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BY

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I

Numbers Simultaneously Polygonal

by

L. B. Wade Anderson, Jr. *

For the purpose of this study, the term, *polygonal number,* will refer only to positive integers and is defined as follows: let $\{a_k\}$ be an arithmetic sequence whose first term is 1 and whose common difference is $m-2$, where m is a positive integer greater than 2. The sequence of partial sums, $\{s_r\}$, associated with $\{a_k\}$ is called a sequence of m-gonal numbers or the sequence of polygonal numbers with m sides. For example, when $m = 3$, the arithmetic sequence to be considered is $\{a_k\} = \{1, 2, 3, \ldots, k, \ldots\}$, and the associated sequence in this case is $\{s_r\} = \{1, 3, 6, \ldots, r(r+1)/2, \ldots\}$ This is the sequence of 3-gonal (triangular) numbers. For simplicity, the rth term of the sequence of m-gonal numbers will be denoted by p_m^r. Table I is a general listing of p_m^r. Table I was obtained using the following well-known formulas for arithmetic sequences and series: $a_k = 1 + (k-1)(m-2)$ and $s = (r/2)(2 + (r-1)(m-2))$.

Historically, the numbers were named polygonal because they can describe, for a given m, a nest of regular polygons of m sides having a common vertex and with $r = 1, 2, 3, \ldots$ points for each side. The diagrams shown below in Figure I illustrate polygons which are representative of the first four triangular, square, and pentagonal numbers.

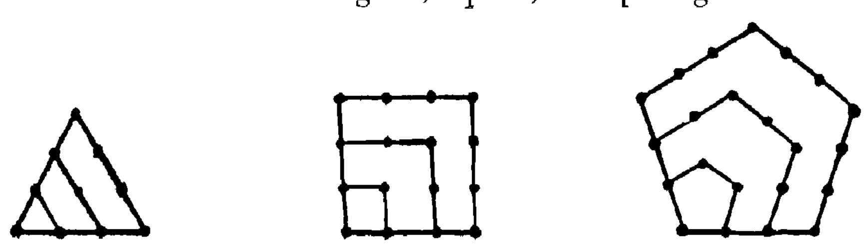

FIGURE I

POLYGONS ILLUSTRATING THE FIRST FOUR TRIANGULAR, SQUARE, AND PENTAGONAL NUMBERS

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TABLE I

POLYGONAL NUMBERS

	1	2	3	4	5	6	...	r
3-gonal	1	3	6	10	15	21	...	$\frac{r(r+1)}{2}$
4-gonal	1	4	9	16	25	36	...	r^2
5-gonal	1	5	12	22	35	51	...	$\frac{r(3r-1)}{2}$
6-gonal	1	6	15	28	45	66	...	$r(2r-1)$
7-gonal	1	7	18	34	55	81	...	$\frac{r(5r-3)}{2}$
8-gonal	1	8	21	40	65	96	...	$r(3r-2)$
9-gonal	1	9	24	46	75	111	...	$\frac{r(7r-5)}{2}$
10-gonal	1	10	27	52	85	126	...	$r(4r-3)$
.	.	.	.	.	.	.	...	.
.	.	.	.	.	.	.	...	.
.	.	.	.	.	.	.	...	.
m-gonal	1	m	3m-3	6m-8	10m-15	15m-24	...	$\frac{r^2(m-2)-r(m-4)}{2}$
.	.	.	.	.	.	.	...	.
.	.	.	.	.	.	.	...	.
.	.	.	.	.	.	.	...	.

If P denotes the set of all polygonal numbers, it is apparent that P is the set of all positive integers except 2. An integer, w, will be called simultaneously polygonal if, and only if, there exist integers r and q such that for distinct integers m and n it is true that $w = p_m^r = p_n^q$. Let the set of all simultaneously polygonal numbers be denoted by P^2. The following facts immediately present themselves: (1) P^2 is a proper subset of P. (2) $1 \epsilon P^2$, since for all $m > 2$, $p_m^1 = 1$. (3) If $w \epsilon P^2$ then exactly one of the following hold: (i) $r = q = 1$ or (ii) $r \neq q$, where $w = p_m^r = p_n^q$. An investigation of the possible ways a given number may be polygonal helps to determine the nature of the set P^2. Let w be any integer. If w is the rth m-gonal number, then $w = p_m^r = (r/2)[2 + (r-1)(m-2)]$ and hence $2w = r[2 + (r-1(m-2)]$. Now, if any·given w is to be polygonal, then 2 w will have to be expressed as a product of two factors. One of these factors is r, and the other is $2 + (r-1)(m-2)$. The following theorem shows that r must be the smaller factor.

Theorem 1: If $w = p_m^r$, then $r < 2 + (r-1)(m-2)$.

Proof: By definition, $m - 2 \geq 1$. Hence, by multiplying each member of the inequality by $r - 1$, $(r-1)(m-2) \geq r-1$ is obtained. Adding 2 to both members yields $(r-1)(m-2) + 2 \geq r + 1$. And it follows that $r < 2 + (r-1)(m-2)$. QED

It is now clear that the smaller factor of 2 w is r, and that by subtracting 2 from the larger factor, the product $(r-1)(m-2)$ is obtained. Using the preceding fact, m is easily determined. As an example of this method, the problem of deciding the number of ways 36 may be polygonal is examined. The first step is to express 2 x 36 as a product of two factors in all possible ways:

$$2 \times 36 = 3 \times 24 = 4 \times 18 = 6 \times 12 = 8 \times 9$$

The first factorization, 2 x 36, is then considered. Since r must correspond to the smaller factor, $r = 2$. By subtracting 2 from the larger factor, 34 is obtained. Hence, $(r-1)(m-2)$ must be 34 in this case, and m is, therefore, 36. Thus, this factorization indicates that $36 = p_{36}^2$. Similarly, the factorization 3 x 24 indicates that $r = 3$ and $(r-1)(m-2) = 22$, which implies that $m = 13$. From this factorization, it is concluded that $36 = p_{13}^3$. Not all factorizations are indicative of a manner in which 36 is polygonal. It will be noticed that the factorization, 4 x 18, shows that 36 is never the 4th element of an m-gonal sequence; since in this case $r = 4$, $(r-1)(m-2) = 16$, and since $3 = r - 1$ does not divide 16, there can be no integral value for $m - 2$ and, hence, no value for m. Table II indicates all of the ways in which 36 is polygonal. Hence, 36 is polygonal in exactly four ways. It is possible that some factorizations can be elminated from consideration. The following theorem indicates that consideration need only be given those factorization of 2 x w, where the smallest factor is less than or equal to $\frac{1}{2}(\sqrt{8w+1}-1)$.

Theorem 2: If $w = p^r_m$, then $r \leq \frac{1}{2}(\sqrt{8w+1}-1)$.

Proof: If $w = (r/2)[2 + (r-1)(m-2)]$, then solving for $m-2$, $m-2 = \frac{2(w-r)}{r(r-1)}$. But, also, by definition $m-2 \geq 1$. So, $\frac{2(w-r)}{r(r-1)} \geq 1$ or $2(w-r) \geq r(r-1)$. Hence $w \geq \frac{r(r-1)}{2} + r$ or $2w \geq r^2 + r$ and completing the square $8w + 1 \geq 4r^2 + 4r + 1 = (2r+1)^2$. Thus, $\sqrt{8w+1} \geq 2r+1$; and therefore, $\frac{1}{2}(\sqrt{8w+1}-1) \geq r$. QED

A number that is not an element of P^2 is 26. This is apparent, since $2 \times 26 = 4 \times 13$ are the only factorizations of 2×26. The first factorization shows that $26 = p^2_{26}$, but since 3 does not divide 11, this is the only way 26 is polygonal. It also follows that, if w is any prime, then the only factorization of $2 \times w$ is $2 \times w$ and, hence, $w = p^2_w$. This is the only way w is polygonal. Thus, P^2 contains no primes. Furthermore, the above method reveals that there are twenty-seven composites less than 150 that are not elements of P^2: 4, 8, 14, 20, 26, 32, 38, 44, 50, 56, 62, 68, 74, 77, 80, 86, 98, 110, 116, 119, 122, 125, 128, 134, 140, 143, and 146. The following conjecture seems appropriate at this point:

Conjecture: With the exception of 4, there does not exist a composite integer that is not an element of P^2 that is not congruent to 2 modulo 3. In partial support of his conjecure is the following theorm:

Theorem 3: If w is a composite and is congruent to 0 modulo 3, then w is an element of P^2.

Proof: If $w \equiv 0 \pmod 3$, then there exists a positive integer k such that $w = 3k$ and then $2w = 2(3k)$. Factorizations of 2w include $2(3k)$ and $3(2k)$. The first factorization implies $r = 2$ and $(r-1)(m-2) = 3k-2$ and hence $m-2 = 3k-2$ which implies $m = 3k = w$ or $w = p^2_w$. The second factorization implies $r = 3$ and $(r-1)(m-2) = 2k-2$ and hence $m-2 = k-1$ so that $m = k+1$ and $w = p^3_{k+1}$. Therefore, $w \in p^2$. QED

TABLE II

THE WAYS 36 IS POLYGONAL

Factorization	r	m	Corresponding polygonal number
2 x 36	2	36	p^{2}_{36}
3 x 24	3	13	p^{3}_{13}
4 x 18	not possible for 36 to be 4th m-gonal		
6 x 12	6	4	p^{6}_{4}
8 x 9	8	3	p^{8}_{3}

II

Numbers Both M-Gonal and N-Gonal

This section deals with the determination of integers w such that, for specific values of m and n, there exist integers r and q such that $w = p^r_m = p^q_n$. Although a general treatment of this question may be considered, it necessarily becomes quite involved, and when specific instances are treated, the method will vary somewhat to facilitate brevity. This general approach could, however, be followed in all cases to be considered. From Table I, if $p^r_m = p^q_n$ then $\frac{1}{2}((m-2)r^2-(m-4)r) = \frac{1}{2}((n-2)q^2-(n-4)q)$. Let $a = m-2$ and $b = n-2$. Then, $\frac{1}{2}(ar^2-(a-2)r) = \frac{1}{2}(bq^2-(b-2)q)$, or multiplying by 8a and completing the square the equation becomes: $(2ar-(a-2))^2 = 4abq^2-4a(b-2)q + (a-2)^2$. Upon multiplying by ab and completing the square on q, the following is obtained: (1) $ab(2ar-(a-2))^2 + a^2(b-2)^2 = (2abq-a(b-2))^2 + ab(a-2)^2$. Let $y = 2ar-a+2$, $x = 2abq-a(b-2)$, and $C = a^2(b-2)^2-ab(a-2)^2$. Now, (1) may be written as (2) $x^2-aby^2 = C$.

Hence, the problem is reduced to finding all integral solutions of (2). It is noteworthy that if r, m, q, and n are integral, then x and y must be integers, but that the converse is not true. That is, integral solutions (x, y) of $x^2-aby^2 = C$ will not necessarily indicate an integral solution (r, q) of $p^r_m = p^q_n$. It is also noted that $x = ab+2a$, $y = a+2$ is always a solution of (2) since $r = 1$, $q = 1$ is always a solution for $p^r_m = p^q_n$. This will be called the trivial solution.

To find all solutions of (2), two cases must be considered. First, if ab is a perfect square and $ab = k^2$ for some integer k, then (2) becomes $(x+ky)(x-ky) = C$. Without loss of generality, C is assumed to be positive, for if it is not, (2) may be rewritten as $(ky+x)(ky-x) = -C$. The following theorem will now be established:

Theorem 4: If $ab = k^2$ for some integer k, where $a = m-2$ and $b = n-2$, then there are at most a finite number of solutions (r, q) such that $p^r_m = p^q_n$.

Proof: In equation (2) above $x = abq-a(b-2)$ and $y = 2ar-a+2$. A solution $x > 0$, $ky > 0$ of $C = x^2-(ky)^2 = (x+ky)(x-ky)$ implies a factorization of C in the form $C = de$ where $d = x+ky$ and $e = x-ky$. Hence $d+e = 2x$ and $d-e = 2ky$. It follows that $d \equiv e \pmod 2$. Conversely, there is a solution $x = (d+e)/2$ and $ky = (d-e)/2$.

(i) Since C is the difference of two squares, $C \not\equiv 2 \pmod 4$.

(ii) If $C \equiv 1 \pmod 4$ or $C \equiv 3 \pmod 4$, then C is odd, and both de and e must be odd, so that $d \equiv e \pmod 2$ will be satisfied. If C is not a square, then every factorization of C implies $d \neq e$. There are $\tau(C)$ choices for d where $\tau(C)$ is the number of divisors of C. $\tau(C)$ is even and there are exactly $\tau(C)/2$ choices for $d > e > 0$. If C is a square, there is one and only one factorization of $C = de$ in which $d = e$, which would not lead to a solution. In this case, $\tau(C)$ is odd, and number of solutions (x, ky) is $(\tau(C) - 1)/2$.

(iii) If $C \equiv O \pmod 4$, then C is even, and, hence, d and e must both be even. Let $d = 2D$ and $e = 2E$. Hence, $C/4 = DE$ where $D > E > 0$ and the number of solutions depends exactly on the number of factorizations of $C/4 = DE$. Proceeding as in case (ii), the number of solutions is $\tau(C/4)/2$. QED

If, however, ab is not a square, the following propositions will be needed to find all solutions of (2). The proofs of these results can be found in elementary number theory texts and will not be included, here.

Definition 1: If D is a natural number, not a perfect square, and if (x_1, y_1) is a solution of $x^2 - Dy^2 = 1$, then (x_1, y_1) is a fundamental solution if, and only if, $x_1 > \frac{1}{2}y_1^2 - 1$.

Theorem 5: The fundamental solution of $x^2 - Dy^2 = 1$, where D is not a perfect square, is unique. That is, there is only one solution (x_1, y_1) that satisfies the inequality $x_1 > \frac{1}{2}y_1^2 - 1$.

Theorem 6: If (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = 1$, where D is a natural number, and not a perfect square, then all positive solutions are given by (x_n, y_n) where $x_n + \sqrt{D}\,y_n = (x_1 + \sqrt{D}\,y_1)^n$ for $n = 1, 2, 3, \ldots$.

Theorem 7: If D is a natural number and if $x^2 - Dy^2 = N$ has one solution, then it has infinitely many. In particular, if (u_1, v_1) is a solution of $u^2 - Dv^2 = 1$ and (x_1, y_1) is a solution of $x^2 - Dy^2 = N$, integers x and y determined by $x + y\sqrt{D} = (u_1 + v_1\sqrt{D})(x_1 + y_1\sqrt{D})$ form a solution of $x^2 - Dy^2 = N$.

Examining the Pellian equation $u^2 - Dv^2 = 1$ with (u, v) any solution of the equation and with (x_1, y_1) any solution of $x^2 - Dy_2 = N$, then according to Theorem 7, integers x_2 and y_2 will also be a solution where $x_2 + \sqrt{D}\,y_2 = (u + \sqrt{D}\,v)(x_1 + \sqrt{D}\,y_1)$. The solution (x_2, y_2) is said to be associated with the solution (x_1, y_1). The set of all associated solutions forms a class of solutions. Since there are infinitely many solutions for the Pellian equation, each class will contain infinitely many solutions for $x^2 - Dy^2 = N$. It is possible to tell whether two given solutions (x_i, y_i) and (x_j, y_j) belong to the same class. The necessary and sufficient condition for the two to be associated is that $(x_i x_j - y_i y_j D)/N$ and $(y_i x_j - x_i y_j)/N$ be integers. If S is the class consisting of the solutions (x_i, y_i), then solutions

$(x_1, -y_1)$ also constitute a class which is usually denoted by $\bar{S}$. S and $\bar{S}$ are called conjugate classes and may be distinct or coincide. In the latter case, they are called ambiguous classes. Among all solutions (x, y) in a given class, the fundamental solution is chosen in the following manner: if y_1 is the least non-negative value of y which occurs in S and if S is not ambiguous, then the number x_1 is also determined, for the solution $(-x_1, y_1)$ belongs to the conjugate class $\bar{S}$. If S is ambiguous, a unique x_1 may obtained by prescribing that $x_1 > 0$. In the fundamental solution, the number $|x_1|$ has the least value which is possible for $|x|$ when (x, y) is an element of S. The case $x = 0$ can only occur when the class is ambiguous, and similarly for the case $y = 0$.

Theorem 8: If S is a class of solutions for the equation $x^2 - Dy^2 = N$ where N is a positive integer with (x, y) the fundamental solution of the class S and with (u_1, v_1) the fundamental solution of $u^2 - Dv^2 = 1$; then

(3) $0 \leqq y \leqq (v_1 \sqrt{N}) / \sqrt{2(u_1 + 1)}$ and

(4) $0 < |x| \leqq \sqrt{\tfrac{1}{2}(u_1 + 1) N}$.

If N is a negative integer, $N = -M$. Now inequalities (3) and (4) become

(5) $0 < y \leqq (v_1 \sqrt{M}) / \sqrt{2(u_1 - 1)}$ and

(6) $0 \leqq |x| \leqq \sqrt{\tfrac{1}{2}(u_1 - 1) M}$.

It is now clear from the preceeding theorems that if ab and C are natural numbers and if ab is not a perfect square, the equations, $x^2 - aby^2 = C$ and $x^2 - aby^2 = -C$, have a finite number of classes of solutions. The fundamental solutions of all classes can be found after a finite number of trials by means of the inequalities (3) and (4) or (5) and (6). If (x_1, y_1) is the fundamental solution of the class S, all the solutions (x, y) of S may be obtained from

(7) $x + y\sqrt{ab} = (x_1 + y_1 \sqrt{ab})(u + v\sqrt{ab})$

where (u, v) run through all the solutions of $u^2 - abv^2 = 1$. When an equation has no solutions satisfying the above inequalities, it has no solutions at all.

III

Triangular Numbers

The methods developed in section I are used in this present section to determine the nature of integers that are both triangular and m-gonal for specific values of $m \neq 3$.

The question to be treated initially concerns solutions for $p_4^r = p_3^q$. Here, $m = 4$ and $n = 3$. Thus, equation (2) becomes $x^2 - 2y^2 = 4$ where $x = 4q + 2$ and $y = 4r$. To facilitate solutions, the equivalent equation $Z^2 - 8r^2 = 1$ where $Z = x/2 = 2q + 1$ will be considered. According to Theorem 6, all positive solutions of the above equation are given by $Z + \sqrt{8}\,r = (Z_1 + \sqrt{8}\,r_1)^n, n = 1, 2, 3, \ldots$ where (Z_1, r_1) is the fundamental solution. This solution is readily determined by trial to be $Z = 3$, $r = 1$ which corresponds to $p_4^1 = p_3^1 = 1$. Hence, all solutions are given by

(8) $Z + \sqrt{8}\,r = (3 + \sqrt{8})^n, n = 1, 2, 3, \ldots$

A listing of the first ten solutions (r, q) and the corresponding polygonal number is given in Table III. The following recursion formula may be derived to further simplify the problem of finding solutions: if (Z_n, r_n) is any solution obtained by (8), then $(Z_n + r_n\sqrt{8})(3 + \sqrt{8}) = 3Z_n + 8r_n + (Z_n + 3r_n)\sqrt{8}$. Thus, the solution (Z_{n+1}, r_{n+1}) is given as

(9) $Z_{n+1} = 3Z_n + 8r_n$ and

(10) $r_{n+1} = Z_n + 3r_n$.

Solving equation (10) for Z_n and substituting the expression in equation (9) yields $Z_n = 3r_n - r_{n-1}$. Now, by replacing Z_n in (10), $r_{n+1} = 6r_n - r_{n-1}$. A similar procedure gives $Z_{n+1} = 6Z_n - Z_{n-1}$.

Theorem 9: All solutions of $p_4^r = p_3^q$ may be determined by $r_n = 6r_{n-1} - r_{n-2}$ and $q_n = 6q_{n-1} - q_{n-2} + 2$ where $(r_1, q_1) = (1, 1)$ and $(r_2, q_2) = (6, 8)$.

This follows from above derived formulas and the fact that $Z = 2q + 1$.

Theorem 10: If $p_4^r = p_3^q$ has solution (r_n, q_n) and the next larger solution is (r_{n+1}, q_{n+1}), then $r_n + q_n = q_{n+1} - r_{n+1}$.

It will be necessary to present the following lemma before the proof of Theorem 10 can be established.

Lemma: If $p_4^r = p_3^q$, then $q_n = (f_n + e_n - 2)/4$ and $r_n = (f_n - e_n)/(4\sqrt{2})$, where $f_n = (3 + \sqrt{8})^n$ and $e_n = (3 - \sqrt{8})^n$ for $n = 1, 2, 3, \ldots$.

TABLE III

TRIANGULAR NUMBERS THAT ARE SQUARES

$p_4^r = p_3^q$

r	q	$p_4^r = p_3^q$
1	1	1
6	8	36
35	49	1,225
204	288	41,616
1,189	1,681	1,413,721
6,930	9,800	48,024,900
40,391	57,121	1,631,432,881
235,416	332,928	55,420,693,056
1,372,105	1,940,449	1,882,672,131,025
7,997,214	11,309,768	63,955,431,761,796

Proof: If $Z = 2q + 1$, all solutions may be obtained from $Z + \sqrt{8}r = (3 + \sqrt{8})^n$. Now, $Z + \sqrt{8}\,r = f$, and $Z - \sqrt{8}\,r = e$. Eliminating r, $Z = (f + e)/2$ or $q = (f + e - 2)/4$. By eliminating Z, $r = (f - e)/(4\sqrt{2})$ is obtained. Hence, the lemma is proved.

This lemma allows the following proof for Theorem 10.
Proof: If $f_n = (3 + \sqrt{8})^n$ and $f_{n+1} = (3 + \sqrt{8})^{n+1}$ and e_n and e_{n+1} are defined in a similar fashion, then by the above lemma $q_n = (f_n + e_n - 2)/2$, $r_n = (f_n - e_n)/(4\sqrt{2})$, $q_{n+1} = (f_{n+1} + e_{n+1} - 2)/2$ and $r_{n+1} = (f_{n+1} + e_{n+1})/(4\sqrt{2})$. The theorem now follows from the fact that $f_{n+1} = (3 + \sqrt{8})f_n$ and $e_{n+1} = (3 - \sqrt{8})e_n$. QED

Now, investigations will be directed toward solutions for $p_5^r = p_3^q$ Here, m = 5, n = 3 and, hence, a = 3 and b = 1. Thus, equation (2) becomes $x^2 - 3y^2 = 6$ where $x = 6q + 3$ and $y = 6r - 1$. The equation may be simplified and rewritten as

(11) $y^2 - 3Z^2 = -2$ where $Z = 2q + 1$.

TABLE IV

TRIANGULAR NUMBERS THAT ARE PENTAGONAL

n	(y, Z)	(r, q)	$p_5^r = p_3^q$
1	(5, 3)	(1,1)	1
2	(19,11)	-	-
3	(71,41)	(12,20)	210
4	(265,153)	-	-
5	(989,571)	(165,285)	40,755
6	(3691,2131)	-	-
7	(13775,7953)	(2296,3976)	7,906,276
8	(51409,29681)	-	-
9	(191891,110781)	(31982,55391)	1,534,109,136
.	.	.	.
.	.	.	.
.	.	.	.

NOTE: Solutions (u_n, v_n) are determined by $u_n + v_n\sqrt{3} = (2 + \sqrt{3})^n$ Then, solutions (y, Z) are determined by $y + Z\sqrt{3} = (1 + \sqrt{3})(u_n + v_n\sqrt{3})$. Thus $y = 6r - 1$ and $Z = 2q + 1$ yield solutions (r, q).

The fundamental solution of $u^2 - 3v^2 = 1$ is obtained by trial and is found to be (2, 1). Possible classes of solutions for (11) are determined by its fundamental solutions. These fundamental solutions are found by applying inequalities (5) and (6). Here, $M = 3$, $v_1 = 1$, and $u_1 = 2$. Thus, possibilities for fundamental solutions are $0 < Z \leqq \sqrt{2}/\sqrt{2} = 1$ and $0 \leqq |y| \leqq \sqrt{3}/\sqrt{2}$ or (0, 0), (0, 1), (1, 0), (1, 1). Of these possibilities, only (1, 1) is a solution for (11), and, therefore, there is just one fundamental solution and one class of solutions. All solutions may be determined by (12) $y + Z\sqrt{3} = (1 + \sqrt{3})(u + v\sqrt{3})$ where (u, v) run through all the solutions of $u^2 - 3v^2 = 1$. Once again, not all solutions (y, Z) lead to solution (r, q). A listing of the first five solutions (r, q) appears in Table IV.

The following theorem is useful when solutions (r, q) of $p_5^r = p_3^q$ are required:

Theorem 11: If (u_n, v_n) is a solution of $u^2 - 3v^2 = 1$, then (u_{n+1}, v_{n+1}) will yield a solution (r, q) of $p_5^r = p_3^q$ if, and only if: (i) when $9v_n \equiv 0 \pmod 6$, then $u_n \equiv 1$ (mode 6), or (ii) when $9v_n \equiv 3 \pmod 6$, then $u_n \equiv 4 \pmod 6$.

Proof: According to Theorem 6, $u_{n+1} + v_{n+1}\sqrt{3} = (u_n + \sqrt{3}v_n)(2 + \sqrt{3})$ or $u_{n+1} = 2u_n + 3v_n$ and $v_{n+1} = u_n + 2v_n$. By equation (12) $y_{n+1} + Z_{n+1}\sqrt{3} = (1 + \sqrt{3})(u_{n+1} + \sqrt{3}v_{n+1})$, and, thus, $(6r - 1) + (2q + 1)\sqrt{3} = (5u_n + 9v_n) + (3u_n + 5v_n)\sqrt{3}$. Hence, $r = (5u_n + 9v_n + 1)/6$ and $q = (3u_n + 5v_n - 1)/2$. If r is to be an integer, then $5u_n + 9v_n$ must be congruent to 5 modulo 6. Any multiple of 9 must be congruent to 0 modulo 6 or 3 modulo 6. In the first case, $5u_n$ must be congruent to 5 modulo 6 which implies $u_n \equiv 1 \pmod 6$, and in the later case, $5u_n$ is necessarily congruent to 2 modulo 6 which implies $u_n \equiv 4 \pmod 6$. These conditions are also sufficient for q to be integral. Conversely, if $9v_n \equiv 0 \pmod 6$ and $u_n \equiv 1 \pmod 6$ or $9v_n \equiv 3 \pmod 6$ and $u_n \equiv 4 \pmod 6$, a solution (r, q) is obvious. QED

Determining solutions of $p_6^r = p_3^q$ is a simple matter since equation (2) becomes $x^2 - 4y^2 = 0$ where $x = 8q + 4$ and $y = 8r - 2$. Thus, $q = 2r - 1$ and solutions are obtained. The next theorem follows from the above solution.

Theorem 12: If for an integer, $w > 0$, there exists an r such that $w = p_6^r$, then there exists a q such that $w = p_3^q$.

The converse is obviously not true.

As a final example of triangular numbers that are simultaneously polygonal, the problem of finding solutions for $p_{11}^r = p_3^q$ is treated. For this case, equation (2) becomes: (13) $x^2 - 9y^2 = -360$ where $x = 9(2q + 1)$ and $y = 18r - 7$. In this example. $ab = 9$ is a perfect square and thus is indicative of a finite number of solutions. The exact number of solutions (3y, x) determined by $(3y + x)(3y - x) = 360$ is given by Theorem 4 to be $\tau(360/4)/2 = 6$. These solutions are easily obtained from the six factorizations of 360 where both factors are even and are the following: (91, 89), (47, 43), (33, 27), (23, 13), (21, 9), and (24, 1). Of the above solutions for (13), only $3y = 33$ and $x = 27$ lead to integral values of r and q. This solution implies $r = 1$ and $q = 1$.

Theorem 13: The only triangular number that is 11-gonal is 1.

IV

Square Numbers

The problem of finding numbers that are square and triangular was treated in section III. In this next section, the determination of squares that are polygonal in another specific manner will be the object of investigation.

Finding solutions (r, q) for $p_5^r = p_4^q$ by direct substitution in equation (2) indicates that m = 5, a = 3, n = 4, b = 2, $C = a^2 (b-2)^2 - ab(a-2)^2 = -6$, ab = 6, and, thus, all solutions of $x^2 - 6y^2 = -6$ must be examined. This equation may be simplified somewhat, since $x = 2abq - a(b-2) = 12q$ and $(12q)^2 - 6y^2 = -6$ is equivalent to $y^2 - 24q^2 = 1$: Here $y = 2ar - a + 2 = 6r - 1$. According to Theorem 7, all solutions of $y^2 - 24q^2 = 1$ are given by $y_n + \sqrt{24}\, q_n = (y_1 + \sqrt{24}\, q_1)^n$ for n = 1, 2, . . . where (y_1, q_1) is the fundamental solution of $y^2 - 24q^2 = 1$. By trial, (y_1, q_1) is found to be (5, 1). Hence, all solutions are given by $y_n + \sqrt{24}\, q_n = (5 + \sqrt{24})^n$ for n = 1, 2, ... Table V shows the first nine solutions for the above equation and the corresponding integrals values of r and q. It is, once again, noted that only values of y which yield integral values for $r = \frac{y+1}{6}$ will be indicative of solutions for $p_5^r = p_4^q$.

As a final illustration of the method, all solutions (r, q) of $p_7^r = p_4^q$ will be considered. This case differs from the preceding examples in that there are two classes of solutions. Here, a = 5, b = 2, $y = 10r - 3$, $x = 20q$ and $C = -90$. By direct substitution equation (2) becomes $x^2 - 10y^2 = -90$ or $400q^2 - 10y^2 = -90$ or equivalently $y^2 - 40q^2 = 9$. The fundamental solution of $u^2 - 40v^2 = 1$ is found by trial to be (19, 3), and, thus, according to inequalities (4) and (5), possible fundamental solutions (y, q) for $y^2 - 40q^2 = 9$ must satisfy $0 \leqq q \leqq (9/2\sqrt{10}) < 2$ and $0 < |y| < 3\sqrt{10} < 10$. Thus, the only possible fundamental solutions (y, q) must have q = 0 or q = 1. If $q \leqq 0$, then a solution $(y_1, q_1) = (3, 0)$. If q = 1, a solution is $(y_2, q_2) = (7, 1)$. To see that there are indeed two classes, the expression $\frac{y_1 y_2 - q_1 q_2 40}{9}$ must be examined. Since this expression is not integral, there are two classes of solutions. All solutions (y^n, q^n) associated with (3, 0) may be obtained from $y_n + q_n\sqrt{40} = 3$

$(u_n + v_n\sqrt{40})$ where (u_n, v_n) is a solution of $u^2 - 40v^2 = 1$. All solutions associated with (7, 1) may be obtained from $y_n + q_n\sqrt{40} = (7 + \sqrt{40})(u_n + v_n\sqrt{40})$. Table VI shows the first few solutions.

The following theorem identifies those m-gonal sequences that contain a finite number of squares.

Theorem 14: There are, at most, a finite number of solutions (r, q) for $p_m^r = p_4^q$ if m is of the form $m = 2k^2 - 2$ where k is an integer greater than 1.

Proof: Using equation (2) where b = 2, solutions for $p_m^r = p_4^q$ are given by $x^2 - 2ay^2 = -2a(a-2)^2$. but since b = 2, x =

TABLE V

SOME SOLUTIONS FOR $p_5^r = p_4^q$

n	$y = 6r - 1$	r	q	$p_5^r = p_4^q$
1	5	1	1	1
2	49	-	10	-
3	485	81	99	9,801
4	4,801	-	980	-
5	47,525	7,921	9,701	94,109,401
6	470,499	-	96,030	-
7	4,656,965	776,161	950,599	903,638,458,801
8	46,099,201	-	9,409,960	-
9	456,335,045	76,055,841	93,149,001	8,676,736,387,298,001

NOTE: Solutions (y, q) are determined by $y_n + \sqrt{24}q_n = (5 + \sqrt{24})^n$ and values for r are then obtained from $y = 6r - 1$.

4aq and the above equation can be rewritten as $(4aq)^2 - 2ay^2 = -2a(a-2)^2$ or $16a^2q^2 - 2ay^2 = -2a(a-2)^2$ or equivalently $y^2 - 8aq^2 = (a-2)^2$. According to Theorem 4, there can be at most a finite number of solutions if 8a is a perfect square. Now, 8a is a perfect only if a is the double of a perfect square. Thus, $m = 2k^2 + 2$ implies $a = m - 2k^2$. QED

TABLE VI

SOME SOLUTIONS FOR $p_7^r = p_4^q$

n	(u_n, v_n)	Associated solution	(y_n, q_n)	r_n	$p_7^r = p_4^q$
Fund.	soln.	(3,0)	-	-	-
Fund.	soln.	(7,1)	(7,1)	1	1
1	(19,3)	(3,0)	(57,9)	6	81
1	(19,3)	(7,1)	(253,9)	-	-
2	(721,114)	(3,0)	(2163,342)	-	-
2	(721,114)	(7,1)	(9607,1519)	961	923,561

NOTE: Solutions (u_n, v_n) of $u^2 - 40v^2 = 1$ are determined by $u_n + v_n\sqrt{40} = (19 + 3\sqrt{40})^n$ for $n = 1, 2, \ldots$. Solutions (y_n, q_n) are determined by $y_n + q_n\sqrt{40} = 3(u_n + v_n\sqrt{40})$ if associated solution is (3, 0) and $y_n + q_n\sqrt{40} = (7 + \sqrt{40})(u_n + v_n\sqrt{40})$ if associated solution is (7, 1). Values for r_n are given by $r_n = \frac{y_n + 3}{10}$.

V

Area For Further Study

The objective of this study has been to present a general method for finding numbers polygonal in more than one way. Section II presents such a method that allows the determination of values of r and q such that $p_m^r = p_n^q$ for given values of m and n. The substitution of these values of m and n in equation (2) results in an equation that may be solved, if possible, by finding the fundamental solutions of all classes of solutions through the use of inequalities (4) and (5) or (6) and (7). By examining solutions of these classes, one may determine the values of r and q. A few of the infinitely many theorems that concern particular types of simultaneously polygonal numbers have been stated and proved. There also seems to be no end to the number of available theorems concerning simultaneously polygonal numbers. Each particular pair of values for m and n leads to a multitude of these theorems.

A source of further study seems to lie in the nature of the set P^2. Also, the definition of P^n for n greater than two seems evident and the nature of these sets is completely unknown.