THE [illegible] DECIMAL EXPANSIONS OF REAL NUMBERS AS THEY APPLY TO ALGEBRAIC AND TRANSCENDENTAL NUMBERS

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CHAPTER I

INTRODUCTION

Transcendental numbers is a subject in which relatively little in the way of basic results has been accomplished. The existence of transcendental numbers was not proved until 1851 [7, page 65]. The main results on transcendental numbers depend upon the fact that algebraic numbers cannot be approximated very readily by rationals, while transcendental numbers can be approximated much more readily by rationals [2, page 1].

Presented in this paper are three methods of representing real numbers by expansion as a sequence of integers. These are the decimal expansion, the Cantor expansion and the continued fraction expansion. The characteristics of each are discussed, particularly concerning what can be learned from the expansion regarding the nature of the numbers. The accuracy of approximation to a real number by rationals derived from each of the three expansions is also discussed.

Continued fractions embody two basic concepts of real numbers and give the best rational approximations to real numbers under certain conditions. Liouville's Theorem, which gives conditions for a number to be algebraic, is presented in Chapter IV along with the construction of transcendental numbers from their continued fractions.

A few preliminary definitions and comments are given in this first chapter. The theorems and definitions in this paper are restricted to the real numbers and all algebraic forms are over the real numbers. The reader is assumed to be familiar with the different types of real numbers and their properties. He should also be familiar with the concepts of upper and lower bound and least upper bound and greatest lower bound.

Definition 1.1 An equation will be called a polynomial equation of degree n if and only if it is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

where $a_n \neq 0$ and $a_0, a_1, \dots a_n$ are real numbers. The real numbers $a_0, a_1, \dots a_n$ are called the coefficients of the equation.

Definition 1.2 A number α is called an algebraic number of degree n if and only if α is a root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

of degree n with integer coefficients $a_0, a_1, \dots a_n$ and is not a root of any polynomial equation with integer coefficients of degree less than n.

It is worth noting that a polynomial equation with rational coefficients is equivalent to an equation with integer coefficients, since both sides of the equation may be multiplied by an appropriate integer to get an equation with integer coefficients.

Definition 1.3 A real number α is called a transcendental number if and only if α is not a root of any polynomial equation with integer coefficients (i.e., α is not algebraic).

Definition 1.4 Let α be a real number. Then the notation

$$[\alpha]$$

shall be used to denote the greatest integer not exceeding α.

Then $[\alpha]$ is an integer and $[\alpha] \leq \alpha$; $0 \leq \alpha - [\alpha] < 1$ and if a is an integer $a + [\alpha] = [a + \alpha]$.

The Least Upper Bound Property of the real numbers will be assumed: Every set of real numbers that is bounded from above has a unique least upper bound. Every set of real numbers that is bounded from below has a unique greatest lower bound.

Chapter II

THE DECIMAL EXPANSION

This chapter is devoted to a discussion of the decimal expansion.

Definition 2.1. (Decimal expansion). Let α be a positive real number. Form the sequence

$$d_0.d_1d_2d_3 \ldots$$

in the following way:

Let $d_0 = [\alpha]$.

If $d_0 < \alpha$, let $r_1 = \alpha - d_0$.

Then $d_1 = [10r_1]$.

If $d_1 < 10r_1$, let $r_2 = 10r_1 - d_1$

Then $d_2 = [10r_2]$.

. . .

$d_n = [10r_n]$, $\quad r_{n+1} = 10r_n - d_n$

$d_{n+1} = [10r_{n+1}]$

. . . .

If for some index n, $d_n = 10\,r_n$, then the process must stop and the decimal expansion is finite having n terms. Otherwise the expansion is infinite. In either case, the sequence $d_0.d_1d_2 \ldots$ is called the decimal expansion of α.

The set of rationals

$$s_0 = d_0,\ s_1 = d_0 + \frac{d_1}{10},\ s_2 = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2},\ \ldots$$

$$s_n = s_{n-1} + \frac{d_n}{10^n}$$

are called the partial sums.

By this definition, the decimal expansion of a real number is positive. A negative number would be represented $-(d_0.d_1d_2d_3\ldots)$. Thus, all theorems will be considered for positive real numbers.

Example 2.1 Let $\alpha = \frac{5}{16}$. Then

$$d_0 = \left[\frac{5}{16}\right] = 0, \quad r_1 = \frac{5}{16} - 0 = \frac{5}{16}$$

$$d_1 = \left[\frac{50}{16}\right] = 3, \quad r_2 = \frac{50}{16} - 3 = \frac{2}{16}$$

$$d_2 = \left[\frac{20}{16}\right] = 1, \quad r_3 = \frac{20}{16} - 1 = \frac{4}{16}$$

$$d_3 = \left[\frac{40}{16}\right] = 2, \quad r_4 = \frac{40}{16} - 2 = \frac{8}{16}$$

$$d_4 = \left[\frac{80}{16}\right] = 5, \quad r_5 = \frac{80}{16} - 5 = 0$$

Since $d_4 = 10r_4$, stop. Thus, $\alpha = 0.3125$ and the expansion is finite. The partial sums are:

$$s_0 = 0$$

$$s_1 = 0 + \frac{3}{10} = \frac{3}{10}$$

$$s_2 = 0 + \frac{3}{10} + \frac{1}{10^2} = \frac{31}{100}$$

$$s_3 = 0 + \frac{3}{10} + \frac{1}{10^2} + \frac{2}{10^3} = \frac{312}{1000}$$

$$s_4 = 0 + \frac{3}{10} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{5}{10^4} = \frac{3125}{10,000}.$$

Example 2.2 Let $\alpha = \frac{40}{33}$. Then

$$d_0 = \left[\frac{40}{33}\right] = 1, \quad r_1 = \frac{40}{33} - 1 = \frac{7}{33}$$

$$d_1 = \left[\frac{70}{33}\right] = 2, \quad r_2 = \frac{70}{33} - 2 = \frac{4}{33}$$

$$d_2 = \left[\frac{40}{33}\right] = 1, \quad r_3 = \frac{40}{33} - 1 = \frac{7}{33}$$

$$d_3 = \left[\frac{70}{33}\right] = 2 \quad , \quad r_4 = \frac{70}{33} - 2 = \frac{4}{33}$$

. . .

Then $\alpha = 1.2121$ ... and the expansion is infinite. The partial sums are:

$$s_0 = 1$$

$$s_1 = 1 + \frac{2}{10} = \frac{12}{10}$$

$$s_2 = 1 + \frac{2}{10} + \frac{1}{10^2} = \frac{121}{100}$$

$$s_3 = 1 + \frac{2}{10} + \frac{1}{10^2} + \frac{2}{10^3} = \frac{1212}{1000}$$

. . .

Theorem 2.1. $r_n = 10^{n-1}\alpha - [10^{n-1}\alpha]$.

Proof: By induction. First, note that

$$r_1 = \alpha - d_0 = 10^0\alpha - [10^0\alpha],$$

since $d_0 = [\alpha]$.

Next, let k be an integer such that

$$r_k = 10^{k-1}\alpha - [10^{k-1}\alpha] .$$

Then by definition 2.1,

$$d_k = [10(10^{k-1}\alpha - [10^{k-1}\alpha])]$$

Also by definition 2.1,

$$\begin{aligned} r_{k+1} &= 10r_k - d_k \\ &= (10)\, 10^{k-1}\alpha - (10)\, [10^{k-1}\alpha] - [10(10^{k-1}\alpha - [10^{k-1}\alpha])] \\ &= 10^k\alpha - (10\, [10^{k-1}\alpha] + [10^k\alpha - 10\, [10^{k-1}\alpha]]) \\ &= 10^k\alpha - [10\,[10^{k-1}\alpha] + 10^k\alpha - 10\, [10^{k-1}\alpha]] \\ &= 10^k\alpha - [10^k\alpha] \ . \end{aligned}$$

Thus, the theorem is true for n=1, and if the theorem is true for some integer k, then it is true for $k + 1$. By the axiom of induction, the theorem is true for every integer n.

Theorem 2.2 $d_n = [10^n\alpha] - 10[10^{n-1}\alpha]$.

Proof: By definition 2.1, $r_{n+1} = 10r_n - d_n$. This implies $d_n = 10r_n - r_{n+1}$. By theorem 2.1, $r_n = 10^{n-1}\alpha - [10^{n-1}\alpha]$ and $r_{n+1} = 10^n\alpha - [10^n\alpha]$. Then

$$\begin{aligned} d_n &= 10(10^{n-1}\alpha - [10^{n-1}\alpha]) - 10^n\alpha + [10^n\alpha] \\ &= 10^n\alpha - 10[10^{n-1}\alpha] - 10^n\alpha + [10^n\alpha] \\ &= [10^n\alpha] - 10[10^{n-1}\alpha]. \end{aligned}$$

The proof is finished.

It is seen that the terms d_n of the decimal expansion of a real number α are unique.

Theorem 2.3. α is an upper bound of the partial sums s_n.

Proof: By definition 2.1, $s_n = d_0 + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n}$. By theorem 2.2, $d_n = [10^n\alpha] - 10[10^{n-1}\alpha]$. This implies

$$\begin{aligned} s_n &= [\alpha] + \frac{[10\alpha] - 10[\alpha]}{10} + \ldots + \frac{[10^n\alpha] - 10[10^{n-1}\alpha]}{10^n} \\ &= \frac{10^n[\alpha]}{10^n} + \frac{10^{n-1}[10\alpha] - 10^{n-1}(10)[\alpha]}{10^n} + \ldots + \frac{10[10^{n-1}\alpha] - 10(10[10^{n-2}}{10^n} \\ &\quad + \frac{[10^n\alpha] - 10[10^{n-1}\alpha]}{10^n} \\ &= \frac{10^n[\alpha] + 10^{n-1}[\alpha] - 10^n[\alpha] +}{10^n} \quad \cdots \\ &\quad + \frac{10[10^{n-1}\alpha] - 10^2[10^{n-2}\alpha] + [10^n\alpha] - 10[10^{n-1}\alpha]}{10^n} \end{aligned}$$

(1) $$= \frac{[10^n\alpha]}{10^n} \leq \frac{10^n\alpha}{10^n} = \alpha.$$

Thus, $s_n \leq \alpha$ for every index n and α is an upper bound of the partial sums s_n.

Theorem 2.4. The partial sums s_n satisfy the following inequalities:

$$0 \leq \alpha - s_{n+1} \leq \alpha - s_n < \frac{1}{10^n}.$$

Proof: α is an upper bound of the s_n by theorem 2.3. Then $s_{n+1} \leq \alpha$ which implies $0 \leq \alpha - s_{n+1}$. Since $s_{n+1} = s_n + \frac{d_{n+1}}{10^{n+1}}$ it is seen that $s_n \leq s_{n+1}$, and $\alpha - s_{n+1} \leq \alpha - s_n$.

From (1) it is seen that

$$\alpha - s_n = \alpha - \frac{[10^n \alpha]}{10^n}.$$

This implies

$$\alpha - s_n = \frac{10^n\alpha - [10^n\alpha]}{10^n}.$$

But $10^n\alpha - [10^n\alpha] < 1$ from definition 1.4. Hence $\alpha - s_n < \frac{1}{10^n}$ and the theorem is proved.

Theorem 2.5. Every real number α has one and only one decimal expansion such that α is the least upper bound of the partial sums s_n.

Proof: Given α, theorem 2.2 proves that the decimal expansion $d_0.d_1d_2d_3 \ldots$ is unique. That every decimal expansion corresponds to one and only one real number follows from the uniqueness of the least upper bound, since $d_0.d_1d_2 \ldots < d_0 + 1$ and the s_n are bounded from above by $d_0 + 1$.

To show the uniqueness of the decimal representation of a real number α, suppose that there exist $d_0.d_1d_2 \ldots$ and $\bar{d}_0.\bar{d}_1\bar{d}_2 \ldots$ such that $d_i \neq \bar{d}_i$ for at least one index i, and such that α is the least upper bound of the partial sums s_n and $\bar{s}_n$ (i.e. two different decimal expansions represent the same number α).

Let k be the first index such that $d_k \neq \bar{d}_k$. Then either $d_k < \bar{d}_k$ or $\bar{d}_k < d_k$. Consider first the case that $d_k < \bar{d}_k$. Since d_k and $\bar{d}_k$ are integers, $\bar{d}_k \geq d_k + 1$ and $\bar{s}_k \geq s_k + \frac{1}{10^k}$. Then $\bar{s}_n \geq s_n + \frac{1}{10^k}$ for every $n \geq k$. Since α is the least upper bound of the partial sums s_n and $\bar{s}_n$, this implies

$$\alpha \geq \alpha + \frac{1}{10^k}$$

which is impossible. A similar argument is true if $\bar{d}_k < d_k$. Thus, the original assumption, namely that $d_i \neq \bar{d}_i$ for at least one index i is false. Then $d_i = \bar{d}_i$ for every i and the theorem is proved.

It should be noted that a decimal with infinite repeating 9's such as 1.999 ... is equivalent to 2.000 ... because they have the same least upper bound. The arguments of uniqueness must exclude decimals that end in infinite repeating 9's. Decimals that end in infinite 0's may be considered finite (terminating) decimals.

Theorem 2.6 A real number α is rational if and only if its decimal expansion is either finite or periodic.

Proof: If the decimal expansion is finite, then

$$\alpha = s_n = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n}.$$

Then $\alpha = s_n$ is the sum of rationals and is rational. If the decimal expansion is periodic, such as

$$d_o.d_1d_2 \ldots d_k\bar{d}_1\bar{d}_2 \ldots \bar{d}_p\bar{d}_1\bar{d}_2 \ldots \bar{d}_p \ldots$$

then

$$\alpha = s_k + \frac{\bar{d}_1}{10^{k+1}} + \ldots + \frac{\bar{d}_p}{10^{k+p}} + \frac{\bar{d}_1}{10^{k+p+1}} + \ldots + \frac{\bar{d}_p}{10^{k+2p}} + \ldots$$

$$\alpha = s_k + \frac{10^{p-1}\,\bar{d}_1 + \dots + \bar{d}_p}{10^{k+p}} + \frac{10^{p-1}\,\bar{d}_1 + \dots + \bar{d}_p}{10^{k+2p}} + \dots$$

The sum of the infinite series is $\frac{a}{1-r}$, where a is the first term and r is the common ratio. Thus,

$$\alpha = s_k + \frac{(10^{p-1}\,\bar{d}_1 + \dots + \bar{d}_p)}{1 - \frac{1}{10^p}}$$

$$= s_k + \frac{10^p(10^{p-1}\,\bar{d}_1 + \dots + \bar{d}_p)}{10^p - 1}$$

Then, since s_k is rational and

$$\frac{10^p(10^{p-1}\,\bar{d}_1 + \dots + \bar{d}_p)}{10^p - 1}$$

is rational, α is the sum of rationals and is rational. Thus, a periodic decimal represents a rational number.

Next, it must be shown that if α is rational its decimal expansion will be either finite or periodic. Let $\alpha = \frac{a}{b}$ where a, b are integers and $\frac{a}{b}$ is in lowest terms with $b \geq 1$. By theorem 2.1

$$r_n = 10^{n-1}\frac{a}{b} - \left[10^{n-1}\frac{a}{b}\right]$$

then

$$br_n = 10^{n-1}a - \left[10^{n-1}a\right].$$

Thus, br_n is an integer. Also, $0 \leq r_n < 1$ by definition 1.4. If $r_n = 0$ for some n, then $d_n = [10r_n] = 0$ and the expansion stops and is finite. If $0 < r_n < 1$ for every n, then the expansion is infinite. Then $0 < br_n < b$ and the integer br_n is one of the integers 1,2,3, ... (b-1). Then for some k and some $m = k + L$

$$br_k = br_{k+L}.$$

This implies $r_k = r_{k+L}$ and $r_{k+1} = r_{k+L+1}$ $\cdots$. Since $d_k = [10r_k]$, the above implies $d_{k+s} = d_{(k+L)+s}$ for any index s. Thus, the expansion is periodic. The theorem is proved.

It is seen that every real number can be expressed uniquely by a decimal expansion. Every rational number has a decimal expansion that is either finite or periodic. Thus, it can be determined whether a number is rational or irrational from the decimal expansion. A non-periodic infinite decimal is not rational, but nothing can be said about whether such a number is algebraic or transcendental.

Chapter III

THE CANTOR EXPANSION

This chapter is devoted to a discussion of the Cantor expansion of a real number. The Cantor expansion is seen to be a generalization of the decimal expansion.

Definition 3.1. (Cantor Expansion). Let α be a positive real number. Let b_1, b_2, b_3, ... be a fixed infinite sequence of integers such that $b_n \geq 2$ for every n. Form the sequence of integers

$$c_0 : c_1 c_2 c_3 \ldots$$

in the following way:

Let $c_0 = [\alpha]$

If $c_0 < \alpha$, let $r_1 = \alpha - c_0$.

Then $c_1 = [b_1 r_1]$.

If $c_1 < b_1 r_1$, let $r_2 = b_1 r_1 - c_1$.

. . . .

$c_n = [b_n r_n] \qquad r_{n+1} = b_n r_n - c_n$

$c_{n+1} = [b_{n+1} r_{n+1}]$

. . . .

If for some index n, $b_n r_n = c_n$, then the process stops and the expansion is finite with n terms. Otherwise the expansion is infinite. In either case, the sequence $c_0 : c_1 c_2 c_3 \ldots$ is called the Cantor expansion of α.

The set of rationals

$$s_0 = [\alpha] \quad , \quad s_1 = [\alpha] + \frac{c_1}{b_1}$$

$$s_2 = [\alpha] + \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2}, \ldots,$$

$$s_n = s_{n-1} + \frac{c_n}{b_1 b_2 \cdots b_n}$$

are called the partial sums of the expansion.

As in the case of the decimal expansion, the Cantor expansion is considered for positive real numbers only. A negative number would be represented $-(c_0 : c_1 c_2 \ldots)$.

Example 3.1. Let $b_1 = 2$, $b_2 = 3$, $b_3 = 5$, ... such that $b_n = p_n$ where p_n is the n^{th} prime number. Let $\alpha = \frac{17}{11}$. Then

$$c_0 = \left[\frac{17}{11}\right] = 1 \qquad r_1 = \frac{17}{11} - 1 = \frac{6}{11}$$

$$c_1 = \left[\frac{12}{11}\right] = 1 \qquad r_2 = \frac{12}{11} - 1 = \frac{1}{11}$$

$$c_2 = \left[\frac{3}{11}\right] = 0 \qquad r_3 = \frac{3}{11} - 0 = \frac{3}{11}$$

$$c_3 = \left[\frac{15}{11}\right] = 1 \qquad r_4 = \frac{15}{11} - 1 = \frac{4}{11}$$

$$c_4 = \left[\frac{28}{11}\right] = 2 \qquad r_5 = \frac{28}{11} - 2 = \frac{6}{11}$$

$$c_5 = \left[\frac{66}{11}\right] = 6 \qquad r_6 = \frac{66}{11} - 6 = 0$$

Since $c_5 = b_5 r_5$, stop. Thus, $\alpha = 1{:}10126$ and is a finite expansion. The partial sums s_n are:

$$s_0 = 1$$

$$s_1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$s_2 = 1 + \frac{1}{2} + \frac{0}{(2)(3)} = \frac{3}{2}$$

$$s_3 = 1 + \frac{1}{2} + \frac{0}{(2)(3)} + \frac{1}{(2)(3)(5)} = \frac{46}{30}$$

$$s_4 = 1 + \frac{1}{2} + \frac{0}{(2)(3)} + \frac{1}{(2)(3)(5)} + \frac{2}{(2)(3)(5)(7)} = \frac{324}{210}.$$

$$c_5 = 1 + \frac{1}{2} + \frac{0}{(2)(3)} + \frac{1}{(2)(3)(5)} + \frac{2}{(2)(3)(5)(7)} + \frac{6}{(2)(3)(5)(7)(11)}$$

$$= \frac{3570}{2310}.$$

In the theorems that follow, a fixed infinite sequence of integers b_1, b_2, b_3, ... with $b_n \geq 2$ will be assumed to be given.

Theorem 3.1. $r_n = b_{n-1}\, b_{n-2} \cdots b_1\alpha - [b_{n-1}\, b_{n-2} \cdots b_1\alpha]$.

Proof: By induction. First note that $r_1 = \alpha - c_0 = \alpha - [\alpha]$

and

$$\begin{aligned} r_2 &= b_1 r_1 - c_1 \\ &= b_1\alpha - b_1[\alpha] - [b_1 r_1] \\ &= b_1\alpha - b_1[\alpha] - [b_1\alpha - b_1[\alpha]] \\ &= b_1\alpha - (b_1[\alpha] + [b_1\alpha - b_1[\alpha]]) \\ &= b_1\alpha - [b_1[\alpha] + b_1\alpha - b_1[\alpha]] \\ &= b_1\alpha - [b_1\alpha] \end{aligned}$$

Next, let k be an integer such that

$$r_k = b_{k-1}\, b_{k-2} \cdots b_1\alpha - [b_{k-1}\, b_{k-2} \cdots b_1\alpha]$$

Then

$$\begin{aligned} r_{k+1} &= b_k r_k - c_k \\ &= b_k(b_{k-1}b_{k-2} \cdots b_1\alpha - [b_{k-1}b_{k-2} \cdots b_1\alpha]) \\ &\quad - [b_k(b_{k-1}b_{k-2} \cdots b_1\alpha - [b_{k-1}b_{k-2} \cdots b_1\alpha]] \\ &= b_k b_{k-1} \cdots b_1\alpha - b_k[b_{k-1}b_{k-2} \cdots b_1\alpha] \\ &\quad - [b_k b_{k-1} \cdots b_1\alpha - b_k[b_{k-1}b_{k-2} \cdots b_1\alpha]] \\ &= b_k b_{k-1} \cdots b_1\alpha - [b_k[b_{k-1}b_{k-2} \cdots b_1\alpha] + \\ &\quad + b_k b_{k-1} \cdots b_1\alpha - b_k[b_{k-1}b_{k-2} \cdots b_1\alpha]] \\ &= b_k b_{k-1} \cdots b_1\alpha - [b_k b_{k-1} \cdots b_2 b_1\alpha] \end{aligned}$$

The theorem is proved.

Theorem 3.2. $c_n = [b_n b_{n-1} \cdots b_1 \alpha] - b_n [b_{n-1} b_{n-2} \cdots b_1 \alpha]$

Proof: By definition 3.1, $r_{n+1} = r_n b_n - c_n$. This implies $c_n = b_n r_n - r_{n+1}$. Also by theorem 3.1, $r_n = b_{n-1} b_{n-2} \cdots b_1 \alpha - [b_{n-1} b_{n-2} \cdots b_1 \alpha]$ Then

$$c_n = b_n (b_{n-1} b_{n-2} \cdots b_1 \alpha - [b_{n-1} b_{n-2} \cdots b_1 \alpha])$$

$$- (b_n b_{n-1} \cdots b_1 \alpha - [b_n b_{n-1} \cdots b_1 \alpha])$$

$$= b_n b_{n-1} \cdots b_1 \alpha - b_n [b_{n-1} b_{n-2} \cdots b_1 \alpha] - b_n b_{n-1} \cdots b_1 \alpha$$

$$+ [b_n b_{n-1} \cdots b_1 \alpha]$$

$$= [b_n b_{n-1} \cdots b_1 \alpha] - b_n [b_{n-1} b_{n-2} \cdots b_1 \alpha]$$

The theorem is finished.

Theorem 3.3. $s_n = \dfrac{[b_n b_{n-1} \cdots b_1 \alpha]}{b_n b_{n-1} \cdots b_1}$

Proof: By induction. First note that $s_0 = [\alpha]$ and

$$s_1 = [\alpha] + \frac{c_1}{b_1}$$

$$= [\alpha] + \frac{[b_1 \alpha] - b_1 [\alpha]}{b_1}$$

$$= \frac{b_1 [\alpha] + [b_1 \alpha] - b_1 [\alpha]}{b_1}$$

$$= \frac{[b_1 \alpha]}{b_1}$$

Next, let k be an integer such that

$$s_k = \frac{[b_k b_{k-1} \cdots b_1 \alpha]}{b_k b_{k-1} \cdots b_1}$$

Then
$$s_{k+1} = \frac{[b_k b_{k-1} \cdots b_1 \alpha]}{b_k b_{k-1} \cdots b_1} + \frac{c_{k+1}}{b_{k+1} b_k \cdots b_1}$$
$$= \frac{[b_k b_{k-1} \cdots b_1 \alpha]}{b_k b_{k-1} \cdots b_1} + \frac{[b_{k+1} b_k \cdots b_1 \alpha] - b_{k+1}[b_k b_{k-1} \cdots b_1 \alpha]}{b_{k+1} b_k \cdots b_1}$$
$$= \frac{b_{k+1}[b_k b_{k-1} \cdots b_1 \alpha] + [b_{k+1} b_k \cdots b_1 \alpha] - b_{k+1}[b_k b_{k-1} \cdots b_1 \alpha]}{b_{k+1} b_k \cdots b_1}$$
$$= \frac{[b_{k+1} b_k \cdots b_1 \alpha]}{b_{k+1} b_k \cdots b_1}$$
The theorem is proved.

Theorem 3.4. α is an upper bound of the partial sums s_n.

Proof: By theorem 3.3.
$$s_n = \frac{[b_n b_{n-1} \cdots b_1 \alpha]}{b_n b_{n-1} \cdots b_1} \leq \frac{b_n b_{n-1} \cdots b_1 \alpha}{b_n b_{n-1} \cdots b_1} = \alpha .$$
Thus, $s_n \leq \alpha$ for every index n and α is an upper bound of the partial sums.

Theorem 3.5. The partial sums s_n satisfy the following inequalities:
$$0 \leq \alpha - s_{n+1} \leq \alpha - s_n < \frac{1}{b_n b_{n-1} \cdots b_1}$$

Proof: From theorem 2.4, $s_{n+1} \leq \alpha$. Thus, $0 \leq \alpha - s_{n+1}$.
then $s_{n+1} = s_n + \frac{c_{n+1}}{b_{n+1} b_n \cdots b_1}$ implies $s_n \leq s_{n+1}$.
This implies $\alpha - s_{n+1} \leq \alpha - s_n$. To show $\alpha - s_n < \frac{1}{b_n b_{n-1} \cdots b_1}$
consider
$$\alpha - s_n = \alpha - \frac{[b_n b_{n-1} \cdots b_1 \alpha]}{b_n b_{n-1} \cdots b_1}$$
$$= \frac{b_n b_{n-1} \cdots b_1 \alpha - [b_n b_{n-1} \cdots b_1 \alpha]}{b_n b_{n-1} \cdots b_1}$$

But $b_n b_{n-1} \ldots b_1\alpha - [b_n b_{n-1} \ldots b_1\alpha] < 1$ by definition 1.4. Hence

$$\alpha - s_n < \frac{1}{b_n b_{n-1} \ldots b_1}$$

and the theorem is proved.

Theorem 3.6. Every real number α has one and only one Cantor expansion such that α is the least upper bound of the partial sums s_n.

Proof: Given α, theorem 3.2 guarantees that the Cantor expansion corresponds to one and only one real number α follows from the uniqueness of the least upper bound, since $c_0:c_1c_2c_3 \ldots < c_0 + 1$ shows that the partial sums s_n are bounded from above by $c_0 + 1$.

To show the uniqueness of the Cantor expansion of a real number α, suppose that there exist

$$c_0:c_1c_2c_3 \ldots \quad \text{and} \quad \bar{c}_0:\bar{c}_1\bar{c}_2\bar{c}_3 \ldots$$

such that $c_i \neq \bar{c}_i$ for at least one index i and such that α is the least upper bound of the partial sums s_n and $\bar{s}_n$ (i.e. two different Cantor expansions represent the same number α). Let k be the first index such that $c_k \neq \bar{c}_k$. Then either $c_k < \bar{c}_k$ or $\bar{c}_k < c_k$. Consider first the case that $c_k < \bar{c}_k$. Since c_k and $\bar{c}_k$ are integers, $\bar{c}_k \geq c_k + 1$ and

$$\bar{s}_k \geq s_k + \frac{1}{b_k b_{k-1} \ldots b_1}.$$

Then

$$\bar{s}_n \geq s_n + \frac{1}{b_k b_{k-1} \ldots b_1}$$

for every $n \geq k$. Since α is the least upper bound of the partial sums s_n and $\bar{s}_n$, this implies

$$\alpha \geq \alpha + \frac{1}{b_k b_{k-1} \ldots b_1}.$$

This is impossible. A similar argument is true if $\bar{c}_k < c_k$. Thus, the original assumption that $c_i \neq \bar{c}_i$ for at least one index i is false. Then $c_i = \bar{c}_i$ for every i and the theorem is proved.

It should be noted that as in the case of the decimal expansion there is an exception. If a particular infinite Cantor expansion has after a certain point $c_n = b_n - 1$, then the terminating expansion

$$(c_0 + 1):c_1c_2 \ldots c_{n-1} \text{ and } c_0:c_1c_2 \ldots c_nc_{n+1} \ldots$$

have the same least upper bound. All arguments of uniqueness must therefore exclude Cantor expansions that are infinite and have after a certain point $c_n = b_n - 1$.

A test for rationals analogous to that for the decimal expansion does not exist for the Cantor expansion. For a discussion of this see Stefan Drobot's book Real Numbers [1, page 28]. The Cantor expansion depends essentially upon the base-sequence chosen. However, sufficient conditions xist for a Cantor expansion to represent an irrational number.

Theorem 3.7. Let α be a real number. Let a base-sequence $b_1b_2 \ldots$ be chosen such that each prime number divides infinitely many of the b_n's. Then, if the Cantor expansion $c_0:c_1c_2 \ldots$ of α is infinite, α is irrational.

Proof: By contradiction. Suppose $\alpha = \frac{p}{q}$ where p and q are integers with $q \geq 1$ and the base-sequence is such that each prime number divides infinitely many of the b_n's and the expansion is infinite.

By theorem 3.2

$$c_n = \left[b_n b_{n-1} \cdots b_1 \frac{p}{q}\right] - b_n \left[b_{n-1} b_{n-2} \cdots b_1 \frac{p}{q}\right].$$

Let N be the index such that the product $b_N b_{N-1} \cdots b_1$ is divisible by q(= qR). Then for every $m > N$,

$$c_m = \left[b_m b_{m-1} \cdots b_N b_{N-1} \cdots b_1 \frac{p}{q}\right] - b_m \left[b_{m-1} \cdots b_N \cdots b_1 \frac{p}{q}\right]$$

$$= \left[b_m b_{m-1} \cdots b_{N+1} \cdot RP\right] - b_m \left[b_{m-1} \cdots b_{N+1} \cdot RP\right].$$

Since the b_i are integers and R and P are integers, $c_m = 0$. This contradicts that the expansion was infinite. Thus, the opposite of the original assumption is true, namely that α is irrational. This completes the proof.

It is possible to choose a base-sequence such that every rational number will have a finite expansion and every irrational number will have an infinite expansion.

Theorem 3.8. There exists a base-sequence such that every rational has a finite Cantor expansion and every irrational number has an infinite expansion.

Proof: Choose the base-sequence in the following way: $b_1 = 2$, $b_2 = 3$, $b_3 = 5$, ... $b_n = P_i$ where the P_i's are chosen as indicated in table 1.

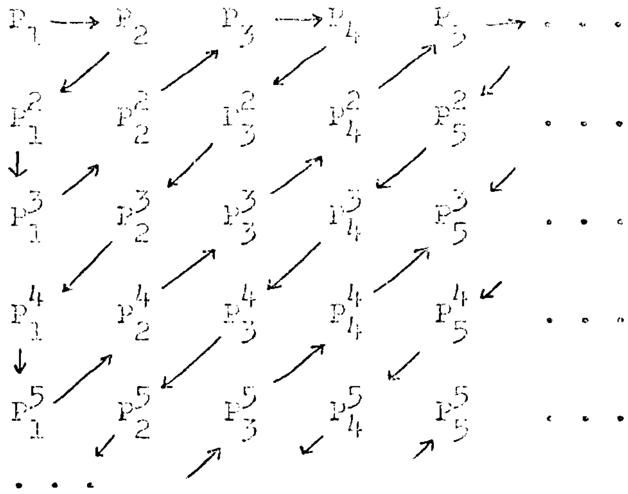

Table 1

P_n is the nth prime number.

Then every prime number divides infinitely many of the b_n's and by theorem 3.7 if the expansion is infinite, α is irrational. If the expansion is finite, then α is not irrational, and thus is rational. Hence every rational number has a finite expansion and every irrational number has an infinite expansion.

The Cantor expansion is seen to be a generalization of the decimal expansion. There is an improvement in determining the nature of a number in that if the proper base-sequence is chosen, the expansion of a rational number will be finite and the expansion of an irrational number will be infinite. Nothing can be said, however about whether a number is algebraic or transcendental. The practicality of the Cantor expansion for determining the rationality of a real number might be questioned. Admittedly, it is more cumbersome than the decimal expansion.

Chapter IV

THE CONTINUED FRACTION EXPANSION

Definition 4.1. (Continued Fraction). Let α be a positive real number. Form the sequence $\{a_0; a_1 a_2 \dots\}$ in the following way: Let $r_0 = \alpha$. Then $a_0 = [r_0]$

If $a_0 < r_0$, let $r_1 = \dfrac{1}{r_0 - a_0}$.

Then $a_1 = [r_1]$

If $a_1 < r_1$, let $r_2 = \dfrac{1}{r_1 - a_1}$.

. . .

$a_n = [r_n] \qquad r_{n+1} = \dfrac{1}{r_n - a_n}$

$a_{n+1} = [r_{n+1}]$

. . .

If for some index n, $a_n = r_n$, then the process stops and has n terms. Otherwise the expansion is infinite. In either case the sequence $\{a_0; a_1 a_2 \dots\}$ is called the continued fraction expansion of α. The continued fraction can also be represented in the following way:

$$\{a_0; a_1 a_2 \dots\} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n + \dots}}}}$$

Definition 4.2. The rational numbers

$$f_0 = a_0 \ , \ f_1 = a_0 + \frac{1}{a_1} \ , \quad f_2 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \ , \ . \ . \ . \ . \ ,$$

$$f_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

Are called the convergents of the continued fraction.

Example 4.1. Let $\alpha = \frac{15}{11}$. Then,

$$a_0 = \left[\frac{15}{11}\right] = 1 \qquad r_1 = \frac{1}{\frac{15}{11} - 1} = \frac{11}{4}$$

$$a_1 = \left[\frac{11}{4}\right] = 2 \qquad r_2 = \frac{1}{\frac{11}{4} - 2} = \frac{4}{3}$$

$$a_2 = \left[\frac{4}{3}\right] = 1 \qquad r_3 = \frac{1}{\frac{4}{3} - 1} = \frac{3}{1}$$

$$a_3 = \left[\frac{3}{1}\right] = 3$$

Since $a_3 = r_3$, stop. Then $\alpha = \left\{1;213\right\}$ and the expansion is finite. The convergents are

$$f_0 = 1$$

$$f_1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$f_2 = 1 + \cfrac{1}{2 + \cfrac{1}{1}} = \frac{4}{3}$$

$$f_3 = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}} = \frac{15}{11}$$

Note that α may be written:

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}$$

Example 4.2. Let $\alpha = \sqrt{3}$. Then

$$a_0 = [\sqrt{3}] = 1 \qquad r_1 = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2}$$

$$a_1 = \left[\frac{\sqrt{3} + 1}{2}\right] = 1 \qquad r_2 = \frac{1}{\frac{\sqrt{3} + 1}{2} - 1} = \sqrt{3} + 1$$

$$a_2 = [\sqrt{3} + 1] = 2 \qquad r_3 = \frac{1}{\sqrt{3} + 1 - 2} = \frac{\sqrt{3} + 1}{2}$$

. . .

Thus, $= 1;121212 \ldots = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}$

and the expansion is infinite. The convergents f_n are

$$f_0 = 1$$

$$f_1 = 1 + \frac{1}{1} = 2$$

$$f_2 = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{5}{3}$$

$$f_3 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}} = \frac{7}{4}$$

$$f_4 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}} = \frac{19}{11}$$

. . .

Theorem 4.1. The convergents are rational numbers $f_n = \frac{p_n}{q_n}$ such that

$$p_0 = a_0 \qquad p_1 = a_0 a_1 + 1 \qquad p_n = p_{n-1}a_n + p_{n-2}$$

$$q_0 = 1 \qquad q_1 = a_1 \qquad q_n = q_{n-1}a_n + q_{n-2}$$

Proof: By induction. First note that $f_0 = a_0 = \frac{a_0}{1}$. Thus $p_0 = a_0$ and $q_0 = 1$. Then

$$f_1 = a_0 + \frac{1}{a_1} = \frac{a_0a_1 + 1}{a_1}.$$

Thus, $p_1 = a_0a_1 + 1$ and $q_0 = a_1$. Also note that

$$f_2 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}$$

$$= a_0 + \frac{a_2}{a_1a_2 + 1}$$

$$= \frac{a_0a_1a_2 + a_0 + a_2}{a_1a_2 + 1}$$

$$= \frac{(a_0a_1 + 1)a_2 + a_0}{a_1a_2 + 1}$$

Thus $p_2 = p_1a_2 + p_0$ and $q_2 = q_1a_2 + q_1$. Let k be an integer such that

$$f_k = \frac{p_k}{q_k} = \frac{p_{k-1}a_k + p_{k-2}}{q_{k-1}a_k + q_{k-2}}$$

Then

$$f_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}} = \frac{p_{k-1}a_k + p_{k-2}}{q_{k-1}a_k + q_{k-2}}$$

Then

$$f_{k+1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k + \cfrac{1}{a_{k+1}}}}}}} = \frac{p_{k-1}\left(a_k + \frac{1}{a_{k+1}}\right) + p_{k-2}}{q_{k-1}\left(a_k + \frac{1}{a_{k+1}}\right) + q_{k-2}}$$

Thus,

$$f_{k+1} = \frac{p_{k-1}a_k + \frac{p_{k-1}}{a_{k+1}} + p_{k-2}}{q_{k-1}a_k + \frac{q_{k-1}}{a_{k+1}} + q_{k-2}}$$

$$= \frac{\dfrac{p_{k-1}a_k a_{k+1} + p_{k-1} + p_{k-2}a_{k+1}}{a_{k+1}}}{\dfrac{q_{k-1}a_k a_{k+1} + q_{k-1} + q_{k-2}a_{k+1}}{a_{k+1}}},$$

$$= \frac{(p_{k-1}a_k + p_{k-2})\ a_{k+1} + p_{k-1}}{(q_{k-1}a_k + q_{k-2})\ a_{k+1} + q_{k-1}}$$

But $p_{k-1}a_k + p_{k-2} = p_k$ and $q_{k-1}a_k + q_{k-2} = q_k$ by the inductive hypotheses. So

$$f_{k+1} = \frac{p_{k+1}}{q_{k+1}} = \frac{p_k a_{k+1} + p_{k-1}}{q_k a_{k+1} + q_{k-1}}$$

The theorem is proved.

Theorem 4.2. $q_n p_{n-1} - q_{n-1}\ p_n = (-1)^n$

Proof: By induction. First note that

$$q_1 p_0 - q_0 p_1 = a_0 a_1 - 1(a_0 a_1 + 1) = -1 = (-1)^1 .$$

Then let k be an integer such that $q_k p_{k-1} - q_{k-1}p_k = (-1)^k$.

Then, using theorem 4.1

$$\begin{aligned} q_{k+1}p_k - q_k p_{k+1} &= (q_k a_{k+1} + q_{k-1})\ p_k - q_k\ (p_k a_{k+1} + p_{k-1}) \\ &= q_k a_{k+1} p_k + q_{k-1}p_k - q_k a_{k+1} p_k - q_k p_{k-1} \\ &= -(q_k p_{k-1} - q_{k-1}p_k) \\ &= (-1)\ (-1)^k \\ &= (-1)^{k+1} . \end{aligned}$$

Thus, $q_n p_{n-1} - q_{n-1}p_n = (-1)^n$ for every n.

Theorem 4.3. $q_n p_{n-2} - q_{n-2} p_n = (-1)^{n-1} a_n$.

Proof: Using theorem 4.1.

$$q_n p_{n-2} - q_{n-2} p_n = (q_{n-1} a_n + q_{n-2}) p_{n-2} - q_{n-2}(p_{n-1} a_n + p_{n-2})$$

$$= q_{n-1} a_n p_{n-2} + q_{n-2} p_{n-2} - q_{n-2} p_{n-1} a_n - q_{n-2} p_{n-2}$$

$$= q_{n-1} a_n p_{n-2} - q_{n-2} p_{n-1} a_n$$

$$= (q_{n-1} p_{n-2} - q_{n-2} p_{n-1}) a_n.$$

By theorem 4.2, $q_{n-1} p_{n-2} - q_{n-2} p_{n-1} = (-1)^{n-1}$. So $q_n p_{n-2} - q_{n-2} p_n = (-1)^{n-1} a_n$ and the theorem is proved.

Theorem 4.4. Every convergent $\frac{p_n}{q_n}$ is in lowest terms.

Proof: By contradiction. $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$ from theorem 4.2. Suppose that for some n, $\frac{p_n}{q_n}$ is not in lowest terms. Then let k be an integer such that $k > 1$ and k divides p_n and q_n. Then k divides $q_n p_{n-1} - q_{n-1} p_n$ which implies that k divides $(-1)^n$. This is impossible. Thus, $\frac{p_n}{q_n}$ is in lowest terms.

Theorem 4.5. The convergents with even indices f_{2k} increase with k and the indices with odd indices f_{2L+1} decrease with L.

Proof: Consider

$$f_{n+2} - f_n = \frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}$$

$$= \frac{q_n p_{n+2} - q_{n+2} p_n}{q_{n+2} q_n}$$

$$= \frac{(-1)(q_{n+2} p_n - q_n p_{n+2})}{q_{n+2} q_n}$$

$$= \frac{(-1)(-1)^{n+1} a_{n+2}}{q_{n+2} q_n}$$

$$f_{n+2} - f_n = \frac{(-1)^{n+2} a_{n+2}}{q_{n+2} q_n}$$

using theorem 4.3 to get $q_n + p_n - q_n p_{n+2} = (-1)^{n+2} a_{n+2}$. If n is even ($n = 2k$), then

$$f_{2k} - f_{2k-2} = \frac{(-1)^{2k+2} a_{2k+2}}{q_{2k+2} q_{2k}} > 0.$$

This implies $f_{2k-2} < f_{2k}$. If n is odd ($n = 2L + 1$), then

$$f_{2L+1} - f_{2L+1-2} = \frac{(-1)^{2L+1+2} a_{2L+1-2}}{q_{2L+1-2} q_{2L+1}} < 0.$$

This implies $f_{2L+1} < f_{(2L+1)-2}$ and the theorem is proved.

Theorem 4.6. Any convergent with an even index is less than or equal to any with an odd index ($f_{2k} \leq f_{2L+1}$).

Proof: Consider

$$f_{2m+1} - f_{2m} = \frac{p_{2m+1}}{q_{2m+1}} - \frac{p_{2m}}{q_{2m}}$$

$$= \frac{q_{2m} p_{2m+1} - q_{2m+1} p_{2m}}{q_{2m+1} q_{2m}}$$

$$= \frac{(-1)(q_{2m+1} p_{2m} - q_{2m} p_{2m-1})}{q_{2m+1} q_{2m}}$$

$$= \frac{(-1)^{2m+2}}{q_{2m+1} q_{2m}}$$

$$= \frac{1}{q_{2m+1} q_{2m}} > 0.$$

Thus, $f_{2m} < f_{2m+1}$. Consider f_{2k} and f_{2L+1}. If $k < L$, then $2k < 2L$, and $f_{2k} < f_{2L} < f_{2L+1}$ since $f_{2k} < f_{2L}$ by theorem 4.5. If $L < k$, then $2L + 1 < 2k + 1$ and $f_{2k} < f_{2k+1} < f_{2L+1}$ since $f_{2k+1} < f_{2L+1}$ by theorem 4.5. The theorem is proved.

Theorem 4.7. All denominators $q_n \geq 2^{\frac{n-1}{2}}$.

Proof: By induction. First note that $q_0 = 1 \geq 2^{\frac{0-1}{2}} = \frac{1}{\sqrt{2}}$.

also, $q_1 = a_1$ and since a_1 is an integer, $q_1 \geq 1 = 2^{\frac{1-1}{2}} = 2^0$.
Next, let k be an integer such that $q_k \geq 2^{\frac{k-1}{2}}$. Then by theorem 4.1

$$q_{k+1} = q_k a_n + q_{k-1} \geq 2^{\frac{k-1}{2}} a_n + 2^{\frac{k-2}{2}} (\sqrt{2}a_n + 1).$$

Since $\sqrt{2} \geq 1$ and $a_n \geq 1$ it is seen that

$$q_{k+1} \geq 2^{\frac{k-2}{2}} (2^1) = 2^{\frac{k}{2}}.$$

Thus, $q_n \geq 2^{\frac{n-1}{2}}$ for every index n.

Theorem 4.8. $\alpha = \dfrac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}$ for $n \geq 2$.

Proof: By induction. First note that

$$\frac{p_1 r_2 + p_0}{q_1 r_2 + q_0} = \frac{(a_0 a_1 + 1)\frac{1}{r_1 - a_1} + a_0}{a_1 \frac{1}{r_1 - a_1} + 1}$$

$$= \frac{\dfrac{a_0 a_1 + 1 + a_0 r_1 - a_0 a_1}{r_1 - a_1}}{\dfrac{a_1 + r_1 - a_1}{r_1 - a_1}}.$$

$$= \frac{1 + a_0 \frac{1}{r_0 - a_0}}{\frac{1}{r_0 - a_0}}.$$

$$= \frac{\dfrac{r_0 - a_0 + a_0}{r_0 - a_0}}{\dfrac{1}{r_0 - a_0}}. \qquad = \frac{r_0}{1} = \alpha.$$

Next, let k be an integer such that

$$\alpha = \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1} r_k + q_{k-2}}$$

then

$$\frac{p_k r_{k+1} + p_{k-1}}{q_k r_{k+1} + q_{k-1}} = \frac{p_k \dfrac{1}{r_k - a_k} + p_{k-1}}{q_k \dfrac{1}{r_k - a_k} + q_{k-1}}$$

$$\frac{p_k r_{k+1} + p_{k-1}}{q_k r_{k+1} + q_{k-1}} = \frac{p_k + p_{k-1} r_k - p_{k-1} a_k}{q_k + q_{k-1} r_k - q_{k-1} a_k}$$

$$= \frac{p_{k-1} r_k + (p_k - p_{k-1} a_k)}{q_{k-1} r_k + (q_k - q_{k-1} a_k)}$$

By theorem 4.1, $p_k = p_{k-1}\, a_k + p_{k-2}$ and $q_k = q_{k-1} a_k + q_{k-2}$. This implies $p_k - p_{k-1} a_k = p_{k-2}$ and $q_k - q_{k-1} a_k = q_{k-2}$. Thus,

$$\frac{p_k r_{k+1} + p_{k-1}}{q_k r_{k+1} + q_{k-1}} = \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1}\, r_k + q_{k-2}} = \alpha$$

by the inductive hypothesis. Thus,

$$\alpha = \frac{p_{n-1} r_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}}$$

for $n \geq 2$ and the theorem is finished.

Theorem 4.9. α is the greatest lower bound of the odd convergents and α is the least upper bound of the even convergents. $(f_{2k} \leq \alpha \leq f_{2k+1})$

Proof: Consider

$$\alpha - \frac{p_n}{q_n} = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}} - \frac{p_n}{q_n}$$

$$= \frac{q_n p_{n-1} r_n + q_n p_{n-2} - q_{n-1} r_n p_n - q_{n-2} p_n}{(q_{n-1}r_n + q_{n-2})\, q_n}$$

$$= \frac{(q_n p_{n-1} - q_{n-1}p_n) r_n + (q_n p_{n-2} - q_{n-2} p_n)}{(q_{n-1}r_n + q_{n-2})\, q_n}$$

$$= \frac{(-1)^n r_n + (-1)^{n-1} a_n}{(q_{n-1}r_n + q_{n-2})\, q_n} .$$

Now, $(q_{n-1}r_n + q_{n-2})q_n > 0$ since q_i and $r_i > 0$. If n is even ($n = 2k$) then,

$$\alpha - \frac{p_{2k}}{q_{2k}} = \frac{r_{2k} - a_{2k}}{(q_{2k-1}r_{2k} + q_{2k-2})q_{2k}}$$

Then from definition 4.1, $r_{2k+1} = \frac{1}{r_{2k} - a_{2k}}$ which implies

$r_{2k} - a_{2k} = \frac{1}{r_{2k+1}} > 0$. Thus $\alpha - \frac{p_{2k}}{q_{2k}} > 0$ and α is an upper bound of the even convergents. To see that α is the least upper bound, consider

$$\alpha - \frac{p_{2k}}{q_{2k}} = \frac{1}{(q_{2k-1}r_{2k} + q_{2k-2})\, q_{2k} r_{2k+1}} .$$

Since $r_i > 1$,

$$\alpha - \frac{p_{2k}}{q_{2k}} < \frac{1}{(q_{2k-1} + q_{2k-2})q_{2k}}$$

Then since $q_n \geq 2^{\frac{n-1}{2}}$ it is seen that

$$\alpha - \frac{p_{2k}}{q_{2k}} < \frac{1}{\left(2^{\frac{2k-2}{2}} + 2^{\frac{2k-3}{2}}\right)2^{\frac{2k-1}{2}}} = \frac{1}{2^{\frac{4k-3}{2}} + 2^{\frac{4k-4}{2}}} < \frac{1}{2\cdot 2^{2k}}$$

So $\frac{p_{2k}}{q_{2k}} < \alpha$ and $\alpha - \frac{p_{2k}}{q_{2k}} < \frac{1}{2^{2k+1}}$. Thus the difference may be made as small as is wished, which implies α is the least upper bound of the even convergents.

If n is odd (n = 2L+1), it is seen that

$$\alpha - \frac{p_{2L+1}}{q_{2L+1}} = \frac{a_{2L+1} - r_{2L+1}}{(q_{2L} r_{2L+1} + q_{2L-1}) q_{2L+1}}$$

Then, from definition 4.1, $a_{2L+1} - r_{2L+1} = \frac{-1}{r_{2L+2}}$ and it is seen that $\alpha - \frac{p_{2L+1}}{q_{2L+1}} < 0$. Thus $\alpha < \frac{p_{2L+1}}{q_{2L+1}}$ and α is a lower bound of the odd convergents.

By a similar argument to the case of the even convergents, it is seen that α is the greatest lower bound of the odd convergents. Since the finite case must be included, and for a finite expansion with 2k terms, $f_{2k} = \alpha$. Also, for a finite expansion with 2L+1 terms, $f_{2L+1} = \alpha$. It is then seen that

$$f_{2k} \leq \alpha \leq f_{2L+1} .$$

From the results of theorems 4.5, 4.6 and 4.9 "... it can be seen that the continued fraction reflects in a way, both Dedekind's and Cantor's ideas about the concept of a real number. The fact that all 'even' convergents are less than all the 'odd' convergents reflects the idea of Dedekind's cut. The fact that both the 'even' and the 'odd' convergents tend to the same number reflects Cantor's idea of a real number as a limit of sequences of rationals." [1, page36]

Theorem 4.10. Every real number α has one and only one continued fraction expansion for which α is the least upper bound of the even convergents and the greatest lower bound of the odd convergents.

Proof: Given a real number α, from definition 4.1 it is seen that $r_0 = \alpha$ is unique. Then $a_0 = [r_0]$ is unique. $r_1 = \frac{1}{r_0 - a_0}$ is unique. ... $r_n = \frac{1}{r_{n-1} - a_{n-1}}$ is unique. $a_n = [r_n]$ is unique. Thus, the expansion $\{a_0;a_1a_2 \ldots\}$ is unique.

Given a continued fraction, $\{a_0;a_1a_2 \ldots\}$ there are two cases to consider. First, if the expansion is finite, then

$$\alpha = f_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}$$

which is unique. If the expansion is infinite, then by theorem 4.9 α is the unique least upper bound of the even convergents and the greatest lower bound of the odd convergent.

To show the uniqueness of the representation for a real number α, suppose that there exist

$$\{a_0;a_1a_2 \ldots\} \quad \text{and} \quad \{\bar{a}_0;\bar{a}_1\bar{a}_2 \ldots\}$$

such that $a_i \neq \bar{a}_i$ for at least one index i and α is the least upper bound of the even convergents f_{2n} and $\bar{f}_{2n}$ and α is the greatest lower bound of the odd convergents f_{2L+1} and $\bar{f}_{2L+1}$ (i.e. two different continued fractions represent the same number α).

Let k be the first index such that $a_k \neq \bar{a}_k$. Then either $a_k < \bar{a}_k$ or $\bar{a}_k < a_k$. Consider first the case that $a_k < \bar{a}_k$. Since a_k and $\bar{a}_k$ are integers, this implies $\bar{a}_k \geq a_k + 1$. Then

$$\bar{f}_k \geq f_k + \frac{(-1)^k}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})} .$$

If k is even, $\bar{f}_k \geq f_k + \frac{1}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})}$

and $\bar{f}_{k+2t} \geq f_{k+2t} + \frac{1}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})}$

for every integer $t \geq 0$. Since α is the least upper bound of the even convergents, $\bar{f}_{2n}$ and f_{2n} the above implies

$$\alpha \geq \alpha + \frac{1}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})} .$$

This is impossible. If k is odd, then it is seen that

$$\bar{f}_k \geq f_k + \frac{-1}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})}$$

which implies

$$\bar{f}_{k+2t} \geq f_{k+2t} - \frac{1}{(q_{k-1}(a_k+1) + q_{k-2})(q_{k-1}a_k + q_{k-2})}$$

for every integer $t \geq 0$. Then, since α is the greatest lower bound of the odd convergents f_{2L+1} and $\bar{f}_{2L+1}$, it is seen that

$$\alpha \geq \alpha - \frac{1}{(q_{k-1}(a_{k+1}) + q_{k-2})(q_{k-1}a_k + q_{k-2})}$$

This is impossible. A similar argument will result from the assumption that $\bar{a}_k < a_k$. Thus, the original assumption that $a_i \neq \bar{a}_i$ for at least one index i is false. Hence $a_i = \bar{a}_i$ for every i, and the theorem is proved.

Theorem 4.11. $\left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}}$

Proof: Suppose n is even (n=2k). Then n+1 is odd, and by theorem 4.9

$$\frac{p_{2k}}{q_{2k}} < \alpha .$$

This implies $\left|\alpha - \frac{p_{2k}}{q_{2k}}\right| = \alpha - \frac{p_{2k}}{q_{2k}}$.

Also by theorem 4.9, $\alpha < \frac{p_{2k+1}}{q_{2k+1}}$ which implies $\left|\alpha - \frac{p_{2k+1}}{q_{2k+1}}\right| = \frac{p_{2k+1}}{q_{2k+1}} - \alpha$.

Then $\left|\alpha - \frac{p_{2k}}{q_{2k}}\right| + \left|\alpha - \frac{p_{2k+1}}{q_{2k+1}}\right| = \alpha - \frac{p_{2k+1}}{q_{2k+1}} + \frac{p_{2k}}{q_{2k}} - \alpha$

$$= \frac{-(q_{2k+1}p_{2k} + q_{2k}p_{2k+1})}{q_{2k+1}q_{2k}}$$

$$= \frac{(-1)(q_{2k+1}p_{2k} - q_{2k}p_{2k+1})}{q_{2k+1}q_{2k}}$$

$$= \frac{(-1)^{2k+2}}{q_{2k}q_{2k+1}}$$

Thus, if n is even (n=2k),

$$\left|\alpha - \frac{p_{2k}}{q_{2k}}\right| + \left|\alpha - \frac{p_{2k+1}}{q_{2k+1}}\right| = \frac{1}{q_{2k}q_{2k+1}}$$

If n is odd (n=2L+1), then from theorem 4.9,

$$\alpha < \frac{p_{2L+1}}{q_{2L+1}} \quad \text{and} \quad \frac{p_{2L+2}}{q_{2L+2}} < \alpha$$

Then,

$$\left|\alpha - \frac{p_{2L+1}}{q_{2L+1}}\right| + \left|\alpha - \frac{p_{2L+2}}{q_{2L+2}}\right| = \frac{p_{2L+1}}{q_{2L+1}} - \alpha + \alpha - \frac{p_{2L+2}}{q_{2L+2}}$$

$$= \frac{q_{2L+2}p_{2L+1} - q_{2L+1}p_{2L+2}}{q_{2L+1}q_{2L+2}}$$

$$= \frac{(-1)^{2L+2}}{q_{2L+1}q_{2L+2}}$$

$$= \frac{1}{q_{2L+1}q_{2L+2}}$$

Thus, for every n,

$$\left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} .$$

Theorem 4.12.

$$\frac{a_{n+2}}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

Proof: That

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

follows from theorem 4.11. If n is even (n=2k), then

$$\left|\alpha - \frac{p_{2k}}{q_{2k}}\right| = \alpha - \frac{p_{2k}}{q_{2k}}$$

from theorem 4.11, and

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+2}}{q_{2k+2}} < \alpha$$

from theorem 4.5 and theorem 4.9. Then

$$\frac{p_{2k+2}}{q_{2k+2}} - \frac{p_{2k}}{q_{2k}} < \alpha - \frac{p_{2k}}{q_{2k}} = \left|\alpha - \frac{p_{2k}}{q_{2k}}\right|$$

But,

$$\frac{p_{2k+2}}{q_{2k+2}} - \frac{p_{2k}}{q_{2k}} = \frac{q_{2k}p_{2k+2} - q_{2k+2}p_{2k}}{q_{2k+2}q_{2k}}$$

$$= \frac{(-1)(-1)^{2k+1}a_{2k+2}}{q_{2k+2}q_{2k}}$$

$$= \frac{a_{2k+2}}{q_{2k}q_{2k+2}}$$

using theorem 4.3. Thus,

$$\frac{a_{2k+2}}{q_{2k}q_{2k+2}} < \left|\alpha - \frac{p_{2k}}{q_{2k}}\right| .$$

If n is odd (n=2L+1), then

$$\left|\alpha - \frac{p_{2L+1}}{q_{2L+1}}\right| = \frac{p_{2L+1}}{q_{2L+1}} - \alpha$$

from theorem 4.9 and

$$\alpha < \frac{p_{2L+3}}{q_{2L+3}}$$

from theorem 4.5. Thus,

$$\alpha - \frac{p_{2L+1}}{q_{2L+1}} < \frac{p_{2L+3}}{q_{2L+3}} = \frac{q_{2L+1}p_{2L+3} - q_{2L+3}p_{2L+1}}{q_{2L+1}q_{2L+3}}$$

$$= \frac{(-1)(-1)^{2L+2}a_{2L+3}}{q_{2L+1}q_{2L+3}}$$

Thus,

$$\alpha - \frac{p_{2L+1}}{q_{2L+1}} < \frac{-(a_{2L+3})}{q_{2L+1}q_{2L+3}}$$

This implies

$$\frac{a_{2L+3}}{q_{2L+1}q_{2L+3}} < \frac{p_{2L+1}}{q_{2L+1}} - \alpha = \left|\alpha - \frac{p_{2L+1}}{q_{2L+1}}\right|$$

Thus,

$$\frac{a_{n+2}}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

for every index n.

Theorem 4.13.

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$$

Proof: From theorem 4.12,

$$\frac{a_{n+2}}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right|$$

Then, since $a_n \geq 1$,

$$\frac{1}{q_n q_{n+2}} < \frac{a_{n+2}}{q_n q_{n+2}} \leq \left|\alpha - \frac{p_n}{q_n}\right|.$$

Also from theorem 4.12,

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_{n+1} q_{n+2}} .$$

But since $q_n \leq q_{n+1}$,

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_n q_{n+1}} .$$

Thus,

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right|$$

and the proof is finished.

Theorem 4.14. $\dfrac{1}{(q_n + q_{n+1})q_n} < \left|\alpha - \dfrac{p_n}{q_n}\right| < \dfrac{1}{q_n^2}$.

Proof: Since $q_n \geq q_{n-1}$, theorem 4.12 implies

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} .$$

From theorem 4.12,

$$\frac{a_{n+2}}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right|.$$

But, $\dfrac{a_{n+2}}{q_{n+2}} = \dfrac{a_{n+2}}{q_{n+1}a_{n+2}+q_n} = \dfrac{1}{q_{n+1} + \dfrac{q_n}{a_{n+2}}} \geq \dfrac{1}{q_{n+1} + q_n}$

since $a_{n+2} \geq 1$ and from theorem 3.1, $q_{n+2} = q_{n+1}a_{n+2} + q_n$. Thus,

$$\frac{a_{n+2}}{(q_n+q_{n+1})q_n} \leq \frac{a_{n+2}}{q_n q_{n+2}} < \left|\alpha - \frac{p_n}{q_n}\right|$$

Then, since $a_{n+2} \geq 1$, it is seen that

$$\frac{1}{(q_n+q_{n+1})q_n} \leq \frac{a_{n+2}}{(q_n+q_{n+1})q_n} < \left|\alpha - \frac{p_n}{q_n}\right|$$

and the theorem is finished.

Theorem 4.15. $|q_{n-1}(\quad q_{n-1} - p_{n-1})| < 1$.

Proof: From theorem 4.14 it is seen that

$$\left|\frac{\alpha q_{n-1} - p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_{n-1}^2} .$$

This implies

$$\frac{|q_{n-1}(\alpha q_{n-1} - p_{n-1})|}{q_{n-1}^2} < \frac{1}{q_{n-1}^2} .$$

Thus, $q_{n-1}(\alpha q_{n-1} - p_{n-1}) < 1$. The proof is finished.

Theorem 4.16. A real number α is rational if and only if its continued fraction is finite.

Proof: Suppose α is rational, $\alpha = \frac{p}{q}$ where p and q are integers, $q \geq 1$ and $\frac{p}{q}$ is in lowest terms. By the division algorithm, there exist unique integers band c such that $p = qb + c$ with $b \geq 1$ and $0 \leq c \leq q$. Thus,

$$p = qb_0 + c_1 \text{ and } \frac{p}{q} = b_0 + \frac{c_1}{q} .$$

Then

$$a_0 = b_0 \text{ , and } r_1 = \frac{1}{\frac{c_1}{q}} = \frac{q}{c_1} \text{ where } q > c_1 .$$

Then

$$\frac{q}{c_1} = b_1 + \frac{c_2}{c_1} \text{ , and}$$

$$a_1 = b_1 \text{ , } r_2 = \frac{1}{\frac{c_2}{c_1}} = \frac{c_1}{c_2} \text{ where } c_1 > c_2$$

Then

$$\frac{c_1}{c_2} = b_2 + \frac{c_3}{c_2} \text{ , and}$$

$$a_2 = b_2 \text{ , } r_3 = \frac{1}{\frac{c_3}{c_2}} = \frac{c_2}{c_3} \text{ where } c_2 > c_3$$

. . .

It is seen that $q > c_1 > c_2 \dots > c_n$. Since the c_i are integers, there are only a finite number of c_i such that $q > c_i$. Thus, for some n, $c_n = 0$. Then by definition 4.1, the expansion stops and is finite. (This argument is analogous to Euclid's Algorithm).

If the expansion is finite, $\{a_0; a_1 a_2 \dots a_n\}$ then

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}$$

and α is rational. Thus, the proof is finished.

Theorem 4.17. An irrational number is algebraic of degree 2 if and only if its continued fraction expansion is periodic.

An outline of the proof follows. First prove that if the continued fraction is periodic, α is algebraic of degree 2.

Next, to prove that if α is algebraic of degree 2 the expansion is periodic, let α satisfy the equation

$$A\alpha^2 + B\alpha + C = 0$$

where A, B and C are integers and $A \neq 0$. Show that there are only a finite number (n) of possible r_n's. Then for some k,

$$r_k = r_{k+n}$$

$$r_{k+1} = r_{k+n+1}$$

$$\cdots$$

$$r_{k+L} = r_{(k+n)+L}$$

for every integer L. Then, since $a_i = [r_i]$, $a_{k+L} = a_{(k+n)+L}$ for every integer L and the proof will be complete.

Proof: If the continued fraction is periodic, then starting from some place

$$\alpha = \{a_0; a_1 \ldots a_k \bar{a}_1 \ldots \bar{a}_m \bar{a}_1 \ldots\}$$

then

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k + \cfrac{1}{\bar{a}_1 + \cfrac{1}{\bar{a}_2 + \cdots + \cfrac{1}{\bar{a}_m + \cfrac{1}{\cdots}}}}}}}$$

and $r = \{; \bar{a}_1 \ldots \bar{a}_n \bar{a}_1 \ldots\}$ is the kth remainder.

By theorem 4.8,

(1) $$\alpha = \frac{p_k r + p_{k-1}}{q_k r + q_{k-1}} .$$

Then, since the m^{th} remainder of r is also r,

(2) $$r = \frac{\bar{p}_m r + \bar{p}_{m-1}}{\bar{q}_m r + \bar{q}_{m-1}} .$$

Solving (1) for r,

(3) $$r = \frac{p_{k-1} - \alpha q_{k-1}}{\alpha q_k - p_k} .$$

Substituting the value of r from (3) into (2)

$$\frac{p_{k-1} - \alpha q_{k-1}}{\alpha q_k - p_k} = \frac{\bar{p}_m \dfrac{p_{k-1} - \alpha q_{k-1}}{\alpha q_k - p_k} + \bar{p}_{m-1}}{\bar{q}_m \dfrac{p_{k-1} - \alpha q_{k-1}}{\alpha q_k - p_k} + \bar{q}_{m-1}} .$$

Then,

$$\frac{p_{k-1} - \alpha q_{k-1}}{\alpha q_k - p_k} = \frac{(\bar{p}_{m-1} q_k - \bar{p}_m q_{k-1})\alpha + (\bar{p}_m p_{k-1} - \bar{p}_{m-1} p_k)}{(\bar{q}_{m-1} q_k - \bar{q}_m q_{k-1})\alpha + (\bar{q}_m p_{k-1} - \bar{q}_{m-1} p_k)} .$$

But since p_i, q_i, $\bar{p}_i$ and $\bar{q}_i$ are integers, the equation may be simplified to

$$\frac{a - b\alpha}{c\alpha - d} = \frac{e + f\alpha}{g + h\alpha}$$

where a,b, ..., h are integers. Then

$$(cf - bh)\alpha^2 + (ce - df + bg - ah)\alpha + (-de - ag) = 0.$$

Thus, is the root of an equation of the form

$$A\alpha^2 + B\alpha + c = 0$$

with integer coefficients. Thus, α is algebraic of degree 2.

Next, let α be algebraic of degree 2. Let α satisfy the equation

$$A\alpha^2 + B\alpha + C = 0$$

where A, B, and C are integers and A $\neq$ 0. Let $\{a_0; a_1 a_2 \dots\}$ be its continued fraction and let $\frac{p_i}{q_i}$ be its convergents. Let r_n be its n^{th} remainder.

Then, from theorem 4.3,

$$\alpha = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}$$

and

$$A \cdot \left(\frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}} \right)^2 + B \cdot \left(\frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}} \right) + C = 0.$$

Then

$$A(p_{n-1}^2 r_n^2 + 2p_{n-1}p_{n-2}r_n + p_{n-2}^2) + B\cdot(p_{n-1}q_{n-1}r_n^2 + p_{n-1}r_n$$
$$+ p_{n-2}q_{n-2}) + C\ (q_{n-1}^2 r_n^2 + 2\ q_{n-1}r_n + q_{n-2}^2) = 0$$

Thus,

$$(Ap_{n-1}^2 + B\cdot p_{n-1}q_{n-1} + C\cdot q_{n-1}^2)\ r_n^2 + (2A\cdot p_{n-1}p_{n-2} + B\cdot p_{n-2}q_{n-1} +$$
$$B\cdot p_{n-1}q_{n-2} + 2\ C\cdot q_{n-1}q_{n-2})\ r_n + (A\cdot p_{n-2}^2 + B\cdot p_{n-2}q_{n-2}$$
$$+ C\cdot q_{n-2}^2) = 0.$$

Let

$$A_n = A\cdot p_{n-1}^2 + B\cdot p_{n-1}q_{n-1} + C\cdot q_{n-1}^2$$

$$B_n = 2\ A\cdot p_{n-1}p_{n-2} + B\cdot p_{n-2}q_{n-1} + Bp_{n-1}q_{n-2} + 2\ C\cdot q_{n-1}q_{n-2}$$

$$C_n = A\cdot p_{n-2}^2 + B\cdot p_{n-2}q_{n-2} + C\cdot q_{n-2}^2\ .$$

Then r_n is a root of the quadratic equation

$$A_n r_n^2 + B_n r_n + C_n = 0$$

with integer coefficients A_n, B_n, C_n.

It will be shown that there are only a finite number of solutions possible. First note that $C_n = A_{n-1}$ and $B_n^2 - 4A_nC_n = B^2 - 4AC$.

Thus, if it can be shown that the A_n's are bounded it will follow that $C_n = A_{n-1}$ and $B_n = \sqrt{B^2 - 4AC + 4A_nC_n}$ are bounded. Then, since A_n, B_n and C_n are integers, there are only a finite number of equations

$$A_nr_n^2 + B_nr_n + C_n = 0$$

possible. Thus r_n will be seen to have only a finite number of possible values.

If $A_n = 0$, then $C_n = A_{n-1} = 0$ and $B_n = \sqrt{B^2 - 4AC}$ is bounded by a fixed number independent of n. Then the equation reduces to $B_nr_n = 0$ which has only a finite number of solutions. Next, consider the case that $A_n \neq 0$.

To show that the A_n's are bounded, from theorem 4.15 it is seen that

$$\left| q_{n-1}(q_{n-1}\alpha - p_{n-1}) \right| < 1 .$$

Let

$$q_{n-1}(q_{n-1} \quad - p_{n-1}) = \delta_n .$$

Then $|\delta_n| < 1$ and $p_{n-1} = q_{n-1}\alpha - \dfrac{\delta_n}{q_{n-1}}$

Thus,

$$A_n = A\left(q_{n-1}\alpha - \frac{\delta_n}{q_{n-1}}\right)^2 + B\left(q_{n-1}\alpha - \frac{\delta_n}{q_{n-1}}\right)q_{n-1} + C\, q_{n-1}^2$$

$$= Aq_{n-1}^2\alpha^2 - A\alpha\delta_n + \frac{A\delta_n^2}{q_{n-1}^2} + B\, q_{n-1}^2\alpha - B\delta_n + C\, q_{n-1}^2$$

$$= (A\alpha^2 + B\alpha + C)q_{n-1}^{\ 2} - (2A\alpha + B)\delta_n + A\frac{\delta_n^{\ 2}}{q_{n-1}^2}$$

But $A\alpha^2 + B\alpha + C = 0$ by hypothesis, so

$$A_n = -(2A\alpha + B)\delta_n + A\frac{\delta_n^{\ 2}}{q_{n-1}^2} .$$

then

$$|A_n| = \left| -2(A \quad + B)\delta_n + A\frac{\delta_n^{\ 2}}{q_{n-1}^2} \right|$$

$$\leq |2(A\alpha + B)\delta_n| + \left| A\frac{\delta_n^{\ 2}}{q_{n-1}^2} \right|$$

Then since $|\delta_n| < 1$ and $\frac{1}{q_{n-1}^2} < 1$,

$$|A_n| < |2A\alpha + B| + |A|$$

Since A, B and α are fixed numbers, it is seen that $|A_n|$ is bounded. Thus A_n is bounded by a fixed number independent of n, and the proof is complete.

The continued fraction expansion gives more information about the nature of a number than either the decimal or Cantor expansions. Not only can it be determined if a number is rational or irrational, but it can be determined if an irrational number is algebraic of degree 2. It is not known if the continued fraction can give any information about algebraic numbers of higher degree than 2.

The continued fraction is seen to embody two basic concepts of real numbers (i.e., the Dedekind cut and convergent sequences of rationals). In the next chapter it will be seen that the continued fractions are very good approximations to real numbers.

Chapter V.

APPROXIMATION OF REAL NUMBERS BY RATIONALS

This chapter is devoted to a study of approximations of real numbers by rationals, and how approximations are affected by the number being either algebraic or transcendental.

From theorem 2.4 and theorem 3.5 respectively, the accuracy of approximation of the partial sum of a real number α is given as

$$0 \leq \alpha - \frac{r_n}{t_n} < \frac{1}{t^n}$$

where $t_n = 10^n$ for the decimal expansion and $t_n = b_1 b_2 \cdots b_n$ for the Cantor expansion. For the continued fraction, the accuracy of approximation is given by theorem 4.14 as

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} .$$

Comparing the right side of both inequalities would seem to indicate that the continued fraction expansion gives a better rational approximation than either the decimal or the Cantor, since

$$\frac{1}{q_n^2} < \frac{1}{t_n} \text{ if } q_n = t_n .$$

Theorem 5.1 shows that every rational number with a denominator $b \leq q_n$ is at a greater distance from α than the convergent $\frac{p_n}{q_n}$.

Theorem 5.1. If $\frac{p_n}{q_n}$ is the n^{th} convergent to α, then for any rational $\frac{a}{b} \neq \frac{p_n}{q_n}$ with $1 \leq b \leq q_n$

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{a}{b}\right| .$$

Proof: Consider

$$\left|\alpha - \frac{a}{b}\right| - \left|\alpha - \frac{p_n}{q_n}\right| = \frac{q_n|b\alpha - a| - b|q_n\alpha - p_n|}{bq_n}$$

since $1 \leq b \leq q_n$ implies $bq_n > 0$. Then

$$\left|\alpha - \frac{a}{b}\right| - \left|\alpha - \frac{p_n}{q_n}\right| \geq \frac{b|b\alpha - a| - b|q_n\alpha - p_n|}{q_n b}$$

since $b \leq q_n$. But

$$\frac{b|b\alpha - a| - b|q_n\alpha - p_n|}{q_n b} = \frac{|b\alpha - a| - |q_n\alpha - p_n|}{q_n}$$

(i) Thus, if $|b\alpha - a| > |q_n\alpha - p_n|$, then

$$\left|\alpha - \frac{a}{b}\right| - \left|\alpha - \frac{p_n}{q_n}\right| > 0 \text{ and } \left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{a}{b}\right|.$$

First, suppose $\frac{a}{b} = \frac{p_{n-1}}{q_{n-1}}$ $(\neq \frac{p_n}{q_n})$. Then from theorem 4.12,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

This implies $|q_n\alpha - p_n| < \frac{1}{q_{n+1}}$. Also from theorem 4.12,

$$\frac{a_{n+1}}{q_{n-1}q_{n+1}} < \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right|.$$

This implies

$$\frac{a_{n+1}}{q_{n+1}} < |q_{n-1}\alpha - p_{n-1}|.$$

But

$$\frac{1}{q_{n+1}} \leq \frac{a_{n+1}}{q_{n+1}} \text{ since } a_i \geq 1.$$

So

$$\left|q_n\alpha - p_n\right| < \frac{1}{q_{n+1}} < \left|q_{n-1}\alpha - p_{n-1}\right|$$

If $\frac{a}{b} \neq \frac{p_{n-1}}{q_{n-1}}$, then $\left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| > 0$.

But

$$\left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{\left|aq_{n-1} - bp_{n-1}\right|}{bq_{n-1}}.$$

Thus,

$$\frac{\left|aq_{n-1} - bp_{n-1}\right|}{bq_{n-1}} > 0.$$

Since a, b, p_{n-1} and q_{n-1} are integers, then $\left|aq_{n-1} - bp_{n-1}\right|$ is an integer greater than 0. Thus $\left|aq_{n-1} - bp_{n-1}\right| \geq 1$. Then

$$\left|\frac{a}{b} - \alpha\right| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| \geq \left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| \geq \frac{1}{bq_{n-1}}$$

which implies

(1) $$q_{n-1}\left|b\alpha - a\right| + b\left|q_{n-1}\alpha - p_{n-1}\right| \geq 1.$$

But, from theorem 4.11, and since $b \leq q_n$, it is seen that

$$\frac{1}{q_n q_{n-1}} = \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| + \left|\alpha - \frac{p_n}{q_n}\right|.$$

This implies

$$1 = q_n \cdot \left| q_{n-1}\alpha - p_{n-1} \right| + q_{n-1} \cdot \left| q_n \alpha - p_n \right| .$$

Then, since $b \leq q_n$

(2)
$$1 \geq b \left| q_{n-1}\alpha - p_{n-1} \right| + q_{n-1} \left| q_n \alpha - p_n \right| .$$

Combining (1) and (2) gives

$$q_{n-1} \left| b\alpha - a \right| + b \left| q_{n-1}\alpha - p_{n-1} \right| \geq b \left| q_{n-1}\alpha - p_{n-1} \right|$$

$$+ q_{n-1} \left| q_n \alpha - p_n \right| .$$

Which implies

$$\left| b\alpha - a \right| \geq \left| q_n \alpha - p_n \right| .$$

In order to get (i) it is sufficient to prove that $\left| b\alpha - a \right| = \left| q_n\alpha - p_n \right|$ is impossible.

Suppose α is irrational, then

$$b\alpha - a = \pm (q_n \alpha - p_n)$$

implies $b = q_n$ and $a = p_n$ which contradicts the assumption that $\frac{a}{b} \neq \frac{p_n}{q_n}$. If α is rational and if $\frac{p_n}{q_n}$ is the last convergent, then

$$0 = \left| \alpha - \frac{p_n}{q_n} \right| .$$

This implies

$$0 = \left| q_n \alpha - p_n \right| .$$

Then if $\left| q_n\alpha - p_n \right| = \left| b\alpha - a \right|$, it is seen that $0 = \left| b\alpha - a \right|$ which implies $0 = \left| \alpha - \frac{a}{b} \right|$. Then $\alpha = \frac{a}{b} = \frac{p_n}{q_n}$, contradicting $\frac{a}{b} \neq \frac{p_n}{q_n}$.

If $\frac{p_n}{q_n}$ is not the last convergent to α, then let $\frac{p_N}{q_N}$ be the last convergent. Then $|q_n\alpha - p_n| = |b\alpha - a|$ implies that

$$\left| q_n \cdot \frac{p_N}{q_N} - p_n \right| = \left| b \cdot \frac{p_N}{q_N} - a \right|$$

This implies $|q_N p_N - q_N p_n| = |b p_N - a q_N|$

So either $q_N p_N - q_N p_n = b p_N - a q_N$ or

$$q_N p_N - q_N p_n = - b p_N + a\, q_N \; .$$

Then, either $(q_n - b)p_N = (p_n - a)q_N$ or $(q_{n+b})p_N = (p_n - a)q_N$.

Since p_N and q_N have no common divisor by theorem 4.4, the first case implies $q_n - b$ is divisible by q_N. But $q_N > q_n - b$. Thus, $q_n - b$ must equal 0 to be divisible by q_N. Now, $q_n - b = 0$ implies $q_n = b$. Similarly $p_n = a$ which contradicts $\frac{a}{b} \neq \frac{p_n}{q_n}$.

In the case $(q_n + b)p_N = (p_n+a)q_N$, $q_n + b$ must be divisible by q_N which is greater than q_n. Thus $q_n + b > 2q_n$ which contradicts $b \leq q_n$. Thus, $|q_n\alpha - p_n| = |b\alpha - a|$ is impossible and the theorem is proved.

The converse to theorem 5.1 is not true, as the following example shows.

Example 5.1. Let $= \frac{1}{5}$, $p = 1$, $q = 3$. Then $\frac{p}{q} = \frac{1}{3}$.

The only two convergents to α are $f_0 = 0$ and $f_1 = \frac{1}{5}$. But for every rational $\frac{a}{b} \neq \frac{1}{3}$ with $1 \leq b \leq 3$, it is true that $\left|\frac{1}{5} - \frac{1}{3}\right|$

$\left|\frac{1}{5} - \frac{1}{3}\right| < \left|\frac{1}{5} - \frac{a}{b}\right|$

To see this, note that $\left|\frac{1}{5}-\frac{1}{3}\right|=\frac{2}{15}$. Suppose $\frac{2}{15}\geq\left|\frac{1}{5}-\frac{a}{b}\right|$ and $1\leq b\leq 3$. Then $\frac{1}{5}\leq\frac{a}{b}\leq\frac{1}{3}$. This implies $b=3$ and $a=1$, contradicting $\frac{a}{b}\neq\frac{1}{3}$.

Thus, a convergent is always a better approximation than others, but an approximation that is better than others need not be a convergent. To get the implication both ways, the following definition is introduced.

Definition 5.1. (Best approximation). A rational $\frac{p}{q}$ with $q\geq 1$ is called the best approximation to a real number α if and only if for every rational $\frac{a}{b}\neq\frac{p}{q}$ with $1\leq b\leq q$, one has

$$|q\alpha-p|<|b\alpha-a|$$

It should be noted here that

$$|q\alpha-p|<|b\alpha-a| \text{ implies } \left|\alpha-\frac{p}{q}\right|<\left|\alpha-\frac{a}{b}\right|$$

because

$$\left|\alpha-\frac{p}{q}\right|=\frac{1}{q}\cdot|q\alpha-p|<\frac{1}{q}|b\alpha-a|\leq\frac{1}{b}|b\alpha-a|=\left|\alpha-\frac{a}{b}\right|.$$

Theorem 5.2. A rational $\frac{p}{q}$ with $1\leq q\leq B$ is the best approximation to a real number α if and only if $\frac{p}{q}$ is one of the convergents $\frac{p_n}{q_n}$ to with $q_n\leq B$ for $n\geq 1$.

Proof: Let $\frac{p_n}{q_n}$ be the nth convergent to α. It must be proved that for every $\frac{a}{b}\neq\frac{p_n}{q_n}$ with $1\leq b\leq q_n$ it is true that $|q_n\alpha-p_n|<|b\alpha-a|$. This is seen to be true from the argument of theorem 5.2.

Next, let $\frac{p}{q}$ be the best approximation to α. Then for every rational $\frac{a}{b}\neq\frac{p}{q}$ with $1\leq a\leq b$,

(1) $\quad |q\alpha - p| < |b\alpha - a|.$

It must be proved that $\frac{p}{q}$ is one of the convergents $\frac{p_n}{q_n}$ to α. It will first be proved that

$$\frac{p_0}{q_0} \leq \frac{p}{q} \leq \frac{p_1}{q_1} .$$

If $\frac{p}{q} < \frac{p_0}{q_0}$, then since $\alpha > \frac{p_0}{q_0}$ and since $q_0 = 1$ it is seen that $\frac{p}{q} < \frac{p_0}{1}$.

Then

$$|q_0\alpha - p_0| = |\alpha - p_0| < \left|\alpha - \frac{p}{q}\right| \leq q\left|\alpha - \frac{p}{q}\right| = |q\alpha - p|$$

So

$|q_0\alpha - p_0| < |q\alpha - p|$ which contradicts (1).

If $\frac{p}{q} > \frac{p_1}{q_1}$, then $\left|\frac{p}{q} - \frac{p_1}{q_1}\right| > 0$ and $\frac{|q_1 p - qp_1|}{qq_1} > 0.$

This implies that the integer $|q_1 p - qp_1| \geq 1.$

Then, since $\alpha < \frac{p_1}{q_1}$,

$$\left|\alpha - \frac{p}{q}\right| > \left|\frac{p_1}{q_1} - \frac{p}{q}\right| \geq \frac{1}{qq_1}$$

and $|q\alpha - p| > \frac{1}{q_1}$. Then by theorem 4.14, $\frac{1}{q_1^2} > \left|\alpha - \frac{p_1}{q_1}\right|$ which implies $\frac{1}{q_1} > |q_1\alpha - p_1|$. Then $|q\alpha - p| > |q_1\alpha - p_1|$ again contradicting (1). If $\frac{p}{q}$ is equal to either $\frac{p_0}{q_0}$ or $\frac{p_1}{q_1}$ the proof is finished.

If not, suppose $\frac{p}{q}$ is not one of the successive convergents. Then $\frac{p}{q}$ lies between two convergents

$$\frac{p_{n-1}}{q_{n-1}} \quad \text{and} \quad \frac{p_{n+1}}{q_{n+1}}$$

both of which are on the same side of α. Then

$$\frac{1}{qq_{n-1}} \leq \left|\frac{p}{q} - \frac{p_{n-1}}{q_{n-1}}\right| < \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}} .$$

So $\frac{1}{qq_{n-1}} < \frac{1}{q_n q_{n-1}}$ which implies $1 \leq q_n \leq q$. Now,

$$\left|\alpha - \frac{p}{q}\right| \geq \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p}{q}\right| \geq \frac{1}{qq_{n+1}}$$

using theorem 4.2. This implies

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{qq_{n+1}} .$$

Then $|q\alpha - p| \geq \frac{1}{q_{n+1}}$. But $\frac{1}{q_{n+1}} > |q_n\alpha - p_n|$ from theorem 4.12. Thus, $|q\alpha - p| > |q_n\alpha - p_n|$ and $1 \leq q_n < q$. This contradicts (1). Thus, the supposition that $\frac{p}{q}$ is not a convergent is false, and the proof is finished.

To see that the theorem is not true for $n = 0$, consider the following example.

Example 5.2. Let $\alpha = 3 + \frac{1}{2}$. The convergent f_0 to α is $\frac{3}{1}$ but this is not the best approximation because there is another best approximation, $\frac{4}{1}$ with $1 \leq 1 \leq 1$.

Theorem 5.2 leads to the problem of how to recognize whether a rational $\frac{p}{q}$ is one of the convergents to a real number α. This is answered in the next theorem.

Theorem 5.3. If the integers $q \geq 1$ and p satisfy the condition

$$|q\alpha - p| < \frac{1}{2q}$$

then $\frac{p}{q}$ is one of the convergents to α.

Proof: It is sufficient to prove that $\frac{p}{q}$ is the best approximation to α. The proof is by contradiction. Suppose $\frac{p}{q}$ is not the best approximation to α. Then let $\frac{a}{b} \neq \frac{p}{q}$ be the best. Then for $1 \leq b \leq q$, let

$$|b\alpha - a| < |q\alpha - p| < \frac{1}{2q} .$$

Then, $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2bq}$ and $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.

Then $\left|\frac{p}{q} - \frac{a}{b}\right| = \left|(\frac{p}{q} - \alpha) + (\alpha - \frac{a}{b})\right| \leq \left|\alpha - \frac{p}{q}\right| + \left|\alpha - \frac{a}{b}\right|$

and $\left|\frac{p}{q} - \frac{a}{b}\right| < \frac{1}{2bq} + \frac{1}{2q^2} = \frac{b + q}{2bq^2}$.

Then, $|pb - aq| < \frac{b+q}{2q}$. But $\frac{a}{b} \neq \frac{p}{q}$ implies $|b \cdot p - aq| \geq 1$. Therefore $\frac{b+q}{2q} > 1$. This implies $b+q > 2q$ which implies $b > q$. This contradicts $b \leq q$. Hence, $\frac{p}{q}$ is the best approximation to α and $\frac{p}{q}$ is a convergent to α.

Definition 5.2. (Accuracy of approximation) A number α can be approximated with accuracy of degree n if and only if the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

has infinitely many solutions $\frac{p}{q}$.

The next theorem shows the connection between algebraic numbers and approximation by rationals. The theorem is due to J. Liouville in 1851 [1, page 65].

Theorem 5.4. (Liouville's Theorem) If α is an irrational algebraic number of degree n, α cannot be approximated to any degree higher than n.

Proof: Let α be an algebraic number of degree n. Then it must be shown that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^{n+k}}$$

for all rationals $\frac{p}{q}$, except possibly for a finite number of them, for every $k \geq 1$. It is sufficient to prove the inequality for $k = 1$ since $\frac{1}{q^{n+k}} \leq \frac{1}{q^{n+1}}$ for $k > 1$.

Suppose that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n+1}}$. Now, $\left|\frac{p}{q}\right| - |\alpha| \leq \left|\alpha - \frac{p}{q}\right|$ and also $q \geq 1$.

Thus, $\left|\frac{p}{q}\right| - |\alpha| < \frac{1}{q_{n+1}} \leq 1$ since $q_{n+1} \geq 1$. Then $\frac{p}{q} < |\alpha| + 1$.

Since α is an algebraic number of degree $n > 1$, there exists a polynomial

$$P_n(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$$

such that $P_n(\alpha) = 0$ and there is no such polynomial of degree less than n. This implies α is irrational, otherwise α would satisfy a polynomial equation of first degree. It will be shown that $P_n(x)$ has no rational roots. If $x = \frac{p}{q}$ is a root, then, since $P_n(\alpha) = 0$ it is seen that

$$P_n\left(\frac{p}{q}\right) = P_n\left(\frac{p}{q}\right) - P_n(\alpha)$$

$$= A_n\left(\frac{p}{q}\right)^n + A_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + A_1\left(\frac{p}{q}\right) + A_0$$

$$- A_n\alpha^n - A_{n-1}\alpha^{n-1} - \ldots - A_1\alpha - A_0$$

$$= A_n\left(\left(\frac{p}{q}\right)^n - \alpha^n\right) + A_{n-1}\left(\left(\frac{p}{q}\right)^{n-1} - \alpha^{n-1}\right) + \dots + A_1\left(\frac{p}{q} - \alpha\right)$$

(1) $$= \left(\alpha - \frac{p}{q}\right)\Bigg(A_n\left(\alpha^{n-1} + \frac{p}{q}\alpha^{n-2} + \dots + \left(\frac{p}{q}\right)^{n-1}\right) +$$

$$+ A_{n-1}\left(\alpha^{n-2} + \frac{p}{q}\alpha^{n-3} + \dots + \left(\frac{p}{q}\right)^{n-2}\right)$$

$$+ \dots + A_2\left(\alpha + \frac{p}{q}\right) + A_1\Bigg)$$

Then, either $\alpha - \frac{p}{q} = 0$ which is impossible since α is irrational or

$$A_n\left(\alpha^{n-1} + \frac{p}{q}\alpha^{n-2} + \dots + \left(\frac{p}{q}\right)^{n-1}\right) + A_{n-1}\left(\alpha^{n-2} + \frac{p}{q}\alpha^{n-3} + \dots\right.$$

$$\left. + \left(\frac{p}{q}\right)^{n-2}\right) + \dots + A_2\left(\alpha + \frac{p}{q}\right) + A_1 = 0.$$

But this is a polynomial equation of degree n-1 in α which contradicts the assumption that α is an algebraic number of degree n. Thus $P_n\left(\frac{p}{q}\right) \neq 0$. Then for any rational $\frac{p}{q}$,

(2) $$\left|P_n\left(\frac{p}{q}\right)\right| = \frac{1}{q^n}\left|A_n p^n + A_{n-1}p^{n-1}q + \dots + A_1 p q^{n-1} + A_0 q^n\right| \geq \frac{1}{q^n}$$

because $\left|A_n p^n + A_{n-1}p^{n-1}q + \dots + A_0 q^n\right| \geq 1$ since A_i are integers, P is an integer and q is an integer implies

$$\left|A_n p^n + A_{n-1}p^{n-1}q + \dots + A_0 q^n\right| \text{ is an integer } \geq 1.$$

Thus, for any rational $\frac{p}{q}$, $\left|P_n\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^n}$.

Next, consider all $\frac{p}{q}$ for which $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$. It must be that $\left|\frac{p}{q}\right| < |\alpha| + 1$. Also, it must be that for those $\frac{p}{q}$, using (1) and (2)

$$\frac{1}{q^n} \leq \left|\alpha - \frac{p}{q}\right| \cdot \left(|A_n| \left(|\alpha^{n-1}| + \left|\frac{p}{q}\right| \cdot |\alpha^{n-2}| + \cdots + \left|\frac{p}{q}\right|^{n-1}\right)\right.$$

$$\left. + \cdots + |A_2|\left(|\alpha| + \left|\frac{p}{q}\right|\right) + |A_1|\right)$$

$$< \left|\alpha - \frac{p}{q}\right| \left(|A_n| \cdot n \cdot (|\alpha| + 1)^{n-1} + \cdots + |A_2| \cdot 2 \cdot (\alpha + 1) + |A_1|\right)$$

since $|\alpha| < |\alpha| + 1 < n \cdot (|\alpha| + 1)$, etc. Denote the quantity $\left(|A_n| \cdot n(|\alpha| + 1)^{n-1}\right.$ $\left. + \cdots + |A_2| 2(|\alpha| + 1) + |A_1|\right)$ by b. b is fixed and independent of $\frac{p}{q}$.

Then, if $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n+1}}$ it is seen that $\frac{1}{q^n} < \left|\alpha - \frac{p}{q}\right|$ b.

So

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n+1}} \text{ and } \frac{1}{q^n} < \left|\alpha - \frac{p}{q}\right| b.$$

This implies $\frac{1}{q^n} < \frac{1}{q^{n+1}} \cdot b$, which implies $q < b$. Then since q is an integer, there are only a finite number of values possible for q. Then $\left|\frac{p}{q}\right| < |\alpha| + 1$, which implies $|P| < |q|(|\alpha| + 1)$ and there are only a finite number of values possible for p.

Thus, there are only a finite number of $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n+1}} .$$

The theorem is proved.

This theorem indicates that algebraic numbers cannot be approximated beyond a certain degree for any given algebraic number. If a number can be approximated to any given degree $n > 2$, then the number must be non-algebraic. (i.e., transcendental).

Definition 5.3. A real number is called a Liouville's number if it can be approximated with accuracy of any degree $n > 2$.

The fact that transcendental numbers exist was first proved in 1851 by Liouville [1, page 65] . A proof that transcendental numbers exist follows.

Theorem 5.5 For any given exponent $n > 2$ there always exist (irrational) numbers α such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

has infinitely many solutions $\frac{p}{q}$.

Proof: Given n, the number α can be constructed from its continued fraction expansion. Let a_0 be any integer. Choose $a_1, a_2, \ldots$ such that

$$a_{k+1} = q_k^{n-2} \text{ where } \frac{p_k}{q_k} \text{ is the } k^{\text{th}} \text{ convergent.}$$

Then by theorem 4.1 and theorem 4.12,

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}} = \frac{1}{q_k(q_k a_{k+1} + q_{k-1})} < \frac{1}{q_{k+1}^2 a_{k+1}} \qquad \frac{1}{q_k^n}$$

for all convergents $\frac{p_k}{q_k}$ to α. Thus,

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k^n} \text{ for infinitely many rationals } \frac{p}{q} .$$

The continued fraction expansion is an improvement over the decimal and the Cantor expansion in that it is possible to determine whether a number is algebraic of degree 2.

Chapter VI.

CONCLUSION

It is easily seen that the set of algebraic real numbers is countable. Let the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

have integer coefficients. Since the integers are countable, there are only a countable number of such equations of degree n. Each equation of degree n has exactly n roots. Thus the set of all roots to polynomial equations with integer coefficients (i. e., the algebraic numbers) is countable. Since the real numbers are not countable it follows that the non-algebraic (transcendental) numbers are not countable. Thus, there are more transcendental numbers(in a sense) than algebraic numbers.

The problem of determining whether a particular real number is algebraic or transcendental is very difficult. Using the facts established in this paper, the best procedure to determine if a particular number is algebraic or transcendental would seem to be as follows.

(1) Derive the decimal expansion. If the decimal expansion is either finite or periodic, the number is rational.

(2) If the decimal expansion is infinite and not periodic, then derive the continued fraction expansion. If the continued fraction expansion is finite, then the number is rational. If the expansion is infinite and periodic, the number is algebraic of degree 2.

(3) If the continued fraction is infinite and not periodic, then attempt to approximate the number by rationals. Using definition 5.2 and theorem 5.4, determine if it is possible for the number to be algebraic of some degree n. If not, then the number is transcendental.

The fact that the Cantor expansion terminates for rational numbers if a proper base-sequence is chosen suggests an area for further study. An algebraic number of degree 2 can be written in the form

$$a_1\sqrt{b_1} + a_2\sqrt{b_2} + \dots + a_n\sqrt{b_n} + c$$

where a_i, b_i, and c are integers. The question then arises: Is there a possibility of choosing a base-sequence involving the square roots of the integers (or primes) such that the Cantor expansion of an algebraic number of degree 2 will be finite using that expansion? If this is possible, then perhaps the results can be generalized to algebraic numbers of higher degree.

An area for further study suggested by the continued fractions is whether they can give any information about algebraic numbers of higher degree than 2. If not, perhaps a study of the reason for the difficulty would suggest another type of expansion that would determine if a number is algebraic of higher degree than 2.

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