point compactification of the space. An important aspect of a one-point compactification of an arbitrary topological space is the relative ease of the construction.

DEFINITION 1.1 $(Y', \mathcal{T}')$ is a one-point compactification of $(X, \mathcal{T})$ provided $(Y', \mathcal{T}')$ is a compactification of $(X, \mathcal{T})$ and $Y' - X$ is a singleton set.

2. THE TRIVIAL ONE-POINT COMPACTIFICATION

Several constructions of a one-point compactification are available. One of these constructions is a one-point compactification of any topological space $(X, \mathcal{T})$. This construction of a one-point compactification is done in a trivial manner and may be called the trivial one-point compactification. It is formed by adding a distinct new single element ∞ to X and by adding only one new open set to $\mathcal{T}$. The open set to be added is $X \cup \{\infty\}$. This space is clearly compact since the only open set containing the point ∞ is the set $X \cup \{\infty\}$, which also covers the entire space. This new space is clearly not T_0. Thus very few relationships carry over from the old space to the new compact space. Hence, better constructions of a one-point compactification are needed.

3. ALEXANDROFF ONE-POINT COMPACTIFICATION

DEFINITION 3.1 The Alexandroff one-point compactification of a topological space $(X, \mathcal{T})$ is the set $X' = X \cup \{\infty\}$, where ∞ is any element which is not a member of X. The topology of X' consists of the open sets of X and all subsets of the form $H \cup \{\infty\}$ such that the complement of H in X is compact and closed in X. The Alexandroff one-point compactification of $(X, \mathcal{T})$ is denoted by $(X', \mathcal{T}')$.

The rest of this section will show the relative importance of the construction of the Alexandroff one-point compactification.

THEOREM 3.1 Let $(X', \mathcal{T}')$ be the Alexandroff one-point compactification of the topological space $(X, \mathcal{T})$. $(X', \mathcal{T}')$ is a compact topological space.

PROOF. Since $\emptyset$ belongs to $\mathcal{T}$, $\emptyset$ belongs to $\mathcal{T}'$. The null set is compact and closed in X, hence $X \cup \{\infty\}$ or X' belongs to $\mathcal{T}'$. Let A and B be any two distinct elements of $\mathcal{T}'$. If A and B both belong to $\mathcal{T}$ then $A \cap B$ belongs to $\mathcal{T}$ and hence to $\mathcal{T}'$, also arbitrary unions of elements in $\mathcal{T}$ belong to $\mathcal{T}$ and hence to $\mathcal{T}'$. If A and B both do not belong to $\mathcal{T}$ then let $A = C_X(P) \cup \{\infty\}$ and $B = C_X(Q) \cup \{\infty\}$ where P and Q are compact closed subsets of $(X, \mathcal{T})$. Then, $A \cup B = (C_X(P) \cup \{\infty\}) \cup (C_X(Q) \cup \{\infty\}) = (C_X(P) \cup C_X(Q)) \cup \{\infty\} = C_X(P \cap Q) \cup \{\infty\}$. But since the intersection of two compact closed subsets is compact and closed, it is seen that $A \cup B$ belongs to $\mathcal{T}'$. The intersection of any collection of compact closed sets is compact and closed. Thus it follows the union of any collection of elements in $\mathcal{T}' - \mathcal{T}$ belong to $\mathcal{T}'$. Now $A \cap B = (C_X(P) \cup \{\infty\}) \cap (C_X(Q) \cup \{\infty\}) = [C_X(P) \cap (C_X(Q) \cup \{\infty\})] \cup [\{\infty\} \cap (C_X(Q) \cup \{\infty\})] = [(C_X(P) \cap C_X(Q)) \cup (C_X(P) \cap \{\infty\})] \cup \{\infty\} = (C_X(P) \cap C_X(Q)) \cup \{\infty\} = C_X(P \cup Q) \cup \{\infty\}$. But since the union of two compact closed subsets is compact and closed, it is seen that $A \cap B$ belongs to $\mathcal{T}'$.

Suppose $A = C_X(P) \cup \{\infty\}$ and B belongs to $\mathcal{T}$, then $A \cup B = (C_X(P) \cup \{\infty\}) \cup B = (C_X(P) \cup \{\infty\}) \cup C_X(C_X(B)) = (C_X(P) \cup C_X(C_X(B))) \cup \{\infty\} = C_X(P \cap C_X(B)) \cup \{\infty\}$. Since B is open in X, $C_X(B)$ is closed in X and hence, since P is closed and compact, $P \cap C_X(B)$ is closed and compact in X. Thus $A \cup B = C_X(P \cap C_X(B)) \cup \{\infty\}$ belongs to $\mathcal{T}'$. Now $A \cap B = (C_X(P) \cup \{\infty\}) \cap B = (C_X(P) \cap B) \cup (\{\infty\} \cap B) = C_X(P) \cap B$. Since P is closed in X, $C_X(P)$ is open in X and hence $C_X(P) \cap B$ belongs to $\mathcal{T}$. Therefore, $A \cap B$ belongs to $\mathcal{T}'$. Thus $(X', \mathcal{T}')$ satisfies the requirements for a topological space.

Let $\{O_i\}$ be any open cover of X'. Since each O_i is open in X', each O_i is either an open set in $\mathcal{T}$ or is of the form $C_X(P) \cup \{\infty\}$ where P is a closed compact subset of X. Since $\{O_i\}$ covers X', some fixed O_j in $\{O_i\}$ contains the point ∞. It follows that O_j is of the form $C_X(K) \cup \{\infty\}$ where K is a closed compact subset of X. Considering that K is compact there exists a finite subcover of $\{O_i\}$ that covers K. Note that $K \cup O_j = K \cup (C_X(K) \cup \{\infty\}) = X \cup \{\infty\} = X'$. Therefore, the finite subcover of K together with O_j covers X'. Thus X' is compact.

THEOREM 3.2 The relative topology on X in $(X', \mathcal{T}')$, the Alexandroff one-point compactification, is $\mathcal{T}$.

PROOF. Let $(X', \mathcal{T}')$ be the Alexandroff one-point compactification. By the definition of the Alexandroff one-point compactification $\mathcal{T} \leq \mathcal{T}'$.

Assume that O is any open set in X'. Therefore O is either of the form $C_X(K) \cup \{\infty\}$ or O is in $\mathcal{T}$. If O is in $\mathcal{T}$ then $O \cap X = O$ is in the relative topology. If O is of the form $C_X(K) \cup \{\infty\}$ then $O \cap X = (C_X(K) \cup \{\infty\}) \cap X = C_X(K)$. Since K is closed and compact in X, $C_X(K)$ is open in X and is in $\mathcal{T}$. Thus the relative topology on X in $(X', \mathcal{T}')$ is $\mathcal{T}$.

THEOREM 3.3 $(X, \mathcal{T})$ is not compact if and only if X is a dense subset of $(X', \mathcal{T}')$.

PROOF. Assume that X is not compact. Since X is not compact its complement in X, $\emptyset$, union with $\{\infty\}$ is not in $\mathcal{T}'$. Therefore $\{\infty\}$ is not in $\mathcal{T}'$ and hence is not open in X'. Thus every open set in X' containing ∞ must contain some point of X, and hence X is dense in X'.

Assume that X is a dense subset of X'. Since X is dense in X' every open set in X' containing ∞ must contain some point of X. Therefore $\{\infty\}$ is not in $\mathcal{T}'$. Since $\{\infty\}$ is not in $\mathcal{T}'$, $\emptyset$ cannot be the complement of a closed and compact set in X. Thus X is not compact.

COROLLARY 3.1 $(X, \mathcal{T})$ is compact if and only if the point ∞ is not an isolated point of $(X', \mathcal{T}')$.

PROOF. The proof follows from the preceding theorem.

THEOREM 3.4 The Alexandroff one-point compactification $(X', \mathcal{T}')$ of a non-compact space $(X, \mathcal{T})$ is a one-point compactification.

PROOF. The result follows from Theorems 3.1, 3.2, and 3.3.

The Alexandroff one-point compactification is thus a one-point compactification. The next theorem shows why if any one-point compactification is to be used, the Alexandroff one-point compactification is the one considered most often.

THEOREM 3.5 If any one-point compactification $(Y, \mathcal{T}_0)$ of $(X, \mathcal{T})$ is a T_2-space, then $(Y, \mathcal{T}_0)$ is precisely the Alexandroff compactification.

PROOF. Assume that Y is a T_2-space, then $\{\infty\}$ is closed in Y and hence X is open in Y. If Q is any open set in $\mathcal{T}$, Q is an open set of X as a subspace of Y. Therefore there exists an open set M of Y such that $M \cap X = Q$, but this implies that Q is also open in Y. Let N be any open set in Y such that $\infty \notin N$. Therefore $N \subset X$ and hence, N is open in X as a subspace of Y. But this indicates that N is also open in $(X, \mathcal{T})$.

Let O be any open set in Y such that $\infty \in O$. Set H to be the complement of O in X. Now $H = C_X(O) = C_{X'}(O)$. Then H is closed and compact in X. Thus, if $\infty \in O \in \mathcal{T}_0$ then O is of the form $C_X(H) \cup \{\infty\}$ where H is closed and compact in X.

Let K be any closed and compact subset of X. It must be shown that $C_X(K) \cup \{\infty\}$ is open in Y. Certainly the compact subset K is closed in the Hausdorff space Y. Thus $C_Y(K)$ is open in Y. But $C_Y(K) = C_X(K) \cup \{\infty\}$.

Thus if a topological space has a Hausdorff one-point compactification, then that compactification is precisely the Alexandroff one-point compactification.

4. PROPERTIES OF THE ALEXANDROFF ONE-POINT COMPACTIFICATION

Most of the relationships between $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ are concerned with the separation axioms. Several other relationships partially rely on these separation axioms. Thus a natural building of theorems might be observed. Throughout the rest of this chapter $(X', \mathcal{T}')$ will denote the Alexandroff one-point compactification of the topological space $(X, \mathcal{T})$.

THEOREM 4.1 $(X', \mathcal{T}')$ is T_1 if and only if $(X, \mathcal{T})$ is T_1.

PROOF. Assume that X is a T_1-space. Let a and b be distinct elements of X' such that $a = \infty$ and $b \in X$. Since $\{b\}$ is a singleton set in X, it is seen to be closed and compact in X. Therefore $C_X\{b\} \cup \{\infty\}$ is open in X'. Thus the complement of $C_X\{b\} \cup \{\infty\}$ in X', namely $\{b\}$, is closed in X'. Since X is open in X', the complement of X in X', namely $\{\infty\}$, is closed in X'. Thus singleton sets are closed in X' and X' is T_1.

Assume that X' is a T_1-space. T_1 is a hereditary property. Since the relative topology of X in $(X', \mathcal{T}')$ is $\mathcal{T}$, it is seen that X is a T_1-space.

THEOREM 4.2 $(X', \mathcal{T}')$ is T_2 if and only if $(X, \mathcal{T})$ is T_2 and locally compact.

PROOF. Assume that X' is a T_2 space. Since the relative topology of X as a subspace of X' is $\mathcal{T}$, then $(X, \mathcal{T})$ is a subspace of X'. X is T_2 since T_2 is a hereditary property. Let a be any element of X. Since X' is T_2 there exists disjoint open sets A and B in X' such that $a \in A$ and $\infty \in B$. Because A and B are disjoint $\infty \notin cl_{X'}(A)$. Now $cl_X(A) = cl_{X'}(A) \cap X = cl_{X'}(A)$ is a closed subset of X', and is therefore compact. Hence, X is locally compact since every point in X has a compact neighborhood.

Assume that X is T_2 and locally compact. Let a and b be distinct elements of X'. If $a, b \in X$, there exists disjoint open sets A and B in X such that $a \in A$ and $b \in B$. By definition A and B are also in $\mathcal{T}'$. Suppose $a = \infty$ and $b \in X$. Since X is locally compact there exists a compact neighborhood N of b in X. By the definition of a neighborhood there exists an open set O in X such that $b \in O \subset N$. Since X is T_2, N is closed. Therefore $C_X(N) \cup \{\infty\} \in \mathcal{T}'$. Now $C_X(N) \cup \{\infty\}$ and O are disjoint open sets in X', and $a = \infty \in C_X(N) \cup \{\infty\}$. Thus X' is T_2.

Since every compact Hausdorff space is normal the following corollary holds.

COROLLARY 4.1 If $(X, \mathcal{T})$ is a locally compact T_2-space, then $(X', \mathcal{T}')$ is normal.

THEOREM 4.3 If X is connected and not compact, then X' is connected.

PROOF. Assume that X is connected and not compact. Let O be any subset of X' that is both open and closed in X'. It may be assumed that $\infty \notin O$. Therefore $O \subset X$ and $O \in \mathcal{T}$. Since O is also closed in X', $X' - O$ is open in X'. $(X' - O) \cap X$ is thus open in X and is in $\mathcal{T}$. Note that $O \cap [(X' - O) \cap X] = \emptyset$ and $O \cup [(X' - O) \cap X] = X$. Hence O and $(X' - O) \cap X$ are complements in X, and O is both open and closed in X.

But since X is connected, O is either $\emptyset$ or X. If $O = X$, then X is both open and closed in X' and $\{\infty\}$ is both open and closed in X', but by corollary 3.1, this contradicts the fact that X is not compact. Therefore

$O \neq \emptyset$ and the only subsets of X' that are both open and closed are $\emptyset$ and X'. Thus X' is connected.

The following example shows that the converse of the preceding theorem is not true.

EXAMPLE 4.1 Let Q denote the set of rational numbers with the usual topology. Q is neither compact nor connected. It will be shown that Q', the Alexandroff one-point compactification of Q, is connected.

Let O be any subset of Q' that is both open and closed in Q'. It may be assumed that O is of the form $C_Q(P) \cup \{\infty\}$ where P is closed and compact in Q. However, since O is open and closed in Q', $C_{Q'}(O)$ is also open and closed in Q', but $C_{Q'}(O) = P$. Therefore P is both open and compact in Q. Since the only set in Q that is both compact and open in Q is $\emptyset$, Q' is seen to be connected.

THEOREM 4.4 If X' is second countable, then X is second countable.

PROOF. Assume that X' is second countable. By theorem 3.2 $(X, \mathcal{T})$ is seen to be a subspace of $(X', \mathcal{T}')$. Since second countability is hereditary, $(X, \mathcal{T})$ is second countable.

THEOREM 4.5 If X is a second countable T_2-space, then X' is second countable.

PROOF. Assume that X is a second countable T_2-space. Since X is second countable there exists a countable base β for the topology $\mathcal{T}$ on X. The topology $\mathcal{T}'$ consists of $\mathcal{T}$ and all sets of the form $C_X(K) \cup \{\infty\}$ where K is compact in X, a T_2-space. Since there is a countable bases for $\mathcal{T}$ it is only necessary to show that there exists a countable bases for the sets in $\mathcal{T}'$ that contain ∞.

Let P be the complement in X of the compact subset K of X. Since X is T_2, K is closed in X and hence, P is open in X. Therefore, there exists a countable base β^* for the complements in X of the compact subsets of X. Now union each member of β^* with $\{\infty\}$ and denote this by β^{**}. It is easily seen that β^{**} generates all open sets in $\mathcal{T}'$ that contain the point ∞. Therefore, the countable collection $\beta \cup \beta^{**}$ is seen to generate $\mathcal{T}'$. Thus X' is second countable.

THEOREM 4.6 $(X, \mathcal{T})$ is separable if and only if $(X', \mathcal{T}')$ is separable.

PROOF. Assume that X is separable, then there exists a subset G of X such that G is countable and $cl_X(G) = X$. Let $G' = G \cup \{\infty\}$ and note that G' is countable. Let p be any point in X' and let 0 be any open set in X' that contains p. If 0 contains the point ∞, then 0 contains a point of G'. If 0 does not contain the point ∞, then 0 is also an open set in $\mathcal{T}$. Hence 0 contains a point of G and thus a point of G'. Therefore $cl_{X'}(G') = X'$ and X' is separable.

Assume that X' is separable, then there exists a subset G' of X' such that G' is countable and

$cl_{X'}(G') = X'$. Consider the set G such that $G = G' - \{\infty\}$. Thus G is a countable subset of X. Let p be any point of X and let O be any open set in $\mathcal{T}$ that contains p. Since O is in $\mathcal{T}$ it is also in $\mathcal{T}'$ and hence, O contains a point of G'. But since $\infty \notin O$, O must contain a point of G. Thus $cl_X(G) = X$ and X is separable.

THEOREM 4.7 If $(X, \mathcal{T})$ is a locally compact separable metrizable space, then $(X', \mathcal{T}')$ is a separable metrizable space.

PROOF. Assume that X is a locally compact separable metrizable space. Since X is metrizable and separable, X is normal and second countable. By corollary 4.1, since X is locally compact and T_2, X' is normal. By theorem 4.5, since X is a second countable T_2-space, X' is second countable. Hence, X' is a second countable normal space and by Urysohn's metrization theorem, X' is metrizable. Since X is separable, by theorem 4.6, X' is separable. Thus X' is a separable metrizable space.

The reader interested in necessary and sufficient conditions for a topological space to be metrizable is referred to Kelly [10].

THEOREM 4.8 If the set A is a nowhere dense subset of X', then $A \cap X$ is a nowhere dense subset of X.

PROOF. Let A be a nowhere dense subset of X'. Let O be any non-empty open set in $\mathcal{T}$. Now O is also in $\mathcal{T}'$ and $0 \not\subset cl_{X'}A$. Since $cl_X(A \cap X) \subset cl_{X'}(A \cap X) \subset cl_{X'}(A)$ it follows that $0 \not\subset cl_X(A \cap X)$. Thus $A \cap X$ is nowhere dense in X.

THEOREM 4.9 If the set A is a nowhere dense subset of X, then A is a nowhere dense subset of X'.

PROOF. Let A be a nowhere dense subset of X. Let 0 be any non-empty open subset of X, then $0 \not\subset cl_X(A)$. Note that since A is a subset of X, $cl_{X'}(A) \subset cl_X(A) \cup \{\infty\}$. Therefore if $0 \in \mathcal{T}$ and $0 \not\subset cl_X(A)$, then $0 \not\subset cl_{X'}(A)$. Let H be any open set of X' that contains the point ∞. H is of the form $C_X(P) \cup \{\infty\}$ where P is a closed compact subset of X. Therefore $C_X(P)$ is open in X and hence, $C_X(P) \not\subset cl_X(A)$. Thus $H = C_X(P) \cup \{\infty\} \not\subset cl_{X'}(A)$ and A is a nowhere dense subset of X'.

THEOREM 4.10 Let $(X, \mathcal{T})$ be a non-compact topological space. $(X, \mathcal{T})$ is first category if and only if $(X', \mathcal{T}')$ is first category.

PROOF. Assume that X is first category. Therefore X is the countable union of nowhere dense subsets. The set $\{\infty\}$ is closed in X' but not open because X is not compact, it follows that $\{\infty\}$ is a nowhere dense subset of X'. By theorem 4.9, X' is the countable union of nowhere dense subsets. Therefore X' is first category.

Assume that X' is first category. X' is the countable union of nowhere dense subsets. Restrict each of these nowhere dense subsets to X. Then by theorem 4.8, X is seen to be the countable union of nowhere dense subsets. Thus X is first category.

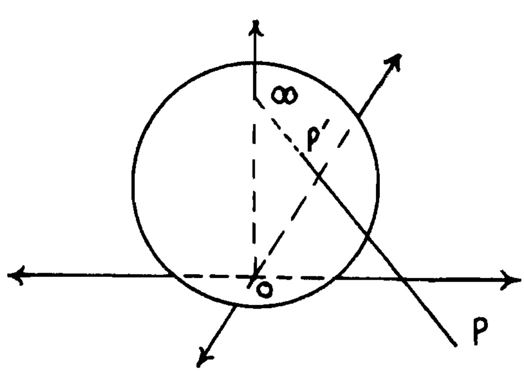
COROLLARY 4.2 Let $(X, \mathcal{T})$ be a non-compact topological space. $(X, \mathcal{T})$ is second category if and only if $(X', \mathcal{T}')$ is second category.

PROOF. The result is immediate from the preceding theorem.

5. EXAMPLES OF THE ALEXANDROFF ONE-POINT COMPACTIFICATION

The following are some examples of the Alexandroff one-point compactification.

EXAMPLE 5.1 Consider the Alexandroff one-point compactification of the complex plane C. The point ∞, called the point of infinity, is added to C and the new space is denoted by C_∞. The open sets of C_∞ consists of the open sets of C and the complements in C_∞ of closed and bounded subsets of C. C_∞ is often called the extended complex plane.



The extended complex plane may be visualized as a sphere of an arbitrary fixed radius tangent to the complex plane at the origin. The tangent point is referred to as the south pole and the north pole is considered to be the point of infinity. The indicated projection from the axis of the sphere to the plane sets up a homeomorphism from the sphere less the north pole, to the complex plane. The sphere is usually called the Riemann sphere. (The Alexandroff one-point compactification of the Euclidean plane may be demonstrated in a like manner.)

The following example may be found in Burgess [2].

EXAMPLE 5.2 Consider the space R, the real number line, with the usual topology $\mathcal{T}$. The Alexandroff one-point compactification of R consists of adjoining the point ∞ to R and of adding to $\mathcal{T}$ the subsets of the form $0 \cup \{\infty\}$ where $C_R(0)$ is closed and bounded in R.

It follows that if the open set S contains ∞ then there exists points a and b of R such that $a \leq b$ and the set $\{x \in R \mid x < a \text{ or } x > b\}$ is a subset of S. Hence, a sequence of real numbers $\{x_i\}_1^{\infty}$ converges to ∞ if and only if given $N > 0$ there exists an integer n such that $|x_i| > N$, for all $i > n$. Thus ∞ can be considered as a point at infinity.

EXAMPLE 5.3 Let X be any infinite set with the discrete topology. Let 0 be any open set of X' containing the point ∞. Since $C_X(0)$ must be compact, $C_X(0)$ must contain all but a finite number of points of X, the finite subsets of X being the only compact subsets.

EXAMPLE 5.4 Consider the topological space $(0, 1]$ with the usual topology. The Alexandroff one-point compactification of $(0, 1]$ yields the space $[0, 1]$. However, the function defined on $(0, 1]$ by $f(x) = \sin(\frac{1}{x})$ is bounded and continuous but the function cannot be extended continuously to $[0, 1]$.

The problem of extending a continuous function defined on X to a continuous function defined on Y where Y is a compactification of X is discussed in the next chapter.

CHAPTER III

STONE-ČECH COMPACTIFICATION

This chapter is concerned with the Stone-Čech compactification, its construction and properties. The Stone-Čech compactification is a compactification of an arbitrary completely regular space, which in this paper implies T_1. Since spaces that are locally compact and T_2 are also completely regular, it is possible to construct a T_2 Alexandroff one-point compactification as well as the Stone-Čech compactification. However, the Stone-Čech compactification has a very useful property which the Alexandroff one-point compactification fails to have, the property of being able to extend specific continuous functions defined on X to continuous functions defined on βX. βX will denote the Stone-Čech compactification of the topological space $(X, \mathcal{T})$ where X is completely regular.

1. PRELIMINARIES

DEFINITION 1.1 Let X be a non-empty set and let $\{X_i\}$ be a non-empty collection of topological spaces. For each i let f_i be a mapping from X to X_i. The weakest topology on X such that $f_i: X \to X_i$ is continuous for each i is called the weak topology generated by the f_i's.

DEFINITION 1.2 Let $\{X_i: i \in I\}$ be a non-empty collection of topological spaces, indexed by the indices i of the index set I. The product of the topological spaces X_i is denoted by $\prod\{X_i: i \in I\}$. The topology on $\prod\{X_i: i \in I\}$ is the weak topology generated by the projections of p_i of $\prod\{X_i: i \in I\}$ onto X_i, for all i in I.

DEFINITION 1.3 Let S be a subspace of Y and let f be a function of S to the space X. If h is a function of Y to X such that $h(t) = f(t)$ for all t in S, h is called an extension of f.

Let $C(X, R)$ denote the collection of all bounded continuous real-valued functions defined on X.

LEMMA 1.1 Let $(X, \mathcal{T})$ be a completely regular topological space. The weak topology on X generated by $C(X, R)$ equals the given topology on X.

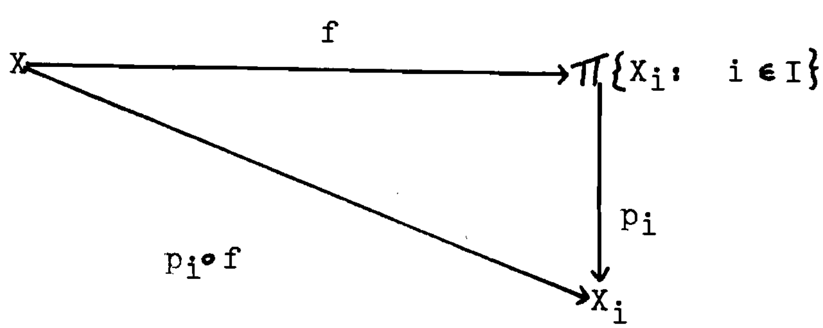
PROOF. Let $\mathcal{T}$ be the given topology and let $\mathcal{T}_w$ be the weak topology on X generated by $C(X, R)$. Let O be any open set in $\mathcal{T}_w$. Let $p \in O$, then there exists $f_1, f_2, \ldots, f_n \in C(X, R)$ and $O_1, O_2, \ldots, O_n$ open in R such that $p \in \bigcap_1^n f_i^{-1}(O_i) \subset O$ since O is in $\mathcal{T}_w$. Each f_i is continuous, hence $\bigcap_1^n f_i^{-1}(O_i)$ is in $\mathcal{T}$. Therefore $O \in \mathcal{T}$ and we have $\mathcal{T}_w \leq \mathcal{T}$.

Let Q be any open set in $\mathcal{T}$ and let p be any point in Q. Set $F = X - Q$ and note that F is closed. Since X is a completely regular space there exists a continuous function g such that $g: X \to [0, 1]$, $g(p) = 0$, and $g(F) = 1$. Note that $g \in C(X, R)$. Since $(-\frac{1}{2}, 1)$ is open in R, $g^{-1}(-\frac{1}{2}, 1)$ is in $\mathcal{T}_w$. $g^{-1}(-\frac{1}{2}, 1) = Q$ since $g(Q) \subset [0, 1)$ and $g(F) = 1$. Hence, $\mathcal{T} \leq \mathcal{T}_w$ and therefore the weak topology generated by $C(X, R)$ equals the given topology on X.

LEMMA 1.2 The relative topology on a subspace of a product space is the weak topology generated by the restriction of the projections to that subspace.

PROOF. Let $P = \prod\{X_i : i \in I\}$ and let Y be a subspace of P. Now $S = \{p_i^{-1}(O_i) \cap Y : O_i$ is open in $X_i, i \in I\}$ is a subbase for the relative topology on Y. But $S = \{(p_i|_Y)^{-1}(O_i) : O_i$ is open in $X_i, i \in I\}$ and thus S is also a subbase for the weak topology on Y generated by the restriction of the projections to the subspace Y. Hence, the two topologies are identical.

LEMMA 1.3 If f is a mapping of a topological space X into a product space $\prod\{X_i : i \in I\}$, then f is continuous if and only if $p_i \circ f$ is continuous for each projection $p_i, i \in I$.



PROOF. Assume that f is a continuous function. Each p_i is continuous by the definition of the topology on $\prod\{X_i: i \in I\}$. Hence, each $p_i \circ f$ is a composite mapping of two continuous functions and is continuous.

Assume that each $p_i \circ f$ is continuous. Let O_i be any open set in X_i, $i \in I$. Since $p_i \circ f$ is continuous for each $i \in I$, $f^{-1}(p_i^{-1}(O_i))$ is open in X. Now $p_i^{-1}(O_i)$ is an arbitrary subbasic open set in $\prod\{X_i: i \in I\}$, and thus it follows that f is continuous.

2. CONSTRUCTION OF βX

This section is concerned with the construction of the Stone-Čech compactification βX of X.

THEOREM 2.1 Let $(X, \mathcal{T})$ be a completely regular topological space. There exists a compact T_2-space βX such that:

(a) X is homeomorphic to a dense subspace of βX;

(b) every bounded continuous real-valued function defined on X has a unique extension to a bounded continuous real-valued function defined on βX.

PROOF. (a) Let $C(X, R)$ be the set of all bounded continuous real-valued functions defined on X. Index the functions in $C(X, R)$ by the indices i, of the index set Δ, hence $C(X, R) = \{f_i: i \in \Delta\}$. Let I_i be the smallest closed interval containing the range of the function f_i. Each I_i is a compact T_2-space and hence their product $I = \prod\{I_i: i \in \Delta\}$ is also a compact T_2-space by Tychonoff's theorem. Since I is compact and T_2, I is normal and hence completely regular. Every subspace of I is completely regular and to show that X is a subspace of I it has to be assumed that X is completely regular.

Define the mapping $F: X \to I$ where $F(x)$ is equal to the point in I whose ith coordinate is the real number $f_i(x)$. For each $i \in \Delta$, $p_i \circ F = f_i$, and thus by lemma 1.3, F is continuous. Now $C(X, R)$ separates the points of X since X is completely regular. Therefore F is one-to-one and X can be embedded into I as the set $F(X)$. Instead of using $F(X)$, X can now be thought of as a subset of I. Thus X is seen to have its own given topology and the topology that it has as a subspace of I. Since F is continuous, the subspace topology on X is weaker than the original topology. The following argument shows, without resorting to this fact, that the two topologies are identical.

$C(X, R)$ is the set of all restrictions to X of the projections p_i of I onto each I_i. By lemma 1.2 the relative topology on X, as a subspace of I, is equal to the weak topology generated by the restrictions of the projections p_i to X. Since X is a completely regular space, by lemma 1.1 the weak topology generated by $C(X, R)$ equals the given topology. Thus the given topology of X is equal to its relative topology as a subspace of I. Therefore X is a subspace of I. Since I is a compact T_2-space, $\overline{X}$ is a compact T_2-space. Let βX denote $\overline{X}$. Thus X is homeomorphic to a dense subset of βX.

(b) For each $i \in \Delta$, f_i is the restriction of the projection p_i to X. Thus p_i extends f_i to I. Let $\overline{p_i}$ denote the restriction of p_i to $\overline{X}$. Hence, each f_i has an extension to a continuous bounded function defined on $\beta X = \overline{X}$. Since X is dense in the T_2-space $\overline{X}$ it follows that this extension is unique. Therefore, not only is the Stone-Čech compactification of an arbitrary completely regular space compact and T_2, but every bounded continuous real-valued function defined on X can be extended to a bounded continuous real-valued function defined on βX. Another important aspect of the Stone-Čech compactification is shown in the following theorem. This property helps characterize βX as will be shown later in this chapter.

THEOREM 2.2 Every continuous function from X to a compact T_2-space Y can be extended uniquely to a continuous function defined on βX to Y.

PROOF. Let Y be any compact T_2-space, then Y is completely regular. Let βY be the Stone-Čech compactification of Y and let L be the embedding map of Y into βY. Index the functions in C(Y, R) by the indices i of the index set Δ. Then βY is a subspace of the product space $\prod\{I_i : i \in \Delta\}$, where I_i is the smallest closed interval containing the range of the bounded continuous real-valued function g_i defined on Y. Note that I_i is a compact T_2-space.

Since Y is compact, Y is homeomorphic to βY. Therefore $L(Y) = \beta Y$ and hence we can fully identify Y by βY. Let y be some point of Y, then $y = \langle y_i \rangle$ where the ith coordinate is the point $g_i(y)$, $i \in \Delta$. Let f be any continuous function from X to Y. Therefore, if x is a point of X, then $f(x) = \langle f_i(x) \rangle$ where the ith coordinate is equal to $g_i(f(x))$. For each $i \in \Delta$, $p_i \circ f = f_i$, and thus f_i a bounded continuous real-valued function from X to I_i. Therefore there exists a unique extension $\hat{f}_i$ of f_i such that $\hat{f}_i$ is a bounded continuous real-valued function from βX to I_i. Define the function $\hat{f}$ from βX to $\prod\{I_i : i \in \Delta\}$ by $\hat{f}(x) = \langle \hat{f}_i(x) \rangle$, for any x in βX. Then for each $i \in \Delta$, $p_i \circ \hat{f} = \hat{f}_i$. Since $\hat{f}_i$ is continuous, $p_i \circ \hat{f}$ is continuous for each projection p_i. Thus by lemma 1.3, $\hat{f}$ is a continuous function from βX to $\prod\{I_i : i \in \Delta\}$.

Let $x \in X$, then $\hat{f}(x) = \langle \hat{f}_i(x) \rangle = \langle f_i(x) \rangle = f(x)$. Therefore $\hat{f}$ is a continuous extension of f. Since X is dense in the T_2-space βX, it follows that $\hat{f}$ is unique. Therefore, it must only be shown that $\hat{f}$ maps βX strictly to βY and hence to Y.

Let p be any point in βX, then $\hat{f}(p)$ is a point in $\prod\{I_i : i \in \Delta\}$. Let O be any open set in $\prod\{I_i : i \in \Delta\}$ that contains the point $\hat{f}(p)$. Then $\hat{f}^{-1}(O)$ is an open set in βX that contains the point p. Since X is dense in βX, $\hat{f}^{-1}(O)$ contains a point x of X. And, since $\hat{f}$ is a unique continuous extension of f, $f(x) \in O$. Thus every open set containing $\hat{f}(p)$ contains a point of βY. It then follows that $\hat{f}(p)$ is a point of closure of βY and hence $\hat{f}(p) \in \beta Y$. Therefore, $\hat{f}(\beta X) \subset \beta Y$.

The next theorem shows that βX is unique. That is, βX is unique in the sense that if T is any other compact T_2-space such that X is dense in T, and every continuous function from X to any compact T_2-space Y can be extended uniquely to a continuous function from T to Y, then βX and T are homeomorphic.

THEOREM 2.3 Let $(X, \mathcal{T})$ be a completely regular topological space. Let T be a topological space satisfying:

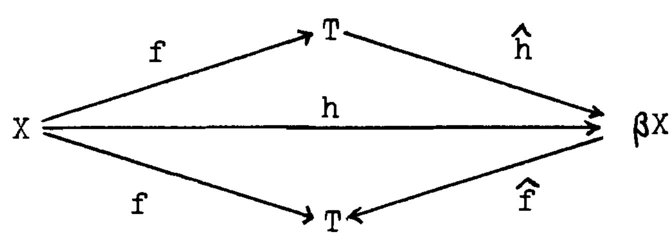
(a) T is a compact T_2-space;

(b) X is homeomorphic to a dense subspace of T;

(c) Each continuous map from X to a compact T_2-space can be extended to a continuous map from T.

Then T is homeomorphic to βX.

PROOF. Let f denote the embedding map from X into T and h the embedding map from X into βX. Let $\hat{f}$ denote the extension of f to βX and $\hat{h}$ the extension of h to T. It follows that the following diagram commutes.



The unique continuous extension of h to βX is the identity on βX, $i_{\beta X}$, and therefore $\hat{h} \cdot \hat{f} = i_{\beta X}$. In a similar manner $\hat{f} \circ \hat{h} = i_T$, the identity on T. Thus $\hat{f}$ and $\hat{h}$ are both one-to-one and onto, and $\hat{h}^{-1} = \hat{f}$. Since $\hat{h}$ and $\hat{f}$ are continuous, it follows that T and βX are homeomorphic.

3. PROPERTIES OF βX

This section contains some of the relationships between X and βX. A number of other relationships lie beyond the scope of this paper and require extensive background material. The reader is referred to the bibliography for additional material. Each topological space considered in this section is assumed to be completely regular.

THEOREM 3.1 Let S be a subspace of X, then every bounded continuous real-valued function f defined on S has a bounded continuous extension to X if and only if f has a bounded continuous extension to βX.

PROOF. Let S be a subspace of X and let f be a bounded continuous real-valued function defined on S. Assume that $\hat{f}$ is a bounded continuous extension of f on X. By the construction of βX there exists a bounded continuous extension $\hat{f}'$ of $\hat{f}$ defined on βX. It is easily seen that $\hat{f}'$ is a bounded continuous extension of f defined on βX.

Assume that f has a bounded continuous extension h defined on βX. Denote the restriction of h on X by f'. It is seen that f' is a bounded continuous extension of f defined on X.

THEOREM 3.2 Every bounded continuous real-valued function defined on the subspace S of X can be extended to a bounded continuous real-valued function defined on X if and only if $cl_{\beta X}(S) = \beta S$.

PROOF. Assume that every bounded continuous real-valued function f on S has a bounded continuous extension $\hat{f}$ on X. By the preceding theorem f has a bounded continuous extension $\hat{f}'$ on βX. It is thus seen that f has a bounded continuous extension defined on $cl_{\beta X}(S)$, which is unique. Since S is dense in $cl_{\beta X}(S)$ and $cl_{\beta X}(S)$ is compact and T_2, it follows by theorem 2.2 and theorem 2.3 that $cl_{\beta X}(S) = \beta S$.

Assume that $cl_{\beta X}(S) = \beta S$. Thus by theorem 2.1 every bounded continuous real-valued function f defined on S has a unique bounded continuous extension $\hat{f}$ defined on βS and hence on $cl_{\beta X}(S)$. Then by Tietze's extension theorem $\hat{f}$ has a bounded continuous real-valued extension $\hat{f}'$ defined on βX. Now by theorem 3.1 it follows that every bounded continuous real-valued function defined on S has a bounded continuous extension defined on X.

COROLLARY 3.1 If K is a compact set in X, then every bounded continuous real-valued function defined on K can be extended to a bounded continuous real-valued function on X.

PROOF. Assume that K is a compact subset of X. Thus $cl_{\beta X}(K) = K$ and $\beta K = K$. Therefore $cl_{\beta X}(K) = \beta K$ and by theorem 3.2 every bounded continuous real-valued function defined on K has a bounded continuous extension defined on X.

THEOREM 3.3 If S is open and closed in X, then $cl_{\beta X}(S)$ and $cl_{\beta X}(X - S)$ are disjoint complementary open sets in βX.

PROOF. Assume that S is open and closed in X, then $X - S$ is open and closed in X. There exists a bounded continuous real function f defined on X such that $f(S) = 0$ and $f(X - S) = 1$. Every bounded continuous real-valued function defined on X has a unique bounded continuous extension defined on βX. Let $\hat{f}$ be the unique bounded continuous extension of f. Therefore $\hat{f}(S) = 0$ and $\hat{f}(X - S) = 1$. It then follows that $\hat{f}(cl_{\beta X}(S)) = 0$ and $\hat{f}(cl_{\beta X}(X - S)) = 1$. Since X is dense in βX it follows that $cl_{\beta X}(S) \cup cl_{\beta X}(X - S) = \beta X$. Thus $cl_{\beta X}(S)$ and $cl_{\beta X}(X - S)$ are disjoint complementary open subsets of βX.

COROLLARY 3.2 An isolated point of X is an isolated point of βX.

PROOF. Assume that p is an isolated point of X. Therefore, since X is T_1, $\{p\}$ is both open and closed in X. By theorem 3.3 $cl_{\beta X}\{p\}$ is both open and closed in βX. However, since βX is T_2 then $cl_{\beta X}\{p\} = \{p\}$. Thus p is an isolated point of βX.

COROLLARY 3.3 βX is connected if and only if X is connected.

PROOF. Assume that βX is connected. Let A be a proper subset of X such that A is open and closed in

X. By theorem 3.3, $cl_{\beta X}(A)$ is both open and closed in βX and $cl_{\beta X}(X - A)$ is disjoint from $cl_{\beta X}(A)$. It then follows that $cl_{\beta X}(A)$ is a proper subset of βX. But this contradicts that βX is connected. Thus X must be connected.

Assume that X is connected. Let B be a proper subset of βX such that B is both open and closed. Hence $\beta X - B$ is both open and closed. Since X is dense in βX it follows that both B and $\beta X - B$ contain points of X. Thus $B \cap X$ is a proper subset of X that is both open and closed in X, but this contradicts the hypothesis that X is connected. Hence βX is connected.

The corollary to the next theorem is very important. The theorem itself shows an important relationship between X and βX.

THEOREM 3.4 X is open in βX if and only if X is locally compact.

PROOF. Assume that X is locally compact. Let $p \in X$ and let K be a compact neighborhood of p in X. Therefore there exists an open set O in X such that $p \in O \subset K$. K is compact in βX and hence closed since βX is T_2. Since O is open in X, there exists an open set H of βX such that $H \cap X = O \subset K$. It is clear that $cl_{\beta X}(H \cap X) \subset cl_{\beta X}(H) \cap cl_{\beta X}(X) = cl_{\beta X}(H)$. Let x be a point in $cl_{\beta X}(H)$, then every open set N in βX that contains x, contains a point of H. Now $N \cap (H \cap X) = (N \cap H) \cap X$ and $N \cap H$ is a non-empty open set. But since X is dense in

βX, every non-empty open set has a non-empty intersection with X. Hence, $N \cap (H \cap X) \neq \emptyset$ and it follows that $x \in cl_{\beta X}(H \cap X)$ and $cl_{\beta X}(H \cap X) = cl_{\beta X}(H)$. Therefore, $H \subset cl_{\beta X}(H) = cl_{\beta X}(H \cap X) \subset K \subset X$ and X is seen to be open.

Suppose that X is open in βX. Since βX is a compact T_2-space, βX is regular. Let $p \in \beta X$, then every neighborhood of p contains a closed neighborhood of p. X, being open, is a neighborhood for each p in X. Therefore, each p in X has a compact neighborhood in X since closed sets are compact in βX. Therefore X is locally compact.

COROLLARY 3.4 Each open set in X is open in βX if and only if X is locally compact.

PROOF: The result is immediate from the preceding theorem.

Let (Y, g) and (Z, k) denote T_2-compactifications of X where g and k are the embedding mappings. Then Y is said to be larger than Z, denoted by $Y \geq Z$, if and only if there exists a continuous function f defined on Y to Z such that $f \circ g = k$.

Let (Y, g) denote any T_2-compactification of X and let $(\beta X, h)$ denote the Stone-Čech compactification of X. By theorem 2.2 there exists a continuous extension $\hat{g}$ of g from βX to Y such that $\hat{g} \circ h = g$. Therefore βX is referred to as the maximal T_2-compactification of X.

EXAMPLE 3.1 Let N represent the set of natural numbers with the discrete topology. Hence, N is completely regular and it is possible to construct the Stone-Čech compactification βN of N. Let S be any subset of N. Every bounded real-valued function defined on S may be extended to a bounded continuous function defined on N, since every real-valued function defined on N is continuous. Thus by theorem 3.2 $cl_{\beta N}(S) = \beta S$. Any subset S of N is both open and closed in N, therefore, by theorem 3.3 $cl_{\beta X}(S)$ is open in βN. Since N is locally compact, by theorem 3.4 N is open in βN. By corollary 3.2, each point of N is an isolated point in βN. Since $\beta N - N$ is closed, it is compact.

EXAMPLE 3.2 Let Q denote the rational numbers with the usual topology. Thus Q is completely regular. Let βQ denote the Stone-Čech compactification of Q. Since Q is not connected, by corollary 3.3 βQ is not connected. By theorem 3.4, since Q is not locally compact, Q is not open in βQ. N and Q have the same cardinality, therefore, there exists a bijection f from N to $Q \subset \beta Q$. Since N is a discrete space, f is continuous. Therefore, by theorem 2.2, there exists a continuous extension $\hat{f}$ of f from βN to βQ. But, $\hat{f}(\beta N)$ is compact in βQ and is therefore closed in βQ. Note that $Q \subset \hat{f}(\beta N)$. Thus, it follows that $\hat{f}(\beta N) = \beta Q$. Therefore βQ is a continuous image of βN.

Since βQ is compact and T_2, every neighborhood of a point in βQ contains a compact neighborhood of the point. However, since Q is not locally compact, no compact neighborhood can be entirely in Q. Therefore, no neighborhood of a point in βQ can be entirely in Q. Thus $\beta Q - Q$ is dense in βQ. Suppose $\beta Q - Q$ is a finite set, it then follows that βQ is not T_2. Hence, $\beta Q - Q$ must be an infinite set. Now since $\hat{f}(\beta N) = \beta Q$ where $\hat{f}$ is defined above, $\beta N - N$ must be an infinite set. It then follows that neither βN or βQ can be one-point compactifications.

CHAPTER IV

A DIFFERENT CONSTRUCTION OF βX

A different construction of the Stone-Čech compactification of completely regular spaces is the main purpose of this chapter. The construction to be developed uses the concepts of Z-ultrafilters. The first part of the chapter will include definitions and basic theorems concerning filters, ultrafilters, zero-sets, Z-filters, and Z-ultrafilters. The reader interested in a more complete discussion of filters and ultrafilters is referred to Thron [14]. The rest of the chapter will be devoted to the construction of the compactification.

1. FILTERS AND ULTRAFILTERS

DEFINITION 1.1 A family $\mathfrak{F}$ of subsets of a set X is called a filter on X if and only if it satisfies the following:

(a) The family $\mathfrak{F}$ is non-empty;

(b) If $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;

(c) If $A \subset B$ and $A \in \mathfrak{F}$, then $B \in \mathfrak{F}$;

(d) $\emptyset \notin \mathfrak{F}$.

EXAMPLE 1.1 Let $\mathfrak{N}_p$ denote the collection of all neighborhoods in the topological space $(X, \mathcal{T})$ that contain the point p in X. $\mathfrak{N}_p$ is a filter on X, it is called the neighborhood filter of the point p.

DEFINITION 1.2 A non-empty family B of subsets of a set X, such that B does not contain the empty set and the intersection of any two members of B contains a member of B, is called a filter base on X.

DEFINITION 1.3 A non-empty family $\mathcal{E}$ is called a filter subbase if the intersection of any finite number of elements in $\mathcal{E}$ is non-empty.

THEOREM 1.1 If $\mathcal{E}$ is a filter subbase, then the family β of all finite intersections of elements in $\mathcal{E}$ is a filter base. If β is a filter base, then the family $\mathfrak{F}$ of all super-sets of the sets in β is a filter.

PROOF. The proof follows from the definitions.

DEFINITION 1.4 Let $(X, \mathcal{T})$ be a topological space and $\mathfrak{F}$ a filter on X. $\mathfrak{F}$ is said to converge to the point p in X if and only if $\mathfrak{F}$ contains every neighborhood of p. The point p is said to be a limit point of the filter $\mathfrak{F}$. It is also said that the filter $\mathfrak{F}$ converges to the point p. The point t is said to be a cluster point of the filter $\mathfrak{F}$ if and only if for each neighborhood N_t of t and any $A \in \mathfrak{F}$, $A \cap N_t \neq \emptyset$.

DEFINITION 1.5 If $\mathfrak{U}$ is a filter on X such that no other filter $\mathfrak{F}$ on X properly contains $\mathfrak{U}$, then $\mathfrak{U}$ is called an ultrafilter on X.

EXAMPLE 1.2 Let S_a denote the collection of all subsets in X that contain the point a of X. S_a is an ultrafilter on X.

Let $\{\mathfrak{U}_i\}$ denote the set of all filters on X that contains the filter $\mathfrak{F}$ on X. It is easily shown that the union of all members of $\{\mathfrak{U}_i\}$ is a filter. By applying Zorn's lemma, the union of all members of $\{\mathfrak{U}_i\}$ is seen to be an ultrafilter on X that contains $\mathfrak{F}$. Hence, every filter on X is contained in an ultrafilter on X.

The next few theorems will be used in the construction of βX.

THEOREM 1.2 A filter $\mathfrak{U}$ is an ultrafilter on X if and only if $A \cup B \in \mathfrak{U}$ implies that either $A \in \mathfrak{U}$ or $B \in \mathfrak{U}$.

PROOF. Let $\mathfrak{U}$ be an ultrafilter on X. Assume that $A \cup B \in \mathfrak{U}$ such that $A \notin \mathfrak{U}$ and $B \notin \mathfrak{U}$. Define $\mathfrak{F}$ to be the family of all subsets H of X such that $H \cup A \in \mathfrak{U}$. Since $A \cup B \in \mathfrak{U}$, $\mathfrak{F}$ is non-empty, and $\emptyset \cup A = A \notin \mathfrak{U}$, hence $\emptyset \notin \mathfrak{U}$. If $H \cup A \in \mathfrak{U}$ and $H \subset O$, then $O \cup A \supset H \cup A \in \mathfrak{U}$ which implies that $O \in \mathfrak{F}$. If H and O are in $\mathfrak{F}$, then $H \cup A \in \mathfrak{U}$ and $O \cup A \in \mathfrak{U}$. Thus $(H \cup A) \cap (O \cup A) = (H \cap O) \cup A \in \mathfrak{U}$ and $H \cap O \in \mathfrak{F}$. Therefore $\mathfrak{F}$ is a filter. Let $C \in \mathfrak{U}$, then $A \cup C \supset C \in \mathfrak{U}$ implies that $A \cup C \in \mathfrak{U}$ and hence $C \in \mathfrak{F}$. But since $B \in \mathfrak{F}$ and $B \notin \mathfrak{U}$, $\mathfrak{F}$ properly contains $\mathfrak{U}$ which contradicts the fact that $\mathfrak{U}$ is an ultrafilter. Thus, if $A \cup B \in \mathfrak{U}$, either $A \in \mathfrak{U}$ or $B \in \mathfrak{U}$.

Assume that $A \cup B \in \mathfrak{U}$ implies that either A or B is in $\mathfrak{U}$. Now $X \in \mathfrak{U}$ since X is a super-set for all sets. Therefore, for every subset A of X either A or $X - A$ is in $\mathcal{U}$. Suppose some filter $\mathcal{E}$ properly contains $\mathcal{U}$, then $\mathcal{E}$ contains some subset B of X such that $B \notin \mathcal{U}$. But if $B \notin \mathcal{U}$, then $X - B \in \mathcal{U}$. Now $B \in \mathcal{E}$ and $\mathcal{U} \subseteq \mathcal{E}$. Hence $X - B \in \mathcal{E}$ and $\emptyset = B \cap X - B \in \mathcal{E}$, but this is a contradiction. Thus $\mathcal{U}$ must be an ultrafilter.

THEOREM 1.3 A filter $\mathcal{U}$ is an ultrafilter on X if and only if $A \cap B$ is non-empty for all $B \in \mathcal{U}$ implies that $A \in \mathcal{U}$.

PROOF. Assume that $\mathcal{U}$ is an ultrafilter on X. Let $A \subset X$ such that $A \cap B \neq \emptyset$, for all B in $\mathcal{U}$. $\mathcal{U} \cup \{A\}$ is then a filter subbase for a filter $\mathcal{E}$ that contains $\mathcal{U}$ and the set A. Since $\mathcal{U}$ is an ultrafilter, $\mathcal{E} = \mathcal{U}$ and $A \in \mathcal{U}$.

Let $\mathcal{U}$ be a filter on X that is not an ultrafilter. Then some filter $\mathcal{E}$ properly contains $\mathcal{U}$. Hence there exists a set A in $\mathcal{E}$ that is not contained in $\mathcal{U}$. Let B be any set in $\mathcal{U}$, then B is in $\mathcal{E}$ and $A \cap B \neq \emptyset$. Therefore the statement that $A \cap B$ is non-empty for all B in $\mathcal{U}$ implies that $A \in \mathcal{U}$, is denied. The theorem is thus proved.

The last two theorems of this section help show some of the uses of filters in topology and analysis.

THEOREM 1.4 Let f be a function from X to Y and let $\mathcal{E}$ be a filter on X. For all $A \in \mathcal{E}$ the collection $f(A)$, denoted by $f(\mathcal{E})$, forms a filter base on Y.

PROOF. Since $\mathcal{E}$ is non-empty, $f(\mathcal{E})$ is non-empty. Since $f(A) \neq \emptyset$ for any $A \in \mathcal{E}$, the null set is not in $f(\mathcal{E})$. Let A and B be in $\mathcal{E}$, then $A \cap B \in \mathcal{E}$. Now $f(A \cap B) \subset f(A) \cap f(B)$. Hence, $f(\mathcal{E})$ is a filter base on Y.

THEOREM 1.5 Let f be a function from $(X, \mathcal{T})$ to (Y, Σ). Then f is continuous at the point t in X if and only if for every filter $\mathcal{E}$ on X that converges to t, the filter $\mathfrak{F}$ generated by the filter base $f(\mathcal{E})$ converges to $f(t)$.

PROOF. Assume that f is a continuous function from X to Y. Let N_{ft} be any neighborhood of $f(t)$. Since f is continuous, $f^{-1}(N_{ft})$ is a neighborhood of t in X. Since $\mathcal{E}$ converges to t, $f^{-1}(N_{ft})$ is in $\mathcal{E}$ and hence N_{ft} is in $\mathfrak{F}$, since $N_{ft} \supset f(f^{-1}(N_{ft}))$. Thus $\mathfrak{F}$ converges to $f(t)$.

Suppose f is not continuous at t, then there exists a N_{ft} such that $f^{-1}(N_{ft})$ does not contain any neighborhood of t in X. $\mathfrak{N}_t$, the neighborhood filter of t, is a filter on X that converges to t. Let N_t be any neighborhood of t, then $f(N_t) \not\subset N_{ft}$. Thus N_{ft} is not an element of $\mathfrak{F}$, the filter generated by $f(\mathfrak{N}_t)$, and hence $\mathfrak{F}$ does not converge to $f(t)$. Therefore, if the filter $\mathfrak{F}$ on Y generated by $f(\mathcal{E})$, where $\mathcal{E}$ is any filter on X that converges to t, converges to $f(t)$, then f must be continuous.

2. ZERO-SETS AND PRINCIPAL Z-ULTRAFILTERS

In this section the concepts of zero-sets and principal Z-ultrafilters are introduced. Zero-sets and principal Z-ultrafilters are used in the construction of βX, the Stone-Čech compactification of a completely regular space. Definitions and the basic theorems used in the construction will be given in this section. The reader is referred to Gillman and Jerison [5] for a more complete discussion of zero-sets and Z-ultrafilters.

DEFINITION 2.1 If $A = \{x \in X: \ f(x) = 0\}$ for some continuous real-valued function f on X, then A is called a zero-set.

Since f is continuous, A is closed and thus all zero-sets are closed. Therefore, the finite union and arbitrary intersection of zero-sets are closed.

THEOREM 2.1 Finite unions and finite intersections of zero-sets are zero-sets.

PROOF. Assume that A and B are zero-sets. Let $A = \{x \in X: \ f(x) = 0\}$ and $B = \{x \in X: \ g(x) = 0\}$, then $A \cup B = \{x \in X: \ f(x) = 0 \text{ or } g(x) = 0\}$ and $A \cap B = \{x \in X: \ f(x) = 0 \text{ and } g(x) = 0\}$. Define h by $h(x) = f(x) \cdot g(x)$, note that h is continuous. The zero-set of the function h is $C = \{x \in X: \ f(x) \cdot g(x) = 0\}$. Hence, $A \cup B = C$ and $A \cup B$ is therefore a zero-set. Define k by $k(x) = |f(x)| + |g(x)|$, note that k is continuous. The zero-set of the function k is $D = \{x \in X: \ |f(x)| + |g(x)| = 0\}$. Hence $A \cap B = D$ and therefore $A \cap B$ is a zero-set.

Since zero-sets are closed and finite intersections and unions of zero-sets are zero-sets, a relationship between zero-sets and closed sets of a topological space might be questioned. The following theorem gives such a relationship between closed subsets and zero-sets of a completely regular space.

THEOREM 2.2 Let $(X, \mathcal{T})$ be a completely regular space. Then the zero-sets of X forms a base for the closed subsets of X.

PROOF. Let B be a closed subset of X, then there exists a continuous real-valued function f on X such that $f(B) = 0$. Let $K = \{x \in X: f(x) = 0\}$, clearly $B \subset K$ with K being a zero-set. For every point p in $K - B$ there exists a continuous real-valued function f_p such that $f_p(B) = 0$ and $f_p(p) \neq 0$. For each such f_p let $K_p = \{x \in X: f_p(x) = 0\}$ and denote the collection of all such zero-sets by $\{K_p: p \in K - B\}$. Then $B = (\bigcap\{K_p: p \in K-B\}) \cap K$. Hence, the collection of all zero-sets of X forms a base for the closed subsets of X.

DEFINITION 2.2 Let Z denote the collection of all zero-sets in X. A filter $\mathcal{E}$ on X and an ultrafilter $\mathcal{U}$ on X intersected with Z are called Z-filters and Z-ultrafilters, respectfully.

DEFINITION 2.3 Let x be a fixed point in X, then the family of all zero-sets of X that contain the point x is called a principal Z-ultrafilter on X, and is denoted by $\mathcal{U}_x$.

THEOREM 2.3 Let $(X, \mathcal{T})$ be completely regular and let $\mathcal{U}_x$ be a principal Z-ultrafilter on X. Then $\mathcal{U}_x$ is a Z-ultrafilter.

PROOF. Since X is a T_1-space, the singleton set $\{x\}$ is closed. Hence, some zero-set contains the point x and $\mathcal{U}_x$ is non-empty. The empty set is not in $\mathcal{U}_x$ since $x \notin \emptyset$. If A and B are zero-sets where $A \in \mathcal{U}_x$ and $A \subset B$, then by the definition of $\mathcal{U}_x$, $B \in \mathcal{U}_x$. If A and B are both in $\mathcal{U}_x$, then $x \in A \cap B$. By theorem 2.1, $A \cap B$ is a zero-set and hence, $A \cap B \in \mathcal{U}_x$. Thus $\mathcal{U}_x$ is a Z-filter on X.

Suppose Q is a zero-set that does not contain the point x. Since Q is closed and X is completely regular, there exists a continuous real-valued function f such that $f(Q) = 1$ and $f(x) = 0$. Therefore there exists a zero-set K_x that contains the point x but does not contain any point of Q. Since $K_x \in \mathcal{U}_x$ and $K_x \cap Q = \emptyset$, it is clear that $\mathcal{U}_x$ is not properly contained in any Z-filter. Thus $\mathcal{U}_x$ is a Z-ultrafilter.

3. CONSTRUCTION OF βX

A different construction of βX is developed in this section. It will be shown, however, that this construction will satisfy the characteristics of the Stone-Čech compactification and by theorem 2.3 of chapter three will be homeomorphic to the construction developed in the last chapter.

DEFINITION 3.1 βX will denote the set of all Z-ultrafilters on X.

DEFINITION 3.2 C_A is the set $\{\mathcal{U} \in \beta X: A \in \mathcal{U}\}$, where A is any zero-set in X.

THEOREM 3.1 Let $(X, \mathcal{T})$ be a completely regular topological space. The collection $\{C_A: A \text{ is a zero-set in } X\}$ is a base for the closed sets of some topology on βX.

PROOF. Let $\mathcal{U}$ be any Z-ultrafilter on X. Since there exists zero-sets having empty intersection, a Z-ultrafilter cannot contain every zero-set. Let K be a zero-set that is not contained in $\mathcal{U}$, hence $\mathcal{U} \notin C_K$. Thus for any point in βX there exists a C_A that does not contain the point.

Let $\mathcal{U} \in \beta X$ such that $\mathcal{U} \notin C_A \cup C_B$. By theorem 2.1, $C = A \cup B$ is a zero-set. Now $C_C = \{\mathcal{U} \in \beta X: C \in \mathcal{U}\}$ and by theorem 1.2 $\mathcal{U} \notin C_C$, since if $C \in \mathcal{U}$ then either A or B is in $\mathcal{U}$. It is now only necessary to show that $C_A \cup C_B \subset C_C$. Let $\mathcal{U} \in C_A \cup C_B$, then $\mathcal{U}$ either contains A or B. $\mathcal{U}$ must therefore contain $A \cup B$ and $\mathcal{U}$ is thus in C_C. Hence, $\{C_A: A \text{ is a zero-set in } X\}$ forms a base for the closed sets for a topology on βX.

THEOREM 3.2 Let $(X, \mathcal{T})$ be a completely regular space and let βX be the collection of all Z-ultrafilters on X. If the collection $\{C_A: A \text{ is a zero-set in } X\}$ is a base for the closed sets for a topology on βX, then

(a) X is homeomorphic to a dense subset of βX;

(b) βX is compact and T_2;

(c) If (Y, s) is any compact T_2-space and $f\colon X \to Y$ is continuous, then there exists a unique continuous extension of f from βX to Y.

PROOF. (a) Define the mapping $h\colon X \to \beta X$ by $h(p) = \mathcal{U}_p$, where $\mathcal{U}_p$ is the principal Z-ultrafilter ascribed to the point p of X. If $\mathcal{U}_p \neq \mathcal{U}_t$ then clearly $p \neq t$ and h is well-defined. To show that h is continuous, let C_A be any basic closed set of βX, then $\cup\{\mathcal{U}_a\colon a \in A\} \subset C_A$ and therefore $A = h^{-1}(\cup\{\mathcal{U}_a\colon a \in A\}) \subset h^{-1}(C_A)$. Let $p \in h^{-1}(C_A)$. Now $h(p) = \mathcal{U}_p$, $\mathcal{U}_p \in C_A$, and hence, $A \in \mathcal{U}_p$. Therefore $p \in A$ and $h^{-1}(C_A) = A$. A is closed since A is a zero-set. Therefore the inverse of a basic closed set is closed under the mapping h and thus h is a continuous function.

Let a and b be distinct points of X. It has been shown that there exists a zero-set K that contains the point a but not the point b. K is therefore an element of $\mathcal{U}_a$ but not an element of $\mathcal{U}_b$. Therefore, $h(a) \neq h(b)$ and h is a one-to-one mapping.

If A is a zero-set of X, then $h(A) = h(X) \cap C_A$. Since C_A is closed in βX, $h(X) \cap C_A$ is closed in $h(X)$. Hence, zero-sets map to closed subsets of $h(X)$. Since the collection of all zero-sets forms a closed base for the topology on X, it follows that closed sets of X map to closed sets of $h(X)$. Therefore h is a closed mapping of

X to $h(X)$. Thus h is a continuous one-to-one closed mapping of X onto $h(X)$ and hence X is homeomorphic to $h(X)$.

To show that $h(X)$ is dense in βX observe that X is contained in all Z-ultrafilters on X. This implies that $C_X = \beta X$. If K is a closed set of βX that contains $h(X)$, then K is the intersection of a collection of basic closed sets each containing $h(X)$. Note that $h(X)$ is the set of all principal Z-ultrafilters on X. If C_H contains $h(X)$, then C_H contains as a subset the set of all principal Z-ultrafilters on X. Now $C_H = \{\mathcal{U} \in \beta X : H \in \mathcal{U}\}$ and this implies that H is an element of every principal Z-ultrafilter. But the only set contained in every principal Z-ultrafilter is X. Thus $H = X$ and C_X is the only closed set of βX that contains $h(X)$. Therefore $\overline{h(X)} = C_X = \beta X$ and $h(X)$ is dense in βX.

(b) To show that βX is compact let $\{C_\alpha : \alpha \in \Delta\}$ be any collection of closed subsets of βX with the finite intersection property. Each C_α is the intersection of a family of basic closed sets, denoted by $C_\alpha = \bigcap \{C_{A(\alpha,\beta)} : \beta \in \Lambda_\alpha\}$. Therefore the family $\{C_{A(\alpha,\beta)} : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$ also has the finite intersection property. Hence, so does the family of zero-sets $\{A(\alpha,\beta) : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$. Therefore $\{A(\alpha,\beta) : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$ satisfies the definition of a subbase for a Z-filter. Let $\mathcal{E}$ be the Z-filter generated by the subbase and let $\mathcal{U}$ be a Z-ultrafilter that contains $\mathcal{E}$. Therefore $\mathcal{U}$ contains each zero-set in $\{A(\alpha,\beta) : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$ and $\mathcal{U}$ is in every basic closed set in $\{C_{A(\alpha,\beta)} : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$.

It then follows that $\mathcal{U}$ is contained in each C_α. Therefore $\mathcal{U} \in \bigcap \{C_\alpha : \alpha \in \Delta\}$. Since each class of closed subsets of βX which satisfies the finite intersection property has a non-empty intersection, βX is compact.

To show that βX is a T_2-space, let $\mathcal{E}$ and $\mathcal{F}$ be distinct points of βX. Since $\mathcal{E}$ and $\mathcal{F}$ are distinct Z-ultrafilters there exists a zero-set A in $\mathcal{E}$ that is not in $\mathcal{F}$, and hence a zero-set B in $\mathcal{F}$ such that $A \cap B = \emptyset$. Note C_A and C_B are disjoint. Since A and B are zero-sets of X let $A = \{x \in X: f'(x) = 0\}$ and let $B = \{x \in X: g'(x) = 0\}$. Define the function k by

$$k(x) = \frac{1}{2} - \frac{|f'(x)|}{|f'(x)| + |g'(x)|}$$

thus k is a continuous function. Define the function f by

$$\begin{aligned} f(x) &= k(x) \quad \text{if } k(x) < 0 \\ &= 0 \quad \text{if } k(x) \geq 0. \end{aligned}$$

Denote the zero-set of f by F. Define the function g by

$$\begin{aligned} g(x) &= k(x) \quad \text{if } k(x) > 0 \\ &= 0 \quad \text{if } k(x) \leq 0. \end{aligned}$$

Denote the zero-set of g by G. The functions f and g are continuous.

$C_A \subset \beta X - C_G$ since $A \cap G = \emptyset$, and $C_B \subset \beta X - C_F$ since $B \cap F = \emptyset$. Therefore the points $\mathcal{E}$ and $\mathcal{F}$ are in the open sets $\beta X - C_G$ and $\beta X - C_F$, respectively. Suppose some point $\mathcal{U}$ of βX is in $\beta X - C_F$, then $F \notin \mathcal{U}$. But since $X \in \mathcal{U}$ and $F \cup G = X$, then $G \in \mathcal{U}$ since $\mathcal{U}$ is a Z-ultra-filter. Therefore $\mathcal{U} \in C_G$ and $\mathcal{U} \notin \beta X - C_G$. Thus $\beta X - C_G$

and $\beta X - C_F$ are disjoint. βX is therefore a T_2-space.

(c) To show that any continuous function f from X to Y, a compact T_2-space, has a unique continuous extension from βX to Y it is only necessary to show that there exists at least one continuous extension. The fact that X is homeomorphic to a dense subset of βX and that Y is a T_2-space implies that any continuous extension will be unique.

Let f be a continuous function from X to Y. If C is a zero-set in Y, then $f^{-1}(C)$ is a zero-set in X. Let $\mathcal{U} \in \beta X$ and for notation purposes let $E(f, \mathcal{U}) = \{\text{zero-sets } A \subset Y : f^{-1}(A) \in \mathcal{U}\}$. To show that $E(f, \mathcal{U})$ is a Z-filter, note that $f^{-1}(Y) = X$ a zero-set in $\mathcal{U}$, hence $E(f, \mathcal{U})$ is non-empty. Let B and C be elements in $E(f, \mathcal{U})$, therefore $f^{-1}(B) \in \mathcal{U}$ and $f^{-1}(C) \in \mathcal{U}$. Since $\mathcal{U}$ is a Z-ultrafilter on X, $f^{-1}(B) \cap f^{-1}(C) \in \mathcal{U}$ and hence $f^{-1}(B \cap C) \in \mathcal{U}$ which implies that $B \cap C \in E(f, \mathcal{U})$. Let $H \in E(f, \mathcal{U})$ and let O be a zero-set in Y such that $H \subset O$, then $f^{-1}(H) \subset f^{-1}(O)$ and $f^{-1}(H) \in \mathcal{U}$. Therefore $f^{-1}(O)$ is an element of $\mathcal{U}$. This implies that $O \in E(f, \mathcal{U})$. The null set is not an element of $E(f, \mathcal{U})$ since $f^{-1}(\emptyset) \notin \mathcal{U}$. Thus $E(f, \mathcal{U})$ is a Z-filter on Y. Suppose $A \cup B \in E(f, \mathcal{U})$, then $f^{-1}(A \cup B) \in \mathcal{U}$. But $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and since $\mathcal{U}$ is a Z-ultrafilter, either $f^{-1}(A)$ or $f^{-1}(B)$ is in $\mathcal{U}$. Hence, if $A \cup B \in E(f, \mathcal{U})$ then either A or B is in $E(f, \mathcal{U})$.

Since Y is compact, every class of closed subsets of Y with non-empty finite intersection, has a non-empty intersection. Hence, $\bigcap\{A:\ A \in E(f,\mathcal{U})\}$ is non-empty. Assume that the intersection contains two distinct points p and q. Since Y is a compact T_2-space it is completely regular. Therefore there exists disjoint open sets P and Q of Y that contain p and q respectively and whose complements are zero-sets. Hence, $(Y - P)\cup(Y - Q) = Y \in E(f,\mathcal{U})$ and this implies that either $Y - P$ or $Y - Q$ is in $E(f,\mathcal{U})$. Since $p \notin Y - P$ and $q \notin Y - Q$, p and q cannot both be in $\bigcap\{A:\ A \in E(f,\mathcal{U})\}$. Thus $\bigcap\{A:\ A \in E(f,\mathcal{U})\}$ contains a unique point which will be denoted by $\hat{f}(\mathcal{U})$. Therefore $\hat{f}$ can be defined as a function from βX to Y.

Let $a \in X$, then the corresponding point in βX is $\mathcal{U}_a$. Now $\hat{f}(\mathcal{U}_a) = p$, where $p \in Y$ and $p \in \bigcap\{A:\ A \in E(f,\mathcal{U}_a)\}$. Note p is the only point in $\bigcap\{A:\ A \in E(f,\mathcal{U}_a)\}$. Then $E(f,\mathcal{U}_a) = \{\text{zero-sets } A \subset Y:\ f^{-1}(A) \in \mathcal{U}_a\}$ and this implies that $f(a)$ is in every zero-set in $E(f,\mathcal{U}_a)$. Hence, $f(a) = p$ and $\hat{f}$ is an extension of f.

To show that $\hat{f}$ is continuous it will first be shown that $cl_{\beta X}(A) = C_A$, where A is any zero-set in X. Let A be a zero-set in X and let h be the embedding map of X into βX. Now $h(A) = \{\mathcal{U}_a:\ a \in A\} \subset \{\mathcal{U} \in \beta X:\ A \in \mathcal{U}\} = C_A$. Therefore, $cl_{\beta X}(A) \subset C_A$. Let C_B contain $h(A) = \{\mathcal{U}_a:\ a \in A\}$. Then $C_B \cap h(X) = \{\mathcal{U}_b:\ b \in B\}$. Hence, $A \subset B$ and if C_B contains $h(A)$ then $C_A \subset C_B$. It follows then that $cl_{\beta X}(A) = C_A$.

Let O be any open set in Y and let p be a point in O such that there exists a $\mathcal{U} \in \beta X$ where $\hat{f}(\mathcal{U}) = p$.

It will be shown that there exists an open set in βX that contains $\mathcal{U}$ and is contained in $\hat{f}^{-1}(O)$, thus showing that $\hat{f}$ is continuous.

Since Y is completely regular, there exists a zero-set-neighborhood F that contains $\hat{f}(\mathcal{U})$ and is contained in O. Let Q denote the open set contained in F that contains the point $\hat{f}(\mathcal{U})$. There exists a continuous real-valued function k such that $k(Y - Q) = 0$ and $k(\hat{f}(\mathcal{U})) = 1$. Therefore there exists a zero-set $F' \supset Y - Q$ such that the complement of F' in Y is an open set contained in F and containing the point $\hat{f}(\mathcal{U})$. Note that $F \cup F' = Y$ and hence $f^{-1}(F) \cup f^{-1}(F') = X$. It then follows that $cl_{\beta X}(f^{-1}(F)) \cup cl_{\beta X}(f^{-1}(F')) = \beta X$.

If $\mathcal{U} \in cl_{\beta X}(f^{-1}(F')) = C_{f^{-1}(F')}$, then $f^{-1}(F') \in \mathcal{U}$. Hence, $F' \in E(f, \mathcal{U})$ and $\hat{f}(\mathcal{U}) \in F'$. But this is impossible and, therefore $\mathcal{U} \notin cl_{\beta X}(f^{-1}(F'))$. Hence $\beta X - cl_{\beta X}(f^{-1}(F'))$ is an open set H in βX that contains the point $\mathcal{U}$. Let $\mathcal{U}'$ be any element of $H \subset cl_{\beta X}(f^{-1}(F))$, then $f^{-1}(F) \in \mathcal{U}'$. Now $\hat{f}(\mathcal{U}') = \bigcap \{A: A \in E(f, \mathcal{U}')\}$, hence $\hat{f}(\mathcal{U}') \in F$. Thus $\hat{f}(H) \subset F < O$ and $\hat{f}$ is a continuous function.

CHAPTER V

SUMMARY

This paper covered the concepts of a one-point compactification and the Stone-Čech compactification. The first chapter introduced the paper, presented the purpose of the paper and the problem under consideration, and listed definitions basic to the entire paper. Chapter two developed the one-point or Alexandroff one-point compactification. The second chapter also developed several relationships between $(X, \mathcal{T})$ and $(X', \mathcal{T}')$. Chapter three developed the Stone-Čech compactification and several relationships between X and βX. It also demonstrated the uniqueness of the Stone-Čech compactification. The fourth chapter used Z-ultrafilters to construct the Stone-Čech compactification. Chapter four also introduced some basic notions of filters, ultrafilters, Z-filters, and Z-ultrafilters.

The possibilities for further study in this area are many. Several relationships concerning the Stone-Čech compactification lie beyond the scope of this paper. Other types of compactifications have been developed. The Wallman compactification is noted in Gillman and Jerison [5] to be equivalent to the Stone-Čech compactification if and only if $(X, \mathcal{T})$ is normal. Thus other types of compactifications, see Gillman and Jerison [5] and Thron [4] for examples, lend themselves to further study.

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