AN INVESTIGATION OF THE TRANSFER FUNCTION ON FINITE GROUPS

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CHAPTER I

INTRODUCTION

Homomorphisms are important in the study of groups. To testify to this there are several homomorphism and isomorphism theorems. These theorems are used repeatedly in proving other theorems.

THE PROBLEM

The transfer is an elusive homomorphism. To determine the images of the transfer, in many instances, requires much work, and the method prescribed by the definition reveals little about the transfer. However, the transfer is a homomorphism that is valuable in the study of groups.

Although computing the transfer requires considerable work, this work is justified as the transfer is used in proving Burnside's Theorem. In addition, since each group determines a set of transfers, the transfer could be a way of characterizing groups.

ORGANIZATION OF THE THESIS

In this thesis, only finite groups will be considered; it is assumed that the reader has knowledge of finite group theory. In order to understand the development of the transfer, it is essential that the reader has worked with

Sylow's Theorem and the isomorphism theorems.

The transfer is presented so that no previous knowledge of it is needed. Definitions, theorems and their proofs are provided. To aid the reader, examples accompany the theorems as well as the definition of the transfer.

The following list of theorems should be familiar to the reader. However, they are presented here for review and as a reference, since all of them are used in proofs of theorems in Chapter II.

<u>Theorem 1.1 (Lagrange's Theorem)</u>

If S is a subgroup of a finite group G, then $[G:S]$, the index of S in G, is equal to the order of G divided by the order of S.

<u>Theorem 1.2</u>

If S is a normal subgroup of a group G, then the cosets of S in G form a group, denoted G/S, of order $[G:S]$.

<u>Theorem 1.3 (An Isomorphism Theorem)</u>

Let f be a homomorphism from a group G onto a group H, with kernel K. Then K is a normal subgroup of G and G/K is isomorphic to H.

<u>Theorem 1.4 (Sylow's Theorem)</u>

Let G be a finite group with a p-Sylow subgroup P. All p-Sylow subgroups of G are conjugate to P and the number of these is a divisor of G and is congruent to one modulo p.

CHAPTER II

THE TRANSFER

This chapter investigates the transfer, and includes its definition, its properties and its use in Burnside's Theorem. In addition, some information concerning commutators is included as an underlying concept connected to the transfer.

COMMUTATORS

A few basic concepts about commutators and commutator subgroups are essential for the development of the transfer. The commutator subgroup of a group G is a normal subgroup, and the factor group of G with respect to the commutator subgroup is an Abelian group. These two properties will be the main concern of this section.

<u>Definition</u>

If a and b are elements of a group G, the <u>commutator</u> of a and b, $[a,b]$, is $a^{-1}b^{-1}ab$. If H and K are subgroups of G then $[H,K]$ will denote the subgroup of G generated by the set of all $[h,k]$, such that h is in H and k is in K. The <u>commutator subgroup</u> G^1 is the subgroup $[G,G]$.

Since the definition allows a and b to be inverses, the identity is always a commutator. However, the set of all

$[h,k]$ such that h is in a subgroup H and k is in a subgroup K is not necessarily a subgroup itself. Therefore $[H,K]$ may contain elements which are not commutators.

The following theorem is supplied to provide the reader with a better understanding of commutators. Also, parts (i) and (iii) will be used to prove the two following theorems.

Theorem 2.1

If a, b and c are elements of a group G, then

(i) $[a,b] = e$ if and only if $ab = ba$,

(ii) $[a,b]^{-1} = [b,a]$,

(iii) $b^{-1}ab = a[a,b]$,

(iv) $[a,bc] = [a,c]c^{-1}[a,b]c$,

(v) $[ab,c] = b^{-1}[a,c]b[b,c]$,

(vi) $b^{-1}[[a,b^{-1}],c]bc^{-1}[[b,c^{-1}],a]ca^{-1}[[c,a^{-1}],b]$
$= e$.

Proof. (Part (i)). By definition $[a,b] = a^{-1}b^{-1}ab$, and if $a^{-1}b^{-1}ab = e$ then $ab = ba$. Conversely, if $ab = ba$, then $a^{-1}b^{-1}ab = e$. Hence $[a,b] = e$, and (i) holds. Each of the other conclusions follows similarly by direct computation using the definition.

Notice that if G is an Abelian group then $ab = ba$ for all a and b in G. Hence, by Theorem 2.1 (i), $G^1 = \{e\}$.

<u>Theorem 2.2</u>

The commutator subgroup G^1 is normal in G.

<u>Proof</u>. Let a be an element in G^1 and let b be an element in G. By (iii) of Theorem 2.1, $b^{-1}ab = a[a,b]$. Now, a and $[a,b]$ are elements of G^1 so that their product must be in G^1. Therefore $b^{-1}ab$ is in G^1 and G^1 is normal in G.

Since G^1 is a normal subgroup of G, there exists a factor group G/G^1, by Theorem 1.2. The existence of G/G^1 makes the following theorem possible.

<u>Theorem 2.3</u>

If G is a group then G/G^1 is Abelian.

<u>Proof</u>. Let x and y be elements of G, then xG^1 and yG^1 are elements of G/G^1. Consider the commutator $[xG^1,yG^1]$; since G^1 is a normal subgroup,

$$[xG^1,yG^1] = (xG^1)^{-1}(yG^1)^{-1}(xG^1)(yG^1)$$
$$= x^{-1}y^{-1}xyG^1 = G^1 \quad .$$

G^1 is the identity of G/G^1, and therefore G/G^1 is Abelian by Theorem 2.1 (i).

DEFINITION OF TRANSFER

This section will be devoted to defining and explaining a special homomorphism, the transfer.

In a group G with subgroup H, a complete set of coset representatives $S = \{x_1,x_2,x_3,\ldots,x_n\}$ is a set such that for any x_i in S, x_iH is a coset of H. Also, if $x_iH = x_jH$ then $x_i = x_j$, and if y is an element of G then there exists one and only one x in S such that y is in the coset xH.

Notation

If $\{g_1, g_2, g_3, \ldots, g_n\}$ is a subset of a group G, then πg_i is the product $g_1 g_2 g_3 \cdots g_n$.

Definition

Let H be a subgroup of a group G. Let S $= \{x_1, x_2, x_3, \ldots, x_n\}$ be a complete set of left coset representatives of H in G. If y is an element of G then for each x_i there exists an h_i in H and an x_j in S such that $yx_i = x_jh_i$. T is the transfer from G into H/H^1 if and only if $T(y) = (\pi h_i)H^1$.

The cosets of H partition G so that every element yx_i is an element of one of the cosets of H. Thus, by the definition of coset, yx_i is the product of some coset representative x_j and some h_i in H. Therefore, there exists an h_i for each x_i such that $yx_i = x_jh_i$. Hence the definition is valid. When H is an Abelian group the transfer can be considered as a function from G into H as $H^1 = \{e\}$.

For an example of the transfer, let G be the cyclic group of order six, $\{e, a, a^2, a^3, a^4, a^5\}$. Let H be the subgroup $\{e, a^2, a^4\}$. Since H is Abelian, $H^1 = \{e\}$ so that the transfer is a mapping from G into H. Let the cosets of H in G be eH and aH, then $\{e, a\}$ is a set of representatives.

Let $x_1 = e$ and $x_2 = a$. Thus, when $y = a^3$, $yx_1 = a^3e = a^3$ and $yx_2 = a^3a = a^4$. Now, a^3 is in aH and a^4 is in eH, or $a^3 = aa^2$ and $a^4 = ea^4$. Hence, $h_1 = a^2$ and $h_2 = a^4$, and $\pi h_i = h_1h_2 = a^2a^4 = e$. Therefore $T(y) = e$.

Table I shows all possible forms of the equation $yx_i = x_jh_i$. For each y there are two such equations, one for $x_1 = e$ and one for $x_2 = a$. For each x_i there is associated an x_j and an h_i.

TABLE I

THE POSSIBLE VALUES FOR THE EQUATION $yx_i = x_jh_i$

	x_i	x_j	h_i		x_i	x_j	h_i
$y = e$	e	e	e	$y = a$	e	a	e
	a	a	e		a	e	a^2
$y = a^2$	e	e	a^2	$y = a^3$	e	a	a^2
	a	a	a^2		a	e	a^4
$y = a^4$	e	e	a^4	$y = a^5$	e	a	a^4
	a	a	a^4		a	e	e

The table does not contain the image of any y under the transfer. However, for each y it gives the appropriate h_1 and h_2 so that T(y) can be determined by taking their product. For $y = a^5$, $h_1 = a^4$ and $h_2 = e$ so that $T(y) = \pi h_i = h_1h_2 = a^4e = a^4$. Thus $T(e) = T(a^3) = e$, $T(a) = T(a^4) = a^2$, and $T(a^2) = T(a^5) = a^4$.

ELEMENTARY PROPERTIES

The transfer has two important characteristics as a mapping. First, the transfer T from G into H/H^1 can be modified so that the mapping is from G/G^1 into H/H^1. By this modification, if K is a subgroup of H which is a subgroup of G, then the mapping from G/G^1 into K/K^1 is the composite function from G/G^1 into H/H^1 and from H/H^1 into K/K^1. This is the transitive property. Secondly, the transfer is also a homomorphism.

The equation $yx_i = x_j h_i$ contains a secondary mapping. For each x_i there is a unique x_j, for which y is fixed. Hence, the equation $yx_i = x_j h_i$ can be written $yx_i = (Ax_i)h_i$ where A is a function from the set of coset representatives $\{x_1, x_2, x_3, \ldots, x_n\}$ into itself such that $Ax_i = x_j$. In the previous example $y = e$ corresponds to the identity permutation, and $y = a$ corresponds to the permutation (ea). In Appendix A other permutations of the transfer can be found.

<u>Theorem 2.4</u>

The function A is a permutation.

<u>Proof</u>. By definition, it must be shown that A is one-to-one. If

$$Ax_i = Ax_j$$

then

$$\begin{aligned} x_i^{-1}x_j &= x_i^{-1}(y^{-1}y)x_j = (x_i^{-1}y^{-1})(yx_j) \\ &= (yx_i)^{-1}(yx_j) \quad . \end{aligned}$$

Now,

$$(yx_i)^{-1} = ((Ax_i)h_i)^{-1}$$

and

$$(yx_j) = (Ax_j)h_j$$

so that

$$\begin{aligned}(yx_i)^{-1}(yx_j) &= ((Ax_i)h_i)^{-1}(Ax_j)h_j \\ &= h_i^{-1}((Ax_i)^{-1}(Ax_j))h_j \quad .\end{aligned}$$

However,

$$Ax_i = Ax_j \quad ,$$

and hence

$$h_i^{-1}((Ax_i)^{-1}(Ax_j))h_j = h_i^{-1}h_j \in H \quad .$$

Thus $x_iH = x_jH$, and, since x_i and x_j are elements of a complete set of representatives, A is one-to-one. Therefore A is a permutation.

For every group G and subgroup H there exists a transfer from G into H/H^1. This follows directly from the definition of transfer.

Theorem 2.5

If H is a subgroup of a group G then there exists exactly one transfer from G into H/H^1.

Proof. It remains to be shown that there is only one transfer from G into H/H^1; or equivalently, the transfer is determined independently of the choice of the coset representatives. Let $yz_i = (Bz_i)a_i$ and $yx_i = (Ax_i)h_i$, where a_i and h_i are in H and $\{x_1,x_2,x_3,\ldots,x_n\}$ and $\{z_1,z_2,z_3,\ldots,z_n\}$

are complete coset representative sets. Now, every z_i is in some coset x_jH so that $z_i = x_jc_i$. For each x_i there exists an x_j and a permutation D such that $x_i = Dx_j$. Also, a permutation of the x_i's must necessarily permute the h_i's and so $h_i = Dh_j$. Thus $yx_i = y(Dx_j) = (ADx_j)(Dh_j)$. By using this equation and the equation $z_i = (Dx_i)c_i$, it follows that

$$yz_i = y((Dx_i)c_i) = (y(Dx_i))c_i$$
$$= ((ADx_i)(Dh_i))c_i \quad .$$

Now, (ADx_i) can be replaced as

$$(D^{-1}ADz_i) = (D(D^{-1}ADx_i)(D^{-1}ADc_i)$$
$$= (ADx_i)(D^{-1}ADc_i)$$

which implies that

$$(ADx_i) = (D^{-1}ADz_i)(D^{-1}ADc_i)^{-1} \quad .$$

Thus

$$(ADx_i)(Dh_i)c_i = (D^{-1}ADz_i)(D^{-1}ADc_i)^{-1}(Dh_i)c_i \quad ,$$

and

$$yz_i = (D^{-1}ADz_i)(D^{-1}ADc_i)^{-1}(Dh_i)c_i \quad .$$

However, $yz_i = (Bz_i)a_i$ so that

$$(Bz_i)a_i = (D^{-1}ADz_i)(D^{-1}ADc_i)^{-1}(Dh_i)c_i \quad .$$

Since (Bz_i) and $(D^{-1}ADz_i)$ are uniquely determined,

$$(Bz_i) = (D^{-1}ADz_i)$$

and

$$a_i = (D^{-1}ADc_i)^{-1}(Dh_i)c_i \quad .$$

Hence

$$(\pi a_i)H^1 = (\pi (D^{-1}ADc_i)^{-1}(Dh_i)c_i)H^1 \quad .$$

Now, since H/H^1 is Abelian and H^1 is a normal subgroup of H, the products may be taken in any order. Also, D and $D^{-1}AD$ are permutations so that

$$(\pi a_i)H^1 = (\pi\ (D^{-1}ADc_i)^{-1}(Dh_i)c_i))H^1$$
$$= (\pi c_i)^{-1}(\pi\ h_i)(\pi c_i)H^1 = (\pi h_i)H^1 \quad .$$

Hence a group and a subgroup determine a transfer. Therefore when two or more transfers are being considered the following notation will be used.

Notation

If H is a subgroup of G then $T_{G,H}$ is the transfer from G into H/H^1.

Theorem 2.6

The transfer is a homomorphism.

Proof. Let $yx_i = (Ax_i)h_i$, and $zx_i = (Bx_i)c_i$ where z and y are elements of G, c_i and h_i are in H, and A and B are permutations. Then

$$(yz)x_i = y(zx_i) = y((Bx_i)c_i) = (y(Bx_i))c_i$$
$$= (ABx_i)(Bh_i)c_i \quad .$$

Hence

$$T(yz) = (\pi(Bh_i)c_i)H^1 \quad ,$$

and since H/H^1 is Abelian,

$$(\pi(Bh_i)c_i)H^1 = (\pi h_i)H^1(\pi c_i)H^1 = T(y)T(z) \quad .$$

Therefore

$$T(yz) = T(y)T(z)$$

and T is a homomorphism.

The next property to be considered is that of transitivity. In general the transfer is not transitive. However, when the subgroup H of the transfer $T_{G,H}$ is Abelian the image of G under $T_{G,H}$ is a subset of H and so, if K is a subgroup of H, there is a transfer $T_{H,K}$ and a transfer $T_{G,K}$ such that $T_{G,K} = T_{H,K}T_{G,H}$. This is the transitive property.

Through the following theorem, a function greatly related to the transfer for which the transitive property holds can be defined. This new function is the same as the transfer when the subgroup of the transfer is Abelian.

Theorem 2.7

The function $T^*_{G,H}$, from G/G^1 into H/H^1, defined by $T^*_{G,H}(yG^1) = T_{G,H}(y)$, for all elements yG^1, is a homomorphism.

Proof. It first must be shown that $T^*_{G,H}$ is well defined. If h and k are elements of G, such that $hG^1 = kG^1$ implies that $T^*_{G,H}(hG^1) = T^*_{G,H}(kG^1)$, then the mapping is well defined. $T^*_{G,H}(hG^1) = T^*_{G,H}(kG^1)$ if and only if G^1 is contained in the kernel of $T_{G,H}$. Let $K = \mathrm{Ker}(T_{G,H})$. By Theorem 1.3, G/K is isomorphic to H/H^1, and hence Abelian. By Theorem 2.1 (i), since the identity of G/K is K, $K = [aK,bK]$ for all elements a and b in G. Now, K is a normal subgroup of G so that $[aK,aK] = [a,b]K$. Thus, all commutators of G are in K. But, K is a subgroup of G and hence G^1 is contained in K. Therefore $T^*_{G,H}$ is well defined.

Let yG^1 and zG^1 be elements of G/G^1. Since $T_{G,H}$ is a homomorphism it follows that

$$T^*_{G,H}(yzG^1) = T_{G,H}(yz) = T_{G,H}(y)T_{G,H}(z)$$
$$= T^*_{G,H}(yG^1)T^*_{G,H}(zG^1) \quad .$$

Therefore $T^*_{G,H}$ is a homomorphism.

<u>Theorem 2.8 (Transitivity of the Transfer)</u>

If G is a group and K is a subgroup of H which is a subgroup of G, then $T^*_{G,K} = T^*_{H,K}T^*_{G,H}$.

<u>Proof</u>. Let $\{x_1,x_2,x_3,\ldots,x_n\}$ be a complete set of coset representatives of H in G, and let $\{z_1,z_2,z_3,\ldots,z_m\}$ be a complete set of coset representatives of K in H. Let y be an element of G and $yx_i = (Ax_i)h_i$ where h_i is in H and A is a permutation of the x_i's. Since G is the union of all cosets of the form x_iH and H is the union of all cosets of the form z_jK, it follows that G is the union of all cosets x_iz_jK. Thus, for each i and j there is a permutation A_i and an element k_{ij} of K such that $h_iz_j = (A_iz_j)k_{ij}$.

Hence

$$yx_iz_j = (yx_i)z_j = ((Ax_i)h_i)z_j = (Ax_i)(h_iz_i)$$
$$= (Ax_i)((A_iz_j)k_{ij}) = (Ax_i)(A_iz_j)k_{ij} \quad .$$

Hence

$$T^*_{G,K}(yG^1) = T_{G,K}(y) = (\pi_{ij}k_{ij})K^1 \quad ,$$

and

$$T^*_{H,K}T^*_{G,H}(yG^1) = T^*_{H,K}T_{G,H}(y) = T^*_{H,K}((\pi_i h_i)H^1$$
$$= T_{H,K}(\pi_i h_i) = \pi_i(T_{H,K}h_i)$$
$$= \pi_i(\pi_j k_{ij})K^1 = (\pi_{ij}k_{ij})K^1 \quad .$$

The last equality is justified, since K/K^1 is Abelian.

Therefore

$$T^*_{G,K} = T^*_{H,K}T^*_{G,H} \quad .$$

CHARACTERISTICS

Computing the transfer is often time consuming. Some of this work can be eliminated by using the theorems in this section. These theorems are also valuable in proving other theorems. In particular, Theorem 2.9 is needed for the proof of Burnside's Theorem.

<u>Theorem 2.9</u>

Let G be a group and H a subgroup of G. Let T be the transfer from G into H/H^1. Then, for each y in G, there exists a subset $S^* = \{x_1, x_2, x_3, \ldots, x_r\}$ of a complete set S of coset representatives of H such that

(i) $x_i^{-1}y^{n_i}x_i$ H for proper choices of n_i and x_i, $i = 1,2,\ldots,r$,

(ii) $\Sigma n_i = [G:H]$,

(iii) $T(y) = (\pi(x_i^{-1}y^{n_i}x_i))H^1$.

<u>Proof</u>. For each element y in G, there exists a

permutation A of S such that $yx_i = (Ax_i)h_i$ where x_i is an element of S and h_i is an element in H. If A is the identity permutation, then $(Ax_i)h_i = x_ih_i$ for all x_i, so that $h_i = x_i^{-1}yx_i$. Thus $x_i^{-1}yx_i$ is an element of H, and

$$T(y) = (\pi h_i)H^1 = (\pi(x_i^{-1}yx_i))H^1 \quad .$$

Also, in this case each $n_i = 1$ and it follows that

$$\Sigma n_i = [G:H] \quad .$$

Otherwise, let $(x_{i1}x_{i2}x_{i3}\ldots x_{it})$ be any cycle of A of length n_i. Then

$$yx_{i1} = x_{i2}h_{i1} \quad , \qquad yx_{i2} = x_{i3}h_{i2} \quad ,$$
$$yx_{i3} = x_{i4}h_{i3} \quad , \ldots , \; yx_{i(t-1)} = x_{it}h_{i(t-1)},$$
$$yx_{it} = x_{i1}h_{it} \quad .$$

Thus

$$x_{i2}^{-1}yx_{i1} = h_{i1} \quad , \qquad x_{i3}^{-1}yx_{i2} = h_{i2} \quad ,$$
$$x_{i4}^{-1}yx_{i3} = h_{i3} \quad , \ldots ,$$
$$x_{it}^{-1}yx_{i(t-1)} = h_{i(t-1)} \quad ,$$
$$x_{i1}^{-1}yx_{it} = h_{it} \quad ,$$

so that

$$(x_{i1}^{-1}yx_{it})(x_{it}^{-1}yx_{i(t-1)})\ldots(x_{i2}^{-1}yx_{i1})$$
$$= h_{it}h_{i(t-1)}\ldots h_{i1} \quad .$$

Therefore

$$(x_{i1}^{-1}y^tx_{i1}) = h_{it}h_{i(t-1)}\ldots h_{i1} \quad ,$$

and it follows that $x_i^{-1}{}^{n_i}x_i \in H$, and so (i) holds. Since H/H^1 is Abelian and H^1 is normal in H, the product πh_i may be taken in any order so that

$$\pi h_i = \pi(x_i^{-1}y^{n_i}x_i) \quad .$$

Thus (iii) holds. Now, since A is a permutation, and therefore onto, the sum of the lengths of the cycles of A (including cycles of length one) is the number of elements in the complete set of coset representatives. Therefore, by Lagrange's Theorem, the sum of the lengths of the cycles of A is $[G:H]$.

In itself, this theorem does not aid in the calculation of the transfer. However, from it the following three theorems are proved, the first two of which simplify the transfer for special cases.

<u>Theorem 2.10</u>

Let H be an Abelian subgroup of a group G. Let $[G:H] = n$ and let T be a transfer from G into H/H^1. If $G = HK$, H is a subset of C(K), the centralizer of K, and y is any element of G, then $T(y) = y^n$.

<u>Proof</u>. Since $G = HK$, K contains a complete set of coset representatives of H. Let $yk_i = (Ak_i)h_i$ where k_i is an element of K, and h_i is an element of H. Then, by Theorem 2.9,

$$T(y) = (\pi(k_i^{-1}y^{n_i}k_i))H^1 \quad .$$

Hence $k_i^{-1}y^{n_i}k_i \in H$ and $y^{n_i} \in k_iHk_i^{-1}$. But, H is a subset of C(K) so that $k_iHk^{-1} = Hk_ik_i^{-1} = H$. Now, y^{n_i} is in H and so it commutes with all elements of K, and therefore

$$(\pi(k_i^{-1}y^{n_i}k_i))H^1 = (\pi(y^{n_i}))H^1 = y^nH^1 \quad .$$

But H^1 is the identity, as H is Abelian, so that $T(y) = y^n$.

<u>Theorem 2.11</u>

Let $[G:H] = n$ and let T be a transfer from G into H/H^1. If H is contained in Z(G), the center of G, and y is an element of G, then $T(y) = y^n$.

<u>Proof</u>. Since H is a subset of Z(G), $H^1 = \{e\}$ so that T can be considered as a transfer from G into H. By Theorem 2.10,

$$x_i^{-1}y^{n_i}x_i \in H$$

and

$$T(y) = \pi(x_i^{-1}y^{n_i}x_i) \quad .$$

Hence

$$y^{n_i} \in x_iHx_i^{-1} \quad .$$

But, H commutes with all elements of G since it is a subset of Z(G) and so

$$x_iHx_i^{-1} = H.$$

Thus y^{n_i} is in H and it too commutes with all elements of G so that

$$\pi(x_i^{-1}y^{n_i}x_i) = \pi y^{n_i} = y^n \quad .$$

Therefore

$$T(y) = y^n \quad .$$

In $A_4 \otimes Z_2$ (Appendix A) the center is $\{1,3\}$. Consider the transfer from G into Z(G). Theorem 2.11 says that for each y in G, $T(y) = y^n$ where $n = [G:H] = 12$. However, all of the elements of $A_4 \otimes Z_2$ have orders which divide 12. Therefore, $T(y) = 1$ for all y in G.

<u>Theorem 2.12</u>

Let $[G:H] = n$ and let T be a transfer from G into H/H^1. If G = HK and the intersection of H and K is the identity then K is a subset of the kernel of T, Ker(T).

<u>Proof</u>. Since G = HK and the intersection of H and K is the identity, K is a complete set of coset representatives of H. Let y be an element of K, if k_i is a representative from K; then

$$k_i^{-1} y^{n_i} k_i \in K \quad .$$

However, by Theorem 2.9(i),

$$k_i^{-1} y^{n_i} k_i \in H \quad .$$

Hence

$$k_i^{-1} y^{n_i} k_i = e \quad .$$

Therefore

$$T(y) = (\pi(k_i^{-1} y^{n_i} k_i))H^1 = eH^1 \quad ,$$

and K is contained in Ker(T).

For an example of this theorem, consider $A_4 \otimes Z_2$ again (Appendix A). The subgroups $H = \{1,3,9,12,14,16,18,20\}$ and $K = \{1,2,4\}$ satisfy the hypothesis of the theorem. The transfer $T_{G,H}$ has kernel $\{1,2,4,10,11,12,15,16,17,18,21,24\}$ which contains K. Notice that the subgroups $\{1,10,17\}$, $\{1,11,21\}$, and $\{1,15,24\}$ could each replace K, and so each is a subset of the kernel.

<u>Theorem 2.13</u>

Let T be a transfer from a group G into the factor group H/H^1. If x and y are conjugates in G then $T(x) = T(y)$.

<u>Proof</u>. Let g be an element of G such that $g^{-1}xg = y$. Then

$$T(y) = T(g^{-1}xg) = T(g^{-1})T(x)T(g)$$
$$= (T(g))^{-1}T(x)T(g)$$

as T is a homomorphism. Since H/H^1 is Abelian,

$$(T(g))^{-1}T(x)T(g) = T(x)(T(g))^{-1}T(g) = T(x) \quad .$$

The fact that the images of conjugates are the same decreases the work of computing the transfer. In $A_4 \otimes Z_2$ (Appendix A) two thirds of the work is eliminated by Theorem 2.13.

<u>Theorem 2.14</u>

Let G be an Abelian group with p-Sylow subgroup H.

If T is a transfer from G into H then T is onto.

Proof. Since G is Abelian, H is Abelian and so T can be considered to be a mapping into H. Now, for every y in H,

$$yx_i = x_i y \quad ,$$

as G is Abelian. Thus $T(y) = y^n$ where $n = [G:H]$. Suppose g and h are in H such that $T(g) = T(h)$. Then $g^n = h^n$, which implies that

$$g^n(h^n)^{-1} = e \quad ,$$

so that

$$(gh^{-1})^n = e \quad .$$

H is a p-Sylow subgroup so that $(p,n) = 1$, and so $gh^{-1} = e$ or $g = h$. Therefore T is onto.

Requiring G to be Abelian seems to be a stronger restriction than needed. However, the theorem cannot be proved by only requiring the subgroup H to be Abelian. In fact, the theorem with such a change is false. To prove this, refer to $A_4 \otimes Z_2$ (Appendix A). The subgroup $\{1,3,9,12,14,16,18,20\}$ is Abelian and the transfer to this subgroup is not onto. The image of the transfer is $\{1,3\}$.

BURNSIDE'S THEOREM

So far the transfer is little more than an interesting homomorphism. However, in this section, it will be used to characterize the structure of some groups.

Theorem 2.15

If P is a p-Sylow subgroup of G, and L and M are self conjugate subsets of P which are conjugate in G, then they are conjugate in $N(P)$.

Proof. At first, the conclusion seems obvious. However, for L and M to be conjugates in $N(P)$ there must exist an element y in $N(P)$ such that $y^{-1}Py = M$.

By definition of conjugate, there exists a g in G such that $g^{-1}Lg = M$. Let t be an element of P; then $g^{-1}tg$ is an element of $g^{-1}Pg$, and

$$(g^{-1}tg)^{-1}(g^{-1}Lg)(g^{-1}tg) = (g^{-1}t^{-1}g)(g^{-1}Lg)(g^{-1}tg)$$
$$= g^{-1}(t^{-1}Lt)g = g^{-1}Lg \quad .$$

This last equality is justified as L is self conjugate in P. Hence $g^{-1}Lg$ is a self conjugate subset of $g^{-1}Pg$. Thus, M is a self conjugate subset of $g^{-1}Pg$, and so $g^{-1}Pg$ is contained in $N(M)$. Hence, by Sylow's Theorem, there exists a z in $N(M)$ such that $z^{-1}(x^{-1}Lx)z = P$. Therefore, $z^{-1}(x^{-1}Lx)z = z^{-1}Mz = M$, and since zx is in $N(P)$, the theorem is proved.

The next theorem is usually presented as part of the proof of Burnside's Theorem. However, it characterizes the transfer for certain types of subgroups and so is valuable in itself.

Theorem 2.16

Let P be a p-Sylow subgroup of a group G, and let T

be a transfer from G into P/P^1. If $N(P) = C(P)$, then

(i) the intersection of Ker(T) and P is the identity,

(ii) T is onto P.

<u>Proof</u>. Since $C(P) = N(P)$ which contains P, P is Abelian. Thus, consider T as a transfer from G into P. Let $\{x_1,x_2,x_3,\ldots,x_n\}$ be a complete set of coset representatives and let y be an element of P. By Theorem 2.9,

$$x_i^{-1}y^{n_i}x_i \in P \quad \text{and} \quad T(y) = \pi(x_i^{-1}y^{n_i}x_i) \quad .$$

Since

$$x_i(x_i^{-1}y^{n_i}x_i)x_i^{-1} = y^{n_i} \quad ,$$

it follows that $x_i^{-1}y^{n_i}x_i$ and y^{n_i} are conjugates in G. Now, each of these is a self conjugate subset of P since P is Abelian. Hence, by Theorem 2.15, there exists a z in N(P) such that $z^{-1}y^{n_i}z = x_i^{-1}y^{n_i}x_i$. But $z^{-1}y^{n_i}z = y^{n_i}$. Therefore, $x_i^{-1}y^{n_i}x_i = y^{n_i}$ and $T(y) = \pi y^{n_i}$. Let $[G:P] = n$. P is a p-Sylow subgroup of G and so $(n,p) = 1$. Hence $y^n \neq e$ if $y \neq e$. Therefore the intersection of Ker(T) and P is the identity. Conclusion (i) is proved. Now, if z is an element of P such that $T(y) = T(z)$, then $y^n = z^n$ and hence $(yz^{-1})^n = e$. This implies that $yz^{-1} = e$, or that $y = z$. Therefore the transfer is onto.

Definition

A subgroup K of a group G is a complement of a subgroup H if and only if G = HK and the intersection of H and K is the identity.

Theorem 2.17 (Burnside's Theorem)

Let G be a group, and let P be a p-Sylow subgroup of G. If N(P) = C(P), then P has a normal complement in G.

Proof. From Theorem 2.16, it follows that the intersection of Ker(T) and P is the identity. Also, by Theorem 2.16, the transfer is onto P. Thus, by an isomorphism theorem (Theorem 1.3), the factor group G/Ker(T) is isomorphic to P. Thus G = Ker(T)P, and therefore Ker(T) is a normal complement of P.

$A_4 \otimes Z_2$ (Appendix A) can also be used to illustrate this theorem. Let the p-Sylow subgroup P be any one of the subgroups $\{1,2,4\}$, $\{1,10,17\}$, $\{1,11,21\}$, or $\{1,15,24\}$. Each of these has its normalizer equal to its centralizer, and each has the subgroup $\{1,3,9,12,14,16,18,20\}$ as its normal complement.

CHAPTER III

CONCLUSION

The transfer is only a small part of group theory, and this thesis hardly reveals anything about the transfer compared to what is still unknown about it.

SUMMARY

In this thesis, a brief discussion of commutators and commutator subgroups provided enough background information to develop the transfer. The transfer was developed starting with the definition. From the definition a few elementary theorems followed. The theorems presented next were intended to characterize the transfer, and to give the reader a better understanding of it. Chapter II was concluded by Burnside's Theorem. Burnside's Theorem is the most important theorem of this thesis and is one of the major theorems of group theory.

SUGGESTIONS FOR FURTHER STUDY

There remains a great deal to be studied about transfers. As defined, the transfer is an into mapping. However, in many instances the transfer is onto. Determining when the transfer is onto would relate the transfer to the structure of groups. Also, a deeper study of the transfer could reveal an easier way of computing it.

In preparing this thesis, other possible topics of study have come to attention.

1) How are the transfer and Burnside's Theorem related to the solvability of groups?

2) Can the class equation be used in connection with the transfer?

3) How can the transfer be used in the study of Abelian groups?

4) What is the importance of commutators and commutator subgroups in the study of groups?

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APPENDIX

APPENDIX A

THE TRANSFERS OF $A_4 \otimes Z_2$

Seven of the transfers of $A_4 \otimes Z_2$ are presented as an aid to the reader. These transfers provide examples to theorems 2.4, 2.11, 2.12, 2.14, and 2.17.

In connection with Theorem 2.13, the conjugate classes of $A_4 \otimes Z_2$ are $\{2,10,21,24\}$; $\{4,11,15,17\}$; $\{5,7,22,23\}$; $\{6,8,13,19\}$; $\{9,14,20\}$; and $\{12,16,18\}$. The center is $\{1,3\}$.

For the transfer onto $\{1,2,4\}$ the set $S = \{1,3,7,8,9,10,11,12\}$ is used as the set of coset representatives. When $y = 1$, $T(y) = 1$, and the permutation of S is the identity permutation. For $y = 2$, $T(y) = 4$, and (1 8 9)(10 11 12) is the permutation.

	x_i	Ax_i	h_i	Permutation
$y = 2$	1	1	2	
	3	3	2	
	7	8	1	
	8	9	1	(7 8 9)(10 11 12)
	9	7	1	
	10	11	1	
	11	12	1	
	12	10	1	

$$T: A_4 \otimes Z_2 \longrightarrow \{1,2,4\}$$

y	T(y)	Permutation
1	1	I
2	4	(7 8 9)(10 11 12)
3	1	(1 3)(7 10)(8 11)(9 12)
4	2	(7 9 8)(10 12 11)
5	4	(1 3)(7 11 9 10 8 12)
6	2	(1 3)(7 12 8 10 9 11)
7	4	(1 7 12 3 10 9)(8 11)
8	2	(1 8 12 3 11 9)(7 10)
9	1	(1 9)(3 12)(7 11)(8 10)
10	4	(1 10 12)(3 7 9)
11	2	(1 11 12)(3 8 9)
12	1	(1 12)(3 9)(7 8)(10 11)
13	2	(1 7 11 3 10 8)(9 12)
14	1	(1 7)(3 10)(8 12)(9 11)
15	2	(1 10 11)(3 7 8)
16	1	(1 10)(3 7)(8 9)(11 12)
17	2	(1 12 10)(3 9 7)
18	1	(1 11)(3 8)(7 9)(10 12)
19	2	(1 9 10 3 12 7)(8 11)
20	1	(1 8)(3 11)(7 12)(9 10)
21	4	(1 12 11)(3 9 8)
22	4	(1 9 11 3 12 8)(7 10)
23	4	(1 8 10 3 11 7)(9 12)
24	4	(1 11 10)(3 8 7)

$T: A_4 \otimes Z_2 \longrightarrow \{1,2,4\}$

Image of T	Ker(T)
$\{1,2,4\}$	$\{1,3,9,12,14,16,18,20\}$

$T: A_4 \otimes Z_2 \longrightarrow \{1,10,17\}$

Image of T	Ker(T)
$\{1,10,17\}$	$\{1,3,9,12,14,16,18,20\}$

$T: A_4 \otimes Z_2 \longrightarrow \{1,11,21\}$

Image of T	Ker(T)
$\{1,11,21\}$	$\{1,3,9,12,14,16,18,20\}$

$T: A_4 \otimes Z_2 \longrightarrow \{1,15,24\}$

Image of T	Ker(T)
$\{1,15,24\}$	$\{1,3,9,12,14,16,18,20\}$

$T: A_4 \otimes Z_2 \longrightarrow \{1,12,16,18\}$

Image of T	Ker(T)
$\{1\}$	$A_4 \otimes Z_2$

$T: A_4 \otimes Z_2 \longrightarrow \{1,3,9,12,14,16,18,20\}$

Image of T	Ker(T)
$\{1,3\}$	$\{1,2,4,10,11,12,15,16,17,18,21,24\}$

$T: A_4 \otimes Z_2 \longrightarrow \{1,2,4,10,11,12,15,16,17,18,21,24\}$

Image of T	Ker(T)
$\{H^1, 2H^1, 4H^1\}$, where $H^1 = \{1,12,16,18\}$	$\{1,3,9,12,14,16,18,20\}$

The Multiplication Table of $A_4 \otimes Z_2$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	2	4	5	1	6	3	8	9	7	11	12	10	20	23	18	24	16	21	14	22	15	13	19	17
3	3	5	1	6	2	4	10	11	12	7	8	9	15	16	13	14	19	20	17	18	22	21	24	23
4	4	1	6	2	3	5	9	7	8	12	10	11	22	19	21	17	24	15	23	13	18	20	14	16
5	5	6	2	3	4	1	11	12	10	8	9	7	18	24	20	23	14	22	16	21	13	15	17	19
6	6	3	4	5	1	2	12	10	11	9	7	8	21	17	22	19	23	13	24	15	20	18	16	14
7	7	13	10	14	15	16	17	18	2	19	20	5	12	21	9	22	3	23	1	24	6	4	11	3
8	8	20	11	23	18	24	16	21	4	14	22	6	10	15	7	13	5	19	2	17	3	1	12	9
9	9	22	12	19	21	17	24	15	1	23	13	3	11	18	8	20	6	14	4	16	5	2	10	7
10	10	15	7	16	13	14	19	20	5	17	18	2	9	22	12	21	1	24	3	23	4	6	8	11
11	11	18	8	24	20	23	14	22	6	16	21	4	7	13	10	15	2	17	5	19	1	3	9	12
12	12	21	9	17	22	19	23	12	3	24	15	1	8	20	11	18	4	16	6	14	2	5	7	10
13	13	14	15	7	16	10	18	2	17	20	5	19	24	11	23	8	22	6	21	4	9	12	1	3
14	14	7	16	13	10	15	2	17	18	5	19	20	4	1	6	3	8	9	11	12	23	24	21	22
15	15	16	13	10	14	7	20	5	19	18	2	17	23	8	24	11	21	4	22	6	12	9	3	1
16	16	10	14	15	7	13	5	19	20	2	17	18	6	3	4	1	11	12	8	9	24	23	22	21
17	17	12	19	21	9	22	3	23	13	1	24	15	5	6	2	4	10	11	7	8	16	14	20	18
18	18	24	20	11	23	8	22	6	14	21	4	16	19	9	17	12	15	1	13	3	10	7	5	2
19	19	9	17	22	12	21	1	24	15	3	23	13	2	4	5	6	7	8	10	11	14	16	18	20
20	20	23	18	8	24	11	21	4	16	22	6	14	17	12	19	9	13	3	15	1	7	10	2	5
21	21	17	22	12	19	9	13	3	23	15	1	24	14	7	16	10	18	2	20	5	11	8	6	4
22	22	19	21	9	17	12	15	1	24	13	3	23	16	10	14	7	20	5	18	2	8	11	4	6
23	23	8	24	20	11	18	4	16	21	6	14	22	1	2	3	5	9	7	12	10	19	17	15	13
24	24	11	23	18	8	20	6	14	22	4	16	21	3	5	1	2	12	10	9	7	17	19	13	15

Lattice of $A_4 \otimes Z_2$