A STUDY OF THE FIBONACCI NUMBER SEQUENCE RELATIVE TO NUMBER THEORY

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CHAPTER I

THE FIBONACCI NUMBER SEQUENCE

The Fibonacci number sequence encompasses an amazingly large segment of the fields of art and architecture while nature has many examples. Some flowers even have their petals arranged in rows such that the number of petals in each row belongs to the Fibonacci number sequence.

Obviously, the domain and range of the Fibonacci number sequence is quite large. Segments of more advanced work may be found in the Fibonacci Quarterly while the more basic properties may be found in Fibonacci Numbers, by N. N. Vorob'ev.

This study was confined to the field of number theory in conjuction with the experimentalist approach. An attempt to develop a concrete foundation of the basic properties with respect to number theory was made. Some of these properties of the Fibonacci number sequence were obvious while others were quite difficult to ascertain.

The plan of investigation was observation of certain patterns which seemed to exist, creation of an hypothesis, and either proof or rejection of the hypothesis. Many of the proofs were done by mathematical induction.

Fibonacci numbers originated from the terms of a recurrent sequence which was developed by the thirteenth century mathematician Leonardo Fibonacci.

One among many of Fibonacci's achievements was his part in the introduction of the Hindu-Arabic numeral system to the western world.

An elementary problem which illustrates the Fibonacci number sequence is the time honored "rabbit problem": "Someone placed a pair of rabbits in a certain place, enclosed on all sides by a wall, to find out how many pairs of rabbits will be born there in the course of one year, it being assumed that every month a pair of rabbits produces another pair, and that rabbits begin to bear young in two months after their own birth."[1] From this information one may construct a chart as follows:

End of Month	Number of Pairs	Number of Pairs Reproducing
0	1	1
1	2	1
2	3	2
3	5	3
4	8	5
5	13	8
6	21	13
7	34	21
8	55	34
9	89	55
10	144	89
11	233	144
12	377	233

From the chart one should note that at the end of the

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[1]N. N. Vorob'ev, Fibonacci Numbers, p. 2.

first month there were two pairs of rabbits, one pair was able to reproduce. This one pair reproduced one pair so that at the end of the second month there were three pairs, two of which were capable of reproduction. These two pairs each reproduced one more pair which resulted in there being five pairs of rabbits at the end of the third month.

Three pairs of rabbits at the end of the third month were capable of reproduction. The result was three more pairs of rabbits for a total of eight pairs after four months. Hence, at the end of the fourth month there were eight pairs of rabbits of which five pairs reproduced during the next month.

This process continued until at the end of twelve months there were 377 pairs of rabbits in the enclosed yard. 376 of the 377 were born during the one year.

If one examined the number of pairs column in the chart and treated it as a sequence of numbers, then it should not have been difficult to have seen that this sequence of numbers could have been generated by the equation $a_n = a_{n-1} + a_{n-2}$ where n is greater than or equal to three with $a_1 = 1$ and $a_2 = 2$.

Resulting from the defining equation of the sequence, the sum of any two adjacent Fibonacci numbers was a Fibonacci number. Likewise, the difference between any two adjacent Fibonacci numbers was a Fibonacci number. These two facts were the starting point of this study. Other properties followed. But, not without much time and effort.

The numbers chosen for the first two terms of this sequence could have been 0 and 1, 1 and 1, or 1 and 2. Each of these generated the same sequence except that one of the sequences contained 0 as an element while the sequence with 1 and 2 as the first two terms had only one 1.

In this paper $a_0 = 1$ and $a_1 = 1$ were the first two terms of the sequence. The advantage or disadvantage in so denoting the first two terms as such is slight. Many sources use $a_1 = 1$ and $a_2 = 1$.

The expression of many of the properties of the Fibonacci number sequence which concern the subscript of the Fibonacci number sequence depend upon how the first two terms are defined. For instance $a_{n+m} = a_n a_m + a_{n-1} a_{m-1}$ for when $a_0 = 1$ and $a_1 = 1$ are the first two terms. But, this does not hold when $a_1 = 1$ and $a_2 = 1$ are the first two terms of the sequence. In the latter case, $a_{n+m} = a_{n-1} a_m + a_n a_{m+1}$, and this does not hold true for the $a_0 = 1$ and $a_1 = 1$ case. Thus, one must be careful about notation when comparing works concerning the Fibonacci number sequence.

The first twenty terms of the Fibonacci number sequence are as follows: $a_0=1$, $a_1=1$, $a_2=2$, $a_3=3$, $a_4=5$, $a_5=8$, $a_6=13$, $a_7=21$, $a_8=34$, $a_9=55$, $a_{10}=89$, $a_{11}=144$, $a_{12}=233$, $a_{13}=377$, $a_{14}=610$, $a_{15}=987$, $a_{16}=1597$, $a_{17}=2584$, $a_{18}=4181$, and a_{19}=6765. Additional Fibonacci numbers along with many other facts are listed in the appendix. This supplementary material may prove worth-while to the interested individual who likes to experiment with numbers.

CHAPTER II

BASIC PROPERTIES

In this chapter some of the more basic and easily discovered properties relative to the Fibonacci number sequence are derived.

The first pattern of the Fibonacci number sequence which was noticed: two odd numbers, one even number, two odd numbers, one even number, ... This alternating sequence results from the fact that an odd number plus an odd number is equal to an even number, and that an odd number plus an even number is equal to an odd number.

Since the first two terms of the sequence are both odd numbers, the third term must be an even number. From this result the sequence was charted as below: (let OD stand for odd and E for even)

$a_0 + a_1 = a_2$, $a_1 + a_2 = a_3$, $a_2 + a_3 = a_4$, $a_3 + a_4 = a_5$
OD + OD = E, OD + E = OD, E + OD = OD, OD + OD = E

$a_4 + a_5 = a_6$, $a_5 + a_6 = a_7$, $a_6 + a_7 = a_8$, $a_7 + a_8 = a_9$
OD + E = OD, E + OD = OD, OD + OD = E, OD + E = OD

$a_8+a_9 = a_{10}$, $a_9+a_{10} = a_{11}$, $a_{10}+a_{11} = a_{12}$, $a_{11}+a_{12}= a_{13}$
E +OD = OD, OD+OD = E, OD + E = OD, E + OD = OD

This process continues on forever. The resulting sequence is odd, odd, even, odd, odd, even, ...

It will be noted later that a Fibonacci number is divisible by 2 if and only if its subscript can be put in the form 3b-1, where b is a positive integer.

Of prime concern was the sum of the first n terms. $a_0 + a_1 = 1+1 = 2$, $a_0 + a_1 + a_2 = 1+1+2 = 4$, $a_0 + a_1 + a_2 + a_3 = 1+1+2+3 = 7$, $a_0 + a_1 + a_2 + a_3 + a_4 = 1+1+2+3+5 = 12$, $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1+1+2+3+5+8 = 20$, and $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1+1+2+3+5+8+13 = 33$. But, $33 = 34-1$, $20 = 21-1$, $12 = 13-1$, $7 = 8-1$, $4 = 5-1$, and $1 = 3-1$.

This may be generalized into the formula $a_0 + a_1 + a_2 + \ldots + a_n = a_{n+2} - 1$. Proof: The formula has already been shown for the n = 1 case. Assuming the n = k case one has $a_0 + a_1 + a_2 + \ldots + a_k = a_{k+2} - 1$. By adding a_{k+1} to each side of the equation and then considering the fact that $a_{k+3} = a_{k+2} + a_{k+1}$ one should have the result $a_0 + a_1 + a_2 + \ldots + a_k + a_{k+1} = a_{k+3} - 1$. This last equation is the n=k+1 case.

Translated into words the above general equation says the sum of the Fibonacci numbers up to and including a_n is equal to 1 less than the a_{n+2} Fibonacci number. One should note that n was substituted for k+1 in the latter equation.

Now consider the sum of a finite number of Fibonacci numbers whose subscripts are even numbers and whose subscripts are odd numbers.

$a_0 + a_2 = 1+2 = 3$, $a_0 + a_2 + a_4 = 1+2+5 = 8$, $a_0 + a_2 + a_4 + a_6 = 1+2+5+13 = 21$, and $a_0 + a_2 + a_4 + a_6 + a_8 = 55$.

But, $a_3 = 3$, $a_5 = 8$, $a_7 = 21$, and $a_9 = 55$. In each case the sum is equal to a Fibonacci number whose subscript is 1 greater than the subscript of the last Fibonacci number in the sum. Now each of the even subscripts can be put in the form 2n, where n is a non-negative integer, and each of the odd subscripts can be put in the form 2n+1. This implies that $a_0 + a_2 + \ldots + a_{2n} = a_{2n+1}$ is the general equation for the sum of a finite number of alternate Fibonacci numbers, starting with a_0, whose subscripts are even integers greater than or equal to 0.

Proof: Suppose $a_0 + a_2 + \ldots + a_{2k} = a_{2k+1}$ is true, where k is a non-negative integer. By adding a_{2k+2} to each side of the equation and remembering that $a_{2k+3} = a_{2k+1} + a_{2k+2}$, one has $a_0 + a_2 + \ldots + a_{2k} + a_{2k+2} = a_{2k+3}$. This is equivalent to the n=k+1 case $a_0 + a_2 + \ldots + a_{2(k+1)} = a_{2(k+1)+1}$.

The next proof concerns the sum of a finite number of alternate Fibonacci numbers, starting with a_1, whose subscripts are odd integers greater than or equal to 1. This proof is very similar to the preceding proof.

Taking into account that $a_1 + a_3 = 4$, $a_1 + a_3 + a_5 = 12$, $a_1 + a_3 + a_5 + a_7 = 33$, and $a_1 + a_3 + a_5 + a_7 + a_9 = 88$, while noting that $88 = a_{10} - 1$, $33 = a_8 - 1$, $12 = a_6 - 1$, and $4 = a_4 - 1$; it is likely that $a_1 + a_3 + \ldots + a_{2n+1} = a_{2n+2} - 1$.

Proof: Assume the n=k case, $a_1 + a_3 + \ldots + a_{2k+1} = a_{2k+2} - 1$. By adding a_{2k+3} to each side of the equation, one has the equation $a_1 + a_3 + \ldots + a_{2k+1} + a_{2k+3} = a_{2k+3} + a_{2k+2} - 1$. Since $a_{2k+4} = a_{2k+3} + a_{2k+2}$ and $2(k+1)+1 = 2k+3$ in conjunction with $2(k+1)+2 = 2k+4$ are true, one may have the result $a_1 + a_3 + \ldots + a_{2(k+1)+1} = a_{2(k+1)+2} - 1$.

A logical step in the development of theory concerning the Fibonacci number sequence is the relationship between the Fibonacci number and the subscript of the Fibonacci number.

One might ask: Does a Fibonacci number with a subscript in the form i+j, where i and j are non-negative integers, have a relationship with certain Fibonacci numbers whose identity depends upon the values of i and j? This is a question one might ponder. An important tool in the development of other properties might be found.

Consider the Fibonacci numbers in the form a_{i+j}, where i and j are non-negative integers. The first few Fibonacci numbers in this form may quite easily be put in the form $a_n + a_m$, where n and m are non-negative intergers.

$a_{2+0} = a_{1+1} = 2 = 1+1 = a_1 + a_1 = a_1 + a_0 = a_0 + a_0$.

$a_{3+0} = a_{2+1} = 3 = 2+1 = a_2 + a_1 = a_2 + a_0$.

$a_{4+0} = a_{3+1} = a_{2+2} = 5 = 3+2 = a_3 + a_2$.

$a_{5+0} = a_{4+1} = a_{3+2} = 8 = 5+3 = a_4 + a_3$.

$a_{6+0} = a_{5+1} = a_{4+2} = a_{3+3} = 13 = 8+5 = a_5 + a_4$.

$a_{7+0} = a_{6+1} = a_{5+2} = a_{4+3} = 21 = 13+8 = a_6 + a_5.$

$a_{8+0} = a_{7+1} = a_{6+2} = a_{5+3} = a_{4+4} = 34 = 21+13 = a_7 + a_6.$

$a_{9+0} = a_{8+1} = a_{7+2} = a_{6+3} = a_{5+4} = 55 = 34+21 = a_8 + a_7.$

$a_{10+0} = a_{9+1} = a_{8+2} = a_{7+3} = a_{6+4} = a_{5+5} = 89 = 55+34 = a_9 + a_8.$

$a_{11+0} = a_{10+1} = a_{9+2} = a_{8+3} = a_{7+4} = a_{6+5} = 144 = 89+55 = a_{10} + a_9.$

$a_{12+0} = a_{11+1} = a_{10+2} = a_{9+3} = a_{8+4} = a_{7+5} = a_{6+6} = 233 = 144+89 = a_{11} + a_{10}.$

After a close scrutiny of the above, the first fact that is evident is nothing more than the basic building apparatus for the Fibonacci number sequence $a_n = a_{n-1} + a_{n-2}$, where n is an integer greater than or equal to 2. But, the slow increase of the values indicates that the numbers above might be the sums of other combinations of Fibonacci numbers such as the sums of products of Fibonacci numbers.

Let one go back to Fibonacci numbers a_{1+1}, a_{2+1}, ..., a_{5+1} and try a somewhat different approach to possible additive combinations that might lead to a general equation concerning Fibonacci numbers in the general form a_{i+j}.

Consider the Fibonacci numbers in the form a_{i+j} as the sum of two numbers, each of which is a product of two Fibonacci numbers.

$a_{2+0} = a_{1+1} = 2 = 1+1.$

$a_{3+0} = a_{2+1} = 3 = 1+2.$

$a_{4+0} = a_{3+1} = a_{2+2} = 5 = 1+4 = 2+3.$

$a_{5+0} = a_{4+1} = a_{3+2} = 8 = 1+7 = 2+6 = 3+5 = 4+4.$

$a_{6+0} = a_{5+1} = a_{4+2} = a_{3+3} = 13 = 1+12 = 2+11 = 3+10 = \ldots = 6+7.$

But, when one considers which numbers used above are not the product of Fibonacci numbers, some of the possible additive combinations are eliminated.

$1 = a_0a_0 = a_0a_1 = a_1a_1$, $2 = a_0a_2 = a_1a_2$, $3 = a_0a_3 = a_1a_3$, $4 = a_2a_2$, $5 = a_0a_4 = a_1a_4$, $6 = a_2a_3$, $9 = a_3a_3$, and $10 = a_2a_4$; while 7, 8, 11, and 12 are not products of Fibonacci numbers.

The only sums not eliminated are 1+1, 1+2, 1+4, 2+3, 2+6, 3+5, 4+4, 3+10, and 4+9.

The possible products whose sum is a_2 are $a_0a_0 + a_0a_0$, $a_0a_0 + a_0a_1$, $a_0a_1 + a_0a_1$, $a_0a_1 + a_1a_1$, and $a_1a_1 + a_1a_1$. Note that $a_ma_n + a_ia_j$ will be considered identical to $a_ia_j + a_ma_n$, while a_ma_n and a_na_m are also considered identical. In other words, commutativity for addition and multiplication, with respect to the type of Fibonacci number combinations now being considered, will be assumed.

Now the possible products may be translated into the general forms $a_{i+j} = a_{i-2}a_j + a_{i-2}a_j$, or $a_{i-2}a_j + a_{i-2}a_{j+1}$, or $a_{i-2}a_{j+1} + a_{i-2}a_{j+1}$, or $a_{i-2}a_{j+1} + a_{i-1}a_{j+1}$, or $a_{i-2}a_j + a_{i-1}a_j$, or $a_{i-1}a_j = a_{i-1}a_j$, or $a_{i-1}a_j + a_{i-1}a_{j+1}$, or $a_{i-1}a_{j+1} + a_{i-1}a_{j+1}$, or $a_{i-2}a_{j+1} + a_{i-1}a_j$, or $a_{i-2}a_j + a_{i-1}a_{j+1}$.

By considering the general forms obtainable from both a_{2+0} and a_{1+1}, and eliminating the contradictory ones, one obtains $a_{i+j} = a_{i-1}a_j + a_{i-1}a_j$.

Letting $i=4$ and $j=2$, $a_{i+j} = a_3a_2 + a_3a_2 = 6+6 = 12$.

But, $a_6 = 13$. Needless to say, the facts do not correlate. One of the underlying assumptions was that i and j would be considered non-negative integers. The same process will be tried. But, with i and j as positive integers instead of non-negative integers. One should note that this change eliminates the a_{2+0} case.

By applying the a_{1+1} possibilities to the a_{3+3} ones, one may have the following:

(1) $a_{i+j} = a_i a_j + a_{i-1} a_{j-1} = a_3 a_1 + a_2 a_0 = 3+2 = 5,$

(2) $a_{i+j} = a_i a_j + a_i a_{j-1} = a_3 a_1 + a_3 a_0 = 3+3 = 6,$

(3) $a_{i+j} = a_i a_j + a_{i-1} a_j = a_3 a_1 + a_2 a_1 = 3+2 = 5,$

(4) $a_{i+j} = a_i a_{j-1} + a_i a_{j-1} = a_3 a_0 + a_3 a_0 = 3+3 = 6,$

(5) $a_{i+j} = a_{i-1} a_j + a_{i-1} a_j = a_2 a_1 + a_2 a_1 = 2+2 = 4,$

(6) $a_{i+j} = a_i a_{j-1} + a_{i-1} a_j = a_3 a_0 + a_2 a_1 = 3+2 = 5,$

(7) $a_{i+j} = a_{i-1} a_{j-1} + a_i a_{j-1} = a_2 a_0 \; a_3 a_0 = 2+3 = 5,$

(8) $a_{i+j} = a_{i-1} a_{j-1} + a_{i-1} a_j = a_2 a_0 + a_2 a_1 = 2+2 = 4,$

(9) $a_{i+j} = a_{i-1} a_{j-1} + a_{i-1} a_{j-1} = a_2 a_0 + a_2 a_0 = 2+2 = 4,$ and

(10) $a_{i+j} = a_i a_j + a_i a_j = a_3 a_1 + a_3 a_1 = 3+3 = 6.$

Since $a_4 = 5$, (2), (4), (5), (8), (9), and (10) are eliminated as a possible general equation for a_{i+j}. (1), (3), (6), and (7) are left as possibilities.

Applying the remaining possibilities to another Fibonacci number seems to be the next logical step to take. a_7 is an easy Fibonacci number to operate on. Let $a_7 = a_{3+4}$.

The following results may be obtained by applying the remaining possibilities to a_{4+3}.

(1) $a_{i+j} = a_i a_j + a_{i-1}a_{j-1} = a_4 a_3 + a_3 a_2 = 15+6 = 21$,

(3) $a_{i+j} = a_i a_j + a_{i-1}a_j = a_4 a_3 + a_3 a_3 = 15+9 = 24$,

(6) $a_{i+j} = a_i a_{j-1} + a_{i-1}a_j = a_4 a_2 + a_3 a_3 = 10+9 = 19$, and

(7) $a_{i+j} = a_{i-1}a_{j-1} + a_i a_{j-1} = a_3 a_2 + a_4 a_2 = 6+10 = 16$.

Since $a_7 = 21$, the general equation whould be $a_{i+j} = a_i a_j + a_{i-1}a_{j-1}$, where i and j are positive integers.

The proof of $a_{n+m} = a_n a_m + a_{n-1}a_{m-1}$, where n and m are positive integers, is as follows: Let n = 1 and m = 1. $a_{1+1} = a_1 a_1 + a_0 a_0 = 1+1 = 2$. $a_2 = 2$ verifies the k= 1 case. The k = 2 case is verified by $a_{1+2} = a_1 a_2 + a_0 a_1 = 3$. Now the k = n+m-1 and k = n+m cases will be assumed, and the k = m+n+1 case will be shown to follow.

From the assumptions above $a_{n+(m-1)} = a_n a_{m-1} + a_{n-1}a_{m-2}$ and $a_{n+m} = a_n a_m + a_{n-1}a_{m-1}$. Adding these equations yields $a_{n+(m-1)} + a_{n+m} = a_n a_{m-1} + a_{n-1}a_{m-2} + a_n a_m + a_{n-1}a_{m-1}$. Since $a_{n+(m-1)} + a_{n+m} = a_{n+m+1}$ one has $a_{n+m+1} = a_n a_{m-1} + a_{n-1}a_{m-2} + a_n a_m + a_{n-1}a_{m-1}$. Rearranging the terms and factoring gives $a_{n+m+1} = a_n(a_{m-1} + a_m) + a_{n-1}(a_{m-2} + a_{m-1}) = a_n a_{m+1} + a_{n-1}a_m$. This is the k = m+n+1 case.

$a_{i+j} = a_i a_j + a_{i-1}a_{j-1}$, where i and j are positive integers, should prove to be an important tool in the derivation of additional properties.

The basic tools needed for the development of some more refined properties of the Fibonacci number sequence have been derived. A brief summary of these tools is as follows:

$a_n = a_{n-1} + a_{n-2}$, $a_{n-1} = a_n - a_{n-2}$, and $a_{n-2} = a_n - a_{n-1}$, where n is an integer greater than 1 while $a_0 = a_1 = 1$.

$a_0 + a_1 + \ldots + a_n = a_{n+2} - 1$, $a_0 + a_2 + \ldots + a_{2n} = a_{2n+1} - 1$, and $a_1 + a_3 + \ldots + a_{2n+1} = a_{2n+2} - 1$, where n is a non-negative integer.

$a_{i+j} = a_i a_j + a_{i-1} a_{j-1}$, where i and j are positive integers.

CHAPTER III

REFINED PROPERTIES

Consider the sequence $(a_0)^2$, $(a_1)^2$, $(a_2)^2$, ..., $(a_n)^2$, where n is a non-negative integer. If n=9, the sequence is 1, 1, 2, 4, 9, 25, 64, 169, 441, 1156, 3025.

The sum of adjacent terms gives 1+1 = 2, 1+2 = 3, 2+4 = 6, 4+9 = 13, 9+25 = 34, 25+64 = 89, 64+169 = 233, 169+441 = 610, 441+1156 = 1597, and 1156+3025 = 4181. But, 2 = a_2, 5 = a_4, 13 = a_6, 34 = a_8, 89 = a_{10}, 233 = a_{12}, 610 = a_{14}, 1597 = a_{16}, and 4181 = a_{18}.

In other words, $(a_0)^2 + (a_1)^2 = a_2$, $(a_1)^2 + (a_2)^2 = a_4$, $(a_2)^2 + (a_3)^2 = a_6$, $(a_3)^2 + (a_4)^2 = a_8$, $(a_4)^2 + (a_5)^2 = a_{10}$, $(a_5)^2 + (a_6)^2 = a_{12}$, $(a_6)^2 + (a_7)^2 = a_{14}$, $(a_7)^2 + (a_8)^2 = a_{16}$, and $(a_8)^2 + (a_9)^2 = a_{18}$. A general form of this property might be $(a_n)^2 + (a_{n+1})^2 = (a_{2n+2})$.[2]

A proof of this comes quite simply from the equation $a_{i+j} = a_i a_j + a_{i-1} a_{j-1}$ by letting i = n+1 and j = n+1. Thus, $a_{2n+2} = a_{n+1} a_{n+1} + a_n a_n = (a_{n+1})^2 + (a_n)^2 = (a_n)^2 + (a_{n+1})^2$. Even though i and j are restricted to the set of positive integers, the equation above holds true when n is 0. The reason is really quite simple. The smallest value i and j may take is 1. If i = 1, then i = 1 = n+1. Thus, n=0.

The last paragraph illustrates that the general equation is valid when n is a non-negative integer.

If one lets n+1 = m in the equation $(a_n)^2 + (a_{n+1})^2 = a_{2n+2}$, one should obtain $(a_{m-1})^2 + (a_m)^2 = a_{2m}$. The latter equation might prove less cumbersome to use.

A somewhat different approach simplifies the mathematical process when considering the difference between adjacent terms of the sequence $(a_0)^2, (a_1)^2, \ldots, (a_n)^2, \ldots$. $a_n - a_{n-1} = a_{n-2}$, and $a_n + a_{n-1} = a_{n+1}$, one is led to the equation $(a_n)^2 - (a_{n-1})^2 = a_{n+1}a_{n-2}$.

When considering the same sequence relative to adjacent terms with respect to multiplication and division, nothing of any significant value was found.

Going through the same process with alternate terms the only property found was concerned with subtraction. $4-1 = 3 = a_3$, $9-1 = 8 = a_5$, $25-4 = 21 = a_7$, $64-9 = 55 = a_9$, $169-25 = 144 = a_{11}$, and $441-64 = 377 = a_{13}$.

Let i=n and j=n-1 in the equation $a_{i+j} = a_i a_j + a_{i-1} a_{j-1}$. $a_{i+j} = a_{2n-1} = a_n a_{n-1} + a_{n-1}a_{n-2}$. By substituting $a_n - a_{n-2}$ for a_{n-1} in the equation, one should have the result $a_{2n-1} = a_n(a_n - a_{n-2}) + (a_n - a_{n-2})a_{n-2} = (a_n)^2 - a_n a_{n-2} + a_n a_{n-2} - a_n a_{n-2} - (a_{n-2})^2 = (a_n)^2 - (a_{n-2})^2$.

The two previously proved equations are quite convenient when one is obtaining the value of a Fibonacci number whose subscript is quite large. For example, in finding a_{100} one would add a_{99} and a_{98}. But, this means adding $a_{98} + a_{97}$

and $a_{97} + a_{96}$. But, $a_{97} = a_{96} + a_{95}$, $a_{96} = a_{95} + a_{94}$, $a_{95} = a_{94} + a_{93}$, ..., $a_3 = a_2 + a_1$. In other words, a long process of additions is necessary in order to find the value of a_{100} using the starting definition of the Fibonacci number sequence.

The process using the equations $a_{2n} = (a_n)^2 + (a_{n-1})^2$ and $a_{2n-1} = (a_n)^2 - (a_{n-2})^2$ might proceed as follows: $a_{100} = (a_{49})^2 + (a_{50})^2$, $a_{49} = (a_{25})^2 - (a_{23})^2$, $a_{50} = (a_{24})^2 + (a_{25})^2$, $a_{25} = (a_{13})^2 - (a_{11})^2$, $a_{24} = (a_{11})^2 + (a_{12})^2$, and $a_{23} = (a_{12})^2 - (a_{10})^2$. $a_{10} = 89$, $a_{11} = 144$, and $a_{12} = 233$.

$a_{49} = 12,586,269,025$ and $a_{50} = 20,365,011,074$.

$(a_{49})^2 = 158,414,167,969,674,450,625$.

$(a_{50})^2 = 414,733,676,044,142,633,467$.

$a_{100} = 573,147,844,013,817,084,101$.

From the sequence of squared Fibonacci numbers, the elements which are seperated by two other terms will be compared with reference to the three basic operations addition, subtraction, and multiplication. 1+9 = 10, 1+25 = 26, 4+64 = 68, 9+169 = 178, 25+441 = 466, and 64+1156 = 1220. 9-1 = 8, 25-1 = 24, 64-4 = 60, 169-9 = 160, 441-25 = 416, and 1156-64 = 1092. 1x9 = 9, 1x25 = 25, 4x64 = 256, 9x169 = 1521, 25x441 = 11,025, and 64x1156 = 73,984.

When considering the results from the addition, at least three general equations may be found. 10 = 2x5 = 13-3 = 8+2, 26 = 2x13 = 34-8 = 21+5, 68 = 2x34 = 89-21 =

55+13, 178 = 2x89 = 233-55 = 144+34, 466 = 2x233 = 610-144 = 377+89, and 1220 = 2x610 = 1597-377 = 987+233 implies $(a_0)^2 + (a_3)^2 = a_2a_4 = a_6 - a_3 = a_5 + a_2$, $(a_1)^2 + (a_4)^2 = a_2a_6 = a_8 - a_5 = a_7 + a_4$, and $(a_2)^2 + (a_5)^2 = a_2a_8 = a_{10} - a_7 = a_9 + a_6$.

In general (1) $(a_n)^2 + (a_{n+3})^2 = 2a_{2n+4}$, (2) $(a_n)^2 + (a_{n+3})^2 = a_{2n+6} - a_{2n+3}$, and (3) $(a_n)^2 + (a_{n+3})^2 = a_{2n+5} + a_{2n+2}$. The proofs of these equations are forthwith.

(1) Let i=j=n+2 in the equation $a_{i+j} = a_ia_j + a_{i-1}a_{j-1}$. $a_{2n+4} = (a_{n+2})^2 + (a_{n+1})^2$. $2a_{2n+4} = 2(a_{n+2})^2 + 2(a_{n+1})^2 = 2(a_{n+3} - a_{n+1})^2 + 2(a_{n+2} - a_n)^2 = (a_{n+3})^2 + (a_n)^2 + (a_{n+3})^2 + (a_n)^2 - 4a_{n+3}a_{n+1} + 2(a_{n+1})^2 + 2(a_{n+2})^2 - 4a_{n+2}a_n = (a_{n+3})^2 + (a_n)^2 + (a_{n+2} + a_{n+1})^2 + (a_n)^2 - 4(a_{n+2} + a_{n+1})a_{n+1} + 2(a_{n+1})^2 + 2(a_{n+2})^2 - 4a_{n+2}a_n = (a_{n+3})^2 + (a_n)^2 + (a_{n+2})^2 + 2a_{n+2}a_{n+1} + (a_{n+1})^2 + (a_n)^2 - 4a_{n+2}a_{n+1} - 4(a_{n+1})^2 + 2(a_{n+1})^2 + 2(a_{n+2})^2 - 4a_{n+2}a_n = (a_{n+3})^2 + (a_n)^2 + 3(a_{n+2})^2 - (a_{n+1})^2 + 2a_{n+2}a_{n+1} + (a_n)^2 -4a_{n+2}a_{n+1} - 4a_{n+2}a_n = (a_{n+3})^2 + (a_n)^2 + (3(a_{n+1})^2 + 6a_{n+1}a_n + 3(a_n)^2 0 - (a_{n+1})^2 + (2(a_{n+1})^2 + 2a_{n+1}a_n) + (a_n)^2 + (-4(a_{n+1})^2 - 4a_{n+1}a_n) + (-4a_{n+1} - 4(a_n)^2) = (a_{n+3})^2 + (a_n)^2 + 5(a_{n+1})^2 - 5(a_{n+1})^2 + 8a_{n+1}a_n - 8a_{n+1}a_n + 4(a_n)^2 - 4(a_n)^2 = (a_{n+3})^2 + (a_n)^2$.

(2) From (1), $(a_n)^2 + (a_{n+3})^2 = 2a_{2n+4} = 2(a_{2n+5} - a_{2n+3}) = 2(a_{2n+6} - a_{2n+4} - a_{2n+3}) = 2a_{2n+6} - 2a_{2n+4} - 2a_{2n+3} = a_{2n+6} - a_{2n+3} + a_{2n+6} - a_{2n+3} - 2a_{2n+4} = a_{2n+6} - a_{2n+3} + a_{2n+5} + a_{2n+4} - a_{2n+3} - 2a_{2n+4} = a_{2n+6} - a_{2n+3} + a_{2n+5} - a_{2n+3} -$

$a_{2n-5} = a_{2n+6} - a_{2n+3}$. Thus, $a_{2n+6} - a_{2n+3} = (a_n)^2 + (a_{n+3})^2$.
(3) From $(a_n)^2 + (a_{n+3})^2 = a_{2n+6} - a_{2n+3}$, one has $(a_n)^2 + (a_{n+3})^2 = a_{2n+5} + a_{2n+2}$ by substituting $a_{2n+5} + a_{2n+4}$ for a_{2n+6} and $a_{2n+4} - a_{2n+2}$ for a_{2n+3}.

Consequently, the addition of terms of the sequence of squared Fibonacci numbers which are separated by two other terms resulted in the following properties: $(a_n)^2 + (a_{n+3})^2 = 2a_{2n+4}$, $(a_n)^2 + (a_{n+3})^2 = a_{2n+6} - a_{2n+3}$, and $(a_n)^2 + (a_{n+3})^2 = a_{2n+5} + a_{2n+2}$.

A quick glance at the product and difference of terms separated by two other terms in the sequence of squared Fibonacci numbers yielded nothing useful at this stage in the development of Fibonacci numbers with reference to number theory.

Next the sum of the first n terms of the sequence of squared Fibonacci numbers will be considered. $(a_0)^2 + (a_1)^2 = 2$, $(a_0)^2 + (a_1)^2 + (a_2)^2 = 6$, $(a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 = 15$, $(a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 = 40$, and $(a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 + (a_5)^2 = 104$.

It should be obvious that $a_1a_2 = 2$, $a_2a_3 = 6$, $a_3a_4 = 15$, $a_4a_5 = 40$, and $a_5a_6 = 104$. This implies the equation $(a_0)^2 + (a_1)^2 + \ldots + (a_n)^2 = a_na_{n+1}$.

The n=1 case has already been shown. Now let one assume the n=k case and then show that the n=k+1 case follows.

The n=k case is $(a_0)^2 + (a_1)^2 + \ldots + (a_k)^2 = a_k a_{k+1}$. Adding a_{k+1} to each side, one should have $(a_0)^2 + (a_1)^2 + \ldots + (a_k)^2 + (a_{k+1})^2 = a_k a_{k+1} + (a_{k+1})^2 = (a_k + a_{k+1})(a_{k+1})$. But, $a_k + a_{k+1} = a_{k+2}$. Hence, $(a_0)^2 + (a_1)^2 + \ldots + (a_{k+1})^2 = a_{k+1}a_{k+2}$. This is the n=k+1 case.

Next the sum of the first n terms of the sequence of the squares of the Fibonacci numbers with odd subscripts was observed. $(a_1)^2 + (a_3)^2 = 10$, $(a_1)^2 + (a_3)^2 + (a_5)^2 = 74$, $(a_1)^2 + (a_3)^2 + (a_5)^2 + (a_7)^2 = 515$, and $(a_1)^2 + (a_3)^2 + (a_5)^2 + (a_7)^2 + (a_9)^2 = 3540$.

An analysis of the above yielded nothing beneficial in reference to this study. This same type of analysis was performed with reference to Fibonacci numbers with even subscripts. The same result was obtained.

An unusual pattern evolves from the product of alternate Fibonacci numbers. $a_0 a_2 = 2$, $a_1 a_3 = 3$, $a_2 a_4 = 10$, $a_3 a_5 = 24$, $a_4 a_6 = 64$, and $a_5 a_7 = 168$. But, $2 = (a_1)^2 + 1$, $3 = (a_2)^2 - 1$, $10 = (a_3)^2 + 1$, $24 = (a_4)^2 - 1$, $64 = (a_5)^2 + 1$, and $168 = (a_6)^2 - 1$.

From this one may reach the tentative conclusion that $a_n a_{n+2} = (a_{n+1})^2 + 1$, where n is an even non-negative integer. n is an odd non-negative integer in the equation $a_n a_{n+2} = (a_{n+1})^2 - 1$.

These two equations might better be expressed in the form $a_{2n} a_{2n+2} = (a_{2n+2})^2 + 1$ and $a_{2n+1} a_{2n+3} = (a_{2n+2})^2 - 1$, where n is an non-negative integer. Either equation of this latter form may be used without regard to whether n is an odd or even number.

The k=n case has already been shown. Going from the k=n case to the k=n+1 case one might have the following:

$a_{2n}a_{2n+2} = (a_{2n+1})^2 + 1,$

$a_{2n}a_{2n+2} - a_{2n+1}a_{2n+1} = 1,$

$a_{2n}a_{2n+2} - a_{2n+1}(a_{2n+2} - a_{2n}) = 1,$

$- a_{2n+2}a_{2n+1} + a_{2n+2}a_{2n} + a_{2n+1}a_{2n} = 1,$

$(a_{2n+2})^2 - (a_{2n+2})^2 - a_{2n+2}a_{2n+1} + a_{2n+2}a_{2n} + a_{2n+1}a_{2n} = 1,$

$(a_{2n+2})^2 - (a_{2n+2} - a_{2n})(a_{2n+2} + a_{2n}) = 1,$

$(a_{2n+2})^2 - a_{2n+1}a_{2n+3} = 1,$

$a_{2n+2}(a_{2n+3} - a_{2n+1}) - a_{2n+1}a_{2n+3} = 1,$

$a_{2n+3}a_{2n+2} - a_{2n+1}a_{2n+2} - a_{2n+1}a_{2n+3} = 1,$

$(a_{2n+3})^2 + a_{2n+3}a_{2n+2} - a_{2n+1}a_{2n+3} - a_{2n+1}a_{2n+2} = (a_{2n+3})^2 +1,$

$(a_{2n+3} - a_{2n+1})(a_{2n+3} + a_{2n+2}) = a_{2n+3} +1,$

$a_{2n+2}a_{2n+4} = (a_{2n+3})^2 +1,$ and

$a_{2(n+1)}a_{2(n+1)+2}=a_{2(n+1)+1} +1,$ for the first equation.

For the second equation one might have

$a_{2n+1}a_{2n+3} = (a_{2n+2})^2 -1,$

$a_{2n+1}(a_{2n+4} - a_{2n+2}) = (a_{2n+2})^2 -1,$

$a_{2n+4}a_{2n+1} - a_{2n+2}a_{2n+1} = (a_{2n+2})^2 -1,$

$a_{2n+4}(a_{2n+3} - a_{2n+2}) - a_{2n+2}a_{2n+1} = (a_{2n+2})^2 -1,$

$a_{2n+4}a_{2n+3} - a_{2n+2}a_{2n+4} - a_{2n+2}a_{2n+1} = (a_{2n+2})^2 -1,$

$a_{2n+4}a_{2n+3} - a_{2n+2}a_{2n+4} - (a_{2n+2})^2 -a_{2n+2}a_{2n+1} = -1,$

$a_{2n+4}a_{2n+3} - a_{2n+2}a_{2n+4} - a_{2n+2}(a_{2n+2} + a_{2n+1}) = -1$,
$a_{2n+4}a_{2n+3} - a_{2n+2}a_{2n+4} - a_{2n+2}a_{2n+3} = -1$,
$(a_{2n+4})^2 + a_{2n+4}a_{2n+3} - a_{2n+2}a_{2n+4} - a_{2n+2}a_{2n+3} = (a_{2n+4})^2 -1$,
$(a_{2n+4} - a_{2n+2})(a_{2n+4} + a_{2n+3}) = (a_{2n+4})^2 -1$, and
$a_{2n+3}a_{2n+5} = (a_{2n+4})^2 -1$. Note that $2n+3 = 2(n+1) +1$,
$2n+5 = 2(n+1) +3$, and $2n+4 = 2(n+1) +2$.

There are many basic facts concerning the Fibonacci number sequence which have little or no value with respect to the number theory oriented approach. The main objective is to develop enough of these facts in order to have sufficient tools to solve some more refined number theory problems.

For example, $(a_n)^2 - (a_{n-3})^2 = a_{2n} - a_{2n-3} - a_{2n-6} - a_{2n-8} - a_{2n-12} - \cdots - a_{2n-16} - \cdots$ until there are k terms, where $k = \frac{1}{2}n +1$, and n is an even positive integer.

CHAPTER IV

MORE REFINED PROPERTIES

An analysis of Fibonacci numbers with odd numbered subscripts brought to light two unexpected results.

In the equation $a_{i+j} = a_i a_j + a_{i-1} a_{j-1}$, let $i = n+1$ and $j = n$. $a_{2n+1} = a_{n+1} a_n + a_n a_{n-1} = a_n(a_{n+1} + a_{n-1}) = (a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1}) = (a_{n+1})^2 - (a_{n-1})^2$, where n is greater than or equal to 1.

If $i = n+1$ and $j = n+2$ in the equation $a_{i+j} = a_i a_j + a_{i-1} a_{j-1}$, one may have $a_{2n+3} = a_{n+1} a_{n+2} + a_n a_{n+1} = a_{n+1}(a_{n+2} + a_n) = (a_{n+2} - a_n)(a_{n+2} + a_n) = (a_{n+2})^2 + (a_n)^2$.

Using the same general equation used above and having $i = n+3$ and $j = n+2$ gives $a_{2n+5} = a_{n+3} a_{n+2} + a_{n+2} a_{n+1} = a_{n+2}(a_{n+3} + a_{n+1}) = (a_{n+3} - a_{n+1})(a_{n+3} + a_{n+1}) = (a_{n+3})^2 - (a_{n+1})^2$.

Consider the general form a_{2n+k}, where k is an odd positive integer. Let $i = j+1$ in the equation $i+j = 2n+k$. Thus $2j+1 = 2n+k$, $2j = 2n+k-1$, and $j = n + \frac{1}{2}k - \frac{1}{2}$. Therefore, $i= n+\frac{1}{2}k+\frac{1}{2}$.

Now one may have $a_{2n+k} =(a_{n+\frac{1}{2}k+\frac{1}{2}})(a_{n+\frac{1}{2}k-\frac{1}{2}})+(a_{n+\frac{1}{2}k-\frac{1}{2}})(a_{n+\frac{1}{2}k-\frac{1}{2}(3)}) =(a_{n+\frac{1}{2}k-\frac{1}{2}})((a_{n+\frac{1}{2}k+\frac{1}{2}}) + (a_{n+\frac{1}{2}k-\frac{1}{2}(3)}))$.
Since $(a_{n+\frac{1}{2}k-\frac{1}{2}}) = (a_{n+\frac{1}{2}k+\frac{1}{2}}) - (a_{n+\frac{1}{2}k-\frac{1}{2}(3)})$ one may have

$a_{2n+k} = ((a_{n+\frac{1}{2}k+\frac{1}{2}})-(a_{n+\frac{1}{2}k-\frac{1}{2}(3)}))((a_{n+\frac{1}{2}k+\frac{1}{2}})+(a_{n+\frac{1}{2}k-\frac{1}{2}(3)})) =$

$a_{2n+k} = (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 - (a_{n+\frac{1}{2}k-\frac{1}{2}(3)})^2$.

Thus, any Fibonacci number whose subscript is an odd positive integer may be denoted by the equation $a_{2n+k} = (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 - (a_{n+\frac{1}{2}k-\frac{1}{2}(3)})^2$, where k is an positive odd integer.

When considering the sum of $a_{2n+1} + a_{2n+3} + a_{2n+5} + \cdots + a_{2n+k}$, where k is an positive odd integer, one may have the results $a_{2n+1} + a_{2n+3} + a_{2n+5} = (a_{n+1})^2 - (a_{n-1})^2 + (a_{n+2})^2 - (a_n)^2 + (a_{n+3})^2 - (a_{n+1})^2 = (a_{n+3})^2 + (a_{n+2})^2 - (a_n)^2 - (a_{n-1})^2$, and $a_{2n+1} + a_{2n+3} + a_{2n+5} + a_{2n+7} = (a_{n+3})^2 + (a_{n+2})^2 - (a_n)^2 -(a_{n-1})^2 + (a_{n+4})^2 - (a_{n+2})^2 = (a_{n+4})^2 + (a_{n+3})^2 - (a_n)^2 - (a_n-1)^2$. Also $a_{2n+1} + a_{2n+3} + a_{2n+5} + a_{2n+7} + a_{2n+9} = (a_{n+4})^2 + (a_{n+3})^2 - (a_n)^2 - (a_{n-1})^2 + (a_{n+5})^2 - (a_{n+3})^2 = (a_{n+5})^2 + (a_{n+4})^2 - (a_n)^2 - (a_{n-1})^2$.

These results suggest the hypothesis $a_{2n+1} + a_{2n+3} + a_{2n+5} + \cdots + a_{2n+k} = (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 + (a_{n+\frac{1}{2}k-\frac{1}{2}})^2 - (a_n)^2 -(a_{n-1})$, where k is a positive odd integer and n is a positive integer.

To prove this hypothesis the n=k case will be assumed and then the n=k+1 case will be shown to follow.

Assume $a_{2n+1} + a_{2n+3} + \cdots + a_{2n+k} = (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 + (a_{n+\frac{1}{2}k-\frac{1}{2}})^2 - (a_n)^2 - (a_{n-1})^2$, where k is a positive odd integer. By taking into consideration that $a_{2n+(k+2)} = (a_{n+\frac{1}{2}k+\frac{1}{2}(3)})^2 - (a_{n+\frac{1}{2}k-\frac{1}{2}})^2$, one may have $a_{2n+(k+2)} = (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 + (a_{n+\frac{1}{2}k-\frac{1}{2}})^2 - (a_n)^2 - (a_{n-1})^2 + (a_{n+\frac{1}{2}k+\frac{1}{2}(3)})^2 - (a_{n+\frac{1}{2}k-\frac{1}{2}})^2 = (a_{n+\frac{1}{2}k+\frac{1}{2}(3)})^2 + (a_{n+\frac{1}{2}k+\frac{1}{2}})^2 - (a_n)^2 - (a_{n-1})^2$.

Using this equation to find $a_1 + a_3 + \ldots + a_{31}$ one might excute the following process. Since 31 = 2n+k, let n = 15 and k = 1. The $a_1 + a_3 + \ldots + a_{31} = (a_{17})^2 + (a_{16})^2 - (a_{14})^2 - (a_{15})^2$ = 6,677,056 + 2,550,409 - 974,169 - 372,000 = 7,881,296.

CHAPTER V

DIVISIBILITY THEOREMS

By examining the divisibility of the smaller Fibonacci numbers with respect to the numbers 2, 3, 4, 5, 6, 7, 8, and 9, the following results were obtained.

INTEGRAL DIVISORS	FIBONACCI NUMBERS WHICH ARE DIVISIBLE BY THE CORRESPONDING INTEGRAL DIVISOR
2	$a_2, a_5, a_8, \ldots, a_{3n-1}, \ldots, a_{59}, a_{62}$
3	$a_3, a_7, a_{11}, \ldots, a_{4n-1}, \ldots, a_{55}, a_{59}$
4	$a_5, a_{11}, a_{17}, \ldots, a_{6n-1}, \ldots, a_{83}, a_{89}$
5	$a_4, a_9, a_{14}, \ldots, a_{5n-1}, \ldots, a_{69}, a_{74}$
6	$a_{11}, a_{23}, a_{35}, \ldots, a_{12n-1}, \ldots, a_{155}, a_{167}$
7	$a_7, a_{15}, a_{23}, \ldots, a_{8n-1}, \ldots, a_{103}, a_{111}$
8	$a_5, a_{11}, a_{17}, \ldots, a_{6n-1}, \ldots, a_{125}, a_{131}$
9	$a_{11}, a_{23}, a_{35}, \ldots, a_{12n-1}, \ldots, a_{119}, a_{131}$

The table above lists various Fibonacci numbers with their respective integral divisors. The properties illustrated may be expressed as divisibility theorems. Since the proofs of these theorems are similar in content and identical in form, only one proof will be given.

First the divisibility theorems will be given. The proof will be in two parts since it is stated in the form "if and only if."

Theorem 1. A Fibonacci number is divisible by 2 if and only if its subscript can be put into the form 3k-1, where k is a positive integer.

Theorem 2. A Fibonacci number is divisible by 3 if and only if its subscript can be put into the form 4b-1, where b is a positive integer.

Theorem 3. A Fibonacci number is divisible by 4 if and only if its subscript can be put into the form 6b-1, where b is a positive integer.

Theorem 4. A Fibonacci number is divisible by 5 if and only if its subscript can be put into the form 5b-1, where b is a positive integer.

Theorem 5. A Fibonacci number is divisible by 6 if and only if its subscript can be put into the form 12j-1, where j is a positive integer.

Theorem 6. A Fibonacci number is divisible by 7 if and only if its subscript can be put into the form 8m-1, where m is a positive integer.

Theorem 7. A Fibonacci number is divisible by 8 if and only if its subscript can be put into the form 6q-1, where q is a positive integer.

Theorem 8. A Fibonacci number is divisible by 9 if and only if its subscript can be put into the form 12t-1, where t is a positive integer.

A proof of one of these theorems will now follow.

Theorem 4. A Fibonacci number is divisible by 3 if and only if its subscript can be put into the form 4b-1, where b is a positive integer.

Part I. Suppose a_n is divisible by 3 when n = 4b-1 in the k = b case. Letting n = 4b and m = 3 in the equation $a_{n+m} = a_n a_m + a_{n-1}a_{m-1}$, one has the result $a_{4b+3} = 3a_{4b} + 2a_{4b-1}$. Let $a_{4b-1} = 3c$, where c is a positive integer. Then $a_{4b+3} = 3a_{4b} + 6c = 3(a_{4b} = 2c)$. The Fibonacci number a_{4b+3} is divisible by 3. Since $4(b+1)-1 = 4b+3$, the k = b+1 case has been demonstrated.

Part II. If a Fibonacci number is divisible by 3, then the Fibonacci number's subscript can be put into the form 4b-1, where b is a positive integer. The contrapositive will be assumed false, and this assumption will lead to a contradiction.

Assume the following statement to be false: If a Fibonacci number's subscript cannot be put into the form 4b-1, where b is a positive integer, then the Fibonacci number is not divisible by 3.

From Part I, a_{4b-1}, where b is a positive integer, is divisible by 3. If a Fibonacci number which is not in the above form, is divisible by 3, it must be in the form a_{4b}, a_{4b+1}, or a_{4b+2}. One should note that a_{4b-1}, a_{4b+3}, a_{4b+7}, a_{4b+11}, ... are all in the form a_{4b-1}. Also a_{4b}, a_{4b+4}, a_{4b+8}, a_{4b+12}, ... are all in the form a_{4b}. Other equivalent forms are a_{4b+1}, a_{4b+5}, a_{4b+9}, ..., and a_{4n+2}, a_{4b+6}, a_{4b+10}, ...

Suppose a_{4b} is divisible by 3. Since a_{4b-1} is also divisible by 3, let $a_{4b} = 3k$ and $a_{4b-1} = 3m$, where k and m are positive integers. Now one should note that $a_{4b-1} + a_{4b} = a_{4b+1}$, and $3k + 3m = a_{4b+1}$. The equation $3(k+m) = a_{4b+1}$ tells one that a_{4b+1} is divisible by 3. From $a_{4b+1} + a_{4b} = a_{4b+2}$, one has $6k + 3m = a_{4b+2}$ and $3(2k+m) = a_{4b+2}$. Thus all Fibonacci numbers are divisible by 3. If Fibonaccinumbers in the form a_{4b} are divisible by 3, then all Fibonacci numbers are divisible by 3. But 1 is a Fibonacci number, and it is not divisible by 3.

Now suppose a_{4b+1} is divisible by 3. Let $a_{4b+1} = 3m$ also $a_{4b-1} = 3n$, where m and n are positive integers. Since $a_{4b-1} + a_{4b} = a_{4b+1}$, one may have $3n + a_{4b} = 3m$ and $a_{4b} = 3(m-n)$. This tells us that a_{4b} is divisible by 3. Let $a_{4b} = 3j$, where j is a positive integer. From $a_{4b} + a_{4b+1} = a_{4b+2}$, one may have $3(j+m) = a_{4b+2}$. This says that a_{4b+2} is divisible by 3. Again, all Fibonacci numbers are divisible by 3.

Last of all, suppose that a_{4b+2} is divisible by 3. Let $a_{4b+2} = 3d$ and $a_{4b-1} = 3e$, where d and e are positive integers. From $a_{4b-1} + a_{4b} = a_{4b+1}$ and $a_{4b+1} = a_{4b+2} - a_{4b}$ one may have $a_{4b-1} + a_{4b} = a_{4b+2} - a_{4b}$. This reduces to $a_{4b-1} + 2a_{4b} = a_{4b+2}$. Let $a_{4b+1} = 3f$ and $a_{4b-1} = 3g$, where f and g are positive integers; thus, $3g + 2a_{4b} = 3f$ and $2a_{4b} = 3(f-g)$. Since 2 and 3 are relatively prime, a_{4b} must be divisible by 3. Then a_{4b+1} must also be divisible by 3. Thus all Fibonacci numbers are divisible by 3.

In this proof, I have demonstrated that if any Fibonacci number not in the form a_{4b-1} is divisible by 3, then all Fibonacci numbers are divisible by 3. This obviously is a contradiction. The conclusion, then, is that only Fibonacci numbers in the form a_{4b-1} are divisible by 3.

CHAPTER VI

CONCLUSION

This study contains a development of the Fibonacci number sequence within the realm of number theory. To create a good foundation with several useful Fibonacci number sequence properties so that an interested student of mathematics could probe into the Fibonacci number sequence with respect to number theory was the purpose of this paper.

One should take special note of the method which was used to discover the properties dealt with in this study. The experimentalist approach is a powerful tool to use when studing mathematics.

A list of the basic properties found in this thesis is as follows:

$a_n = a_{n-1} + a_{n-2}$, where n is a positive integer equal to or greater that 2.

$a_{n+m} = a_n a_m + a_{n-1} a_{m-1}$, where m and n are positive integers.

$a_0 + a_1 + \ldots + a_n = a_{n+2} - 1$, and $a_0 + a_2 + \ldots + a_{2n} = a_{2n+1}$, where n is an non-negative integer.

$a_1 + a_3 + \ldots + a_{2n+1} = a_{2n+2} - 1$, where n is a positive integer.

$(a_n)^2 + (a_{n+1})^2 = a_{2n+2}$, where n is an non-negative integer.

$(a_{m-1})^2 + (a_m)^2 = a_{2m}$, where m is a positive integer.

$a_{2n-1} = (a_n)^2 - (a_{n-2})^2$, where n is a positive integer greater than or equal to 2.

$(a_n)^2 + (a_{n+3})^2 = 2a_{2n+4}$, $(a_n)^2 + (a_{n+3})^2 = a_{2n+6} - a_{2n+3}$, and $(a_n)^2 + (a_{n+3})^2 = a_{2n+5} + a_{2n+2}$, where n is an non-negative integer.

$(a_0)^2 + (a_1)^2 + \ldots + (a_n)^2 = a_n a_{n+1}$, where n is an non-negative integer.

$a_{2n+1} = (a_{n+1})^2 - (a_{n-1})^2$, where n is a positive integer.

$a_{2n+3} = (a_{n+2})^2 - (a_n)^2$, where n is an non-negative integer.

$a_{2n+5} = (a_{n+3})^2 - (a_{n+1})^2$, where n is a positive integer.

The appendix of this paper contains miscellaneous tables which might be an excellant starting point for further study.

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APPENDIX

	$a_n^3+a_{n-1}^3$	$a_{n+1}^3-a_n^3$	$a_{n+2}^2-a_n^2$	$a_{n+1}^2-a_n^2$	$a_n a_{n+1} a_{n+2}$	$a_n^3-a_{n-2}^3$	$\sum a_n$	$\sum a_n^2$
a_0							1	1
a_1	2	0		0			2	2
a_2	9	7	3	3	2	7	4	6
a_3	35	19	8	5	6	26	7	15
a_4	152	98	21	16	30	117	12	40
a_5	637	387	55	39	120	485	20	104
a_6	2709	1685	144	105	520	2072	33	273
a_7	11458	7064	377	272	2184	8749	54	714
a_8	48565	30043	987	715	9282	37107	88	1870
a_9	205679	127071	2584	1869	39270	157114	143	4895
a_{10}	871344	538594	6765	4896	166430	665665	232	12816
a_{11}	3690953	2281015	17711	12815	704880	2819609	376	33552
a_{12}	15635321	9663353	46368	33553	2986128	11944368	609	87841
a_{13}	66231970	40933296	121393	87840	12649104	50596649	986	229970
a_{14}	280563633	173398367	317811	229971	53583010	214311663	1596	602070
a_{15}	1188185803	734523803	832040	602069	226980390	907922170	2583	1576239
a_{16}	5034507976	3111498370	2178309	1576240	961505790	3846022173	4180	4126648
a_{17}	21326515877	13180509531	5702887	4126647	4073001576	16292007901	6764	10803704
a_{18}	90340574445	55833549037	14930352	10803705	17253515288	69014058568	10945	28284465
a_{19}	382688808866	236514685384	39088169	28284464	73087057560	292348234421	17710	74049690
a_{20}	1621095817661	1001892323411	102334155	74049691	309601753890	1238407008795	28656	193864606

n	$\sum a_n^3$	$\sum a_n^4$	$\sum a_n^5$	a_n
0	1	1	1	1
1	2	2	2	1
2	10	18	34	2
3	37	99	277	3
4	162	724	3402	5
5	674	4820	36170	8
6	2871	33381	407463	13
7	12132	227862	4491564	21
8	51436	1564198	49926988	34
9	217811	10714823	553211363	55
10	922780	73457064	6137270812	89
11	3908764	503438760	68054635036	144
12	16558101	3450734281	754774491429	233
13	70140734	23651386922	837042537086	377
14	294121734	162109796922	92830050637086	610
15	1258626537	1111115037483	1029298223070793	987
16	5331629710	7615701104764	11417322172518550	1597
17	22585142414	52198777931900	126619992693837974	2584
18	95672204155	357775783071021	1404237451180502875	4181
19	405273951280	2452231602371646	15573231068749231000	6765
20	1716768021816	16807845698458702	172709782964518145976	10946

a_n	a_n^2	a_n^3	a_n^4	a_n^5	$a_n^2 + a_{n-1}^2$	a_n
1	1	1	1	1	2	a_1
1	1	1	1	1	5	a_2
2	4	8	16	32	13	a_3
3	9	27	81	243	34	a_4
5	25	125	625	3125	89	a_5
8	64	512	4096	32768	233	a_6
13	169	2197	28561	371293	610	a_7
21	441	9261	194481	4084101	1597	a_8
34	1156	39304	1336336	45435424	4181	a_9
55	3025	166375	9150625	503284375	10946	a_{10}
89	7921	704969	62742241	5584059449	28657	a_{11}
144	20736	2985984	429981696	61917364224	75025	a_{12}
233	54289	12649337	2947295521	686719856393	196418	a_{13}
377	142129	53582633	20200652641	7615646045657	514229	a_{14}
610	372100	226981000	138458410000	84459630100000	1346269	a_{15}
987	974169	961504803	949005240561	136668172433707	3524578	a_{16}
1597	2550409	4073003173	6504586067281	10387823149447757	9227465	a_{17}
2584	6677056	17253512704	44583076827136	115202670521319424	24157817	a_{18}
4181	17480761	73087061741	305577005139121	1277617458486664901	63245986	a_{19}
6765	45765225	309601747125	2094455819300625	14168993617568728125	165580141	a_{20}
10946	119814916	1311494070536	14355614096087056	157136551895768914976		