THE ANALYTICAL REPRESENTATIONS OF
POINTS LINES AND CIRCLES
ASSOCIATED WITH A TRIANGLE

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

By

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May 1936

Approved for the Major Department

Approved for the Graduate Council

ACKNOWLEDGMENT

To Dr. Oscar J. Peterson, Head of the Department of Mathematics of the Kansas State Teachers College of Emporia, who patiently directed this study, the writer is indebted and wishes to express his sincere appreciation.

To Dr. Edwin J. Brown, Director of the Graduate Division of the Kansas State Teachers College of Emporia, for his suggestions offered during the writing.

Charles Girod

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CHAPTER I

INTRODUCTION

The past decade has witnessed a revival of interest in the geometry of the triangle. In the United States interest in this field has spread phenomenally, especially since the appearance of the pioneering College Geometry[1] by Nathan Altshiller-Court in 1925, and at the present time courses in college geometry are generally available in our colleges and universities. Now the worker in this field has at hand a considerable body of important material in such books as Johnson's Modern Geometry[2] or Morely's Inversive Geometry and in our mathematical journals.[3]

The possibilities for achievement in this field of geometry seem to be limitless. Few branches of mathematics reward so readily and bountifully the searcher after new truths. It is the purpose of this thesis to provide a means for making more accessible some of the material which gives promise of new and interesting relations.

The methods of modern pure geometry are beautiful in themselves, and it is with a hesitant hand that the writer dares profane them with analytical devices. But it must be recognized that even when the geometer has perfected his techniques and built up a unified structure, which includes such config-

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[1] Nathan Altshiller-Court, College Geometry (Richmond: Johnson Publishing Company, 1925).

[2] Roger A. Johnson, Modern Geometry (Boston: Houghton Mifflin Company, 1929).

[3] Of especial value in this connection is The American Mathematical Monthly, which offers rich suggestions not only in special articles dealing with pure geometry but in the department of Problems and Solutions.

urations as those of Lemoine[4] and Brocard,[5] a sensitive and complete grasp of geometric tools and an active imagination give no assurance that simple properties of concurrence and collinearity or parallelism will not be missed.

It is the purpose of this thesis to present an analytical foundation for the study of the geometry of the triangle. It is hoped that the analytical framework provided here, and summarized in the final chapter, will prove valuable in probing unexplored regions or in extending those regions which have been explored.

It is obvious, of course, that algebraic methods are not always superior to the methods of Euclidean geometry; but at least one very important advantage of algebra over geometry cannot be overlooked. In pure geometry, ideas are cumulative; that is, the geometric structure is built up of interlocking and interrelated pieces, and the geometric tool becomes progressively more intricate. This is less true in algebraic geometry. When the equation of a line has been found, the steps leading to its derivation can generally be ignored without impairing the effectiveness of its use. One cam immediately and automatically select the point of which a given line is the trilinear polar,[6] or possibly indicate a number of points which lie upon the line. Isogonal[7] or isotomic[8] conjugates can be paired without previous recognition of their relationship. When a point is defined by its coordinates it is frequently a simple matter to identify other points with which it is collinear,

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[4] *Cf.* *post*, p.

[5] *Ibid.*, p.

[6] *Ibid.*, p.

[7] *Ibid.*, p.

[8] *Ibid.*, p.

even though the geometric connection be not revealed. Indeed, it has been a temptation to go beyond the scope of this thesis when algebraic forms have suggested ideas which seemed new.

It is thought that the algebraic representations of points and lines and circles given in this thesis will provide a useful and adequate foundation for considerable further study of the geometry of the triangle.

CHAPTER II

TRILINEAR COORDINATES

In an algebraic treatment of geometric properties the choice of a suitable coordinate system is of the utmost importance. Where projective properties only are involved, as in questions having to do with the collinearity of points or the concurrence of lines, general projective coordinates[1] would be most suitable. In this study, however, where many metric relations are given prominent attention, it seems that a more restricted system would be the most useful. The writer has chosen trilinear coordinates[2] as the simplest means of providing an algebraic treatment of the geometry of the triangle. The essential features of this system are presented here in order to avoid a confusion of terms or of implications.

Let A_1, A_2, A_3 be the points of intersection of three nonconcurrent reference lines. The triangle $A_1A_2A_3$ is called the fundamental triangle. The lengths of the sides A_2A_3, A_3A_1, A_1A_2 are denoted by a_1, a_2, a_3 respectively. Let the directed perpendicular distances of any point X from the sides A_2A_3, A_3A_1, A_1A_2 be x_1, x_2, x_3 respectively; then x_1, x_2, x_3, or numbers proportional to them, are called the trilinear coordinates of the point X referred to the triangle $A_1A_2A_3$. These coordinates, x_1, x_2, x_3, are considered positive when the perpendiculars are in the same direction as the perpendiculars from the sides to the opposite angular points of the triangle of reference. Two of these three actual distances, or merely the ratios of

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[1] R. M. Winger, Projective Geometry (Boston: D. C. Heath and Company, 1923), p. 79.

[2] Charles Smith, Conic Sections (London: Macmillan and Company, 1927), p. 341.

the three distances, are sufficient to determine the position of a point.

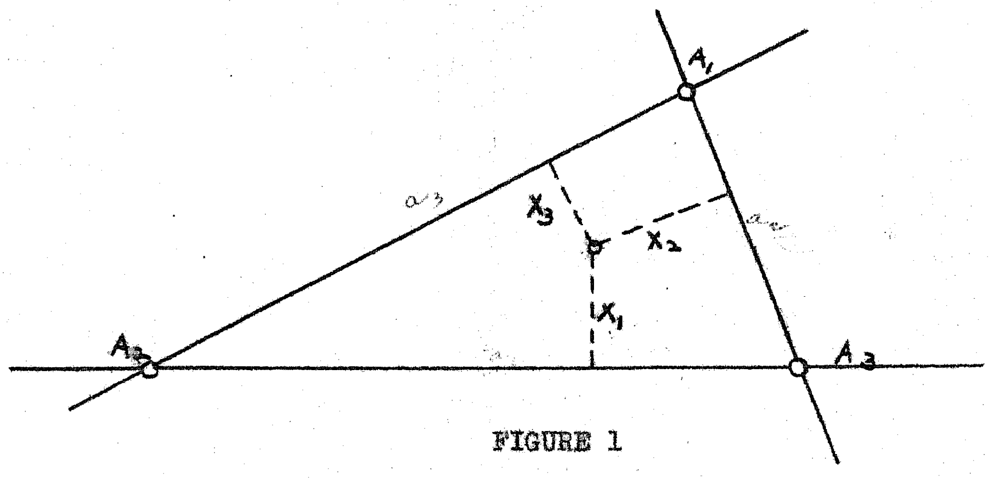

FIGURE 1

The three distances x_1, x_2, x_3 are connected by the relation

$$a_1x_1 + a_2x_2 + a_3x_3 = 2\triangle,$$

where $\triangle$ is the area of the triangle $A_1A_2A_3$. If k is a common multiplier of the coordinates, x_1, x_2, x_3, such that kx_1, kx_2, kx_3, are the actual distances of the point X from the sides of the triangle of reference, then $k(a_1x_1 + a_2x_2 + a_3x_3) = 2\triangle$, or $k = \dfrac{2\triangle}{a_1x_1 + a_2x_2 + a_3x_3}$.

The coordinates of the vertices, A_1, A_2, A_3, are evidently (1, 0, 0), (0, 1, 0) and (0, 0, 1). The equations of the sides, A_2A_3, A_3A_1, A_1A_2 are $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. The equation $m_1x_1 + m_2x_2 + m_3x_3 = 0$ represents a straight line. Its intersections with the sides of the triangle are $(0, m_3, -m_2)$, $(-m_3, 0, m_1)$, $(m_2, -m_1, 0)$, and the equation of the line at infinity, or ideal line, is $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

The equation of a straight line which passes through two given points, X', X'', is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} = 0.$$

The condition that three given points X', X'', X''', be on a straight line is

$$\begin{vmatrix} x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \\ x_1''' & x_2''' & x_3''' \end{vmatrix} = 0.$$

The condition that three straight lines, l, m, n, meet in a point is that the determinant of their associated coefficients is equal to zero. The equations of the lines are:

$$l: \quad l_1x_1 + l_2x_2 + l_3x_3 = 0,$$

$$m: \quad m_1x_1 + m_2x_2 + m_3x_3 = 0,$$

$$n: \quad n_1x_1 + n_2x_2 + n_3x_3 = 0.$$

The condition for their concurrency is

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

Two lines, m, n, are parallel if their point of intersection is a point at infinity; that is, the lines m, n intersect on the ideal line. The condition for this is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

The ideal point on a line is the point whose coordinates satisfy the equation of that line and the ideal line. Let the equation of the given line be $m_1x_1 + m_2x_2 + m_3x_3 = 0$.

The ideal point on this line is

$$\begin{vmatrix} a_2 & a_3 \\ m_2 & m_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ m_3 & m_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ m_1 & m_2 \end{vmatrix}$$

The general equation of a circle[3] is $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + (a_1x_1 + a_2x_2 + a_3x_3)(m_1x_1 + m_2x_2 + m_3x_3) = 0$. The equation of the circumcircle is $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0$; the equation of the ideal line is $a_1x_1 + a_2x_2 + a_3x_3 = 0$; and the equation of the radical axis of the circumcircle and any circle is $m_1x_1 + m_2x_2 + m_3x_3 = 0$.

The center of the general circle is found by solving the equations $F_1:F_2:F_3 = a_1:a_2:a_3$, where

$$F_1 = 2a_1m_1x_1 + (a_1m_2 + a_2m_1 + a_3)x_2 + (a_1m_3 + a_2 + a_3m_1)x_3,$$

$$F_2 = (a_1m_2 + a_2m_1 + a_3)x_1 + 2a_2m_2x_2 + (a_1 + a_2m_3 + a_3m_2)x_3,$$

$$F_3 = (a_1m_3 + a_2 + a_3m_1)x_1 + (a_1 + a_2m_3 + a_3m_2)x_2 + 2a_3m_3x_3.$$

These equations are justified in a later paragraph.

All circles pass through the two circular points at infinity, J, J'. These may be found by solving simultaneously the equations of the ideal line and any circle. Eliminating x_1,

$$a_2a_3x_2^2 + (-a_1^2 + a_2^2 + a_3^2)x_2x_3 + a_2a_3x_3^2 = 0,$$

or $x_2^2 + 2\cos A_1x_2x_3 + x_3^2 = 0.$

Then $x_2:x_3 = -(\cos A_1 \pm i \sin A_1) : 1.$

Similarly, eliminating x_2,

$x_1:x_3 = -(\cos A_2 \pm i \sin A_2) : 1.$

In order that the equations of the ideal line be satisfied it is necessary to use opposite signs with the coefficients of i. The coordinates of the circular points at infinity are then

$$J: \quad (\cos A_2 + i \sin A_2,\ \cos A_1 - i \sin A_1,\ -1),$$

$$J': \quad (\cos A_2 - i \sin A_2,\ \cos A_1 + i \sin A_1,\ -1).$$

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[3] Charlotte Angas Scott, Modern Analytical Geometry (New York: G. E. Stechert and Co., 1924), p. 116.

These may also be written

$$J: \quad (e^{iA_2}, \quad e^{-iA_1}, -1),$$

$$J': \quad (e^{-iA_2}, \quad e^{iA_1}, -1).$$

CHAPTER III

POINTS AND LINES ASSOCIATED WITH A GIVEN POINT

In the study of the properties of a triangle certain point and line configurations present themselves in such an elemental manner that a separate analytical treatment is advisable. The necessity of a consistent and convenient set of notations is immediately apparent.

Let $P(p_1, p_2, p_3)$ be any point. Denote the projections of P upon the sides of the triangle from the opposite vertices by P_1, P_2, P_3. Their coordinates are evidently $(0, p_2, p_3)$, $(p_1, 0, p_3)$, $(p_1, p_2, 0)$.

The equations of the rays through P and the vertices are

$$A_1P: \quad p_3x_2 - p_2x_3 = 0,$$

$$A_2P: \quad p_1x_3 - p_3x_1 = 0,$$

$$A_3P: \quad p_2x_1 - p_1x_2 = 0.$$

The line P_2P_3 meets the side A_2A_3 in the point $P_1'(0,p_2,-p_3)$, which is the harmonic conjugate of P_1 relative to the vertices A_2 and A_3. Similarly, the line P_3P_1 meets the side A_3A_1 in the point $P_2''(-p_1,0,p_3)$, the harmonic conjugate of P_2 relative to the vertices A_3 and A_1; and P_1P_2 meets the side A_1A_2 in the point $P_3'''(p_1,-p_2,0)$, the harmonic conjugate of P_3 relative to A_1 and A_2.

The three points P_1', P_2'', P_3''' are collinear. The line of these points is called the trilinear polar[1] of P. Its equation is

$$p_2p_3x_1 + p_3p_1x_2 + p_1p_2x_3 = 0.$$

The lines $A_1P_1', A_2P_2'', A_3P_3'''$ are respectively the harmonic conjugates

[1] Nathan Altshiller-Court, College Geometry (Richmond: Johnson Publishing Company, 1925), p. 220.

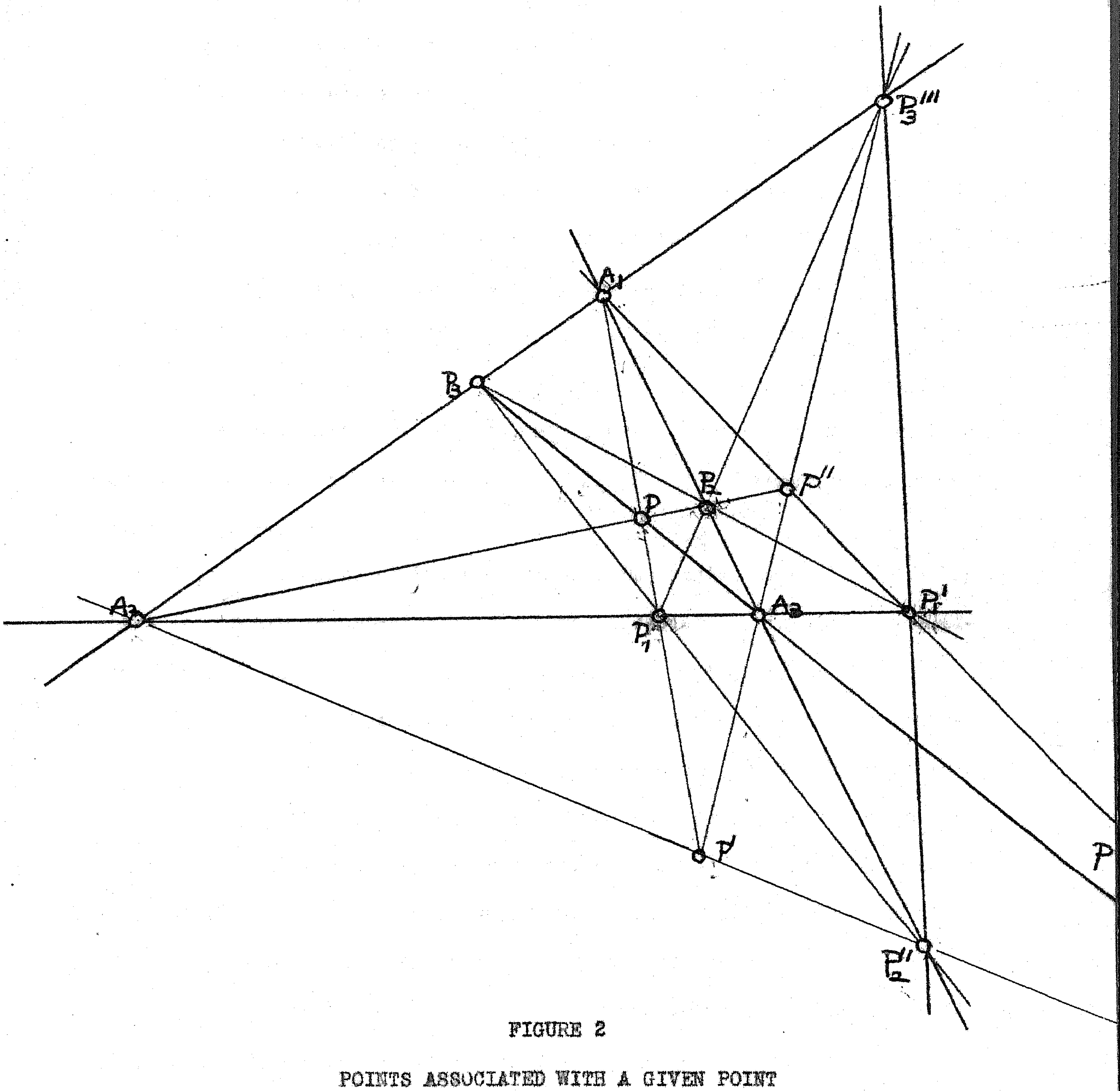

FIGURE 2

POINTS ASSOCIATED WITH A GIVEN POINT

of the rays A_1P, A_2P, A_3P relative to the including sides. Their equations are

A_1P_1': $p_3x_2 + p_2x_3 = 0$,

A_2P_2'': $p_1x_3 + p_3x_1 = 0$,

A_3P_3''': $p_2x_1 + p_1x_2 = 0$.

The lines A_1P, A_2P_2'', A_3P_3''' are concurrent in a point $P'(-p_1,p_2,p_3)$, which is the harmonic conjugate of P relative to A_1 and P_1. Similarly A_1P_1', A_2P, A_3P_3''' are concurrent in a point $P''(p_1,-p_2,p_3)$, the harmonic conjugate of P relative to A_2 and P_2; and A_1P_1', A_2P_2'', A_3P are concurrent in a point $P'''(p_1,p_2,-p_3)$, the harmonic conjugate of P relative to A_3 and P_3.

The four points P, P', P'', P''', are the vertices of a complete quadrangle having the vertices A_1, A_2, A_3 of the reference triangle as diagonal points.[2]

In the summary of the results of this thesis, several of the important points are accompanied by the three associated points of the quadrangular group; as I, I', I'', I''', and M, M', M'', M'''.

The use of subscripts and primes indicated here will be used generally throughout this study with only a few exceptions. The principal exceptions are the use of A_1,A_2,A_3 for the vertices of the fundamental triangle; $B_1,B_2,B_3,B_1',B_2',B_3'$ as the vertices of two triangles to be introduced later; J, J' for the circular points at infinity; and possibly occasional exceptions made advisable by circumstances.

P_a,P_b,P_c will be used to designate the feet of the perpendiculars from P upon the sides of the triangle. Their coordinates are

P_a: $(0,\ p_2 + p_1 \cos A_3,\ p_3 + p_1 \cos A_2)$,

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[2] R. M. Winger, Projective Geometry (Boston: D. C. Heath and Company, 1923) p. 74.

P_b: $(p_1 + p_2 \cos A_3,\ 0,\ p_3 + p_2 \cos A_1)$,

P_c: $(p_1 + p_3 \cos A_2,\ p_2 + p_3 \cos A_1,\ 0)$.

<u>The pedal triangle and circle</u>. It was pointed out in an earlier part of this chapter that the pedal points P_1, P_2, P_3 were the projections of the point $P(p_1, p_2, p_3)$ upon the sides A_2A_3, A_3A_1, A_1A_2 of the triangle of reference. The pedal triangle of P is here defined as the triangle having the points P_1, P_2, P_3 as vertices. The equations of P_2P_3, P_3P_1, P_1P_2 are

$$-p_2p_3x_1 + p_3p_1x_2 + p_1p_2x_3 = 0,$$

$$p_2p_3x_1 - p_3p_1x_2 + p_1p_2x_3 = 0,$$

$$p_2p_3x_1 + p_3p_1x_2 - p_1p_2x_3 = 0.$$

The pedal circle is the circle which passes through P_1, P_2, P_3. Substituting the coordinates of P_1, P_2, P_3 in the general equation of a circle the following relations are obtained

$$p_2m_2 + p_3m_3 = -\frac{a_1p_2p_3}{a_2p_2 + a_3p_3},$$

$$p_1m_1 + p_3m_3 = -\frac{a_2p_3p_1}{a_1p_1 + a_3p_3},$$

$$p_1m_1 + p_2m_2 = -\frac{a_3p_1p_2}{a_1p_1 + a_2p_2}.$$

The values of m_1, m_2, m_3 which determine the pedal circle are:

$$m_1 = \frac{a_1p_2p_3}{p_1(a_2p_2 + a_3p_3)} - \frac{a_2p_3p_1}{p_2(a_3p_3 + a_1p_1)} - \frac{a_3p_1p_2}{p_3(a_1p_1 + a_2p_2)},$$

$$m_2 = -\quad " \quad + \quad " \quad - \quad " \quad ,$$

$$m_3 = -\quad " \quad - \quad " \quad + \quad " \quad .$$

<u>Trilinear</u> polar. Earlier in this chapter the trilinear polar of P was defined to be the line $P_1'P_2''P_3'''$. Its equation is

$$p_2p_3x_1 + p_3p_1x_2 + p_1p_2x_3 = 0$$

or $\dfrac{x_1}{p_1} + \dfrac{x_2}{p_2} + \dfrac{x_3}{p_3} = 0.$

It may also be derived from the formula $\left(\dfrac{\partial f}{\partial x_1}\right)_P x_1 - \left(\dfrac{\partial f}{\partial x_2}\right)_P x_2 - \left(\dfrac{\partial f}{\partial x_3}\right)_P x_3 = 0.$ In this case the curve $f(x_1, x_2, x_3) = 0$ is the degenerate one $x_1x_2x_3 = 0$, and $\left(\dfrac{\partial f}{\partial x_1}\right)_P = p_2p_3$, $\left(\dfrac{\partial f}{\partial x_2}\right)_P = p_3p_1$, $\left(\dfrac{\partial f}{\partial x_3}\right)_P = p_1p_2$, leading to the equation obtained in the paragraph above, $\dfrac{x_1}{p_1} + \dfrac{x_2}{p_2} + \dfrac{x_3}{p_3} = 0.$

<u>Polar with respect to circumcircle.</u> The equation of the circumcircle is $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0$. In this case

$$F_1 = a_2p_3 + a_3p_2$$
$$F_2 = a_3p_1 + a_1p_3$$
$$F_3 = a_1p_2 + a_2p_1$$

and the equation of the polar of P is

$$(a_2p_3 + a_3p_2)x_1 + (a_3p_1 + a_1p_3)x_2 + (a_1p_2 + a_2p_1)x_3 = 0.$$

<u>Polar with respect to the general circle.</u> The general equation of a circle is

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + (a_1x_1 + a_2x_2 + a_3x_3)(m_1x_1 + m_2x_2 + m_3x_3) = 0.$$

Denote by F_1, F_2, F_3 the values of the partial derivatives of the left member with respect to x_1, x_2, x_3 at the point $P(p_1, p_2, p_3)$.

$$F_1 = 2a_1m_1p_1 + (a_1m_2 + a_2m_1 + a_3)p_2 + (a_1m_3 + a_2 + a_3m_1)p_3,$$
$$F_2 = (a_1m_2 + a_2m_1 + a_3)p_1 + 2a_2m_2p_2 + (a_1 + a_2m_3 + a_3m_2)p_3,$$
$$F_3 = (a_1m_3 + a_2 + a_3m_1)p_1 + (a_1 - a_2m_3 + a_3m_2)p_2 + 2a_3m_3p_3.$$

The equation of the polar of P is

$$F_1x_1 + F_2x_2 + F_3x_3 = 0.$$

The center of a circle is the point whose polar is the ideal line.

Identifying the equation of the polar of P with the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, it is found that P is the center of the circle if the equations

$$F_1 : F_2 : F_3 = a_1 : a_2 : a_3$$

are satisfied.

Isogonal conjugates. If two rays through the vertex of an angle make equal angles with its sides they are said to be "isogonal" or "isogonal conjugates."[3] It is evident that two isogonally conjugate rays are symmetrical with regard to the bisector of the angle.

The equations of the lines through the vertices A_1, A_2, A_3 of the triangle of reference and any point $P(p_1, p_2, p_3)$ are

$$p_3x_2 - p_2x_3 = 0,$$
$$p_3x_1 - p_1x_3 = 0,$$
$$p_2x_1 - p_1x_2 = 0.$$

Then the equations of the isogonal conjugates of the rays A_1P, A_2P, A_3P are

$$p_2x_2 - p_3x_3 = 0,$$
$$p_3x_3 - p_1x_1 = 0,$$
$$p_1x_1 - p_2x_2 = 0.$$

It is apparent that these three isogonally conjugate rays are concurrent in a point $Q(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$. This point Q is called the isogonal conjugate of P (Fig. 3). Occasionally $P(p_1, p_2, p_3)$ and $Q(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$ are referred to as inverse points with respect to the triangle.

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[3] Roger A. Johnson, Modern Geometry (Boston: Houghton Mifflin Company, 1929), p. 153.

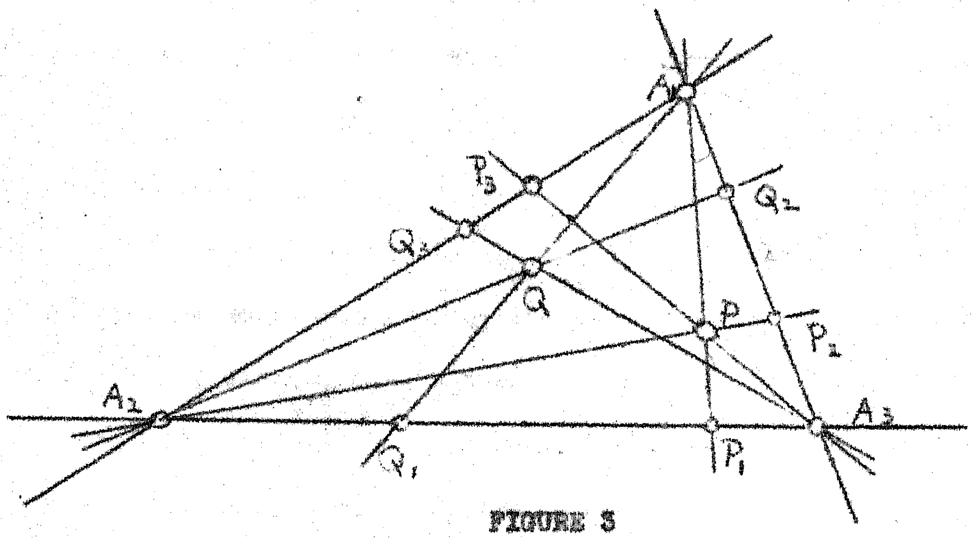

FIGURE 3

Every point not on a side of the given triangle has an actual conjugate. The only self conjugate points are the four equicenters. If P is on the circumcircle, the isogonals of A_1P and A_2P will be parallel. It then follows that the isogonal conjugate of any point P on the circumcircle is at infinity.

<u>Isotomic conjugates</u>. Let $P(p_1,p_2,p_3)$ be any point and let P_1,P_2,P_3 be the projections of P upon the sides of the triangle from the opposite vertices. Let Q_1,Q_2,Q_3 (Fig. 4) be points on the respective sides of the triangle such that,

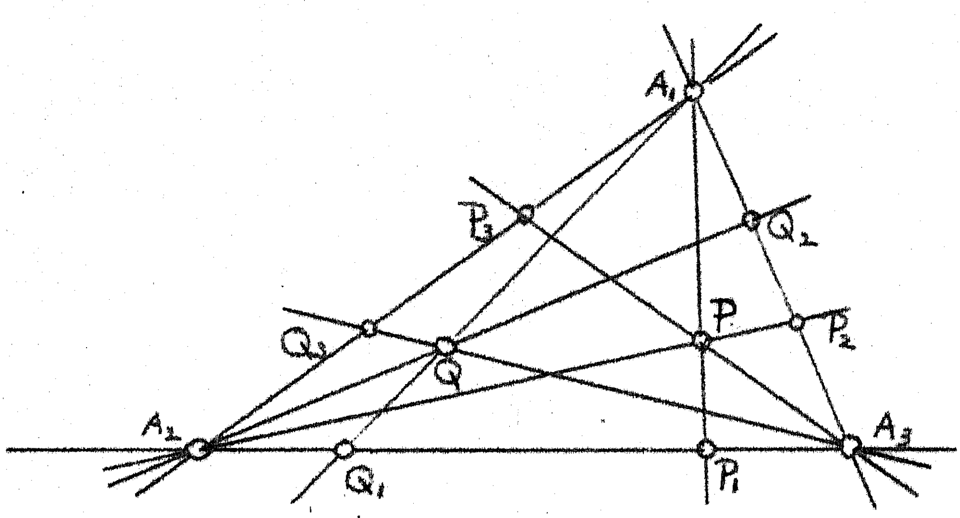

FIGURE 4

considering directed segments on the sides, $A_2P_1 = Q_1A_3$, $A_3P_2 = Q_2A_1$, $A_1P_3 = Q_3A_2$. Then Q_1 divides the directed side A_3A_2 in the same ratio in which P_1 divides A_2A_3; or the ratios in which P_1 and Q_1 divide the side A_2A_3 are reciprocals. The coordinates of P_1 are $(0, p_2, p_3)$. Therefore $A_2P_1 : P_1A_3 = p_3 \csc A_2 : p_2 \csc A_3 = \frac{p_3}{a_2} : \frac{p_2}{a_3} = a_3p_3 : a_2p_2$; and $A_2Q_1 : Q_1A_3 = a_2p_2 : a_3p_3$. The coordinates of Q_1 are

$$(p, a_3p_3 \sin A_3, a_2p_2 \sin A_2),$$

or $$(p, a_3^2p_3, a_2^2p_2).$$

Similarly the coordinates of Q_2 and Q_3 are

$$(a_3^2p_3, 0, a_1^2p_1)$$

and $$(a_2^2p_2, a_1^2p_1, 0).$$

It is evident that the rays A_1Q_1, A_2Q_2, A_3Q_3 are concurrent in the point

$$Q: \left(\frac{1}{a_1^2p_1}, \frac{1}{a_2^2p_2}, \frac{1}{a_3^2p_3}\right).$$

The point Q is called the isotomic conjugate of P.[4]

The only isotomically self-conjugate points are the four points M, M', M'', M''', namely the median point and the three exmedian points.

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[4] *Ibid.*, p. 157.

CHAPTER IV

SPECIAL GROUPS OF POINTS LINES AND CIRCLES

This section deals with several points, lines, and circles of especial interest in connection with the study of the triangle. Geometric properties involving these points are well known, and are generally either derived or suggested in one or more of the better known works in this field. As was suggested in the introduction, the analytic method is not always superior to the purely geometric method in effectiveness, but it is thought that even in the case of those points whose coordinates are not expressible with all the compactness wished for, the forms of these expressions may still suggest further properties.

The incenter and excenters. (A.C.-67; J.-182).[1] Since the incenter is equidistant from the sides of the reference triangle its coordinates are (1, 1, 1). The excenters I', I'', I''' are likewise equidistant from the sides of the reference triangle, and have coordinates (-1, 1, 1), (1, -1, 1), (1, 1, -1).

Median point and exmedian points. (A.C.-59; J.-9). Let M denote the median point of the triangle. Then M_1 is the midpoint of A_2A_3. The coordinates of M_1 are (o, a_3, a_2). The median issued from A_1, that is, the line A M, has the equation $a_2x_2 - a_3x_3 = 0$. Similarly, the equations of the medians issued from A_2 and A_3 are $a_3x_3 - a_1x_1 = 0$ and $a_1x_1 - a_2x_2 = 0$. It is evident that the coordinates of M, the point of intersection of the medians, are $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$.

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[1] A.C.-67, refers to Altshiller-Court's College Geometry, page 67 and J.-182, refers to Johnson's Modern Geometry, page 182. This system of cross references is used throughout Chapter IV.

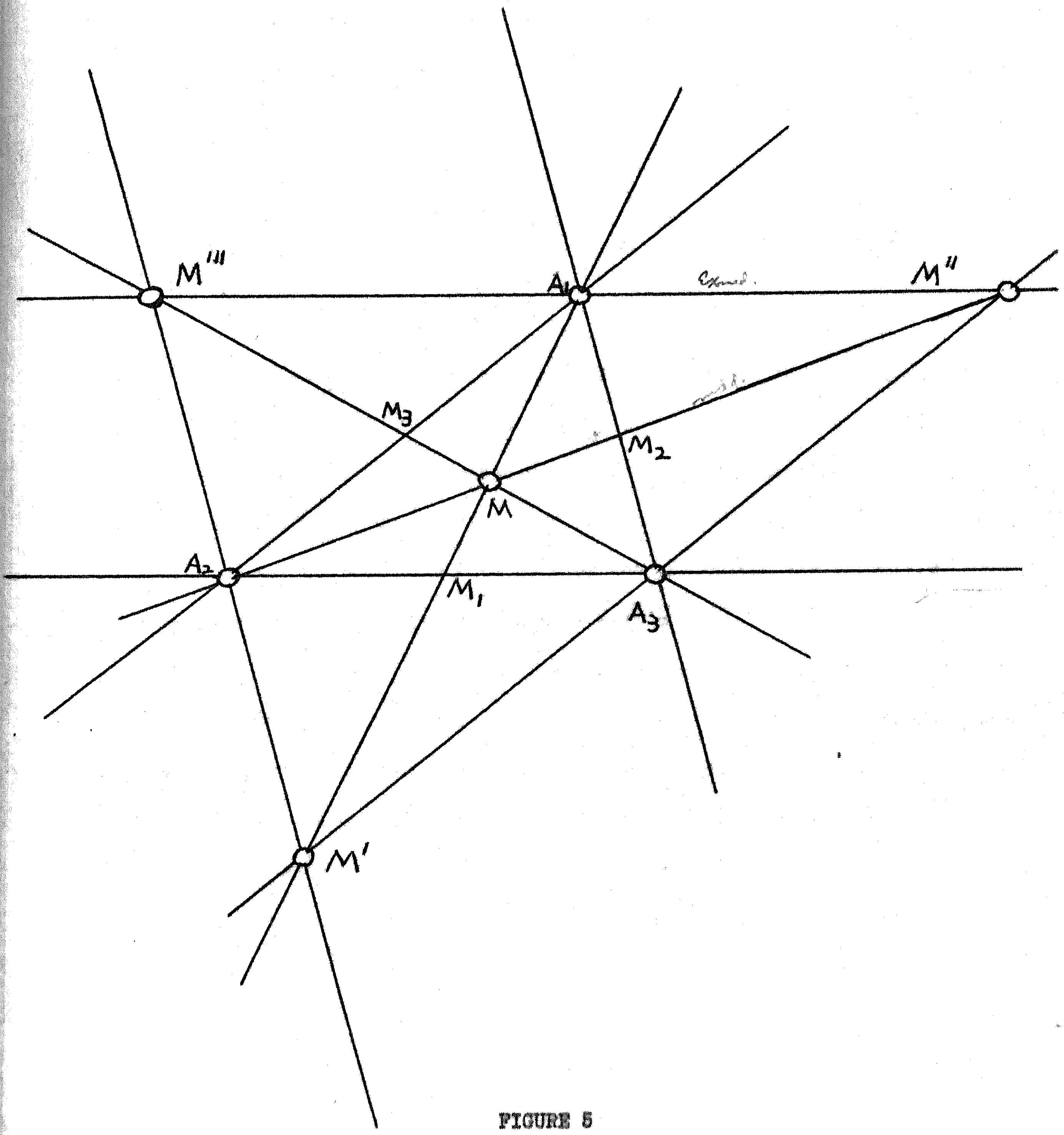

FIGURE 5

MEDIANS AND EXMEDIANS

The lines through the vertices of a triangle parallel to the opposite sides are called exmedians (Fig. 5). They are the harmonic conjugates of the medians with respect to the including sides. The point of concurrency of a median and two exmedians is called an exmedian point.

The equations of the three exmedians are obviously

$$a_2x_2 + a_3x_3 = 0,$$
$$a_3x_3 + a_1x_1 = 0,$$
$$a_1x_1 + a_2x_2 = 0.$$

The coordinates of the exmedian points are $M'\left(-\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right)$, $M''\left(\frac{1}{a_1}, \frac{-1}{a_2}, \frac{1}{a_3}\right)$, $M'''\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{-1}{a_3}\right)$.

<u>Symmedians and exsymmedians</u> (A.C.-222; J.-213). The symmedian point, K, of a triangle is the isogonal conjugate of the median point. A line through a vertex and the symmedian point is called a symmedian (Fig. 6). Since the coordinates of the median point are $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$, the point K has coordinates a_1, a_2, a_3. The equations of the symmedians are therefore $a_3x_2 - a_2x_3 = 0$, $a_1x_3 - a_3x_1 = 0$, $a_2x_1 - a_1x_2 = 0$.

The harmonic conjugate of the symmedians with respect to the including sides is called an exsymmedian (Fig. 6). The exsymmedians are antiparallels of the sides opposite relative to the other two sides. They are also the tangents to the circumcircle at the vertices. Their equations are

$$a_3x_2 + a_2x_3 = 0, \quad a_1x_3 + a_3x_1 = 0, \quad a_2x_1 + a_1x_2 = 0.$$

A symmedian and two exsymmedians are concurrent in a point called an exsymmedian point. The three exsymmedian points are $K'(-a_1, a_2, a_3)$, $K''(a_1, -a_2, a_3)$, $K'''(a_1, a_2, -a_3)$.

<u>The orthocenter</u> (A.C.-82; J.-161). The altitudes of a triangle are concurrent in a point called the orthocenter (Fig. 12). Its coordinates are

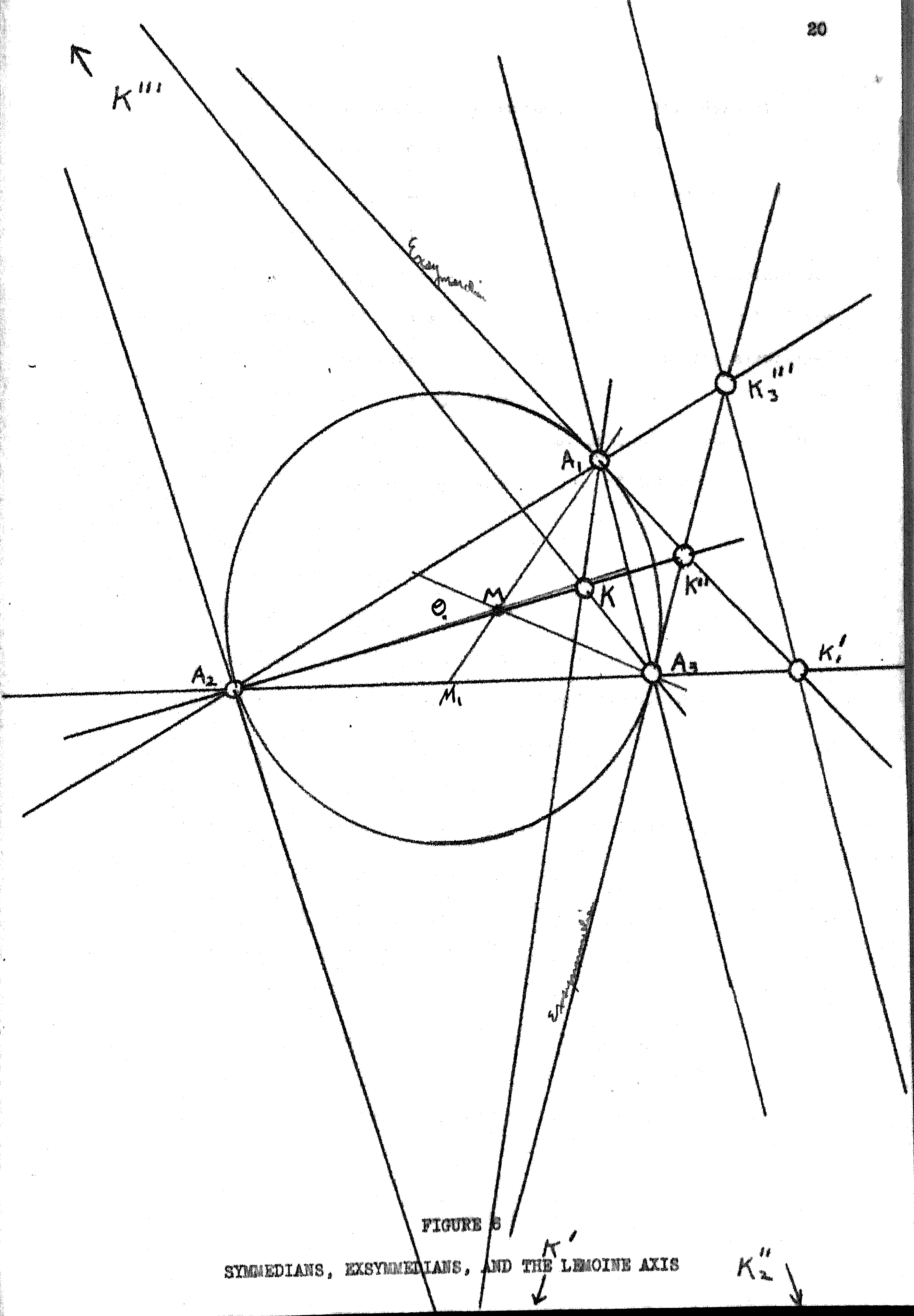

FIGURE 8

SYMMEDIANS, EXSYMMEDIANS, AND THE LEMOINE AXIS obtained by solving simultaneously the equations of the altitudes. The equations of A_1H_1, A_2H_2, A_3H_3, are $x_2 \cos A_2 - x_3 \cos A_3 = 0$, $x_1 \cos A_1 - x_3 \cos A_3 = 0$, $x_1 \cos A_1 - x_2 \cos A_2 = 0$; and the coordinates of H are (sec A_1, sec A_2, sec A_3).

The circumcenter (A.C.-57; J.-161). The perpendicular bisectors of the sides of the triangle of reference are concurrent in a point called the circumcenter (Fig. 12). Its coordinates are obtained by solving simultaneously the equations of the perpendicular bisectors of the sides.

The line M_1O is a line through M_1 parallel to A_1H_1. Since the ideal point on A_1H_1 is (-1, cos A_3, cos A_2), the equation of M_1O is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & a_3 & a_2 \\ -1 & \cos A_3 & \cos A_2 \end{vmatrix} = 0.$$

or $$(a_2^2 - a_3^2)x_1 + a_1a_2x_2 - a_1a_3x_3 = 0.$$

The equations of M_2O and M_3O are found by cyclical permutation to be

$$a_1a_2x_1 + (a_1^2 - a_3^2)x_2 - a_3a_2x_3 = 0,$$

and $$a_1a_3x_1 - a_2a_3x_2 + (a_1^2 - a_2^2)x_3 = 0.$$

The coordinates of the point O, common to these three lines are (Cos A_1, Cos A_2, Cos A_3).

The verbicenter (A.C.-130; J.-149). If L_1 is a point "halfway around the triangle" from A_1, so that

$$\overline{A_1A_2} + \overline{A_2L_1} = \overline{L_1A_3} + \overline{A_3A_1},$$

and if L_2 and L_3 are similarly located, then A_1L_1, A_2L_2, A_3L_3 are concurrent in a point L. This point is sometimes called the verbicenter[2] (Fig. 7).

The coordinates of the verbicenter are obtained by solving simultane-

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[2] The name verbicenter appears in an article on page 65 of the National Mathematics Magazine, November 1935.

obtained by solving simultaneously the equations of the altitudes. The equations of A_1H_1, A_2H_2, A_3H_3, are $x_2 \cos A_2 - x_3 \cos A_3 = 0$, $x_1 \cos A_1 - x_3 \cos A_3 = 0$, $x_1 \cos A_1 - x_2 \cos A_2 = 0$; and the coordinates of H are $(\sec A_1, \sec A_2, \sec A_3)$.

<u>The circumcenter</u> (A.C.-57; J.-161). The perpendicular bisectors of the sides of the triangle of reference are concurrent in a point called the circumcenter (Fig. 12). Its coordinates are obtained by solving simultaneously the equations of the perpendicular bisectors of the sides.

The line M_1O is a line through M_1 parallel to A_1H_1. Since the ideal point on A_1H_1 is $(-1, \cos A_3, \cos A_2)$, the equation of M_1O is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & a_3 & a_2 \\ -1 & \cos A_3 & \cos A_2 \end{vmatrix} = 0.$$

or $$(a_2^2 - a_3^2)x_1 + a_1a_2x_2 - a_1a_3x_3 = 0.$$

The equations of M_2O and M_3O are found by cyclical permutation to be

$$a_1a_2x_1 + (a_1^2 - a_3^2)x_2 - a_3a_2x_3 = 0,$$

and $$a_1a_3x_1 - a_2a_3x_2 + (a_1^2 - a_2^2)x_3 = 0.$$

The coordinates of the point O, common to these three lines are $(\cos A_1, \cos A_2, \cos A_3)$.

<u>The verbicenter</u> (A.C.-130; J.-149). If L_1 is a point "halfway around the triangle" from A_1, so that

$$\overline{A_1A_2} + \overline{A_2L_1} = \overline{L_1A_3} + \overline{A_3A_1},$$

and if L_2 and L_3 are similarly located, then A_1L_1, A_2L_2, A_3L_3 are concurrent in a point L. This point is sometimes called the verbicenter[2] (Fig. 7).

The coordinates of the verbicenter are obtained by solving simultane-

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[2] The name verbicenter appears in an article on page 65 of the National Mathematics Magazine, November 1935.

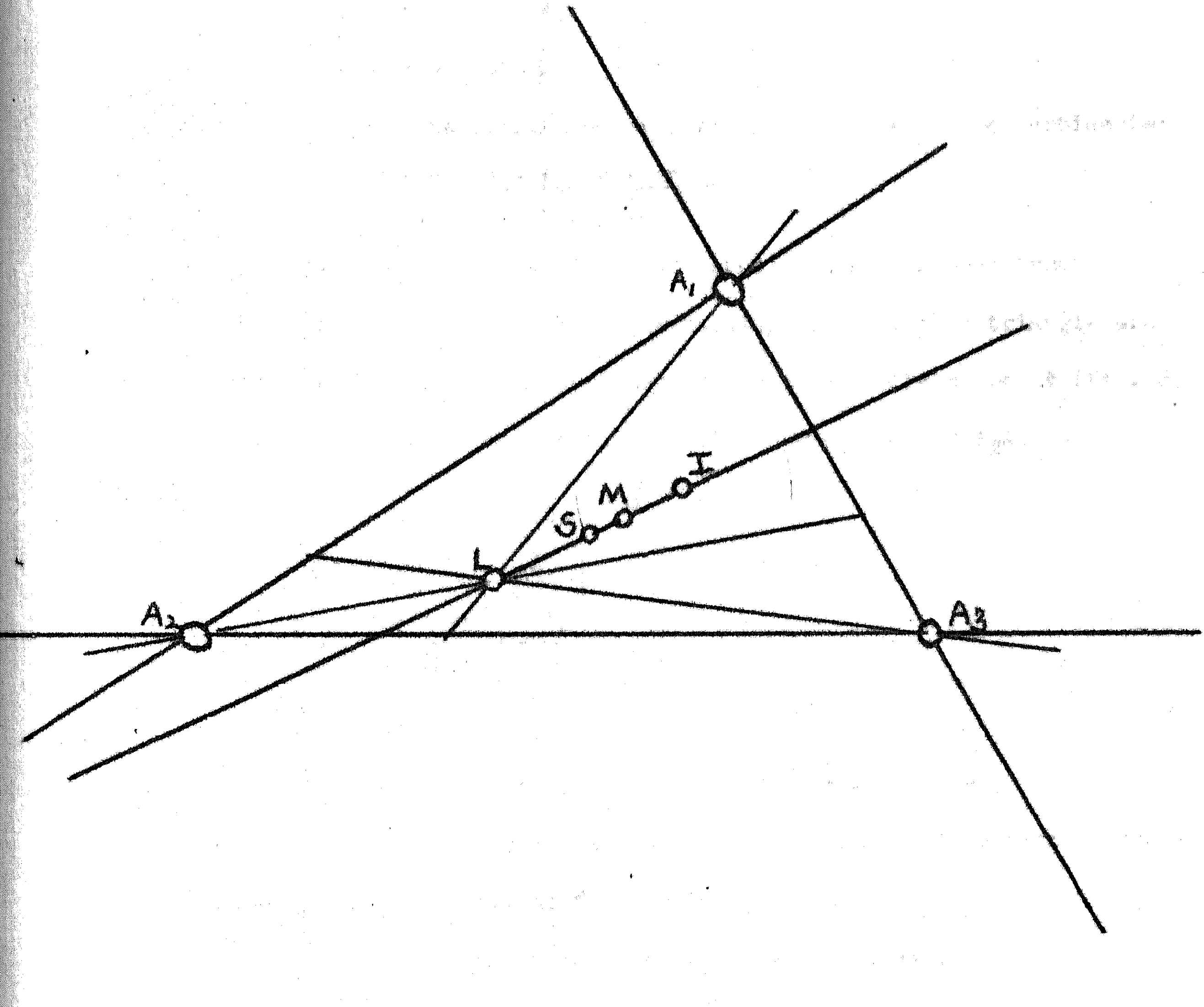

FIGURE 7

VERBICENTER AND LINE LSMI

ously the equations of A_1L_1, A_2L_2, and A_3L_3. The coordinates of L_1 are $\left(0, (s - a_2)\sin A_3, (s - a_3)\sin A_2\right)$, so the equation of A_1L_1 is

$$x_2(s - a_3)\sin A_2 - x_3(s - a_2)\sin A_3 = 0.$$

Similarly, the equations of A_2L_2 and A_3L_3 are

$$x_1(s - a_3)\sin A_1 - x_3(s - a_1)\sin A_3 = 0,$$

and $$-x_1(s - a_2)\sin A_1 + x_2(s - a_1)\sin A_2 = 0.$$

Solving these equations simultaneously, the coordinates of the verbicenter are found to be $\frac{s - a_1}{a_1}, \frac{s - a_2}{a_2}, \frac{s - a_3}{a_3}$.

<u>The Nagel point</u> (A.C.-104). The perpendiculars dropped from the excenters of a triangle upon the corresponding sides of this triangle are concurrent. This point of concurrency is known as the Nagel point (Fig. 9).

The equation of the line through I' perpendicular to A_2A_3 is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ -1 & 1 & 1 \\ 0 & \frac{s - a_2}{a_2} & \frac{s - a_3}{a_3} \end{vmatrix} = 0.$$

which may be written

$$s(a_2 - a_3)x_1 + a_2(s - a_3)x_2 - a_3(s - a_2)x_3 = 0.$$

similarly, the equations of the lines through I'' and I''' perpendicular to the corresponding sides of the fundamental triangle are

$$-a_1(s - a_3)x_1 + s(a_3 - a_1)x_2 + a_3(s - a_1)x_3 = 0,$$

and $$a_1(s - a_2)x_1 - a_2(s - a_1)x_2 + s(a_1 - a_2)x_3 = 0.$$

The Nagel point, which is the point of intersection of these three lines, is

$$\left(\frac{s^2}{a_1a_2a_3}(-a_1 + a_2 + a_3) - \frac{a_2 + a_3}{a_1}, \frac{s^2}{a_1a_2a_3}(a_1 - a_2 + a_3) \quad \frac{a_3 + a_1}{a_2},\right.$$

$$\frac{s^2}{a_1a_2a_3}(a_1 + a_2 - a_3) - \frac{a_1 - a_2}{a_3}\Bigg)$$

which may be written

$$\left(\frac{2s^2}{a_1a_2a_3}(s - a_1) - \frac{a_2 + a_3}{a_1}, \frac{2s^2}{a_1a_2a_3}(s - a_2) - \frac{a_3 + a_1}{a_2},\right.$$

$$\left.\frac{2s^2}{a_1a_2a_3}(s - a_3) - \frac{a_1 + a_2}{a_3}\right).$$

The coordinates of the Nagel point may also be exhibited in a variety of other forms, of which several are given in the summary in Chapter VI.

From these forms the Nagel point is seen to be collinear with the incenter, circumcenter, and the point $\left(\frac{1}{s - a_1}, \frac{1}{s - a_2}, \frac{1}{s - a_3}\right)$. It is also collinear with $(s - a_1, s - a_2, s - a_3)$ and the Spieker center

$$\left(\frac{a_2 + a_3}{a_1}, \frac{a_3 + a_1}{a_2}, \frac{a_1 + a_2}{a_3}\right).$$

<u>The Steiner point</u> (J.-281). If lines are drawn through the vertices of a triangle parallel to the corresponding sides of the first Brocard triangle, they meet at a point on the circumcircle. This point is called the Steiner point (Fig. 8).

The Steiner point can be found by solving simultaneously the equations of A_1S, A_2S, and A_3S. Since these lines are parallel to the sides of the first Brocard triangle, any one of their equations may be found by finding the equation of the line through a vertex and the point at infinity on the corresponding side of the first Brocard triangle.

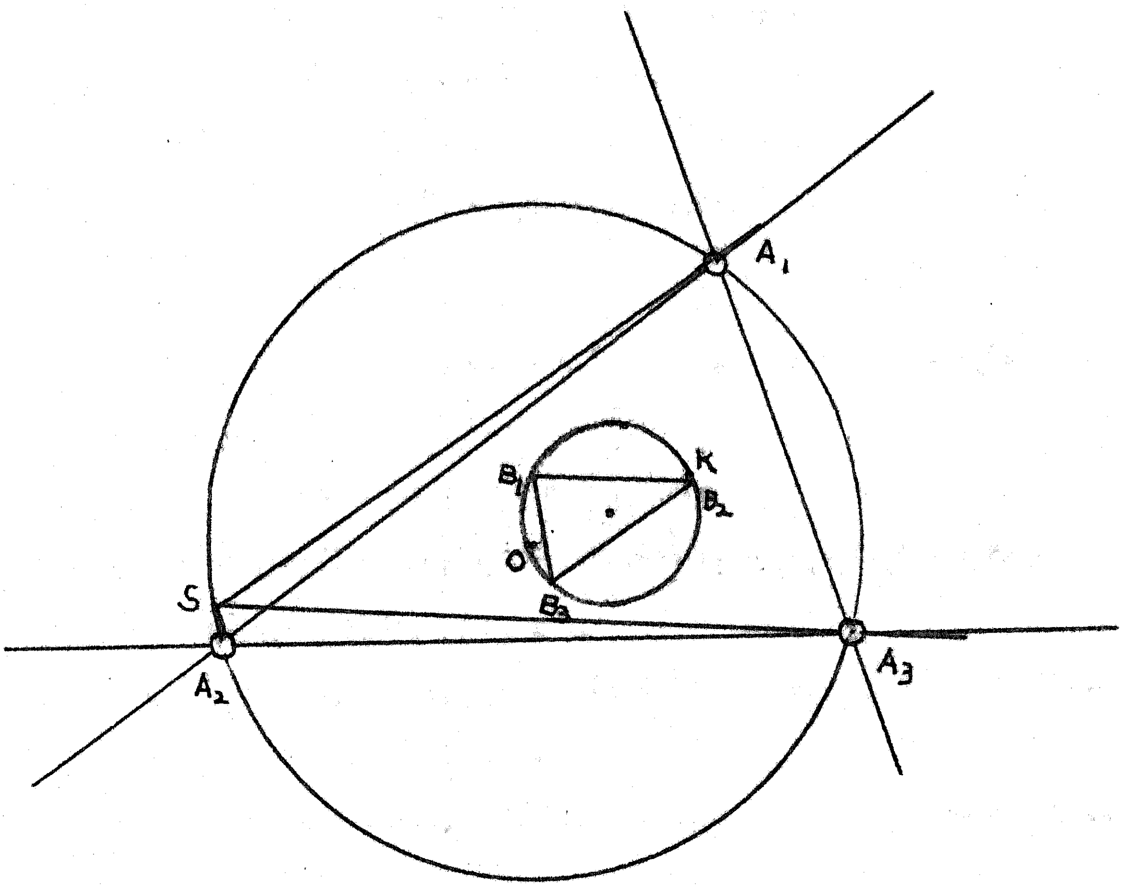

FIGURE 8

THE STEINER POINT

The equation of B_2B_3 is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_3 & a_1a_2a_3 & a_1^2 \\ a_2^3 & a_1^3 & a_1a_2a_3 \end{vmatrix} = 0.$$

which may be written

$$a_1^2(a_1^4 - a_2^2a_3^2)x_1 + a_1a_2(a_3^4 - a_1^2a_2^2)x_2 + a_3a_1(a_2^4 - a_3^2a_1^2)x_3 = 0.$$

The coordinates of the point at infinity of B_2B_3 are

$$\Big(a_2a_3(a_3^4 - a_1^2a_2^2) - a_2a_3(a_2^4 - a_3^2a_1^2),$$
$$a_3a_1(a_2^4 - a_3^2a_1^2) - a_1\, a_2(a_1^4 - a_2^2a_3^2),$$
$$-a_1a_2(a_3^4 - a_1^2a_2^2) + a_1a_2(a_1^4 - a_2^2a_3^2)\Big).$$

The equation of A_1S is

$$\Big(a_1a_2(a_1^4 - a_2^2a_3^2) - a_1a_2(a_3^4 - a_1^2a_2^2)\Big)\, x_2 - \Big(a_3a_1(a_2^4 - a_3^2a_1^2) -$$
$$a_3a_1(a_1^4 - a_2^2a_3^2)\Big)x_3 = 0.$$

Similarly the equations of A_2S and A_3S are

$$\Big(a_1a_2(a_3^4 - a_1^2a_2^2) - a_1a_2(a_2^4 - a_3^2a_1^2)\Big)\, x_1 - \Big(a_2a_3(a_1^4 - a_2^2a_3^2) -$$
$$a_2a_3(a_1^4 - a_2^2a_3^2)\Big)\, x_3 = 0,$$
$$\Big(a_3a_1(a_3^4 - a_1^2a_2^2) - a_3a_1(a_2^4 - a_3^2a_1^2)\Big)\, x_1 - \Big(a_2a_3(a_3^4 - a_1^2a_2^2) -$$
$$a_2a_3(a_3^4 - a_1^2a_2^2)\Big)\, x_2 = 0.$$

The Steiner point, which is the point of concurrency of these three lines, is found to be

$$\left(\frac{1}{a_1(a_2^2 - a_3^2)}, \frac{1}{a_2(a_3^2 - a_1^2)}, \frac{1}{a_3(a_1^2 - a_2^2)}\right).$$

<u>The Gergonne point</u> (A.C.-129; J.-184). The lines from the vertices to the points of contact of the inscribed circle meet in a point, G, called the Gergonne point (Fig. 9). The points G_1, G_2, G_3, are, obviously, the points of contact of the incircle with the sides of the triangle. They are

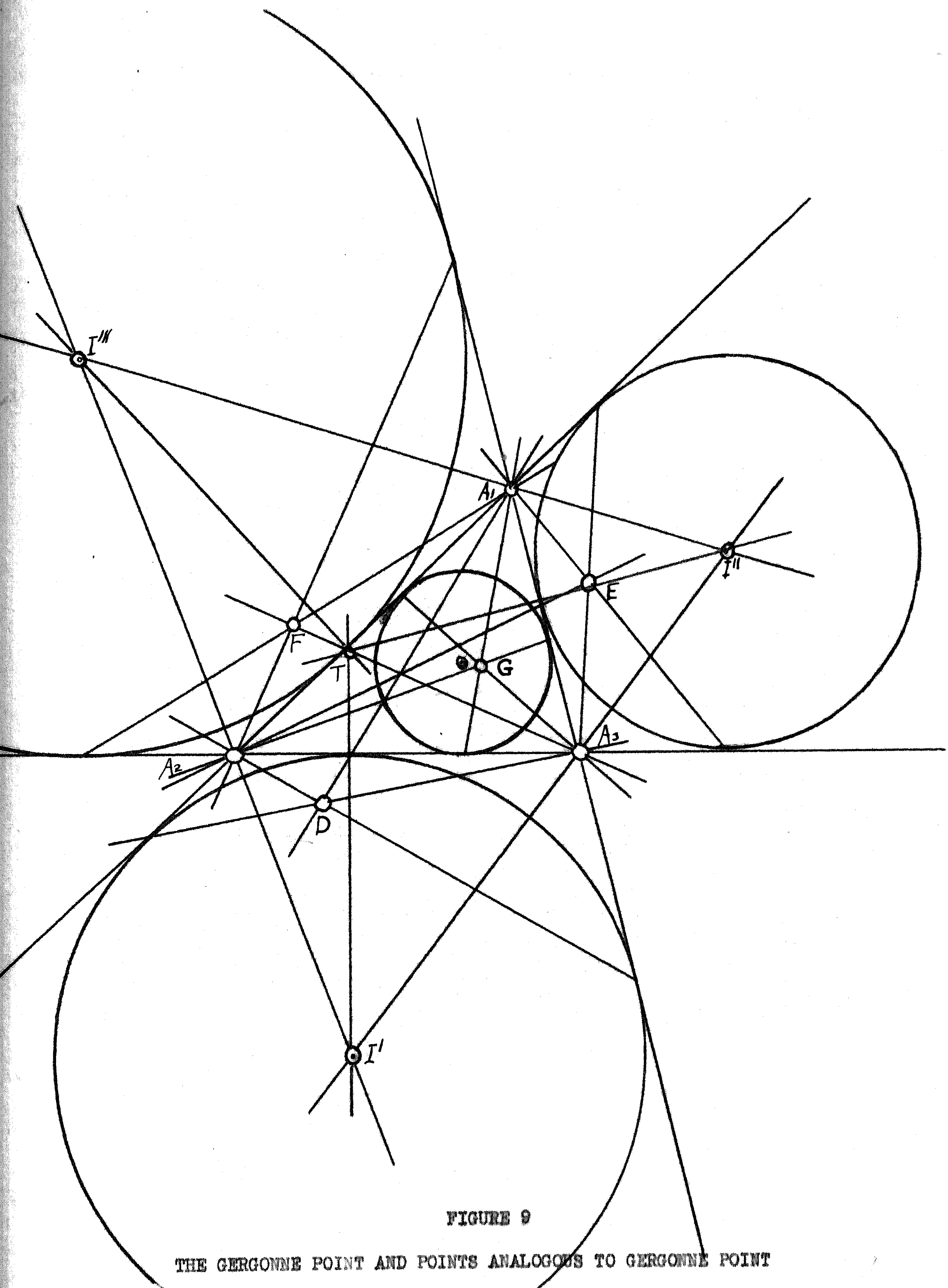

FIGURE 9

THE GERGONNE POINT AND POINTS ANALOGOUS TO GERGONNE POINT the feet of the perpendiculars from the incenter. The coordinates of the ideal point on A_1H_1 are $(-1, \text{Cos } A_3, \text{Cos } A_2)$. Therefore, the equation of the line through I perpendicular to A_2A_3 is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \\ -1 & \text{Cos } A_3 & \text{Cos } A_2 \end{vmatrix} = 0,$$

or $$x_1(\text{Cos } A_2 - \text{Cos } A_3) - x_2(1 + \text{Cos } A_2) + x_3(1 + \text{Cos } A_3) = 0.$$

The coordinates of G_1 are

$$(0,\ 1 + \text{Cos } A_3,\ 1 + \text{Cos } A_2)$$

Similarly, the coordinates of G_2 and G_3 are

$$(1 + \text{Cos } A_3,\ 0,\ 1 + \text{Cos } A_1),$$

and $$(1 + \text{Cos } A_2,\ 1 + \text{Cos } A_1,\ 0).$$

The equations of A_1G_1, A_2G_2, A_3G_3 are

$$x_2(1 + \text{Cos } A_2) - x_3(1 + \text{Cos } A_3) = 0,$$

$$x_1(1 + \text{Cos } A_1) - x_3(1 + \text{Cos } A_3) = 0,$$

$$x_1(1 + \text{Cos } A_1) - x_2(1 + \text{Cos } A_2) = 0.$$

These lines obviously meet in the point G whose coordinates are

$$\left(\frac{1}{1 + \text{Cos } A_1}, \frac{1}{1 + \text{Cos } A_2}, \frac{1}{1 + \text{Cos } A_3}\right),$$

or $$\left(\frac{1}{a_1(s - a_1)}, \frac{1}{a_2(s - a_2)}, \frac{1}{a_3(s - a_3)}\right).$$

<u>Points analogous to the Gergonne point</u> (A.C.-129). The lines joining the vertices of a triangle to the points of contact of an escribed circle with the opposite sides are concurrent. The three escribed circles are then associated with three points analogous to the Gergonne point. These points are here designated by D, E, F (Fig. 9).

The pedal points of D, which are by definition the points of contact of the excircle (I) with A_2A_3, A_3A_1, A_1A_2 are $D_1 = V_1$, D_2, D_3. Their coordinates are

$$D_1: \quad (0, a(s - a_2), a_2(s - a_3))$$

$$D_2: \quad (a_3(s - a_2), 0, -a_1 s)$$

$$D_3: \quad (a_2(s - a_3), -a_1 s, 0)$$

The equations of A_1D_1, A_2D_2, A_3D_3 are:

$$\frac{x_2}{a_3(s - a_2)} = \frac{x_3}{a_2(s - a_3)}, \quad \frac{x_1}{a_3(s - a_2)} = \frac{x_3}{-a_1 s}, \quad \frac{x_1}{a_2(s - a_3)} = \frac{x_2}{-a_1 s}.$$

The coordinates of D are

$$\left(-\frac{1}{a_1 s}, \frac{1}{a_2(s - a_3)}, \frac{1}{a_3(s - a_2)}\right),$$

or $$\left(-\frac{(s - a_2)(s - a_3)}{a_1 s}, \frac{s - a_2}{a_2}, \frac{s - a_3}{a_3}\right).$$

Similarly the coordinates of the points E and F are

$$\left(\frac{s - a_1}{a_1}, -\frac{(s - a_3)(s - a_1)}{a_2 s}, \frac{s - a_3}{a_3}\right),$$

and $$\left(\frac{s - a_1}{a_1}, \frac{s - a_2}{a_2}, -\frac{(s - a_1)(s - a_2)}{a_3 s}\right).$$

<u>Lines through vertices parallel to altitudes.</u> The coordinates of the ideal point of A_1H_1 are $(-1, \text{Cos } A_3, \text{Cos } A_2)$. The equation of the line through A_2 parallel to A_1H_1 is then, $x_1 \text{ Cos } A_2 + x_3 = 0$. Similarly, the equation of the line through A_3 parallel to A_1H_1 is $x_1 \text{ Cos } A_3 + x_2 = 0$. The equations of the lines through A_1 and A_3 parallel to A_2H_2 are $x_2 \text{ Cos } A_1 + x_3 = 0$ and $x_1 + x_2 \text{ Cos } A_3 = 0$; the equations of the lines through A_1 and A_2 parallel to A_3H_3 are $x_2 + x_3 \text{ Cos } A_1 = 0$ and $x_1 + x_3 \text{ Cos } A_2 = 0$.

<u>Lines through vertices parallel to medians.</u> The ideal point on the median A_1M_1 is $(\frac{-2}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})$. The equation of the line through A_2 parallel to A_1M_1 is $a_1x_1 + 2a_3x_3 = 0$ and the equation of the line through A_3 parallel

to A_1M_1 is $a_1x_1 + 2a_2x_2 = 0$. The ideal points on the lines A_2M_2 and A_3M_3 are $(\frac{1}{a_1}, \frac{-2}{a_2}, \frac{1}{a_3})$ and $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{-2}{a_3})$. Similarly the equations of the lines through A_1 and A_3 parallel to A_2M_2 are $a_2x_2 + 2a_3x_3 = 0$ and $2a_1x_1 + a_2x_2 = 0$; and the equation of the lines through A_1 and A_2 parallel to A_3M_3 are $2a_2x_2 + a_3x_3 = 0$ and $2a_1x_1 + a_3x_3 = 0$.

The Euler line (A.C.-95; J.-165). The orthocenter H, the nine point center N, the centroid M, and the circumcenter O, of a triangle lie on a straight line. This line is the Euler line of the triangle (Fig.12). The points are in the order H N M O, with N the midpoint of H O and M a trisection point.

The equation of the Euler line is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \text{Sec } A_1 & \text{Sec } A_2 & \text{Sec } A_3 \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \end{vmatrix} = 0,$$

which may be written

$$\text{Cos } A_1(a_2^2 - a_3^2)x_1 + \text{Cos } A_2(a_3^2 - a_1^2)x_2 + \text{Cos } A_3(a_1^2 - a_2^2)x_3 = 0.$$

The line L M I. The verbicenter L, the Median point M, and the incenter I are collinear, as is evident by inspection of their coordinates,

$$L\left(\frac{s - a_1}{a_1}, \frac{s - a_2}{a_2}, \frac{s - a_3}{a_3}\right),$$

$$M\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right) \text{ and } I\ (1,1,1).$$

The equation of this line is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

which may be written

$a_1(a_2 - a_3)x_1 + a_2(a_3 - a_1)x_2 + a_3(a_1 - a_2)x_3 = 0.$

This line also contains the Spieker center S (Fig. 7). These four points are in the order L S M I with S the midpoint of the segment L I, and M a trisection point.

The line OK (A.C.-245; J.-278). The equation of the line OK, were $O(\cos A_1, \cos A_2, \cos A_3)$ is the circumcenter and $K(a_1, a_2, a_3)$ the symmedian point is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \cos A_1 & \cos A_2 & \cos A_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0,$$

which may be written

$$(a_3 \cos A_2 - a_2 \cos A_3)x_1 + (a_1 \cos A_3 - a_3 \cos A_1)x_2 +$$
$$(a_2 \cos A_1 - a_1 \cos A_2)x_3 = 0,$$

or $$\frac{a_2^2 - a_3^2}{a_1} x_1 + \frac{a_3^2 - a_1^2}{a_2} x_2 + \frac{a_1^2 - a_2^2}{a_3} x_3 = 0.$$

The segment OK is the Brocard diameter (Fig. 10).

The lines M_2M_3, M_3M_1, M_1M_2. The coordinates of M_1, M_2, M_3, which are the pedal points of M, are $(0, a_3, a_2)$, $(a_3, 0, a_1)$, $(a_2, a_1, 0)$. It follows that the equation of $M_2 M_3$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{vmatrix} = 0,$$

or $$-a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

The equations of M_3M_1 and M_1M_2 are,

$$a_1x_1 - a_2x_2 + a_3x_3 = 0$$

and $$a_1x_1 + a_2x_2 - a_3x_3 = 0.$$

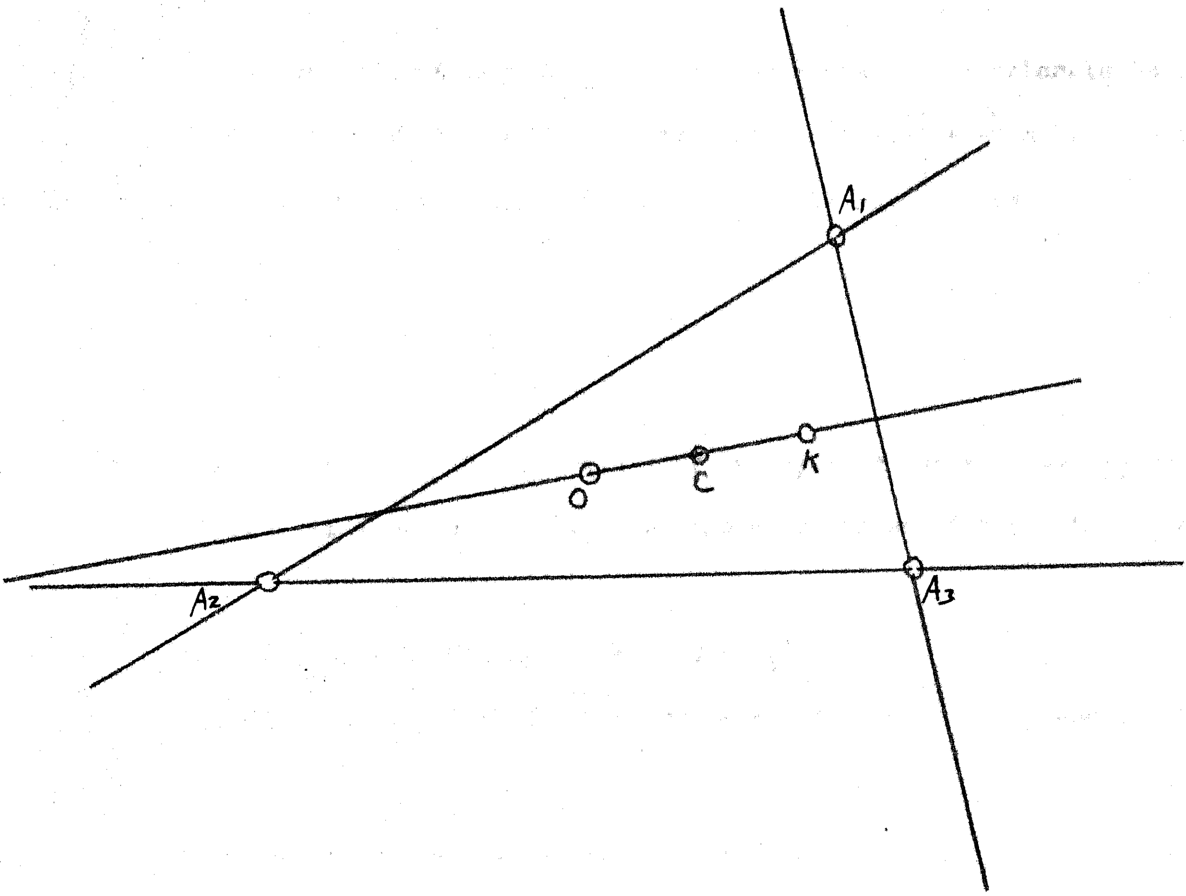

FIGURE 10

THE LINE OCK

The Simson line (A.C.-115; J.-137). The feet of the perpendiculars to the sides of a triangle from a point P are collinear, if and only if the point is on the circumcircle of the triangle. The line through the feet of the perpendiculars to the sides of a triangle from a point on its circumcircle is called the pedal line, or Simson line, of the point with regard to the triangle (Fig. 11).

Let the feet of the perpendiculars on the sides of the triangle be P_a, P_b, P_c. Since the point P is on the circumcircle, $\frac{a_1}{p_1} + \frac{a_2}{p_2} + \frac{a_3}{p_3} = 0$. PP is the line through P and the ideal point of A_1H_1. Its equation is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ p_1 & p_2 & p_3 \\ -1 & \text{Cos } A_3 & \text{Cos } A_2 \end{vmatrix} = 0,$$

or $$(p_2 \text{ Cos } A_2 - p_3 \text{ Cos } A_3)x_1 - (p_3 + p_1 \text{ Cos } A_2)x_2 + (p_2 + p_1 \text{ Cos } A_3)x_3 = 0.$$

Solving this equation with $x_1 = 0$, the coordinates of the point P_a are found to be

$$(0,\ p_2 + p_1 \text{ Cos } A_3,\ p_3 + p_1 \text{ Cos } A_2).$$

By cyclial permutation of subscripts the coordinates of P_b and P_c are seen to be

$$(p_1 + p_2 \text{ Cos } A_3,\ 0,\ p_3 + p_2 \text{ Cos } A_1)$$

and $$(p_1 + p_3 \text{ Cos } A_2,\ p_2 + p_3 \text{ Cos } A_1,\ 0).$$

The equation of the Simson line is then

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & p_2 + p_1 \text{ Cos } A_3 & p_3 + p_1 \text{ Cos } A_2 \\ p_1 + p_2 \text{ Cos } A_3 & 0 & p_3 + p_2 \text{ Cos } A_1 \end{vmatrix} = 0,$$

which may be written

$$(p_2 + p_1 \text{ Cos } A_3)(p_3 + p_2 \text{ Cos } A_1)x_1 + (p_3 + p_1 \text{ Cos } A_2)(p_1 + p_2 \text{ Cos } A_3)x_2 - (p_2 + p_1 \text{ Cos } A_3)(p_1 + p_2 \text{ Cos } A_3)x_3 = 0,$$

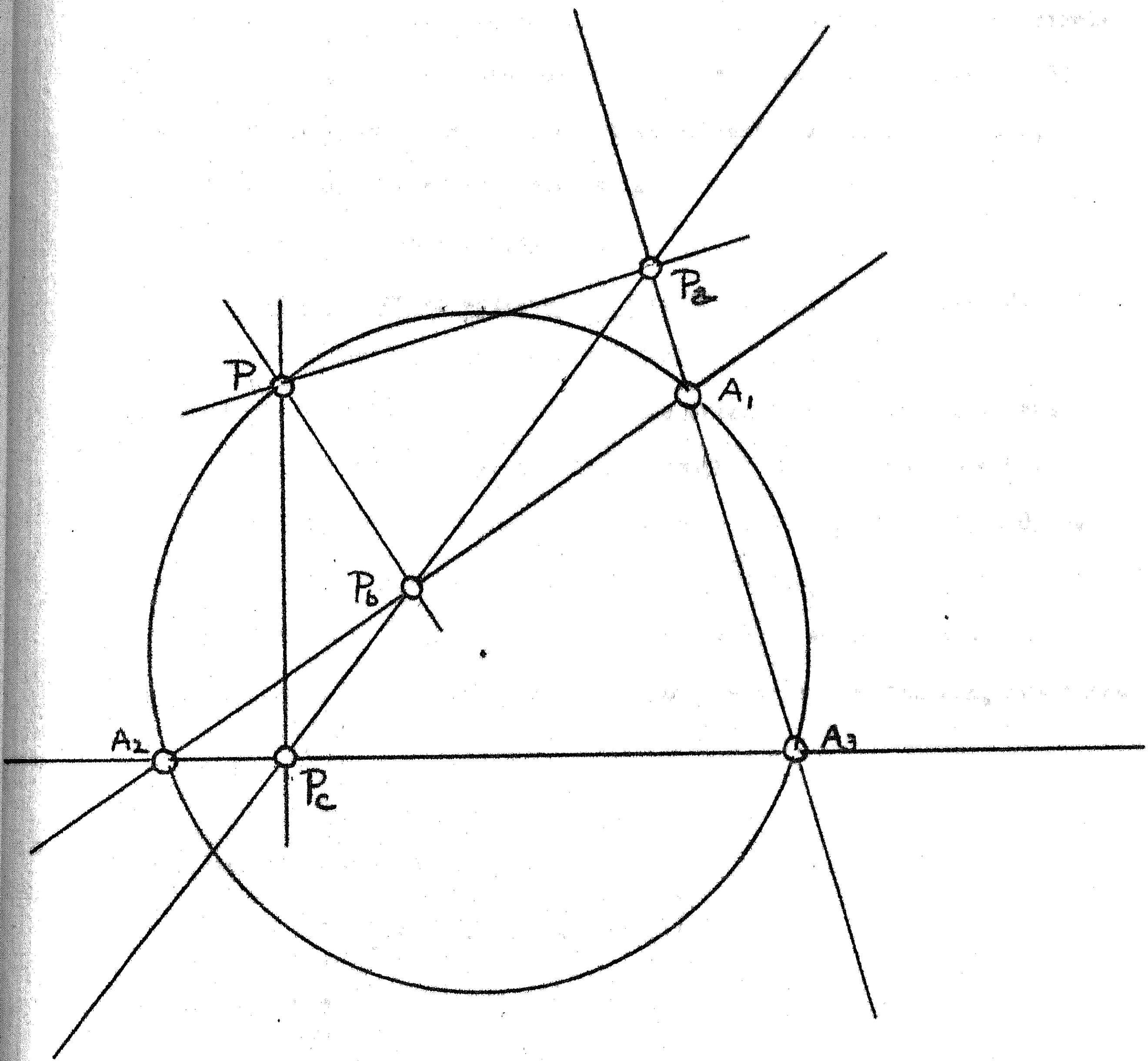

FIGURE 11

THE SIMSON LINE

with the condition that

$$\frac{a_1}{p_1} + \frac{a_2}{p_2} + \frac{a_3}{p_3} = 0.$$

<u>The circumcircle</u> (A.C.-52; J.-161). The circumcircle is the circle which passes through the vertices of the fundamental triangle (Fig. 12). From the coordinates of A_1, A_2, A_3 it is evident that $m_1 = 0$, $m_2 = 0$, $m_3 = 0$ and the equation of the circumcircle is

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0.$$

<u>Inscribed circle; escribed circles</u> (Fig. 9) (A.C.-68; J.-182). The coordinates of the incenter I are obviously (1,1,1). The point of contact of the incircle with the side A_2A_3 of the triangle of reference is the intersection of that side with a line through I parallel to the altitude A_1H_1. The ideal point on this altitude is $(-1, \text{Cos } A_3, \text{Cos } A_2)$, and the equation of the line through I perpendicular to A_2A_3 is

$$x_1(\text{Cos } A_2 - \text{Cos } A_3) - x_2(1 + \text{Cos } A_2) + x_3(1 + \text{Cos } A_3) = 0.$$

The coordinates of the point of contact of the incircle with A_2A_3 are therefore

$$\left(0, \frac{1}{1 + \text{Cos } A_2}, \frac{1}{1 + \text{Cos } A_3}\right),$$

which may also be written

$$\left(0, \frac{1}{a_2(s - a_2)}, \frac{1}{a_3(s - a_3)}\right),$$

or, if preferred,

$$\Big(0, a_3(s - a_3), a_2(s - a_2)\Big).$$

Similarly, the coordinates of the other points of contact with the sides of the triangle are

$$\Big(a_3(s - a_3), 0, a_1(s - a_1)\Big)$$

and $$\Big(a_2(s - a_2), a_1(s - a_1), 0\Big).$$

It is evident that these three points are the pedal points of the point

$$\left(\frac{1}{a_1(s-a_1)},\ \frac{1}{a_2(s-a_2)},\ \frac{1}{a_3(s-a_3)}\right).$$

This point is the Gergonne point G. Accordingly, the three points of contact are here denoted by G_1, G_2, G_3, according to the convention used.

Substituting the coordinates of G_1, G_2, G_3 in the general equation of a circle, the values of m_1, m_2, m_3 are found to be

$$m_1 = \frac{(s-a_1)^2}{a_2a_3},\ m_2 = -\frac{(s-a_2)^2}{a_3a_1},\ m_3 = -\frac{(s-a_3)^2}{a_1a_2}.$$

The equation of the inscribed circle may then be written

$$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2+a_3x_3)\left(a_1(s-a_1)^2x_1 + a_2(s-a_2)^2x_2 + a_3(s-a_3)^2x_3\right) = 0.$$

The equations of the escribed circles may be derived in a similar manner. The points of contact of the excircle (I') with the sides of the triangle of reference are $D_1 = L_1$, D_2, D_3 where $D_1 = L_1$ is the pedal point of the verbicenter on A_2A_3, and D_2, D_3 are the pedal points of D on A_3A_1 and and A_1A_2. The coordinates of L_1, D_2, D_3 are

$$\left(0,\ a_3(s-a_2),\ a_2(s-a_3)\right),$$
$$\left(a_3(s-a_2),\ 0,\ -a_1s\right),$$
$$\left(a_2(s-a_3),\ -a_1s,\ 0\right).$$

Substituting the coordinates of these points in the general equation of a circle and solving for m_1, m_2, m_3, the equation of the excircle (I') is

$$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2 + a_3x_3)\left(a_1s^2x_1 + a_2(s-a_3)^2x_2 + a_3(s-a_2)^2x_3\right) = 0.$$

The equations of the excircles (I'') and I''') are

$$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) -$$
$$(a_1x_1 + a_2x_2 + a_3x_3)\Big(a_1(s - a_3)^2x_1 + a_2s^2x + a_3(s - a_1)^2x_3\Big) = 0.$$

and $a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) -$

$$(a_1x_1 + a_2x_2 + a_3x_3)\Big(a_1(s - a_2)^2x_1 + a_2(s - a_1)^2x_2 + a_3s^2x_3\Big) = 0.$$

The nine point circle and center (A.C.-93; J.-195). The midpoints of the sides of a triangle, the feet of the altitudes, and the midpoints of the segments joining the orthocenter to the vertices of the triangle, lie on a circle. This circle is called the nine point circle (Fig. 12).

Substituting the coordinates of the points $M_1(0, a_3, a_2)$, $M_2(a_3, 0, a_1)$, $M_3(a_2, a_1, 0)$ in the general equation of a circle, the following conditions must be satisfied.

$$a_3m_2 + a_2m_3 = -\frac{a_1}{2},$$

$$a_3m_1 + a_1m_3 = -\frac{a_2}{2},$$

$$a_2m_1 + a_1m_2 = -\frac{a_3}{2},$$

or $$m_1 = -\frac{-a_1^2 + a_2^2 + a_3^2}{4a_2a_3} = -\frac{\cos A_1}{2}$$

$$m_2 = -\frac{a_1^2 - a_2^2 + a_3^2}{4a_3a_1} = -\frac{\cos A_2}{2}$$

$$m_3 = -\frac{a_1^2 + a_2^2 - a_3^2}{4a_1a_2} = -\frac{\cos A_3}{2}$$

The equation of the nine point circle is therefore

$$2(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) -$$
$$(a_1x_1 + a_2x_2 + a_3x_3)(\cos A_1x_1 + \cos A_2x_2 + \cos A_3x_3) = 0,$$

or $$a_1 \cos A_1x_1^2 + a_2 \cos A_2x_2^2 + A_3 \cos A_3x_3^2 -$$
$$(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = 0.$$

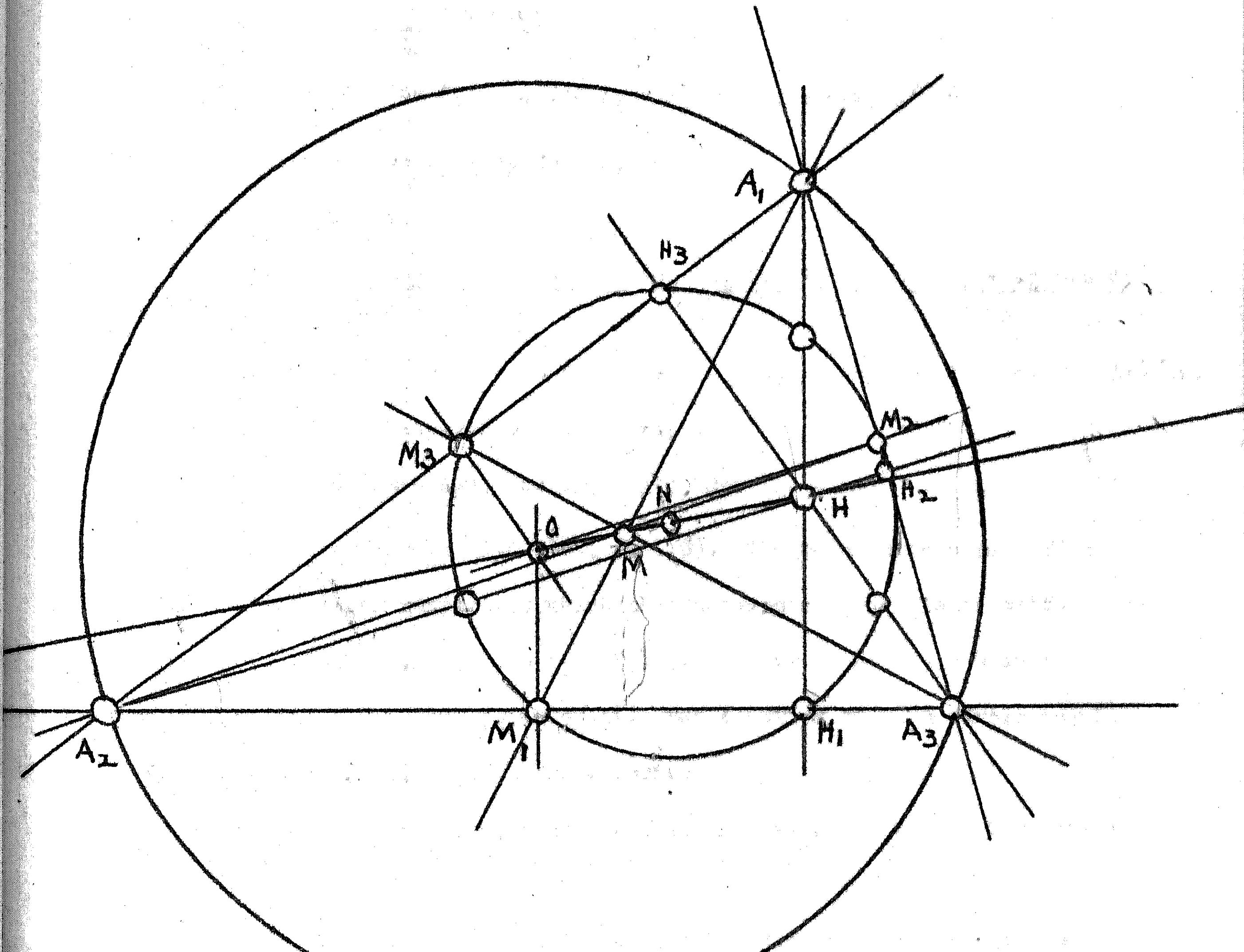

FIGURE 12

NINE POINT CIRCLE, CIRCUMCIRCLE, AND THE EULER LINE HERO

The center $N(n_1, n_2, n_3)$ of the nine-point circle may be easily found from the fact that it divides the segment M O in the ratio -1:3. The distance of M from the side A_2A_3 is $A_2A_3/2R$, and the distance of O from the side A_2A_3 is $R \operatorname{Cos} A_1$. Therefore

$$n_1 = (a_2a_3/2R - R \operatorname{Cos} A_1)/2$$

$$= \frac{R}{2a_1}\ \frac{a_1a_2a_3}{2R^2} - a_1 \operatorname{Cos} A_1 \quad .$$

Recalling that $a_1 \operatorname{Cos} A_1 - a_2 \operatorname{Cos} A_2 - a_3 \operatorname{Cos} A_3 = a_1a_2a_3/2R^2$,

$$n_1 = R(a_2 \operatorname{Cos} A_2 + a_3 \operatorname{Cos} A_3)/2a_1,$$

and the coordinates of N are

$$\frac{a_2 \operatorname{Cos} A_2 + a_3 \operatorname{Cos} A_3}{a_1},\ \frac{a_3 \operatorname{Cos} A_3 + a_1 \operatorname{Cos} A_1}{a_2},\ \frac{a_1 \operatorname{Cos} A_1 + a_2 \operatorname{Cos} A_2}{a_3} .$$

It is easily verified that $2 \operatorname{Cos} (A_2 - A_3) = (a_2 \operatorname{Cos} A_2 - a_3 \operatorname{Cos} A_3)/a_1$, so that the coordinates of N may be written

$$N: \quad \operatorname{Cos} (A_2 - A_3),\ \operatorname{Cos} (A_3 - A_1),\ \operatorname{Cos} (A_1 - A_2) \quad .$$

<u>The Polar circle</u> (A.C.-149; J.-176). The polar circle is defined to be the circle with respect to which the fundamental triangle is self-polar (Fig. 13). That is, the line $x_1 = 0$ is the polar of A_1, $x_2 = 0$ is the polar of A_2, and $x_3 = 0$ is the polar of A_3. The equation of this circle may be readily found from its defining property.

The polars of the vertices, A_1, A_2, A_3, with respect to the general circle,

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + (a_1x_1 + a_2x_2 + a_3x_3)(m_1x_1 + m_2x_2 + m_3x_3) = 0,$$

are

$$2a_1m_1x_1 + (a_1m_2 + a_2m_1 + a_3)x_2 + (a_1m_3 + a_2 + a_3m_1)x_3 = 0,$$

$$(a_1m_2 + a_2m_1 + a_3)x_1 + 2a_2m_2x_2 + (a_1 + a_2m_3 + a_3m_2)x_3 = 0,$$

$$(a_1m_3 + a_2 + a_3m_1)x_1 + (a_1 + a_2m_3 + a_3m_2)x_2 + 2a_3m_3x_3 = 0.$$

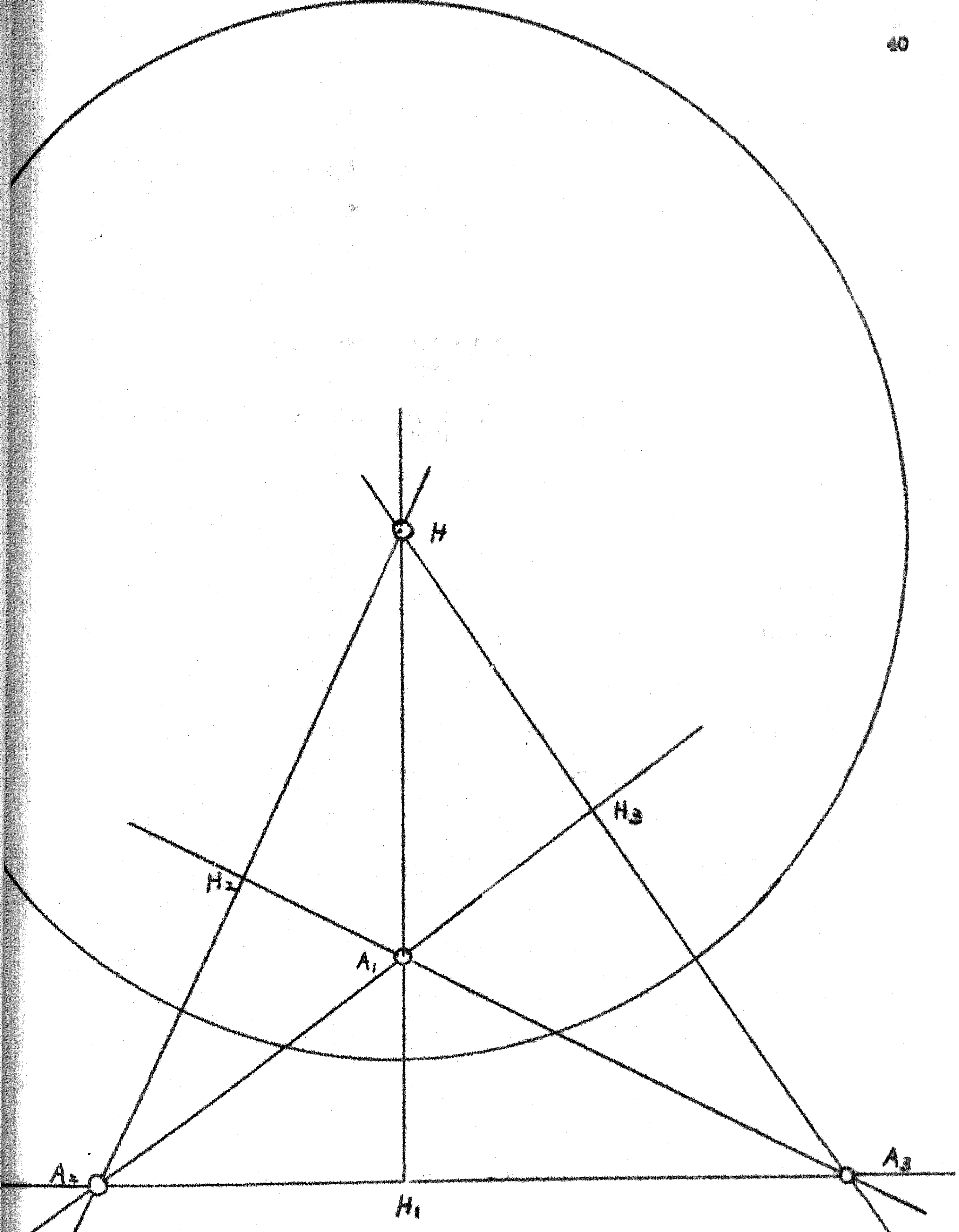

FIGURE 13

THE POLAR CIRCLE

These lines must coincide with $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, so that

$$a_1 + a_2m_3 + a_3m_2 = 0,$$

$$a_1m_3 + a_2 + a_3m_1 = 0,$$

$$a_1m_2 + a_2m_1 + a_3 = 0.$$

This system is satisfied by

$$m_1 = -\frac{-a_1^2 + a_2^2 + a_3^2}{2a_2a_3} = -\mathrm{Cos}\ A_1,$$

$$m_2 = -\frac{a_1^2 - a_2^2 + a_3^1}{2\ a_3a_1} = -\mathrm{Cos}\ A_2,$$

$$m_3 = -\frac{a_1^2 + a_2^2 - a_3^2}{2\ a_1a_2} = -\mathrm{Cos}\ A_3;$$

and the equation of the polar circle is

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 -$$
$$(a_1x_1 + a_2x_2 + a_3x_3)(\mathrm{Cos}\ A_1x_1 + \mathrm{Cos}\ A_2x_2 + \mathrm{Cos}\ A_3x_3) = 0,$$

or $\quad a_1\ \mathrm{Cos}\ A_1x_1^2 + a_2\ \mathrm{Cos}\ A_2x_2^2 + a_3\ \mathrm{Cos}\ A_3x_3^2 = 0.$

The first of these two forms shows that it is coaxal with the circumcircle and the nine-point circle; and the second form shows that the polar circle is real only if one of the angles of the triangle is obtuse.

The Apollonian circles of a triangle (A.C.-234; J.-294). The interior and exterior bisections of the angles A_1, A_2, A_3 of the triangle $A_1A_2A_3$ meet the opposite sides A_2A_3, A_3A_1, A_1A_2 in the points I_1, I_1'; I_2, I_2''; I_3, I_3''' respectively. The circles on I_1I_1', I_2I_2'', I_3I_3''' as diameters are called the Apollonian circles or the circles of Apollonius of the triangle $A_1A_2A_3$ (Fig. 14).

The Apollonian circle with diameter I_1I_1' passes through the vertex A_1. Substituting the coordinates of A_1, I_1 and I_1' in the general equation of a circle, it is found that

$$m_1 = 0,\ m_2 = \frac{a_1a_3}{a_2^2 - a_3^2},\ m_3 = \frac{a_1a_2}{a_2^2 - a_3^2}.$$

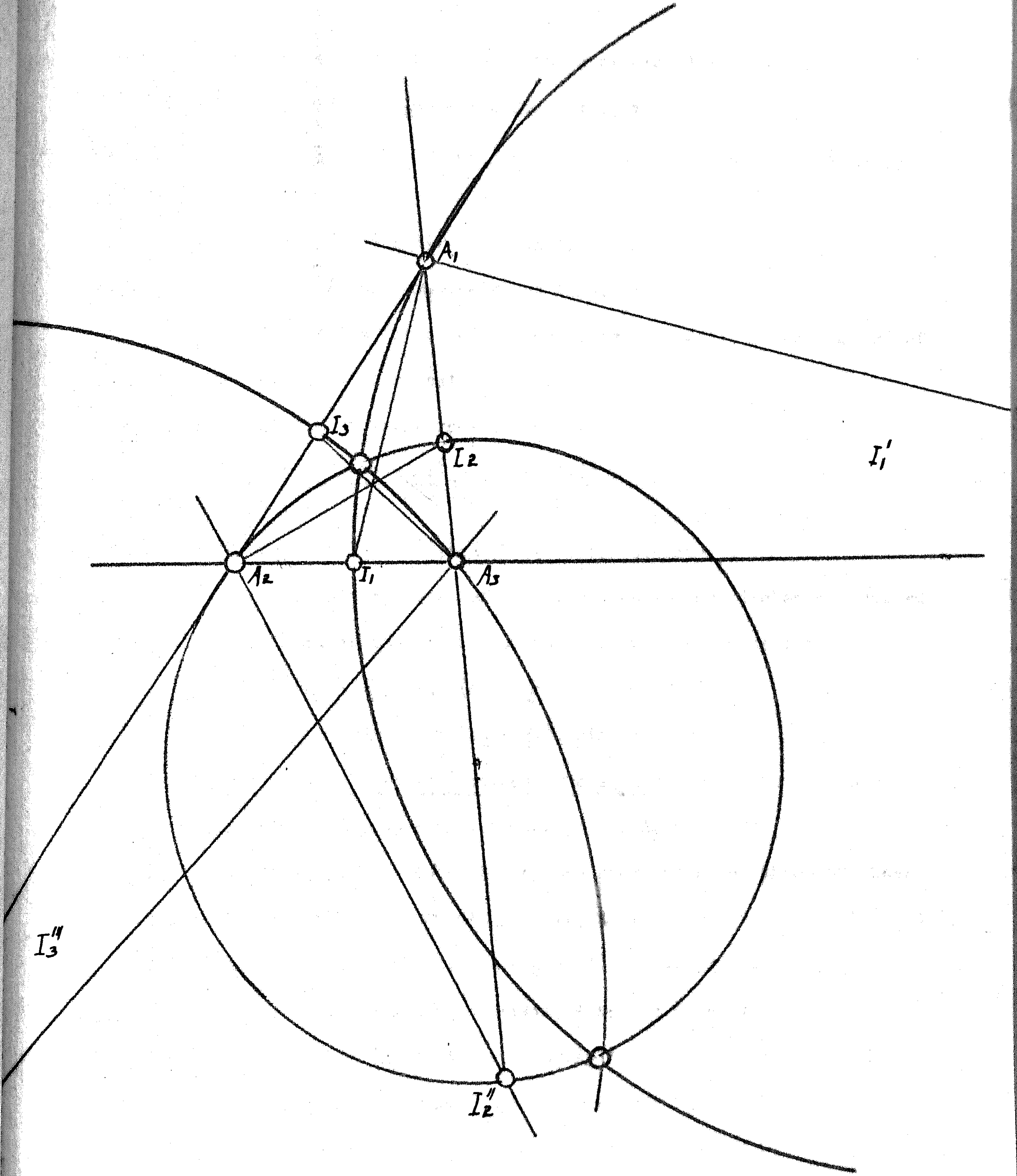

FIGURE 14

THE APOLLONIAN CIRCLES

Accordingly, the equation of the Apollonian circle $(A_1I_1I_1')$ is

$$x_2^2 - x_3^2 - 2 \cos A_2x_3x_1 + 2 \cos A_3x_1x_2 = 0.$$

Similarly the equations of the Apollonian circles $(A_2I_2I_2'')$ and $(A_3I_3I_3''')$ are

$$x_3^2 - x_1^2 - 2 \cos A_3x_1x_2 + 2 \cos A_1x_2x_3 = 0$$

and $$x_1^2 - x_1^2 - 2 \cos A_1x_2x_3 + 2 \cos A_2x_3x_1 = 0.$$

The three Apollonian circles are evidently coaxal, since the sum of their left members is identically zero.

The equation of their radical axis is

$$\frac{a_2^2 - a_3^2}{a_1}x_1 + \frac{a_3^2 - a_1^2}{a_2}x_2 + \frac{a_1^2 - a_2^2}{a_3}x_3 = 0,$$

which is the equation of the line O K (Brocard diameter).

The common points, U, V, of the three Apollonian circles are called the Hessian points, or isodynamic points. Their coordinates are

U: $(\sin (A_1 - \frac{\pi}{3}), \sin (A_2 - \frac{\pi}{3}), \sin (A_3 - \frac{\pi}{3}))$,

V: $(\sin (A_1 - \frac{\pi}{3}), \sin (A_2 - \frac{\pi}{3}), \sin (A_3 - \frac{\pi}{3}))$.

<u>Circles through two vertices and two equicenters.</u> The circle on I I' as diameter (Fig. 15) passes through the points $A_2(0, 1, 0)$, $I(1, 1, 1)$, $A_3(0, 0, 1)$ $I'(-1, 1, 1)$. Substituting the coordinates of three of these points in the general equation of a circle, it is found that $m_1 = -1$, $m_2 = 0$, $m_3 = 0$. The equation of the circle (A_2IA_3I') may be written

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_1(a_1x_1 + a_2x_2 + a_3x_3) = 0.$$

Similarly, the circles (A_3IA_1I'') and (A_1IA_2I''') have the equations

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_2(a_1x_1 + a_2x_2 + a_3x_3) = 0.$$

and $$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_3(a_1x_1 + a_2x_2 + a_3x_3) = 0.$$

In a like manner the equations of the circles through two excenters and the two vertices not collinear with them are,

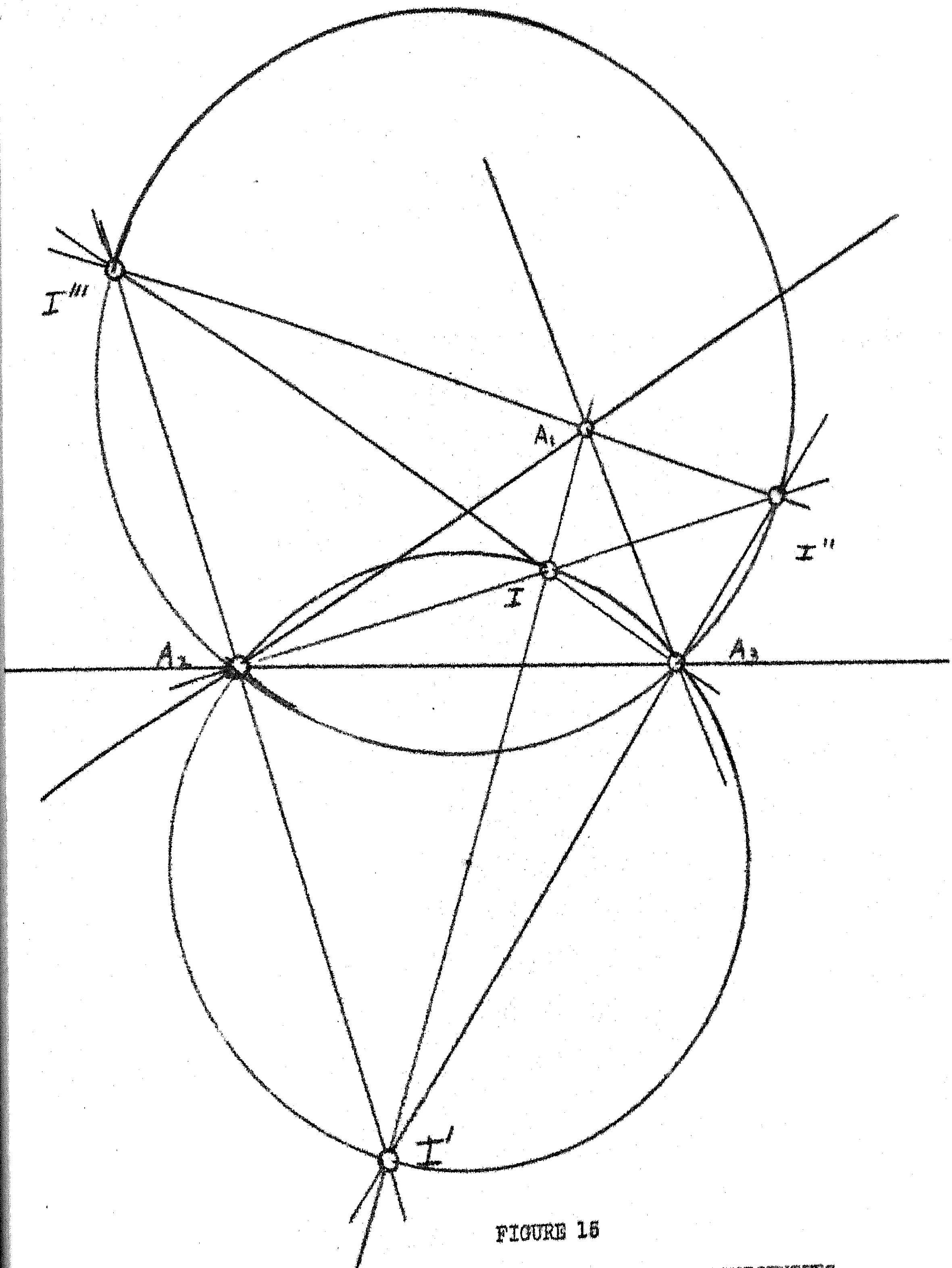

FIGURE 15

CIRCLES THROUGH TWO VERTICES AND TWO EQUICENTERS

$(A_2I'''I''A_3)$: $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_1(a_1x_1 + a_2x_2 + a_3x_3) = 0,$

$(A_3I'I'''A_1)$: $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_2(a_1x_1 + a_2x_2 + a_3x_3) = 0,$

$(A_1I''I'\ A_2)$: $a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_3(a_1x_1 + a_2x_2 + a_3x_3) = 0.$

The Lemoine Circle (A.C.-233; J.-273). The Lemoine Circle is defined by the following theorem.

Theorem. Let lines be drawn through the symmedian point, parallel to the sides of the triangle. They meet the adjacent sides in six points lying on a circle whose center C is the midpoint of KO (Fig. 16).

The equation of the Lemoine Circle may be found by finding the equation of a circle through any three of these points since three points determine a circle.

The ideal point on A_1A_2 is $(-a_2, a_1, 0)$. They the equation of KY_2 is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ a_2 & a_1 & 0 \end{vmatrix} = 0,$$

or $\quad a_3a_1x_1 + a_2a_3x_2 + (a_1^2 + a_2^2)x_3 = 0.$

Solving this equation with $x_1 = 0$ and $x_2 = 0$, the coordinates of Y_2 and Z_1 are found to be $(0, a_1^2 + a_2^2, a_2a_3)$ and $(a_1^2 + a_2^2, 0, a_1a_3)$. The coordinates of the other points are found to be by cyclical premutation $Y_3(a_1a_3, 0, a_2^2 + a_3^2)$; $Y_1(a_3^2 + a_1^2, a_1a_2, 0)$; $Z_2(a_1a_2, a_2^2 + a_3^2, 0)$ and $Z_3(0, a_2a_3, a_3^2 + a_1^2)$.

Substituting the coordinates of Y_1 and Z_2 in the general equation of a circle,

$$\frac{a_1(a_3^2 + a_1^2)}{a_1a_2a_3} m_1 + \frac{a_1^2a_2}{a_1a_2a_3} m_2 = - \frac{a_3^2 + a_1^2}{a_1^2 - a_2^2 - a_3^2},$$

and $$\frac{a_1a_2}{a_1a_2a_3} m_1 + \frac{a_2(a_2^2 + a_3^2)}{a_1a_2a_3} m_2 = - \frac{a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2}$$

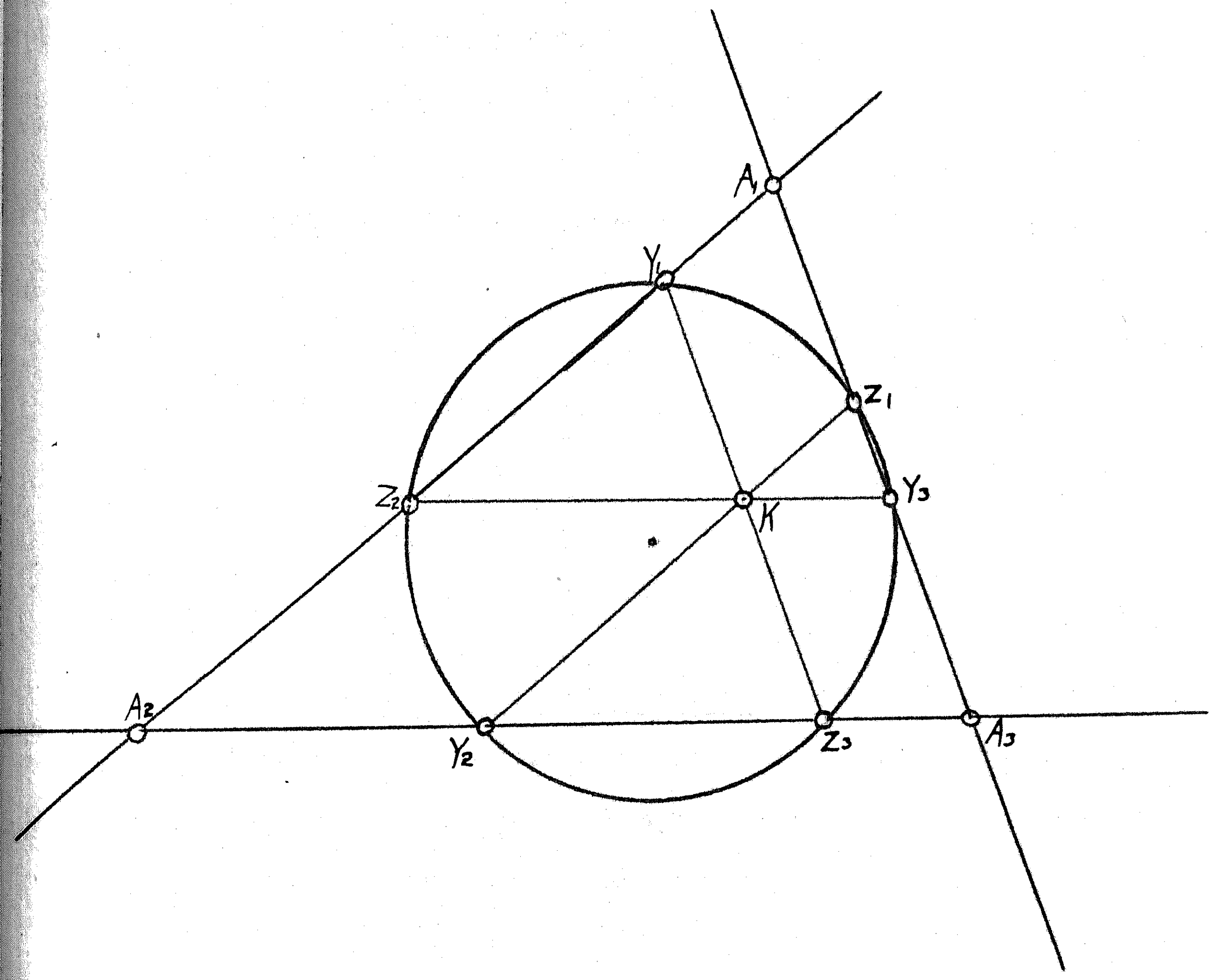

FIGURE 16

FIRST LEMOINE CIRCLE

The values of m_1 and m_2 determined by these equations are $\frac{-a_2a_3(a_2^2 + a_3^2)}{(a_1^2 + a_2^2 + a_3^2)^2}$ and $\frac{-a_3a_1(a_3^2 + a_1^2)}{(a_1^2 + a_2^2 + a_3^2)^2}$; and the value of m_3 is $\frac{-a_1a_2(a_1^2 + a_2^2)}{(a_1^2 + a_2^2 + a_3^2)}$ by the cyclical permutation of subscripts.

The equation of the Lemoine circle is therefore:

$$(a_1^2 + a_2^2 + a_3^2)^2(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = -(a_1x_1 + a_2x_2 + a_3x_3)\Big(-a_2a_3(a_2^2 + a_3^2)x_1 - a_3a_1(a_2^2 + a_1^2)x_2 - a_1a_2(a_1^2 + a_2\ )x_3\Big).$$

<u>Lemoine axis</u>. The intersections of the three exsymmedians with the associated sides are

$K_1'(0, -a_2, a_3)$, $K_2''(a_1, 0, -a_3)$, $K_3'''(-a_1, a_2, 0)$.

These are obviously collinear, the equation of their line being

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} = 0.$$

This line is called the <u>Lemoine axis</u> (Fig. 6). It is obviously the trilinear polar of K.

<u>Second Lemoine Circle (Cosine Circle)</u> (A.C.-233; J.-271). The external symmedians are antiparallels to the opposite sides. Their equations are $a_3x_2 - a_2x_3 = 0$, $a_1x_3 - a_3x_1 = 0$, $a_2x_1 - a_1x_2 = 0$. Lines through the symmedian point K and parallel to these exsymmedians are

$$-a_2a_3x_1 + a_3^2 \cos A_2x_2 + a_2^2 \cos A_3x_3 = 0,$$

$$a_3^2 \cos A_1x_1 - a_3a_1x_2 + a_1^2 \cos A_3x_3 = 0,$$

$$a_2^2 \cos A_1x_1 + a_1^2 \cos A_2x_2 - a_1a_2x_3 = 0.$$

These three lines meet the sides of the fundamental triangle in points which are here denoted by U_1,U_2,U_3; V_1,V_2,V_3; W_1,W_2,W_3. Six of these points, U_2,U_3,V_3,V_1,W_1,W_2, are concyclic having the coordinates

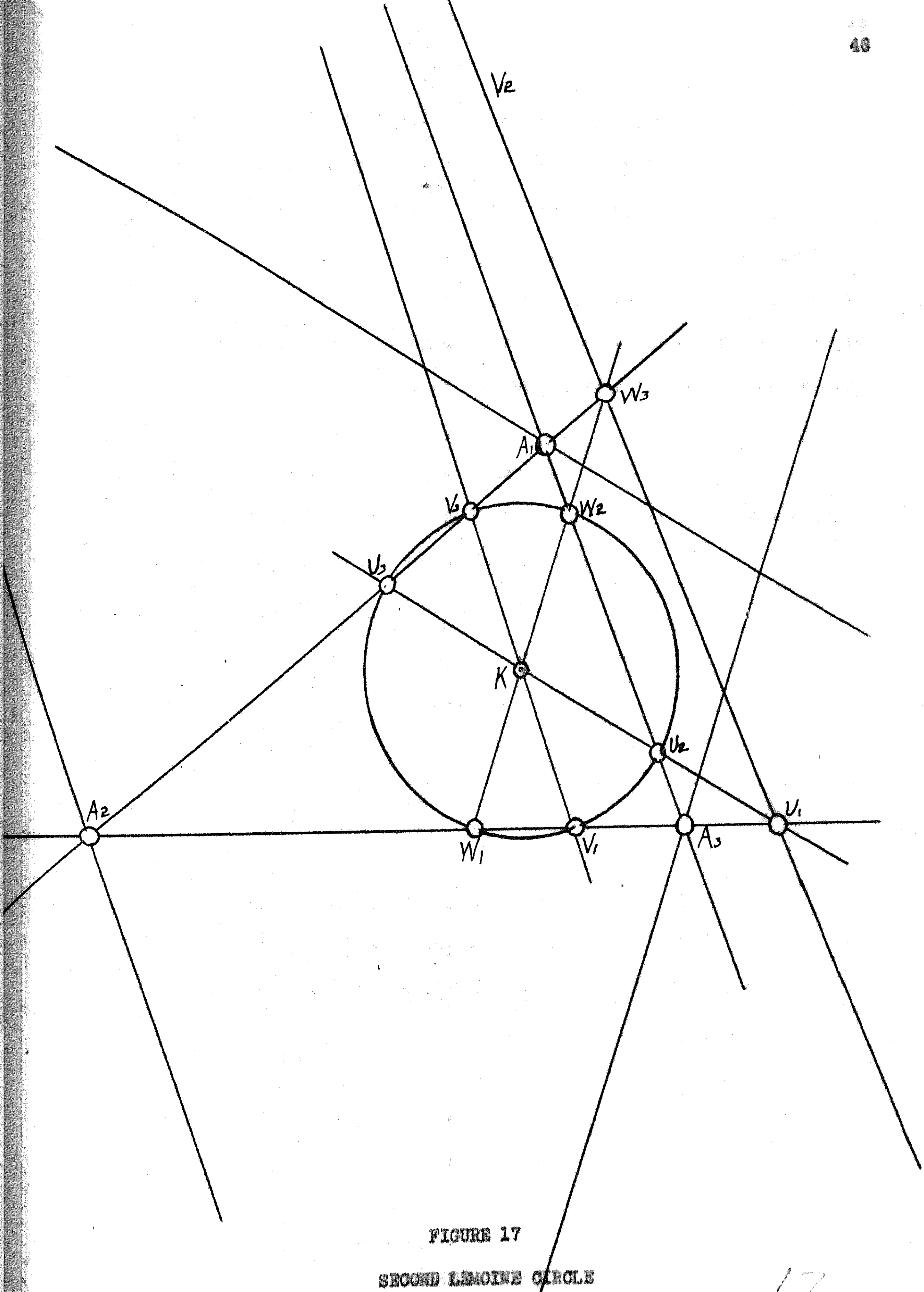

FIGURE 17

SECOND LEMOINE CIRCLE

17

U_2: $(a_2 \cos A_3, 0, a_3)$, U_3: $(a_3 \cos A_2, a_2, 0)$,

V_3: $(a_1, a_3 \cos A_1, 0)$, V_1: $(0, a_1 \cos A_3, a_3)$,

W_1: $(0, a_2, a_1 \cos A_2)$, W_2: $(a_1, 0, a_2 \cos A_1)$.

The circle through the six points mentioned above is called the Second Lemoine circle (Fig. 17). It is sometimes called the Cosine circle because the three segments determined by these points on the sides of the triangle are proportional to the cosines of the corresponding angles of the triangle. That is,

$$V_1W_1 : W_2U_2 : U_3V_3 = \cos A_1 : \cos A_2 : \cos A_3.$$

To find the equations of the Second Lemoine circle assume its equation in the general form, and it is then found that

$$m_1 = \frac{4a_2^2a_3^2 \cos A_1}{(a_1^2 + a_2^2 + a_3^2)^2},$$

$$m_2 = \frac{4a_3^2a_1^2 \cos A_2}{(a_1^2 + a_2^2 + a_3^2)^2},$$

$$m_3 = \frac{4a_1^2a_2^2 \cos A_3}{(a_1^2 + a_2^2 + a_3^2)^2}.$$

The equation of the Second Lemoine circle is

$$(a_1^2 + a_2^2 + a_3^2)^2(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = 4(a_1x_1 + a_2x_2 + a_3x_3)(a_2^2a_3^2 \cos A_1x_1 + a_3^2a_1^2 \cos A_2x_2 + a_1^2a_2^2 \cos A_3x_3).$$

The coordinates of U_1, V_2, W_3 are

U_1: $(0, -a_2^2 \sec A_2, a_3^2 \sec A_3)$,

V_2: $(a_1^2 \sec A_1, 0, -a_3^2 \sec A_3)$,

W_3: $(-a_1^2 \sec A_1, a_2^2 \sec A_2, 0)$.

It is obvious that these points are collinear, the equation of their line being

$$\frac{\cos A_1}{a_1^2} x_1 + \frac{\cos A_2}{a_2^2} x_2 + \frac{\cos A_3}{aa_3^2} x_3 = 0.$$

Spieker circle and Spieker center (J.-227). The Spieker circle, or P circle, is the incircle of the medial triangle $M_1M_2M_3$(Fig. 18). Let S denote the center of this circle. Then S may be easily found, since the lines M_1S, M_2S, M_3S are parallel to the angle bisectors A_1I_1, A_2I_2, A_3I_3.

The ideal point on A_1I_1, and therefore on M_1S, is $(a_2 - a_3, -a_1, -a_1)$. The equation of M_1S is

$$a_1(a_2 - a_3)x_1 + a_2(a_2 + a_3)x_2 - a_3(a_2 + a_3)x_3 = 0.$$

Similarly the equations of M_2S and M_3S are

$$-a_1(a_3 + a_1)x_1 + a_2(a_3 - a_1)x_2 + a_3(a_3 + a_1)x_3 = 0,$$

$$a_1(a_1 + a_2)x_1 \overset{?}{+} a_2(a_1 + a_2)x_2 + a_3(a_1 - a_2)x_3 = 0.$$

The point of intersection of the three lines, which is the Spieker center, has the coordinates

$$S: \quad \frac{a_2 + a_3}{a_1}, \frac{a_3 + a_1}{a_2}, \frac{a_1 + a_2}{a_3}.$$

Let U,V,W denote the points of contact of the Spieker circle with the sides of the triangle $M_1M_2M_3$. Since the ideal point of A_1H_1, and therefore of SU, is $(-1, \cos A_3, \cos A_2)$, the equation of SU is

$$\begin{vmatrix} a_1x_1 & a_2x_2 & a_3x_3 \\ -a_1 & a_2 \cos A_3 & a_3 \cos A_2 \\ a_2 + a_3 & a_3 + a_1 & a_1 + a_2 \end{vmatrix} = 0.$$

This is intersected by the line M_2M_3: $-a_1x_1 + a_2x_2 + a_3x_3 = 0$, in the point

$$U: \quad 1, \frac{s - a_2}{a_2}, \frac{s - a_3}{a_3} \quad .$$

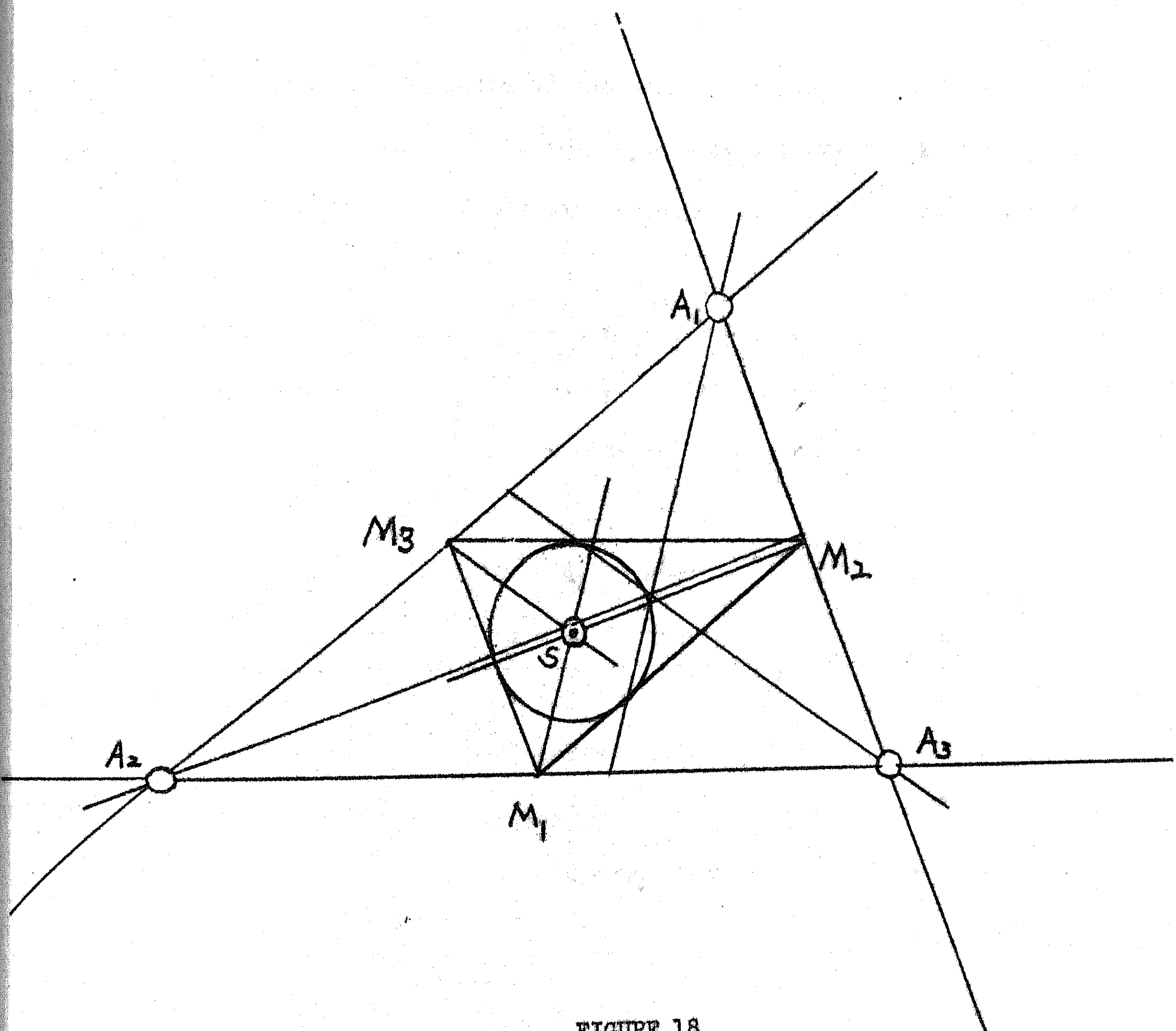

FIGURE 18

THE SPIEKER CIRCLE AND CENTER

Similarly, it is found

$$V:\ \frac{s-a_1}{a_1},\ 1,\ \frac{s-a_3}{a_3}\ ,$$

$$W:\ \frac{s-a_1}{a_1},\ \frac{s-a_2}{a_2},\ 1\ .$$

Assuming the equation of the Spieker circle is the form

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + (a_1x_1 + a_2x_2 + a_3x_3)(m_1x_1 + m_2x_2 + m_3x_3) = 0,$$

the condition that the circle pass through the points U, V, W leads to the relations

$$2a_2a_3m_1 - 2a_3a_1m_2 = (a_1 - a_2)(a_1 + a_2 - a_3),$$

$$2a_3a_1m_2 - 2a_1a_2m_3 = (a_2 - a_3)(-a_1 + a_2 + a_3),$$

$$-2a_2a_3m_1 + 2a_1a_2m_3 = (a_3 - a_1)(a_1 - a_2 + a_3).$$

which yield

$$4a_2a_3m_1 = 4(s - a_2)(s - a_3) - s^2,$$

$$4a_3a_1m_2 = 4(s - a_3)(s - a_1) - s^2,$$

$$4a_1a_2m_3 = 4(s - a_1)(s - a_2) - s^2.$$

The equation of the Spieker circle is then

$$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - \frac{s^2}{4}(a_1x_1 + a_2x_2 + a_3x_3)^2 - (s - a_1)(s - a_2)(s - a_3)(a_1x_1 + a_2x_2 + a_3x_3)\left(\frac{a_1}{s - a_1}x_1 + \frac{a_2}{s - a_2}x_2 + \frac{a_3}{s - a_3}x_3\right) = 0.$$

The equation of the radical axis with the circumcircle is

$$4(s - a_1)(s - a_2)(s - a_3)\left(\frac{a_1}{s - a_1}x_1 + \frac{a_2}{s - a_2}x_2 + \frac{a_3}{s - a_3}x_3\right) = s^2(a_1x_1 + a_2x_2 + a_3x_3).$$

This radical axis is obviously parallel to the line

$$\frac{a_1}{s - a_1}x_1 + \frac{a_2}{s - a_2}x_2 + \frac{a_3}{s - a_3}x_3 = 0,$$

the trilinear polar of the verbicenter.

<u>The Brocard points</u> (A.C.-243; J.-263).

Theorem. In any triangle $A_1A_2A_3$ there is one and only one point such that

$$\angle \Omega A_1A_2 = \angle \Omega A_2A_3 = \angle \Omega A_3A_1 = \omega,$$

and one point Ω' such that

$$\angle \Omega' A_2A_1 = \angle \Omega' A_3A_2 = \angle \Omega' A_1A_3 = \omega.$$

These two points are called the Brocard points of the triangle (Fig. 19).

Let $(A_1A_2A_2)$ denote the circle which passes through the vertices A_1, A_2 and which is tangent to A_2A_3 at A_2. Similarly, let $(A_2A_3A_3)$ denote the circle which passes through the vertices A_2, A_3, and is tangent to A_3A_1 at A_3; and $(A_3A_1A_1)$ denote the circle which passes through the vertices A_3, A_1 and is tangent to A_1A_2 at A_1. Then Ω is the point common to these three circles. These circles are called the direct group of adjoint circles (Fig. 19).

Since the circle $(A_1A_2A_2)$ passes through the vertex (1,0,0) and is tangent to $x_1 = 0$ at (0, 1, 0) its equation is $m_1 = 0$, $m_2 = 0$, $m_3 = -\frac{a_1}{a_2}$

$$a_3a_1x_3^2 + x_1x_3(a_1^2 - a_2^2) - a_2a_3x_1x_2 = 0.$$

The equations of $(A_2A_3A_3)$ and $(A_3A_1A_1)$ are

$$a_1a_2x_1^2 + x_1x_2(a_2^2 - a_3^2) - a_3a_1x_2x_3 = 0,$$

and $$a_2a_3x_2^2 + x_2x_3(a_3^2 - a_1^2) - a_1a_2x_3x_1 = 0.$$

The point Ω common to these three circles is

$$\left(\frac{a_3}{a_2}, \frac{a_1}{a_3}, \frac{a_2}{a_1}\right).$$

In a like manner Ω' is the intersection of three circles, $(A_2A_1A_1)$, $(A_3A_2A_2)$, and $(A_1A_3A_3)$, where $(A_2A_1A_1)$ is the circle which passes through A_1, A_2 and is tangent to A_1A_3 at A_1. These are called the indirect group of

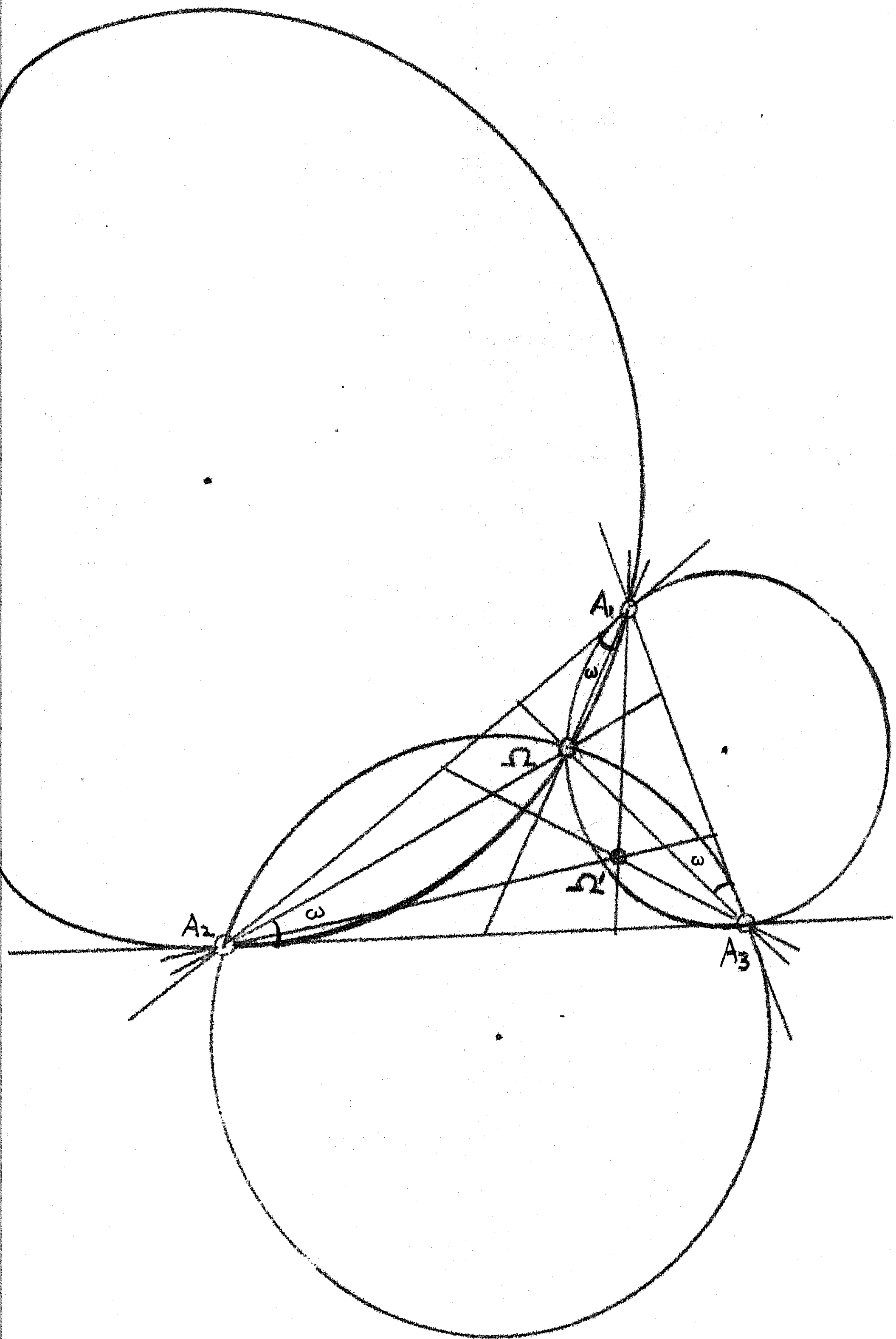

FIGURE 19

THE BROCARD POINTS AND RAYS

adjoint circles.

The equations of $(A_2A_1A_1)$, $(A_3A_2A_2)$, $(A_1A_3A_3)$ are found to be

$a_3a_1x_1^2 - x_3x_1(a_2^2 - a_3^2) - a_1a_2x_2x_3 = 0,$

$a_1a_2x_2^2 - x_1x_2(a_3^2 - a_1^2) - a_2a_3x_3x_1 = 0,$

$a_2a_3x_3^2 - x_2x_3(a_1^2 - a_2^2) - a_3a_1x_1x_2 = 0.$

The point Ω' common to these three circles is $\frac{a_2}{a_3}, \frac{a_3}{a_1}, \frac{a_1}{a_2}$.

The Brocard points, Ω, Ω', are obviously isogonal conjugates.

The Brocard angle (A.C.-245; J.-266). The angle $\Omega A_1A_2 = \Omega' A_1A_3$ is often referred to as the Brocard angle of the triangle, and is usually denoted by ω (Fig. 19).

Theorem. $\cot \omega = \cot A_1 + \cot A_2 + \cot A_3$.

Theorem. $\cot \omega = \dfrac{a_1^2 - a_2^2 - a_3^2}{4\Delta}$

The Brocard rays (J.-266). The lines $A_1\Omega$, $A_2\Omega$, $A_3\Omega$, $A_1\Omega'$, $A_2\Omega'$, and $A_3\Omega'$ are called the Brocard rays (Fig. 19).

The equations of the rays $A_1\Omega$, $A_2\Omega$, $A_3\Omega$ are

$a_2a_3x_2 - a_1^2x_3 = 0,$

$a_2^2x_1 - a_3a_1x_3 = 0,$

$a_1a_2x_1 - a_3x_2 = 0.$

The equations of the rays $A_1\Omega'$, $A_2\Omega'$, $A_3\Omega'$ are

$a_1^2x_2 - a_2a_3x_3 = 0,$

$a_3a_1x_1 - a_2^2x_3 = 0,$

$a_3^2x_1 - a_1a_2x_2 = 0.$

The Brocard Circle (A.C.-245; J.-278). The circle (OK) described on the segment O K as diameter is known as the Brocard circle of $A_1A_2A_3$ (Fig. 20).

The Brocard circle passes through the points O, K, Ω, Ω'. Its equation may be found by finding the equation of the circle through any three of these points.

Substituting the values of $K(a_1, a_2, a_3)$, $\Omega\left(\frac{a_2}{a_3}, \frac{a_3}{a_1}, \frac{a_1}{a_2}\right)$, and $\Omega'\left(\frac{a_3}{a_2}, \frac{a_1}{a_3}, \frac{a_2}{a_1}\right)$ in the general equation of a circle the resulting relations are

$$a_1m_1 + a_2m_2 + a_3m_3 = -\frac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2}$$

$$\frac{a_2}{a_3}m_1 + \frac{a_3}{a_1}m_2 + \frac{a_1}{a_2}m_3 = -1,$$

$$\frac{a_3}{a_2}m_1 + \frac{a_1}{a_3}m_2 + \frac{a_2}{a_1}m_3 = -1.$$

These give

$$m_1 = -\frac{a_2a_3}{a_1^2 + a_2^2 + a_3^2},\; m_2 = -\frac{a_1a_3}{a_1^2 + a_2^2 + a_3^2},\; m_3 = -\frac{a_1a_2}{a_1^2 + a_2^2 + a_3^2}.$$

The equation of the Brocard circle is

$$(a_1^2 + a_2^2 + a_3^2)(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = (a_1x_1 + a_2x_2 + a_3x_3)(a_2a_3x_1 + a_3a_1x_2 + a_1a_2x_3).$$

<u>The Brocard triangles</u> (A.C.-245; J.-277). Let the point of intersection of $A_2\Omega$ and $A_3\Omega'$ be B_1. The intersections of $A_3\Omega$ with $A_1\Omega'$ and $A_1\Omega$ with $A_2\Omega'$ are B_2 and B_3 respectively. The triangle $B_1B_2B_3$ is the so-called First Brocard Triangle (Fig. 20). The coordinates of B_1, B_2, B_3, are found to be $(a_1a_2a_3, a_3^3, a_2^3)$, $(a_3^3, a_1a_2a_3, a_1^3)$, $(a_2^3, a_1^3, a_1a_2a_3)$.

The triangle $B_1'B_2'B_3'$ having for its vertices the points of intersection of (OK) with the symmedians A_1K, A_2K, A_3K of the fundamental triangle is known as the Second Brocard Triangle (Fig. 20). The coordinates of B_1', B_2', B_3' are $(-a_1^2 + a_2^2 + a_3^2, a_1a_2, a_3a_1)$, $(a_1a_2, a_1^2 - a_2^2 + a_3^2, a_2a_3)$, $(a_3a_1, a_2a_3, a_1^2 + a_2^2 - a_3^2)$.

The two Brocard triangles, $B_1B_2B_3$ and $B_1'B_2'B_3'$, are in perspective at Ω.

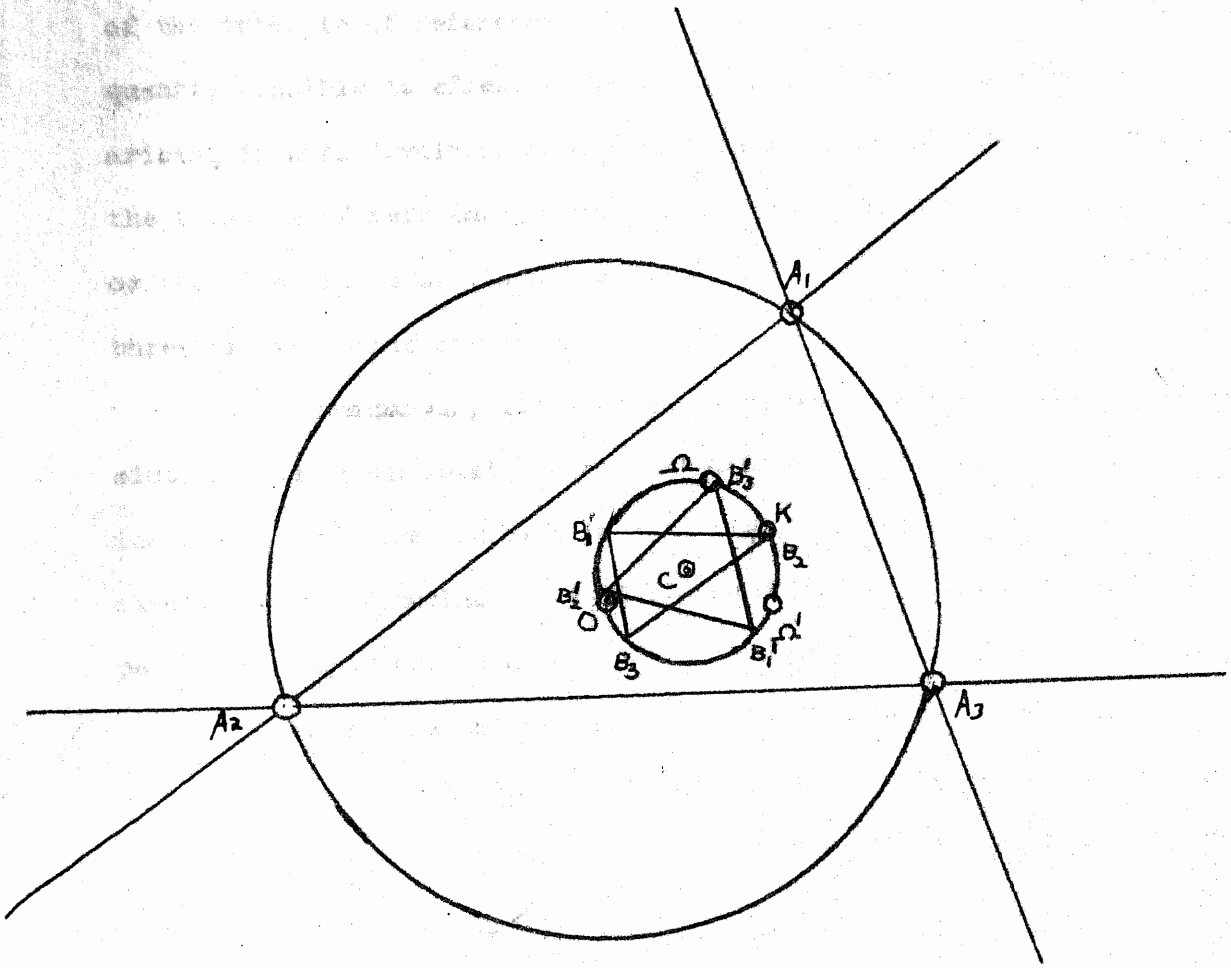

FIGURE 20

THE BROCARD TRIANGLES AND THE BROCARD CIRCLE

CHAPTER V

FORMULAS USED IN SIMPLIFICATION

In the reduction of algebraic expressions encountered in this work it is desirable to have available a list of relations connecting the elements of the triangle of reference. With the aid of these relations it is frequently possible to effect a simplification of rather formidable expressions arising in work involving the actual distances of points from the sides of the triangle of reference, or in the determination of collinearity of points, or the concurrence of lines, or in finding the equation of a circle when three of its points are given.

The accompanying list is not intended to be exhaustive. Many of the simpler well known relations are intentionally omitted, and only a few are included which have not been used explicitly in this work. This list includes several formulas not readily found in the literature, but which are peculiarly important in a study of this kind.

The notations employed in this list are as follows:

1. a_1: The length of the side A_2A_3 of the triangle of reference

   a_2: The length of the side A_3A_1 of the triangle of reference

   a_3: The length of the side A_1A_2 of the triangle of reference

2. a_1': Altitude from A_1

   a_2': Altitude from A_2

   a_3': Altitude from A_3

3. A_1: The angle $A_3A_1A_2$

   A_2: The angle $A_1A_2A_3$

   A_3: The angle $A_2A_3A_1$

4. S: One half the perimeter

5. R: The circumradius

6. r: The inradius

7. r': The exradius associated with A_1

r'': The exradius associated with A_2

r''': The exradius associated with A_3

8. $\triangle$: The area of the triangle $A_1A_2A_3$

Formulas:

1. $s = (a_1 + a_2 + a_3)/2$

2. $s - a_1 = (-a_1 + a_2 + a_3)/2$

   $s - a_2 = (a_1 - a_2 + a_3)/2$

   $s - a_3 = (a_1 + a_2 - a_3)/2$

3. $r = \sqrt{(s - a_1)(s - a_2)(s - a_3)/s}$

4. $R = \dfrac{a_1}{2 \sin A_1} = \dfrac{a_2}{2 \sin A_2} = \dfrac{a_3}{2 \sin A_3}$

5. $\triangle = \frac{1}{2}a_2a_3 \sin A_1 = \frac{1}{2}a_3a_1 \sin A_2 = \frac{1}{2}a_1a_2 \sin A_3$

6. $\triangle = r\,s$

7. $\triangle = r'(s - a_1) = r''(s - a_2) = r'''(s - a_3)$

8. $\triangle = a_1a_2a_3/4R$

9. $\triangle = \sqrt{s(s - a_1)(s - a_2)(s - a_3)}$

10. $4Rrs = a_1a_2a_3$

11. $\sin \dfrac{A_1}{2} = \sqrt{\dfrac{(s - a_2)(s - a_3)}{a_2a_3}}$

    $\sin \dfrac{A_2}{2} = \sqrt{\dfrac{(s - a_3)(s - a_1)}{a_3a_1}}$

    $\sin \dfrac{A_3}{2} = \sqrt{\dfrac{(s - a_1)(s - a_2)}{a_1a_2}}$

12. $\cos \frac{A1}{2} = \sqrt{\frac{s(s - a1)}{a_2 a_3}}$

$\cos \frac{A2}{2} = \sqrt{\frac{s(s - a2)}{a_3 a_1}}$

$\cos \frac{A3}{2} = \sqrt{\frac{s(s - a3)}{a_1 a_2}}$

13. $\tan \frac{A_1}{2} = \frac{r}{s - a_1}$

$\tan \frac{A2}{2} = \frac{r}{s - a_2}$

$\tan \frac{A_3}{2} = \frac{r}{s - a_3}$

14. $rr'r''r''' = \Delta^2$

15. $r' + r'' + r''' = 4R - r$

16. $\frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''} = \frac{1}{r}$

17. $r''r''' + r'''r' + r'r'' = \frac{\Delta^2}{r^2}$

18. $\frac{1}{a_1'} + \frac{1}{a_2'} + \frac{1}{a_3'} = \frac{1}{r}$

19. $a_2 \sin A_3 = a_3 \sin A_2$

$a_3 \sin A_1 = a_1 \sin A_3$

$a_1 \sin A_2 = a_2 \sin A_1$

20. $a_2 \cos A_3 + a_3 \cos A_2 = a_1$

$a_3 \cos A_1 + a_1 \cos A_3 = a_2$

$a_1 \cos A_2 + a_2 \cos A_1 = a_3$

21. $a_2 \cos A_3 - a_3 \cos A_2 = (a_2^2 - a_3^2)/a_1$

$a_3 \cos A_1 - a_1 \cos A_3 = (a_3^2 - a_1^2)/a_2$

$a_1 \cos A_2 - a_2 \cos A_1 = (a_1^2 - a_2^2)/a_3$

22. $a_2 \cos^2 A_3 - a_3^2 \cos^2 A_2 = a_2^2 - a_3^2$

$a_3 \cos^2 A_1 - a_1^2 \cos^2 A_3 = a_3^2 - a_1^2$

$a_1 \cos^2 A_2 - a_2^2 \cos^2 A_1 = a_1^2 - a_2^2$

23. $1 + \cos A_1 = 2\frac{s(s - a_1)}{a_2a_3}$

$1 + \cos A_2 = 2\frac{s(s - a_2)}{a_3a_1}$

$1 + \cos A_3 = 2\frac{s(s - a_3)}{a_1a_2}$

24. $1 - \cos A_1 = \frac{2(s - a_2)(s - a_3)}{a_2a_3}$

$1 - \cos A_2 = \frac{2(s - a_3)(s - a_1)}{a_3a_1}$

$1 - \cos A_3 = \frac{2(s - a_1)(s - a_2)}{a_1a_2}$

25. $a_2(1 + \cos A_3) + a_3(1 + \cos A_2) = 2s$

$a_3(1 + \cos A_1) + a_1(1 + \cos A_3) = 2s$

$a_1(1 + \cos A_2) + a_2(1 + \cos A_1) = 2s$

26. $\cos A_1 + \cos A_2 \cos A_3 = \sin A_2 \sin A_3$

$\cos A_2 + \cos A_3 \cos A_1 = \sin A_3 \sin A_1$

$\cos A_3 + \cos A_1 \cos A_2 = \sin A_1 \sin A_2$

27. $2s \sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = R \sin A_1 \sin A_2 \sin A_3$

28. $\sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{r}{4R}$

29. $\cos \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{s - a_1}{4R}$

$\sin \frac{A_1}{2} \cos \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{s - a_2}{4R}$

$\sin \frac{A_1}{2} \sin \frac{A_2}{2} \cos \frac{A_3}{2} = \frac{s - a_3}{4R}$

30. $\sin A_1 + \sin A_2 + \sin A_3 = \frac{s}{R}$

31. $-\sin A_1 + \sin A_2 + \sin A_3 = \frac{s - a_1}{R}$

$\sin A_1 - \sin A_2 + \sin A_3 = \frac{s - a_2}{R}$

$\sin A_1 + \sin A_2 - \sin A_3 = \frac{s - a_3}{R}$

32. $\cos A_1 + \cos A_2 + \cos A_3 = 1 + \frac{r}{R}$

33. $-\cos A_1 + \cos A_2 + \cos A_3 = \frac{r}{R} \frac{s}{s - a_1} - 1 = 2s \frac{1 - \cos A_1}{a_1} - 1$

$\cos A_1 - \cos A_2 + \cos A_3 = \frac{r}{R} \frac{s}{s - a_2} - 1 = 2s \frac{1 - \cos A_2}{a_2} - 1$

$\cos A_1 + \cos A_2 - \cos A_3 = \frac{r}{R} \frac{s}{s - a_3} - 1 = 2s \frac{1 - \cos A_3}{a_3} - 1$

34. $a_1 \cos A_1 + a_2 \cos A_2 + a_3 \cos A_3 = \frac{2\Delta}{R} = \frac{a_1 a_2 a_3}{2R^2}$

35. $a_2 a_3 \cos A_1 + a_3 a_1 \cos A_2 + a_1 a_2 \cos A_3 = (a_1^2 + a_2^2 + a_3^2)/2$

36. $\cos A_2 \cos A_3 + \cos A_3 \cos A_1 + \cos A_1 \cos A_2 =$

$$\frac{a_2 a_3 + a_3 a_1 + a_1 a_2}{4R^2} - \frac{r}{R} - 1 =$$

$$\frac{\Delta}{R}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) = \frac{r}{R} - 1$$

37. $a_1 \cos A_2 \cos A_3 + a_2 \cos A_3 \cos A_1 + a_3 \cos A_1 \cos A_2 = \frac{\triangle}{R} = \frac{a_1 a_2 a_3}{4R^2}$

38. $a_1 \frac{s - a_2}{a_2} \cdot \frac{s - a_3}{a_3} + a_2 \frac{s - a_3}{a_3} \cdot \frac{s - a_1}{a_1} + a_3 \frac{s - a_1}{a_1} \cdot \frac{s - a_2}{a_2} = s\left(1 - \frac{r}{R}\right)$

39. $\frac{1}{s - a_1} + \frac{1}{s - a_2} + \frac{1}{s - a_3} + \frac{4R + r}{\triangle} = \frac{1}{s}\left(\frac{4R}{r} + 1\right)$

40. $\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \frac{a_3}{s - a_3} = 2\left(\frac{2R}{r} - 1\right)$

CHAPTER VI

TABLES OF RESULTS

TABLE I

SUMMARY OF POINTS

No.	Point	Coordinates
1	A_1	$1, \; 0, \; 0$
2	A_2	$0, \; 1, \; 0$
3	A_3	$0, \; 0, \; 1$
4	B_1	$a_1a_2a_3, \; a_3^3, \; a_2^3$
5	B_2	$a_3^3, \; a_1a_2a_3, \; a_1^3$
6	B_3	$a_2^3, \; a_1^3, \; a_1a_2a_3$
7	B_1^1	$-a_1^2 + a_2^2 + a_3^2, \; a_1a_2, \; a_3a_1$
8	B_2^1	$a_1a_2, \; a_1^2 - a_2^2 + a_3^2, \; a_2a_3$
9	B_3^1	$a_3a_1, \; a_2a_3, \; a_1^2 + a_2^2 - a_3^2$
10	D	$-\frac{(s - a_2)(s - a_3)}{a_1s}, \; \frac{s - a_2}{a_2}, \; \frac{s - a_3}{a_3}$
11	$D_1 \equiv L_1$	$0, \; a_3(s - a_2), \; a_2(s - a_3)$
12	D_2	$-a_3(s - a_2), \; 0, \; a_1s$
13	D_3	$-a_2(s - a_3), \; a_1s, \; 0$
14	E	$\frac{s - a_1}{a_1}, \; -\frac{(s - a_3)(s - a_1)}{a_2s}, \; \frac{s - a_3}{a_3}$
15	E_1	$0, \; -a_3(s - a_1), \; a_2s$

Point	Description	Multiplier (k)
A_1	Vertices of fundamental triangle (Fig. 1)	$2\Delta/a_1$
A_2		$2\Delta/a_2$
A_3		$2\Delta/a_3$
B_1	Vertices of First Brocard triangle (Fig.20)	$2\Delta/a_2a_3(a_1^2 + a_2^2 + a_3^2)$
B_2		$2\Delta/a_1a_3(a_1^2 + a_2^2 + a_3^2)$
B_3		$2\Delta/a_1a_2(a_1^2 + a_2^2 + a_3^2)$
B_1^1	Vertices of Second Brocard triangle (Fig.20)	$2\Delta/a_1(-a_1^2 + 2a_2^2 + 2a_3^2)$
B_2^1		$2\Delta/a_2(2a_1^2 - a_2^2 + 2a_3^2)$
B_3^1		$2\Delta/a_3(2a_1^2 + 2a_2^2 - a_3^2)$
C	Center of Brocard circle	------
D	Points analogous to Gergonne point (Fig. 9)	$2\Delta s/(s^2 - a_2a_3)$
$D_1 \equiv L_1$		$1/2R$
D_2		$1/2R$
D_3		$1/2R$
E		$2\Delta s/(s^2 - a_3a_1)$
E_1		$1/2R$

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
16	$E_2 \equiv L_2$	$a_3(s - a_1), \quad 0, \quad a_1(s - a_3)$
17	E_3	$a_2 s, \quad -a_1(s - a_3), \quad 0$
18	F	$\frac{s - a_1}{a_1}, \quad \frac{s - a_2}{a_2}, \quad -\frac{(s - a_1)(s - a_2)}{a_3 s}$
19	F_1	$0, \quad a_3 s, \quad -a_2(s - a_1)$
20	F_2	$a_3 s, \quad 0, \quad -a_1(s - a_2)$
21	$F_3 \equiv L_3$	$a_2(s - a_1), \quad a_1(s - a_2), \quad 0$
22	G	$\frac{1}{a_1(s - a_1)}, \quad \frac{1}{a_2(s - a_2)}, \quad \frac{1}{a_3(s - a_3)}$
23	G_1	$0, \quad a_3(s - a_3), \quad a_2(s - a_2)$
24	G_2	$a_3(s - a_3), \quad 0, \quad a_1(s - a_1)$
25	G_3	$a_2(s - a_2), \quad 0, \quad a_1(s - a_1)$
26	H	$\sec A_1, \quad \sec A_2, \quad \sec A_3$
27	H_1	$0, \quad \cos A_3, \quad \cos A_2$
28	H_2	$\cos A_3, \quad 0, \quad \cos A_1$
29	H_3	$\cos A_2, \quad \cos A_1, \quad 0$
30	I	$1, \quad 1, \quad 1$
31	I_1	$0, \quad 1, \quad 1$
32	I_2	$1, \quad 0, \quad 1$
33	I_3	$1, \quad 1, \quad 0$
34	I^{1}	$-1, \quad 1, \quad 1$
35	I^{11}	$1, \quad -1, \quad 1$

Point	Description	Multiplier (k)
$E_2 \equiv L_2$		$1/2R$
E_3		$1/2R$
F		$2\Delta/(s^2 - a_1a_2)$
F_1		$1/2R$
F_2		$1/2R$
$F_3 \equiv L_3$		$1/2R$
G	Gergonne point (Fig. 9)	$2\Delta^2/(4R + r)$
G_1		$1/2R$
G_2		$1/2R$
G_3		$1/2R$
H	Orthocenter (Fig. 12)	$2R \cos A_1 \cos A_2 \cos A_3$
H_1		$2\Delta/a_1$
H_2		$2\Delta/a_2$
H_3		$2\Delta/a_3$
I	Incenter (Fig. 15)	r
I_1		$2\Delta/(a_2 + a_3)$
I_2		$2\Delta/(a_3 + a_1)$
I_3		$2\Delta/(a_1 + a_2)$
I^1	Excenters (Fig. 9)	r^1
I^{11}		r^{11}

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
36	I^{111}	$1, \ 1, \ -1$
37	$I_1^{\ 1}$	$0, \ -1, \ 1$
38	$I_2^{\ 11}$	$1, \ 0, \ -1$
39	$I_3^{\ 111}$	$-1, \ 1, \ 0$
40	J	$\cos A_2 + i \sin A_2, \ \cos A_1 - i \sin A_1, \ -1$
41	J	$e^{iA_2}, \ e^{-iA_1}, \ -1$
42	J^1	$\cos A_2 - i \sin A_2, \ \cos A_1 + i \sin A_1, \ -1$
43	J^1	$e^{-iA_2}, \ e^{iA_1}, \ -1$
44	K	$a_1, \ a_2, \ a_3$
45	K_1	$0, \ a_2, \ a_3$
46	K_2	$a_1, \ 0, \ a_3$
47	K_3	$a_1, \ a_2, \ 0$
48	K^1	$-a_1, \ a_2, \ a_3$
49	K^{11}	$a_1, \ -a_2, \ a_3$
50	K^{111}	$a_1, \ a_2, \ -a_3$
51	$K_1^{\ 1}$	$0, \ -a_2, \ a_3$
52	$K_2^{\ 11}$	$a_1, \ 0, \ -a_3$
53	$K_3^{\ 111}$	$-a_1, \ a_2, \ 0$
54	L	$\dfrac{s - a_1}{a_1}, \ \dfrac{s - a_2}{a_2}, \ \dfrac{s - a_3}{a_3}$
55	L_1	$0, \ \dfrac{s - a_2}{a_2}, \ \dfrac{s - a_3}{a_3}$

Point	Description	Multiplier (k)
I^{111}		r^{111}
$I_1{}^{1}$		$2\Delta/(-a_2 + a_3)$
$I_2{}^{11}$		$2\Delta/(-a_3 + a_1)$
$I_3{}^{111}$		$2\Delta/(-a_1 + a_2)$
J	Circular points at infinity	
J		
J^1		
J^1		
K	Symmedian point (Fig. 6)	$2\Delta/(a_1{}^2 + a_2{}^2 + a_3{}^2)$
K_1		$2\Delta/(a_2{}^2 + a_3{}^2)$
K_2		$2\Delta/(a_1{}^2 + a_3{}^2)$
K_3		$2\Delta/(a_1{}^2 + a_2{}^2)$
K^1		$\frac{1}{2}$ tan A_1
K^{11}		$\frac{1}{2}$ tan A_2
K^{111}		$\frac{1}{2}$ tan A_3
$K_1{}^{1}$		$2\Delta/(-a_2{}^2 + a_3{}^2)$
$K_2{}^{11}$		$2\Delta/(-a_3{}^2 + a_1{}^2)$
$K_3{}^{111}$		$2\Delta/(-a_1{}^2 + a_2{}^2)$
L	Verbicenter (Fig. 7)	2r
L_1		$2\Delta/a_1$

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
56	L_2	$\frac{s - a_1}{a_1}$, 0, $\frac{s - a_3}{a_3}$
57	L_3	$\frac{s - a_1}{a_1}$, $\frac{s - a_2}{a_2}$, 0
58	M	$\frac{1}{a_1}$, $\frac{1}{a_2}$ $\frac{1}{a_3}$
59	M_1	0, a_3, a_2
60	M_2	a_3, 0, a_1
61	M_3	a_2, a_1, 0
62	M^1	$-\frac{1}{a_1}$, $\frac{1}{a_2}$, $\frac{1}{a_3}$
63	M^{11}	$\frac{1}{a_1}$, $-\frac{1}{a_2}$, $\frac{1}{a_3}$
64	M^{111}	$\frac{1}{a_1}$, $\frac{1}{a_2}$, $-\frac{1}{a_3}$
65	$M_1{}^1$	0, $-a_3$, a_2
66	$M_2{}^{11}$	a_3, 0, $-a_1$
67	$M_3{}^{111}$	$-a_2$, a_1, 0
68	N	$\frac{a_2 \cos A_2 + A_3 \cos A_3}{a_1}$, $\frac{a_3 \cos A_3 + a_1 \cos A_1}{a_2}$, $\frac{a_1 \cos A_1 + a_2 \cos A_2}{a_3}$
69	N	$\cos (A_2 - A_3)$, $\cos (A_3 - A_1)$, $\cos (A_1 - A_2)$
70	O	$\cos A_1$, $\cos A_2$, $\cos A_3$
71	P	p_1, p_2, p_3
72	Q	q_1, q_2, q_3

Point	Description	Multiplier (k)
L_2		$2\Delta/a_2$
L_3		$2\Delta/a_3$
M	Median point (Fig. 5)	$2\Delta/3$
M_1		$a_1/4R$
M_2		$a_2/4R$
M_3		$a_3/4R$
M^1		2Δ
M^{11}		2Δ
M^{111}		2Δ
${M_1}^1$		
${M_2}^{11}$		
${M_3}^{111}$		
N	Nine point center (Fig. 12)	$R/2$
N		R
O	Circumcenter (Fig. 12)	R
P	Given point (Fig 2)	$2\Delta/(a_1p_1 + a_2p_2 + a_3p_3)$
Q	Given point	$2\Delta/(a_1q_1 + a_2q_2 + a_3q_3)$

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
73	R	$a_2 + a_3, \quad a_3 + a_1, \quad a_1 + a_2$
74	S	$\frac{a_2 + a_3}{a_1}, \quad \frac{a_3 + a_1}{a_2}, \quad \frac{a_1 + a_2}{a_3}$
75	T	$\frac{2s^2}{a_1a_2a_3}(s - a_1) - \frac{a_2 + a_3}{a_1},$
		$\frac{2s^2}{a_1a_2a_3}(s - a_2) - \frac{a_3 + a_1}{a_2},$
		$\frac{2s^2}{a_1a_2a_3}(s - a_3) - \frac{a_1 + a_2}{a_3}$
76	T	$1 + \cos A_1 - \cos A_2 - \cos A_3,$
		$1 - \cos A_1 + \cos A_2 - \cos A_3,$
		$1 - \cos A_1 - \cos A_2 + \cos A_3$
77	T	$- \sin^2 \frac{A_1}{2} + \sin^2 \frac{A_2}{2} + \sin^2 \frac{A_3}{2},$
		$\sin^2 \frac{A_1}{2} - \sin^2 \frac{A_2}{2} + \sin^2 \frac{A_3}{2},$
		$\sin^2 \frac{A_1}{2} + \sin^2 \frac{A_2}{2} - \sin^2 \frac{A_3}{2}$
78	T	$\frac{-a_1}{s - a_1} + \frac{a_2}{s - a_2} + \frac{a_3}{s - a_3},$
		$\frac{a_1}{s - a_1} - \frac{a_2}{s - a_2} + \frac{a_3}{s - a_3},$
		$\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} - \frac{a_3}{s - a_3}$
79	T	$\frac{2R}{\Delta} - \frac{1}{s - a_1}, \quad \frac{2R}{\Delta} - \frac{1}{s - a_2}, \quad \frac{2R}{\Delta} - \frac{1}{s - a_3}$

Point	Description	Multiplier
R		$\Delta/(a_2a_3 + a_3a_1 + a_1a_2)$
S	Speiker center (Fig. 18)	$r/2$
T	Nagel point (Fig. 9)	$2R$
T		R
T		$2R$
T		$r/2$
T		$-\Delta$

TABLE I (continued)

SUMMARY OF POINTS

No.	Points	Coordinates
80	X	$x_1, \quad x_2, \quad x_3$
81	Y	$\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$
82	Z	$\frac{1}{a_1^2 x_1}, \frac{1}{a_2^2 x_2}, \frac{1}{a_3^2 x_2}$
83	Ω	$\frac{a_3}{a_2}, \frac{a_1}{a_3}, \frac{a_2}{a_1}$
84	Ω^1	$\frac{a_2}{a_3}, \frac{a_3}{a_1}, \frac{a_1}{a_2}$

Point	Description	Multiplier (k)
X	Variable point (Fig. 1)	$2\Delta/(a_1x_1 + a_2x_2 + a_3x_3)$
Y	Isogonal conjugate of X	$2\Delta/\left(\frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3}\right)$
Z	Isotomic conjugate of X ?	
Ω	Brocard points (Fig. 19)	$8R\Delta^2/(a_2^2a_3^2 + a_3^2a_1^2 + a_1^2a_2^2)$
Ω'		$8R\Delta^2/(a_2^2a_3^2 + a_3^2a_1^2 + a_1^2a_2^2$

TABLE II

SUMMARY OF LINES

No.	Line	Equation
1	PQ	$x_1(p_2q_3 - p_3q_2) + x_2(p_3q_1 - p_1q_3) + x_3(p_1q_2 - p_2q_1) = 0$
2	A_2A_3	$x_1 = 0$
3	$A_1I = A_1I^1$	$x_2 - x_3 = 0$
4	$A_1I^{11} = A_1I^{111}$	$x_2 + x_3 = 0$
5	$A_1M = A_1M^1$	$a_2x_2 - a_3x_3 = 0$
6	$A_1M^{11} = A_1M^{111}$	$a_2x_2 + a_3x_3 = 0$
7	$A_1K = A_1K^1$	$a_3x_2 - a_2x_3 = 0$
8	$A_1K^{11} = A_1K^{111}$	$a_3x_2 + a_2x_3 = 0$
9	A_1L	$a_2(s - a_3)x_2 - a_3(s - a_2)x_3 = 0$
10	A_1H	$x_2 \cos A_2 - x_3 \cos A_3 = 0$
11	A_1G	$a_2(s - a_2)x_2 - a_3(s - a_3)x_3 = 0$
12	A_2D	$a_3(s - a_2)x_3 - a_1sx_1 = 0$
13	A_3D	$a_1sx_1 + a_2(s - a_3)x_2 = 0$
14	$A_1\Omega$	$a_2a_3x_2 - a_1^2x_3 = 0$
15	$A_1\Omega^1$	$a_1^2x_2 - a_2a_3x_3 = 0$
16	OM_1	$(a_2^2 - a_3^2)x_1 + a_1a_2x_2 - a_1a_3x_3 = 0$
17	IG_1	$(a_2 - a_3)(s - a_1)x_1 + a_2(s - a_2)x_2 - a_3(s - a_3)x_3 = 0$
18	G_2G_3	$-x_1(1+\cos A_1) + x_2(1+\cos A_2) + x_3(1+\cos A_3) = 0$

Line	Description	Ideal Point
PQ		[illegible]
A_2A_3		$0, \quad a_3, \quad -a_2$
$A_1I \equiv A_1I'$		$\frac{a_3 + a_2}{a_1}, \quad -1, \quad -1$
$A_1I'' \equiv A_1I'''$		$\frac{a_3 - a_2}{a_1}, \quad 1, \quad -1$
$A_1M \equiv A_1M'$		$\frac{-2}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$
$A_1M'' \equiv A_1M'''$		$0, \quad -a_3, \quad a_2$
$A_1K \equiv A_1K'$		$-\frac{a_2^2 + a_3^2}{a_1}, \quad a_2, \quad a_3$
$A_1K'' \equiv A_1K'''$		$-\frac{a_2^2 - a_3^2}{a_1}, \quad a_2, \quad a_3$
A_1L		$-1, \quad \frac{s - a_2}{a_2}, \quad \frac{s - a_3}{a_3}$
A_1H		$-1, \quad \cos A_3, \quad \cos A_2$
A_1G		$-a_2a_3, \quad a_3(s - a_3), \quad a_2(s - a_2)$
A_2D		$a_2a_3(s - a_2), \ -a_3a_1(a_3 + a_1), \ a_1a_2s$
A_3D		$a_2a_3(s - a_3), \ a_1a_3s, \ -a_1a_2(a_1 + a_2)$
$A_1\Omega$	Brocard rays (Fig. 19)	$a_2a_3^2 - a_1^2a_2, \ -a_1^3, \ -a_1a_2a_3$
$A_1\Omega'$		$a_3a_1^2 + a_2^2a_3, \ a_1a_2a_3 \ -a_1^3$
OM_1		$-1, \quad \cos A_3, \quad \cos A_2$
IG_1		$-1, \quad \cos A_3, \quad \cos A_2$
G_2G_3		$\frac{-a_2 + a_3}{a_1}, \quad 1, \quad -1$

TABLE II (continued)

SUMMARY OF LINES

No.	Line	Equation
19	$H_2 H_3$	$-x_1 \cos A_1 + x_2 \cos A_2 + x_3 \cos A_3 = 0$
20	I_2I_3	$-x_1 + x_2 + x_3 = 0$
21	$K_1^{I}K_2^{II}K_3^{III}$	$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} = 0$
22	K_2K_3	$-a_2a_3x_1 + a_3a_1x_2 + a_1a_2x_3 = 0$
23	L_2L_3	$-a_1(s - a_2)(s - a_3)x_1 + a_2(s - a_3)(s - a_1)x_2 + a_3(s - a_1)(s - a_2)x_3 = 0$
24	M_2M_3	$-a_1x_1 + a_2x_2 + a_3x_3 = 0$
25	P_2P_3	$p_2p_3x_1 + p_3p_1x_2 + p_1p_2x_3 = 0$
26	$P_1^{I}P_2^{II}P_3^{III}$	$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} = 0$
27		$(a_2p_3 + a_3p_2)x_1 + (a_3p_1 + a_1p_3)x_2 + (a_1p_2 + a_2p_1)x_3 = 0$
28	$P_aP_bP_c$	$(p_2 + p_1 \cos A_3)(p_3 + p_2 \cos A_1)x_1 + (p_3 + p_1 \cos A_2)(p_1 + p_2 \cos A_3)x_2 - (p_2 + p_1 \cos A_3)(p_1 + p_2 \cos A_3)x_3 = 0$

Line	Description	Ideal Point
H_2H_3		$\frac{a_2^2 - a_3^2}{a_1}$, a_2, $-a_3$
I_2I_3		$-a_2 + a_3$, $a_3 + a_1$, $-a_1 - a_2$
$K_1^{1}K_2^{11}K_3^{111}$	Lemoine axis (Fig. 6)	$a_1(a_3^2 - a_2^2)$, $a_2(-a_3^2 + a_1^2)$, $a_3(a_2^2 - a_1^2)$
K_2K_3		$a_1(a_3^2 - a_2^2)$, $a_2(a_3^2 + a_1^2)$, $-a_3(a_1^2 + a_2^2)$
L_2L_3		$\frac{(a_2 - a_3)(s - a_1)}{a_1}$, $s - a_2$, $-(s-a_3)$
M_2M_3		0, a_3a_1, $-a_1a_2$
P_2P_3		$a_3p_3p_1 - a_2p_1p_2$, $-a_3p_2p_3 + a_1p_1p_2$, $a_2p_2p_3 - a_1p_3p_1$
$P_1^{1}P_2^{11}P_3^{111}$	Trilinear polar of P. (Fig. 2)	$p_1(a_2p_2 - a_3p_3)$, $p_2(a_3p_3 - a_1p_1)$, $p_3(a_1p_1 - a_2p_2)$
	Polar of P with respect to circumcircle	
$P_aP_bP_c$	Simson line of P (Fig. 11)	

TABLE II (continued)

SUMMARY OF LINES

No.	Line	Equation
29	HNMO	$\cos A_1(a_2^2 - a_3^2)x_1 - \cos A_2(a_3^2 - a_1^2)x_2 + \cos A_3(a_1^2 - a_2^2)x_3 = 0$
30	KCO	$\frac{a_2^2 - a_3^2}{a_1} x_1 + \frac{a_3^2 - a_1^2}{a_2} x_2 + \frac{a_1^2 - a_2^2}{a_3} x_3 = 0$
31	IOT	$x_1(\cos A_2 - \cos A_3) + x_2(\cos A_3 - \cos A_1) + x_3(\cos A_1 - \cos A_2) = 0$
32	LSMI	$a_1(a_2 - a_3)x_1 + a_2(a_3 - a_1)x_2 + a_3(a_1 - a_2)x_3 = 0$
33	RIK	$x_1(a_2 - a_3) + x_2(a_3 - a_1) + x_3(a_1 - a_2) = 0$

Line	Description	Ideal Point
HNMO	Euler line (Fig. 12)	$-a_1^3 + a_2^3 \cos A_3 + a_3^3 \cos A_2$, $a_1^3 \cos A_3 - a_2^3 + a_3^3 \cos A_1$, $a_1^3 \cos A_2 + a_2^3 \cos A_1 - a_3^3$
KCO	Brocard diameter (Fig. 10)	$a_3(a_1 \cos A_3 - a_3 \cos A_1) - a_2 (a_2 \cos A_1 - a_1 \cos A_2$, $-a_3(a_3 \cos A_2 - a_2 \cos A_3 + a_1 (a_2 \cos A_1 - a_1 \cos A_2$, $a_2(a_3 \cos A_2 - a_2 \cos A_3) - a_1 (a_1 \cos A_3 - a_3 \cos A_1)$
IOT		
LSMI	(Fig. 7)	$(2s/a_1) - 1$, $(2s/a_2) - 1$, $(2s/a_3) - 1$
RIK		$-a_3(a_3 - a_1) + a_2(a_1 - a_2)$, $-a_3(-a_2 + a_3) - a_1(a_1 - a_2)$, $a_2(-a_2 + a_3) + a_1(a_3 - a_1)$

TABLE III

SUMMARY OF CIRCLES

No.	Circle
1	Any circle
2	Circumcircle (Fig. 12)
3	Inscribed circle (Fig. 9)
4	Excircle (I^1) (Fig. 9)
5	Excircle (I^{11}) (Fig. 9)
6	Excircle (I^{111}) (Fig. 9)
7	Nine-point circle (Fig. 12)
8	Polar circle (Fig. 13)
	Circles of Apollonius (Fig. 14)
9	($A_1I_1I_1{}^1$)
10	($A_2I_2I_2{}^{11}$)

No.	Equation
1	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + (a_1x_1 + a_2x_2 + a_3x_3)(m_1x_1 + m_2x_2 + m_3x_3) = 0$
2	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0$
3	$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2 + a_3x_3)\left(a_1(s - a_1)^2x_1 + a_2(s - a_2)^2x_2 + a_3(s - a_3)^2x_3\right) = 0$
4	$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2 + a_3x_3)\left(a_1s^2x_1 + a_2(s - a_3)^2x_2 + a_3(s - a_2)^2x_3\right) = 0$
5	$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2 + a_3x_3)\left(a_1(s - a_3)^2x_1 + a_2s^2x_2 + a_3(s - a_1)^2x_3\right) = 0$
6	$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - (a_1x_1 + a_2x_2 + a_3x_3)\left(a_1(s - a_2)^2x_1 + a_2(s - a_1)^2x_2 + a_3s^2x_3\right) = 0$
7	$a_1 \cos A_1\, x_1^2 + a_2 \cos A_2\, x_2^2 + a_3 \cos A_3\, x_3^2 - (a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = 0$
8	$a_1 \cos A_1\, x_1^2 + a_2 \cos A_2\, x_2^2 + a_3 \cos A_3\, x_3^2 = 0$
9	$x_2^2 - x_3^2 - 2 \cos A_2\, x_3x_1 + 2 \cos A_3\, x_1x_2 = 0$
10	$x_3^2 - x_1^2 - 2 \cos A_3\, x_1x_2 + 2 \cos A_1\, x_2x_3 = 0$

TABLE III (continued)

SUMMARY OF CIRCLES

No.	Circle
11	$(A_3I_3I_3^{111})$
	Circles through two vertices and two equicenters (Fig. 15)
12	$(A_2IA_3I^1)$
13	$(A_3IA_1I^{11})$
14	$(A_1IA_2I^{111})$
	Circles through two excenters and the two vertices not collinear with them (Fig. 15)
15	$(A_2I^{111}I^{11}A_3)$
16	$(A_3I^1I^{111}A_1)$
17	$(A_1I^{11}I^1A_2)$
18	First Lemoine circle (Fig. 16)
19	Second Lemoine circle (Cosine circle) (Fig. 17)

No.	Equation
11	$x_1^2 - x_2^2 - 2\cos A_1\, x_2x_3 + 2\cos A_2\, x_3x_1 = 0$
12	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_1(a_1x_1 + a_2x_2 + a_3x_3) = 0$
13	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_2(a_1x_1 + a_2x_2 + a_3x_3) = 0$
14	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 - x_3(a_1x_1 + a_2x_2 + a_3x_3) = 0$
15	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_1(a_1x_1 + a_2x_2 + a_3x_3) = 0$
16	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_2(a_1x_1 + a_2x_2 + a_3x_3) = 0$
17	$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + x_3(a_1x_1 + a_2x_2 + a_3x_3) = 0$
18	$(a_1^2 + a_2^2 + a_3^2)^2\,(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = -(a_1x_1 + a_2x_2 + a_3x_3)\Big(-a_2a_3(a_2^2 + a_3^2)x_1 - a_3a_1(a_3^2 + a_1^2)x_2 - a_1a_2(a_1^2 + a_2^2)x_3\Big)$
19	$(a_1^2 + a_2^2 + a_3^2)^2\,(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = 4(a_1x_1 + a_2x_2 + a_3x_3)(a_2^2a_3^2 \cos A_1\, x_1 + a_3^2a_1^2 \cos A_2\, x_2 + a_1^2a_2^2 \cos A_3\, x_3)$

TABLE III (continued)

SUMMARY OF CIRCLES

No.	Circle
20	Spieker circle (Fig. 18)
21	Brocard circle (Fig. 20)

No.	Equation
20	$a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - \frac{s^2}{4}(a_1x_1 + a_2x_2 + a_3x_3)^2 + (s - a_1)(s - a_2)(s - a_3)(a_1x_1 + a_2x_2 + a_3x_3)\left(\frac{a_1}{s - a_1}x_1 + \frac{a_2}{s - a_2}x_2 + \frac{a_3}{s - a_3}x_3\right) = 0$
21	$(a_1^2 + a_2^2 + a_3^2)(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = (a_1x_1 + a_2x_2 + a_3x_3)(a_2a_3x_1 + a_3a_1x_2 + a_1a_2x_3)$

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