THE JACOBIAN MATRIX

A Thesis

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by

Sheryl A. Cole

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Marion P. Emerson

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PREFACE

The purpose of this paper is to investigate the Jacobian matrix in greater depth than this topic is dealt with in any individual calculus text.

The history of Carl Jacobi, the man who discovered the determinant, is reviewed in chapter one.

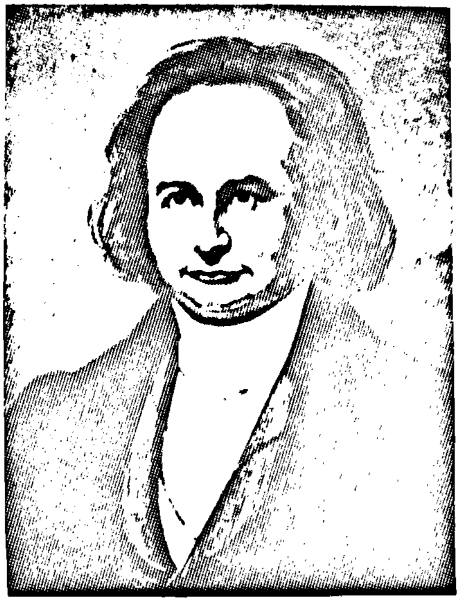
The several illustrations in chapter two demonstrate the mechanics of finding the Jacobian matrix.

In chapter three the magnification of area under a transformation is related to the Jacobian. The amount of magnification under coordinate changes is also discussed in this chapter.

In chapter four a definition of smooth surface area is arrived at by application of the Jacobian.

It is my pleasure to express appreciation to Dr. John Burger for all of his assistance in preparing this paper. An extra special thanks to Stan for his help and patience.

C.G.J. JACOBI (1804-1851)
"Man muss immer umkehren"
(Man must always invert.)



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CHAPTER I

A HISTORY OF JACOBI

Carl Gustav Jacob Jacobi was a prominant mathematician noted chiefly for his pioneering work in the field of elliptic functions.

Jacobi was born December 10, 1804, in Potsdam, Prussia. He was the second of four children born to a very prosperous banker.

Carl Gustav Jacob Jacobi should not be confused with his equally famous older brother Moritz H. Jacobi. Moritz H. Jacobi achieved fame, while still living, as the founder of galvanoplastics. The importance of the ideas and teachings of Carl Gustav was not realized until after his death. Carl's impatience with his brother's popularity during his lifetime once brought out the following statement: "I am not his brother, he is mine." Today his statement aptly describes their relative importance.

Jacobi's first mathematics teacher was an uncle. Under the guidance of his uncle, he was prepared to enter the Potsdam Gymnasium at the age of twelve. After graduating from the Gymnasium, Jacobi was still undecided as to whether to pursue study in the field of mathematics or the field of philosophy.

In 1821 he went to the University of Berlin. It was here that he made his decision to become a mathematician. He spent much time studying work on the masters of mathematics such as Euler and La Grange, as he felt the University instructors had little to offer.

He received his Ph.D. in mathematics from the University of Berlin in 1825 at the age of 21.

Having become "certified", his teaching career began as a lecturer at his alma mater, the University of Berlin. Due to a lack of respect for his own University professors, he attacked his new position with much energy and enthusiasm. His basic teaching philosophy is summed up by the following quotation.

> That irrepresible innovator believed the infallible method to advance mathematics was for domineering professors in the leading universities to drill their own ideas and as little else as possible, into as many advanced students as could be induced to scribble lecture notes. [1. p. 441]

The greatest portion of his lectures were devoted to his personal discoveries and ideas which he was presently investigating.

Some credit the lack of mathematical progress in the twentieth century to his teaching methods. [2. p. 426] The logic behind this feeling is that many poor mathemamaticians do "drill their own ideas, and as little else as possible" into their students. Thus, they offered the student little help and encouragement.

After staying approximately one half year at the University of Berlin, he took a position as lecturer at the University of Königsberg. A year later (1827) he was promoted to assistant professor largely due to a publication of his works on cubic reciprocity. His greatest work FUNDAMENTA NOVA THEORIAE FUNCTIONUM ELLIPTICARUM (New Foundations of the Theory of Elliptic Functions) was completed and published in 1829.

The death of his father in 1832 resulted in the ultimate loss of the family fortune. However, Jacobi continued to work hard in producing works of mathematics. In fact, he worked so diligently that it began to affect his health. Jacobi's physician suggested that he take an active part in politics to get his mind away from his mathematical works. This decision nearly cost him his position of favor with the King of Prussia.

Jacobi soon became the stoolpidgeon for the local liberals and gained the wrath of his only remaining monetary benefactor, the King.

At the age of 45, real poverty had finally struck. He had neither his job nor the family fortune, and a wife and seven children to support. A generous offer from Vienna helped him solve this problem. Germany did not want to lose one of their greatest mathematicians, so the King reconsidered. He returned to his position at the University of Königsberg. He died shortly thereafter (February 18, 1858) of the small pox.

Jacobi, often called 'the great algorist', was second in his field only to Euler. (An algorist is one who develops a way to solve a specific type of problem.) Although he made many contributions to the field of mathematics, his greatest was in the area of elliptic functions. Later, it was found that Guass, working independently, had previously discovered elliptic functions and their double periodicity. Had he published his findings, the total contributions of Jacobi may have been much more advanced.

At approximately the same time Jacobi was working on elliptic functions, so was his elder rival Niels Henrik Abel. Neither of them realized the existence of the other when they first started their work. Abel was credited with first inverting elliptic integrals, thus smoothing the way for further development in that area.

In 1830 the academy awarded Jacobi and Abel the Grand Prize in Mathematics for work done with trancendental functions.

While working with rational numbers Jacobi discovered that an elliptic identity would determine the number of representations of an integer as the sum of two squares. This is found in his "Fundamenta nova", published in 1829.

In 1841, Jacobi first presented the functional determinants which carry his name. DE FORMATIONE ET PROPRIETATIBUS DETERMINANTIUM was a "landmark on the subject of determinants". [3. p. 426]

The basic works of these functional determinants are as follows:

If $y_1, y_2, \ldots y_n$, are functions of $x_1, x_2, \ldots x_n$, then the functional determinant, or Jacobian, is the equations as follows:

$$J = \frac{\partial(y_1, y_2, \ldots . y_n)}{\partial(x_1, x_2, \ldots . x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots\cdots & \\ \vdots & \cdot & & \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots\cdots & \end{vmatrix}$$

This determinant is often more briefly written as: $J(y_i/x_i)$. [4. p. 916-917]

Some properties of the Jacobian are: (a) $J(y_i/x_i)$ $J(x_i/y_i)=1$. (b) $J(y_i/z_i)J(z_i/x_i) = J(y_i/x_i)$. A Definition for the Jacobian is: A Jacobian is a functional determinant formed by the n^2 different coefficients of n given functions of n independent variables. [5. p. 340-341] Sylvester is attributed for having given the name Jacobian to this functional determinant.

With the possible exception of Caucy, Jacobi is generally, considered to have contributed more to the field of functional determinants than any other person.

Additional areas in which Jacobi worked would include continued fractions, Abelian functions, maxima and minima, and the geometric interpretation of the conjugate point. His early works led to a field (late 19th century) later known as elliptic modular functions. He established a theory of dynamics which was not extended until fifty years later by Poincafe.

Jacobi's famous memoirs appeared in "Crelle's Journal For Pure and Applied Sciences" in 1841. Many of his important writings were also published by Crelle's.

It is obvious that Jacobi's theory of elliptic functions was responsible for much of the mathematical analysis in the 18th and 19th century.

Much of the work Jacobi started was continued by Weirstrass and Hermite.

Kronecker and Reimann were former students of Jacobi's who would have to be considered famous mathematicians in their own right.

CHAPTER 2

THE JACOBIAN AND ITS RELATED THEOREMS

The purpose of this chapter is to define a Jacobian and discuss its related theorems. It generalizes the theory of inverses of one variable to transformations defined by a system of equations.

The equations $u = f(x,y)$, $v = g(x,y)$, define a transformation from points in the xy plane to points in the uv plane. The Jacobian matrix for this transformation is the matrix:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The determinant of the above matrix is the Jacobian determinant of the transformation (noted T) or the <u>Jacobian</u> of T. This is also denoted by $\frac{\partial(u,v)}{\partial(x,y)}$. The idea of the Jacobian is easily extended to dimensions greater than two.

The above is the Jacobian of u and v with respect to x and y. That is, the transformation is from the xy plane to the uv plane. If one wishes to define the inverse function from the uv plane to the xy plane, the equations would be in the form $x = F(u,v)$ and $y = G(u,v)$. Then the Jacobian matrix of these equations would be:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

These are inverse transformations (if inverses exist) and their Jacobians are reciprocals.

Some theorems about Jacobians are as follows. In each one must agree that the functions are continuously differentiable.

THEOREM 1. A necessary and sufficient condition that the equations $F(x, y, z, u, v) = 0$, $G(x, y, z, u, v) = 0$ can be solved for x and y (for example) is that $\frac{\partial(u,v)}{\partial(x,y)}$ is not equal to zero in a region R. In other words a necessary and sufficient condition that the equations be a 1-1 transformation is that $J \neq 0$ in any region. In this case J^{-1} exists and is also nonzero--$JJ^{-1} = 1$.

THEOREM 2. If x and y are functions of u and v, while u and v are functions of r and s, then,

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,s)}$$

This is known as the chain rule for Jacobians.

THEOREM 3. If $u = f(x,y)$, and $v = g(x,y)$, then a necessary and sufficient condition that a functional relation of the form $\phi(u,v) = 0$ exists between u and v is that $\frac{\partial(u,v)}{\partial(x,y)}$ be identically zero.

Example 1. GIVEN: $u = 3x^2 - xy$
$v = 2xy^2 + y^3$

$$\frac{\partial u}{\partial x} = 6x-y \qquad \frac{\partial u}{\partial y} = -x$$

$$\frac{\partial v}{\partial x} = 2y^2 \qquad \frac{\partial v}{\partial y} = 4xy + 3y^2$$

$$J = \begin{vmatrix} 6x-y & -x \\ 2y^2 & 4xy + 3y^2 \end{vmatrix} = (6x-y)(4xy + 3y)^2 - (-x)(2y^2)$$
$$= y(24x^2 + 16xy - 3y^2)$$

Since the Jacobian does not equal zero, there exists no functional relation of the form $\phi(u,v) = 0$. This follows by theorem 3. But when $y = 0$, the Jacobian is zero. By Theorem 1, this is not a 1-1 transformation. There exists no inverse.

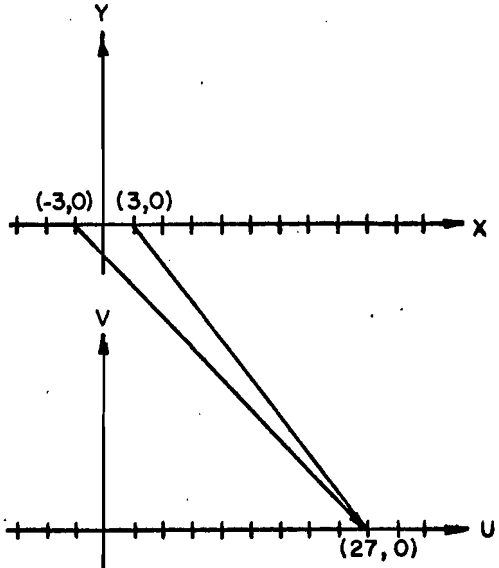
In investigating this particular transformation when $y = 0$. One obtains (by direct substitution) $u = 3x^2$
$v = 0$.

Thus if $y = 0$, so also must v.

$$u = 3x^2 \rightarrow x = \pm\sqrt{u/3}$$

$$\left.\begin{matrix} x = \pm 3 \\ y = 0 \end{matrix}\right\} \xrightarrow{\text{MAPS TO}} \left\{\begin{matrix} u = 27 \\ v = 0 \end{matrix}\right.$$

These points (3,0) and (-3,0) in the xy plane both map to (27,0) in the plane. (See sketch)



As u cannot be a negative number and produce a real value for x, it is obvious that the x axis maps its points only to the positive u axis.

Example 2. GIVEN: $u = x + 2z$
$v = 2y + 3z$
$w = 2x - 2y + z$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial Y} = 0 \qquad \frac{\partial u}{\partial z} = 2$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 2 \qquad \frac{\partial v}{\partial z} = 3$$

$$\frac{\partial w}{\partial x} = 2 \qquad \frac{\partial w}{\partial y} = -2 \qquad \frac{\partial w}{\partial z} = 1$$

$$\text{THUS } J = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} = 0$$

This transformation has no inverse by theorem 1, but there is a functional relationship of the form $\phi(u,v,w) = 0$.

By the following methods one can arrive at this relationship.

by using the first and third equations to eliminate z

$$\begin{array}{r} u = x + \quad 2z \\ -2w = -4x + 4y - 2z \\ \hline u - 2w = -3x + 4y \quad * \end{array}$$

by using the second and third equations to eliminate z

$$\begin{array}{r} v = 2y + 3z \\ -3w = -6x + 6y - 3z \\ \hline v - 3w = -6x + 8y \\ \dfrac{v - 3w}{2} = -3x + 4y \quad * \end{array}$$

By using the two resulting equations (*)

$$u - 2w = \frac{v - 3w}{2} \quad \text{or } 2u - w - v = 0$$

which is a plane in the uvw space.

Example 3. GIVEN: $u = x + 2z$
$v = 2y + 3z$
$w = 2x - 2y + z$

$$J = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

Again this indicates a functional relationship between u and v.

Some 1-1 transformations and their inverses are illustrated below.

Example 4. GIVEN: $u = 2x + 2y$
$v = 5x + 7y$

$$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial u}{\partial y} = 2 \qquad \frac{\partial(u,v)}{\partial(x,y)} = J = \begin{vmatrix} 2 & 2 \\ 5 & 7 \end{vmatrix} = 4$$

$$\frac{\partial v}{\partial x} = 5 \qquad \frac{\partial v}{\partial y} = 7$$

Since this Jacobian will never equal zero, it is a 1-1 transformation. Thus their exists the following inverse:

By using determinants to solve for x and y:

$$x = \frac{\begin{vmatrix} 2 & u \\ 7 & v \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 5 & 7 \end{vmatrix}} \qquad y = \frac{-\begin{vmatrix} 2 & u \\ 5 & v \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 5 & 7 \end{vmatrix}}$$

NOTE: Notice that the determinant in the denominator is actually the Jacobian.

$$x = \frac{-7}{4}u + \frac{-1}{2}v \qquad y = \frac{5}{4}u + \frac{-1}{2}v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = J^{-1} = \begin{vmatrix} \frac{-7}{4} & \frac{1}{2} \\ \frac{5}{4} & \frac{-1}{2} \end{vmatrix} = \frac{1}{4}$$

This inverse could also have been found by finding the inverse matrix associated with J, solving for x and y is not necessary, but is possible.

It is noted that $JJ^{-1} = 1$.

Example 5: GIVEN: $u = 2x + 3y - z$
$v = x - 7y + z$
$w = x + y + 2z$

$$J = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -7 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -41$$

This is also a 1-1 transformation. The inverse can be found by solving for x, y, and z in terms of u, v, and w or by finding the inverse matrix. Since in this case finding the inverse matrix is easiest, one will proceed in that manner.

$$J^{-1} = \begin{vmatrix} \frac{15}{14} & \frac{7}{14} & \frac{4}{14} \\ \frac{1}{41} & \frac{-5}{41} & \frac{3}{41} \\ \frac{-8}{41} & \frac{-1}{41} & \frac{17}{41} \end{vmatrix} = \frac{-1}{41}$$

It is noted again that J and J^{-1} are reciprocals. Assume that one wishes to evaluate the Jacobian of a transformation at a specific point. Given $T(x,y) = (2x + 3y, 5x + 7y)$ it follows that $J = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = 14 - 15 = -1$ In this particular transformation the Jacobian will be -1 regardless of the point chosen, as there are no variables present in the value of the Jacobian.

Following is an example where the Jacobian changes values at different points.

Example 6. GIVEN: $T(x,y) = \left(\frac{x+1}{x+y}, x+y\right)$

$$J = \begin{vmatrix} \frac{y-1}{(x+y)^2} & \frac{-x-1}{(x+y)^2} \\ 1 & 1 \end{vmatrix} = \frac{y-1}{(x+y)^2} - \frac{(-x-1)}{(x+y)^2} =$$

$$\frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

Suppose one wants to determine the value of J at (2,1) and at (1,1). These may be found by evaluating the partial derivative at the point or evaluating J at that point. The value of the Jacobian at (2,1) is 1/3, at (1,1) is 1/2.

The inverse Jacobians at these points are still reciprocals even though the transformation is not 1-1 for all values of x and y. The fact that these transformations are continuous in the interval around these points allows for this.

In the next chapter the relationship of the value of J under the transformation will be investigated.

CHAPTER 3

TRANSFORMATIONS OF AREA

In this chapter the importance of the Jacobian when transforming one area to another area will be shown.

For most transformations, closed regions in one space map to closed regions in the image space. If ΔAxy and ΔAuv denote respective areas in these regions, then:

$$\lim \frac{\Delta Axy}{\Delta Auv} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

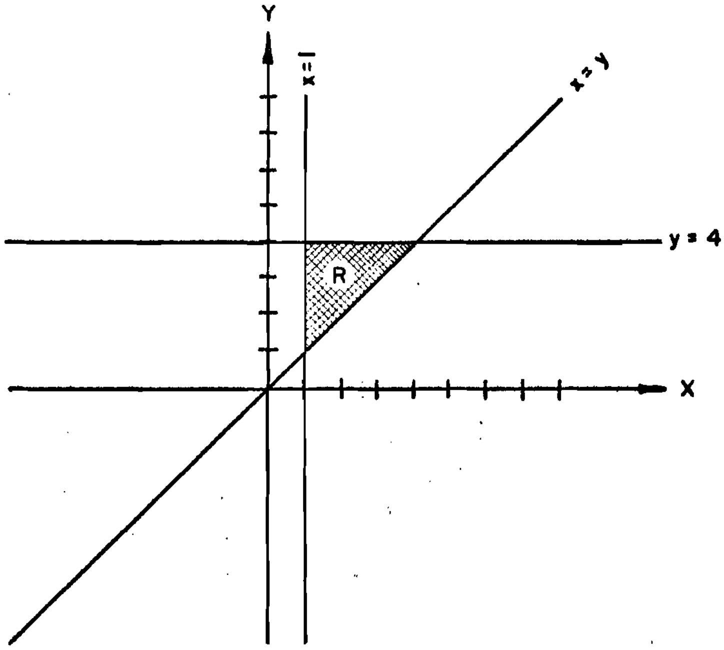
where lim denotes the limit at ΔAxy (or ΔAuv) approaches zero. This Jacobian is often called the <u>Jacobian of the Transformation</u>. [6. p. 108] If the Jacobian of the function is a constant the expression lim is unnecessary as the value of the Jacobian varies with position.

Example 1. $u = \frac{x+y}{2}$

$v = \frac{x-y}{2}$

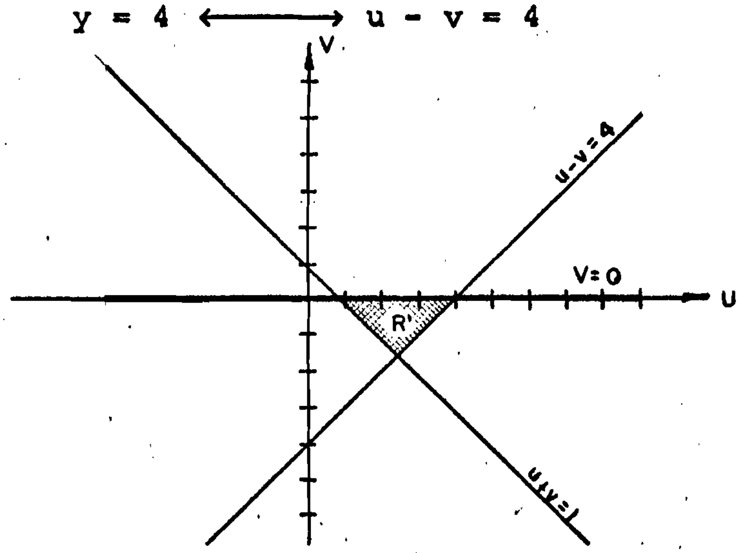
Suppose one wishes to transform the area bounded by the following lines to the uv-plane:

$$x = y$$
$$x = 1$$
$$y = 4 .$$



Each of these lines map as follows: (These may be found by substituting the respective lines into the transformation.)

$$x = y \longleftrightarrow v = 0$$
$$x = 1 \longleftrightarrow u + v = 1$$
$$y = 4 \longleftrightarrow u - v = 4$$



Since both triangles are right triangles, the areas of each may be figured as shown below:

Area of R $= 1/2(3)(3) = 9/2$, Area of R' $= 1/2(3/2\sqrt{2})(3/2\sqrt{2}) = 9/4$.

It is interesting to note that both orthogonality and the type of triangle remain unchanged under the transformation.

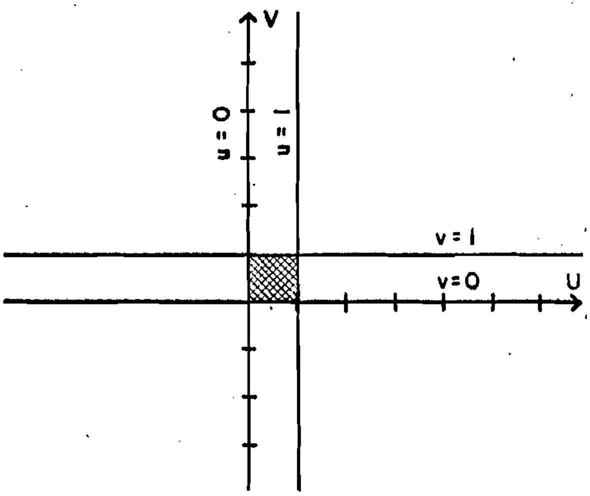
The Jacobian of T from the xy-plane to the uv-plane is

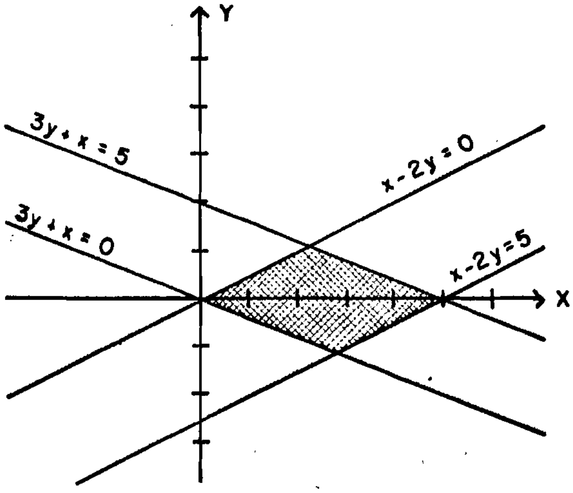
$$\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = 1/2$$

This is also the ratio of the area in the uv-plane to the area in the xy-plane. This might be explained in this manner. The transformation from the xy-plane to the uv-plane "magnifys" the area. The Jacobian tells one the amount of magnification.

Example 2. Given the transformations: $x = 2u + 3v$
$y = u - v$

Find and discuss the region bounded by $u = 0$, $u = 1$, $v = 0$, and $v = 1$.





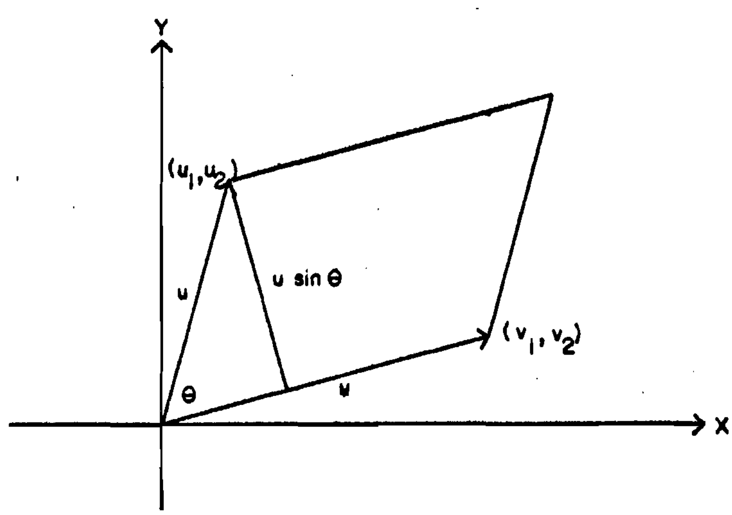
The following are the line transformations:

$$\begin{aligned} u = 0 &\longleftrightarrow 3y + x = 0 \\ v = 0 &\longleftrightarrow x - 2y = 0 \\ u = 1 &\longleftrightarrow 3y + x = 5 \\ v = 1 &\longleftrightarrow x - 2y = 5 \end{aligned}$$

The unit square is mapped to the parallelogram bounded by the above lines with vertices (0,0), (2,1), (5,0), (3,-1). The transformation is from the uv-plane to the xy-plane, so the area in the uv-plane is magnified by 5 (since the Jacobianis 5) and placed in the xy-plane. Thus, the area of the parallelogram is 5 units. All of this can be verified by analytic methods.

If one returns for a moment to look at some properties of vectors, these should help relate why the Jacobian shows magnification.

First one wishes to show that the absolute value of a 2 x 2 determinant whose rows are components of vectos u and v is the area of a parallelogram with u and v as adjacent sides.



One wishes to prove the area of the parallelogram which is (u)(v)sin θ is $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

Proof: The following identities will help in the proof;

$$(u \cdot v) = |u|\,|v| \cos\,\theta$$

$$|(u x v)| = \begin{vmatrix} u & u \\ v^1 & v^2 \\ {}_1 & {}_2 \end{vmatrix}$$

$$(u \cdot v)^2 + |(u x v)|^2 = |u|^2 |v|^2$$

By starting with the last identity, we get

$$\Big[|u|\,|v| \cos\Big]^2 + |(uxv)|^2 = |u|^2 |v|^2$$

$$|(uxv)|^2 = |u|^2|v|^2 - |u|^2|v|^2\cos^2\theta$$

$$= \Big[|u|^2|v|^2\Big]\Big[1 - \cos^2\theta\Big]$$

$$= |u|^2|v|^2 \sin^2\theta$$

$$\begin{vmatrix} u_1 u_2 \\ v_1 v_2 \end{vmatrix} = |u|\,|v| \sin\,\theta$$

This idea can be extended to dimensions greater than 2. That is, a 3 x 3 determinant whose rows are components ov vectors u, v, and w will give the volume of the parallelpiped formed by these vectors. This leads to the following theorem which associates the Jacobian with this area.

THEOREM 4.

Suppose T is a linear transformation of R with matrix A. If T maps an n-dimensional parallelpiped P onto a parallelpiped P*, then

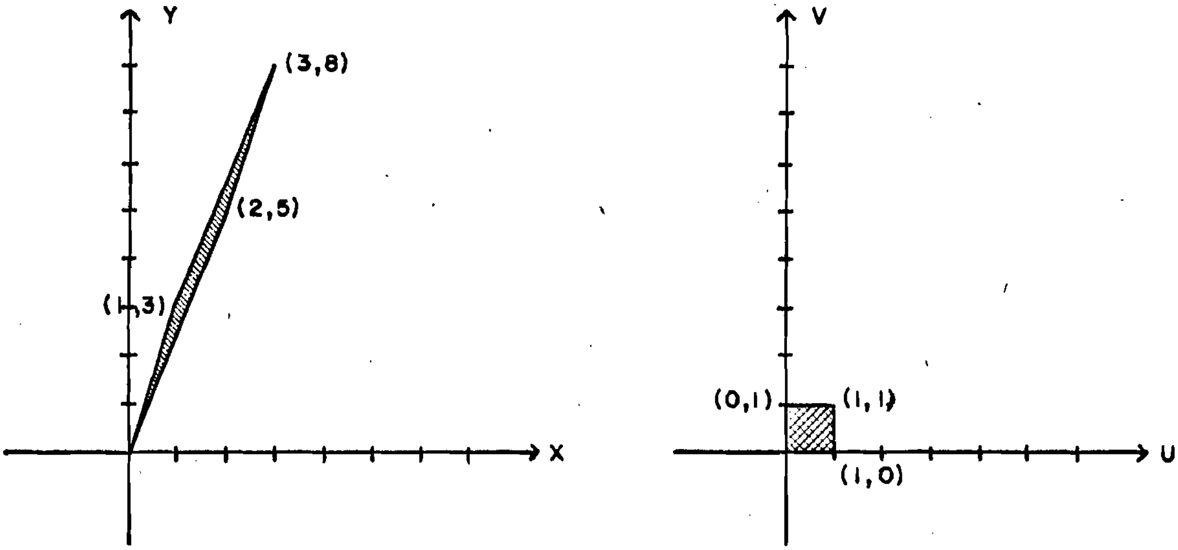
(volume of P*) =, |det(A)| (volume of P)

Proof: The volume of P is |det(B)| where B has as its rows components of the vectors forming P. The edges of P* are rows of matrix AB and its volume is |det(AB)|. Since det(AB) = (det(A)) (det(b)), we get the above statement. [7. p. 401]

If a transformation is not linear, then it can be approximated by a linear transformation which is the Jacobian of the transformation. If it is linear, this matrix A (above) is just the Jacobian of the transformation. Thus one has the Jacobian as the "magnification factor".

Example 3.

Find the linear transformation that takes the following parallogram into the unit square. What is its Jacobian? Its Area?



The transformation is $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

The value of the Jacobian is 1, then the area of the parallelogram is magnified by 1 to get the area of the unit square which also is 1. The area of the parallelogram is 1.

THEOREM 5.

Suppose T is a one-to-one transformation whose domain and range are in R^N and that there is an open set U on which the component functions of T have continuous first-order partial derivatives and the Jacobian of T is nonzero. Let R be a subset of U that is mapped onto a compact set S that has n-D volume. If f is continuous on S, then

$$\int_s f(p)\,dv = \int_r f\big(T(q)\big)\,|J(T)|\,dv \quad [8.\ p.\ 414]$$

This theorem generalizes a little more the role that the Jacobian plays in finding areas.

Example 4.

Given a parallelogram with vertices (0,0), (2,1), (5,0), (3,-1) under a transformation defined as $x = 2u + 3v$ and $y = u - v$. Express $\int_r (x + y)\,dA$ in the form $\iint du\,dv$.

Since this transformation was discussed in example 2 one knows the limits of integration are 0 and 1.

$$\int_r (x+y)\,dA = \int_0^1\int_0^1 (2u + 3v + u - v)(5)\;du\,\partial v$$

$$\int_0^1\int_0^1 (3u + \partial v)(5)\;du\partial v$$

$$\int_0^1\int_0^1 (15u + 10v)\;\partial u\partial v$$

$$\left.\frac{15u^2}{2} + 10uv\right]_0^1$$

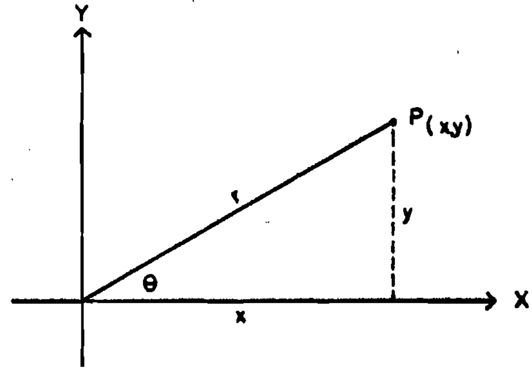
$$\int_0^1\left(\frac{15}{2} + 10v\right)\partial v$$

$$\left.\frac{15v}{2} + \frac{10v^2}{2}\right]_0^1$$

$$12\ 1/2$$

The "local magnification factor" when rectangular coordinates are changed to other coordinate systems follows.

POLAR COORDINATES:



The proper coordinate changes are:

$$x = r \cos \theta$$
$$y = r \sin \theta \ .$$

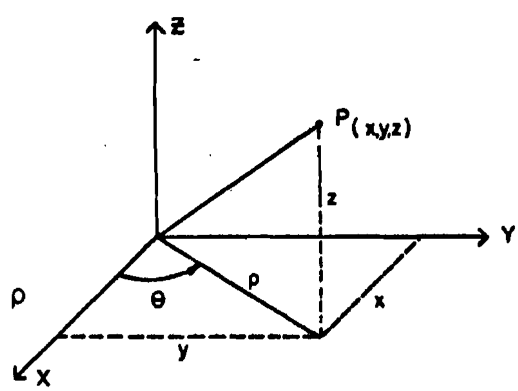
The corresponding Jacobian $\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)$ is;

$$\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \ .$$

CYLINDRICAL COORDINATES:

The transformation for rectangular coordinates to cylindrical coordinates is the same as for polar coordinates except a z coordinate is added as follows:

$$x = \rho \cos \theta \quad y = \rho \sin \theta \quad z = z \ .$$



$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \begin{vmatrix} \cos \theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

The magnification factor is ρ which corresponds to r in the 2-d figure above.

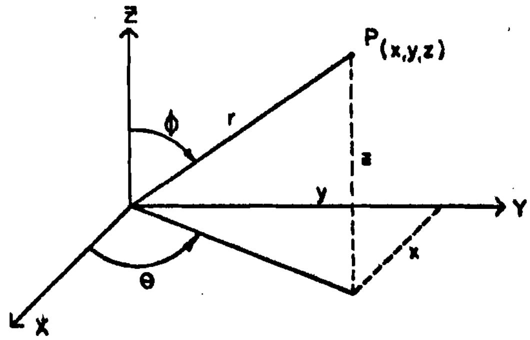
SPHERICAL COORDINATES:

Spherical coordinates and rectangular coordinates are related in the following manner:

$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi \ .$$



The Jacobian of this transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} =$$

$$-r^2 \sin\phi \ .$$

The absolute value of the expression $r^2 \sin \phi$ is the magnification.

CHAPTER 4

SURFACE AREA

Chapter three was concerned mainly with magnification of areas from one coordinate system to another. This chapter will cover the investigation of the Jacobian as it relates to surface areas. A concept of smooth surface areas will be developed.

The relationship of the length of a line segment to its projection on an axis is basic to the concept of the relation of an area and its projection. If the line segment AB (fig. 1) has projection CE on the axis, the length of AB is $\frac{1}{\cos\alpha}$ (CE).

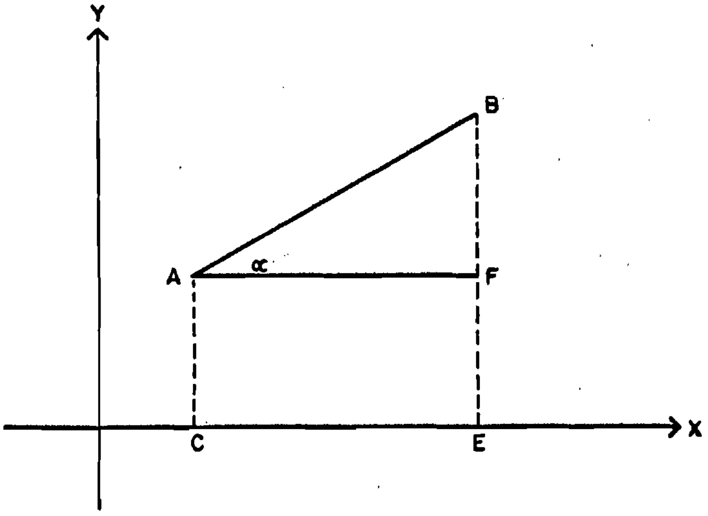


FIGURE 1

One can at this time find another angle equal to α and thus a method of evaluating the angle α. ∇f is the direction of greatest change at a point and is perpendicular to a point on the segment. $\frac{\partial f}{\partial y}$ at this same point would be perpendicular to the x axis. (see fig. 2) The angle between the gradient (∇f) and a unit vector in the direction of $\frac{\partial f}{\partial y}$ is equal to α. So the cos α can be evaluated by the formula $\frac{\partial f}{\partial y}$ divided by the length of ∇f. It might be well to note here that it is necessary to know the identity $|\nabla f| \cos \alpha = \frac{\partial f}{\partial s}$ (directional derivative).

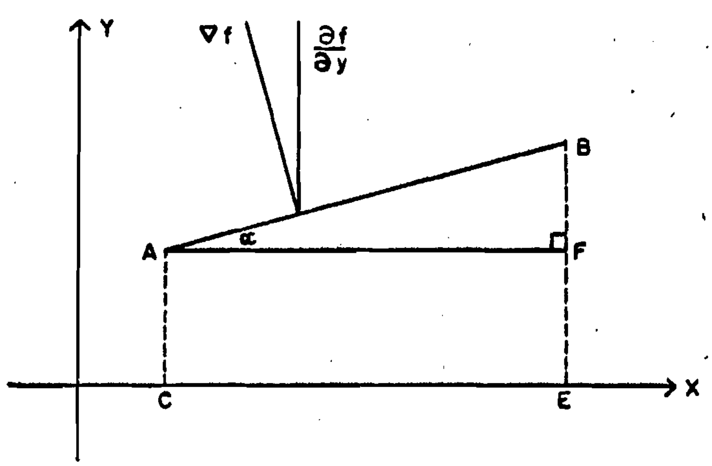


FIGURE 2

One can apply the idea developed above to the area of a small portion of a surface. If a surface has a tangent plane at a point, then a reasonable approximation of the area of the piece of surface is the area of the projection of the tangent plane divided by the cos α, where α is relatively in the same position as discussed in the two dimensional illustration.

One can be a little more explicit in explaining this in reference to figure 3. If one wishes a good approximation to the area of ΔS one can use the area of $\Delta D/\cos\alpha$. So the area of the whole surface is given by the following formula:

$$\text{area of } S = \int_D |\cos\alpha|^{-1}\, dA$$

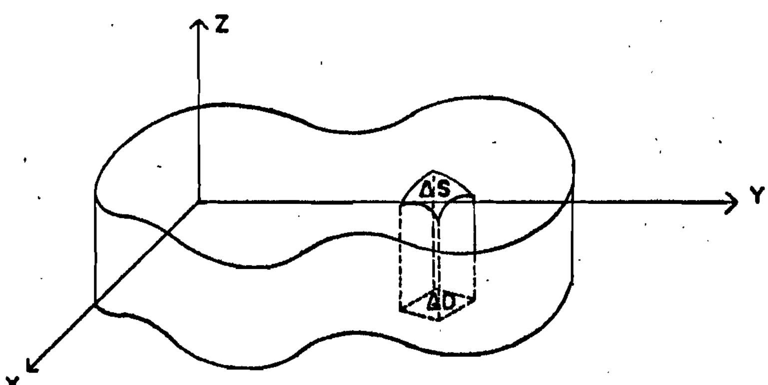


FIGURE 3

The difficulty here is in finding the $\cos\alpha$, but the method for finding $\cos\alpha$ in 2-space (refer to fig. 2) also applies here.

$$\cos\alpha = \frac{\frac{\partial f}{\partial z}}{|\nabla f|} \qquad \text{(Refer to fig. 4)}$$

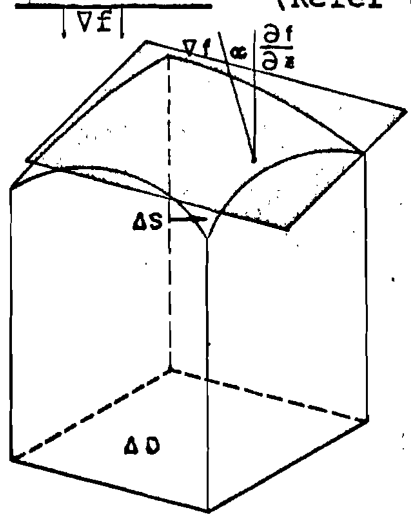


FIGURE 4

All of this is related to the Jacobian. Let one presume figure 5 to be the small piece of the tangent plane in figure 4 projected on the xy plane. Although the axis have been moved this will not change the final result. The area of this projection (ΔD) is the area of a parallelogram with (u_1, u_2, u_3) and (v_1, v_2, v_3) as its adjacent sides. This can be expressed as u x v or

$\begin{vmatrix} u_1 u_2 \\ v_1 v_2 \end{vmatrix}$. (This was determined in chapter 3.)

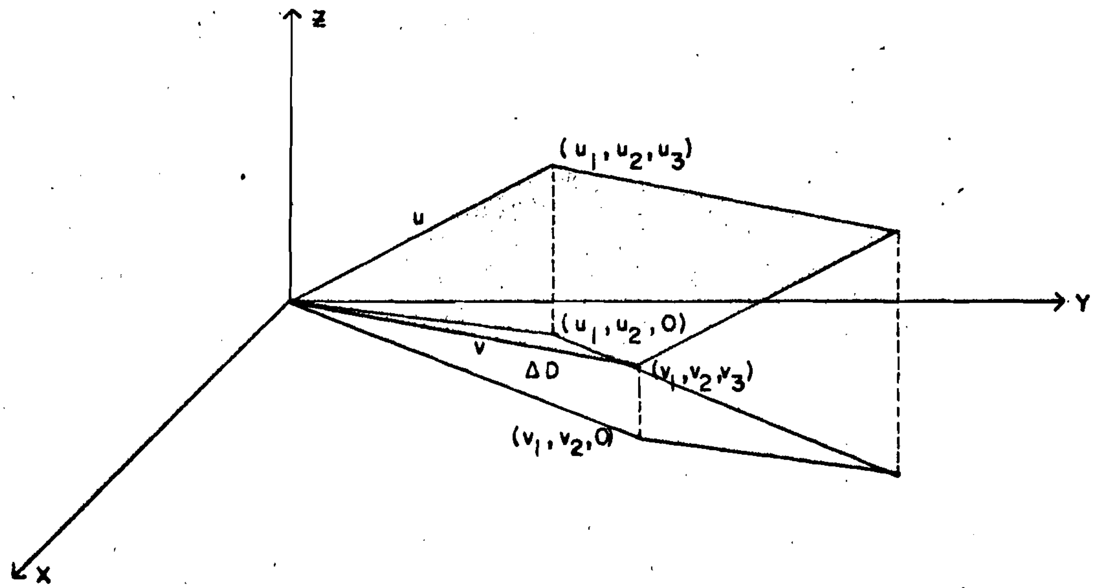


FIGURE 5

u_1 may be thought of as $\frac{\partial u}{\partial y}$ since in the xy plane x is held constant and the movement takes place in the y direction for this coordinate. Likewise, u_2 as $\frac{\partial u}{\partial x}$; v_1 as $\frac{\partial v}{\partial y}$; and v_2 as $\frac{\partial v}{\partial x}$. Then the determinant becomes

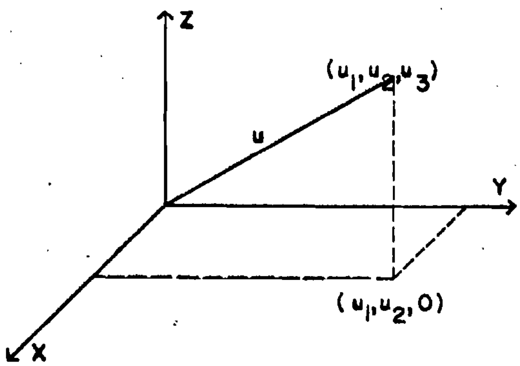


FIGURE 6

$$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial (u,v)}{\partial (x,y)} .$$

A surface is considered to be smooth at a point P if a tangent plane exists at that point and at every point in its neighborhood. This leads to the conclusion that the area of the projection of the tangent plane in each of the xy, yz, and xz planes cannot be zero. Now let one introduce the following notation for these Jacobians.

If a surface is parametrically defined as:
$x = f(u,v) \qquad y = g(u,v) \qquad z = h(u,v)$
then:

These Jacobians have to be nonzero for the surface to be smooth since they represent the area of the projection of the tangent plane. The ratio of these special Jacobians also gives the direction of the normal.

CONCLUSION

Jacobi made many contributions to the mathematical world. Although his work in elliptical functions was probably most important, the discovery of the Jacobian matrix is not to be slighted. In analysis, this matrix is used to evaluate the change in area or volume under a transformation. This is particularly useful when an area or volume is difficult to find before transformation. This matrix is used to approximate surface area and to define smooth surface area.

This paper has only discussed a few of the physical applications of the Jacobian.

FOOTNOTES

FOOTNOTES

1. Bell, Eric T. *Development of Mathematics.* Second edition. New York: McGraw-Hill Book Company, 1945, p. 441.

2. *Ibid.* p. 426.

3. *Ibid.*

4. "Jacobi, C.G.J.", *Van Nostrand's Scientific Encyclopedia.* (3rd ed.), pp. 916-917.

5. "Jacobi, C.G.J.", *Encyclopedia Britanica.* (1968) XII. pp. 340-341.

6. Spiegel, Murray R., *Theory and Problems of Advanced Calculus.* New York: Schaum Publishing. p. 108.

7. *Ibid.* p. 401

8. James, Robert C. *Advanced Calculus.* Belmont, Calif.: Wadsworth Publishing Co., Inc., 1966. p. 414.

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