A STUDY OF HELLY'S THEOREM ON INTERSECTION PROPERTIES OF CONVEX SETS: ITS RELATION TO THEOREMS OF CARATHEODORY AND RADON ON CONVEX COVERS AND ITS APPLICATIONS

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J.M.B.

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CHAPTER I

INTRODUCTION

The present thesis centers around a theorem (called "Helly's theorem") which sets forth conditions under which the intersection of a family of convex sets cannot be empty. Historically, this theorem was discovered by Helly in 1913 and communicated to Radon. It was first published in 1921 by Radon (using Radon's theorem), followed by a proof of König in 1922, and Helly's own proof appeared in 1923. Many proofs of Helly's theorem are now known. In recent years there has been a steady flow of publications devoted to extending Helly's theorem, and many of the results have significant applications in other areas of mathematics.

I. DESIDERATA

It is assumed here that the reader is an advanced student who has completed some courses in higher mathematics. Since the general setting is Euclidean n-space, the completion of a course in finite-dimensional vector spaces is essential. A familiarity with certain fundamental concepts of topology (continuity, interior, closure, boundary) will be very helpful, but genuine topological considerations have been kept in the background.

There is no easily available account of the present subject combining simplicity with generality, however desirable, although Eggleston [2] has made a step in this direction. Most of the important literature that was available for the preparation of this tract, while not intended for large groups of readers, has been developed in a general n-dimensional linear space. In particular, Eggleston [2] and Danzer-Grünbaum-Klee [1] both treat the subject in Euclidean n-space. Valentine [4] has rigorously developed the theory in a topological linear space (that is, a vector space with a Hausdorff topology such that the operations of vector addition and scalar multiplication are considered as continuous functions in all variables jointly). On the other hand, the book by Yaglom-Boltyanskii [5] is a beautiful treatment dealing with the plane case of convex sets, including a stimulating presentation of Helly's theorem and various applications of it. The numerous pictures and examples presented in [5] provide an excellent intuitive background for the understanding of the basic theory.

The bibliography at the end of this tract is not complete in any sense, but it represents an exhaustive one with respect to the available sources. The various results that have been published in the mathematical reviews and journals, mainly those concerning Helly's theorem, were not available.

Nevertheless, the primary references cited at the end of this tract are important standard sources; in fact, each of these sources is referred to in the bibliography of each of the others. The two main sources used in the preparation of this report were those of Eggleston [2]and Danzer-Grünbaum-Klee [1], forming a basis for the present material; so, many of the results presented here can be found in these two references with a more detailed account than is given here. Hadwiger-Debrunner [3] was also a valuable source, and much of the material presented in Chapter V is based on that particular text.

II. THE PROBLEM

Statement of the problem. There is a close relationship between Helly's theorem on the intersection of convex sets and the theorems of Caratheodory and Radon on the convex covers of sets. Linking these two dual aspects of convexity leads to an illuminating interplay of ideas, and the two approaches lead to different generalizations and results. The primary purpose of this inquiry was to (1) to determine the interdependence of these three theorems, and (2) to make a survey of the important applications that have been made of these results, particularly that of Helly, with the ultimate aim being to ascertain the general significance of Helly's theorem. To facilitate this profound objective, some generalizations and variants, or Helly-type theorems, are also presented, illustrating the diversity and utility of Helly's theorem as well as some of the chief methods used in the theory.

Importance of the study. Convexity is a quite active field today. In addition to being important for geometry, it provides efficient methods for the study of mathematics. In particular, it has a stimulating geometric and intuitive appeal when restricted to the plane. The importance of the study of convex sets is evidenced by the use in the Russian schools of several textbooks on convex bodies (see Yaglom-Boltyanskii [5]). Helly's theorem is especially characteristic of the subject, providing an excellent introduction to the theory. In view of the popularity of this theorem and its numerous applications in various other parts of mathematics, it seemed worthwhile to pursue the subject and acquire an appreciation of its true importance. This was one of the main objectives throughout the preparation of this report.

III. ORGANIZATION OF THE THESIS

The contents of this thesis is divided somewhat naturally into six chapters. Chapter II contains some definitions, certain fundamental theorems, and the introduction to some unusual concepts necessary for the understanding of the ensuing material. Chapters III and IV comprise the main results of this paper. In Chapter III the inter-relationship existing between the theorems of Helly, Caratheodory, and Radon is deduced. This "dual" aspect of convexity is the most interesting and unusual result presented in this thesis. Applications of Helly's theorem and related results are fully investigated in Chapter IV. Having developed the main results in these two chapters, Chapter V continues with some generalizations of Helly's theorem, and a selected group of "Helly-type theorems" are presented, all of which shed additional light on the heart of the matter. Finally, the main findings of the paper are summarized in the last chapter, Chapter VI, with some concluding remarks, and some other interesting related problems are indicated.

CHAPTER II

DEFINITIONS OF TERMS USED AND BASIC CONCEPTS

The containing space is taken to be n-dimensional real Euclidean space (with its usual metric) and is denoted by R^n. It is convenient to regard points in R^n as vectors, and vector addition and scalar multiplication are defined coordinatewise. The inner product is important. The distance between points x and y in R^n is $d(x,y) = |x - y|$. The symbol $\emptyset$ is used for the empty set, and 0 is used for the real number zero as well as for the origin of R^n.

I. CONVEXITY

DEFINITION 2.1. The <u>line</u> determined by two points x and y of R^n is the set of all points of the form

$$ax + (1 - a)y \qquad \text{(a real).}$$

The <u>closed segment</u> $[xy]$ joining points x and y of R^n is the set of all points of the form

$$ax + (1 - a)y \qquad (0 \leq a \leq 1),$$

while the <u>open segment</u> (xy) is the set of all points of the form

$$ax + (1 - a)y \qquad (0 < a < 1).$$

Where nothing else is said, the closed segment $[xy]$ will be referred to as, simply, the segment xy. This should cause no confusion.

DEFINITION 2.2. A subset X of R^n is called convex if and only if the line segment xy joining any two points x and y in X is contained in X. A closed and bounded convex subset of R^n with nonempty interior (relative to R^n) is called a convex body. A convex figure is a convex set in the plane.

The simplest examples of convex sets are the empty set, a single point, a segment, a triangle, the whole space, half-planes, lines, rays, and strips between parallel lines.

DEFINITION 2.3. If $x_1, \ldots, x_k$ are k points of R^n, then each point x of the form

$$x = a_1x_1 + \ldots + a_kx_k \quad (a_i \geqslant 0,\ a_1 + \ldots + a_k = 1)$$

is called a convex combination of the k points $x_1, \ldots, x_k$.

Frequently it is more convenient to use a more general form of the convexity condition.

THEOREM 2.4. If X is a convex set and if $x_1, \ldots, x_k$ are k points of X, then every convex combination of $x_1, \ldots, x_k$ also belongs to X.

PROOF. It is trivially true for k = 1. If k = 2, the theorem is just the definition that X is convex. Assume inductively that it is true for k = m and consider a point of the form

$$x = a_1x_1 + \ldots + a_mx_m + a_{m+1}x_{m+1} \quad (a_i \geqslant 0,\ a_1 + \ldots + a_{m+1} = 1).$$

If $a_{m+1} = 1$, then $x = x_{m+1}$ belongs to X. Suppose $a_{m+1} < 1$. Let $t = a_1 + \ldots + a_m = 1 - a_{m+1} > 0$. Then

$$x = t((a_1/t)x_1 + \dots + (a_m/t)x_m) + a_{m+1}x_{m+1}$$
$$= (1 - a_{m+1})((a_1/t)x_1 + \dots + (a_m/t)x_m) + a_{m+1}x_{m+1}.$$

Let $z = (a_1/t)x_1 + \dots + (a_m/t)x_m$. Then

$$x = (1 - a_{m+1})z + a_{m+1}x_{m+1}.$$

By hypothesis, the point z belongs to X. Since X is convex and contains both z and x_{m+1}, it follows that it contains x. Thus the theorem is true when $k = m+1$, hence true for all k. This completes the proof.

Since the closure and the interior of a convex set are also convex, the properties of general convex sets can usually be inferred from those of closed convex sets or from those of open convex sets; for this reason the material in this report is restricted generally to closed sets. The closure, interior, and boundary of a convex set are defined in terms of spherical neighborhoods.

DEFINITION 2.5. The <u>spherical neighborhood</u> (or simply the <u>neighborhood</u>) of the point p with radius r is the set

$$S(p,r) = \{x : x \in R^n,\ d(p,x) < r\}.$$

The <u>closed spherical neighborhood</u> (or <u>closed neighborhood</u>) of the point p with radius r is the set

$$\overline{S}(p,r) = \{x : x \in R^n,\ d(p,x) \leqslant r\}.$$

If the points are restricted to lie in the plane, then the neighborhood $S(p,r)$ is called an <u>open disk</u>, while $\overline{S}(p,r)$ is called a <u>closed disk</u>. In n-space $(n \geqslant 3)$ $S(p,r)$ is called an open n-ball and $\overline{S}(p,r)$ is called a closed n-ball. The terms open cell and closed cell (in R^n) are also used.

THEOREM 2.6. Let X be a convex set with a nonempty interior, denoted by int X, and let x and y be two points of X, where x belongs to int X. Then every point of the segment xy, except possibly y, is an interior point of X.

PROOF. Refer to Figure 1. Let z be any point of the segment xy different from y. Then $z = sx + (1 - s)y$, where $0 < s \leq 1$ (since $z \neq y$). Since $x \in \text{int } X$, there exists $r > 0$ such that $S(x,r) \subset X$. It remains to show $S(z,sr) \subset X$. If $p \in S(z,sr)$, i.e. if

$$\begin{aligned} |p - z| &= |p - (sx - (1 - s)y)| \\ &= |p + (s - 1)y - sx| < sr, \end{aligned}$$

then $|(1/s)p + (s-1/s)y - x| < r$. Hence the point

$$p' = (1/s)p + (s-1/s)y$$

is contained in $S(x,r)$. Since $p = sp' + (1-s)y$, p is on the segment $p'y \subset X$. This completes the proof.

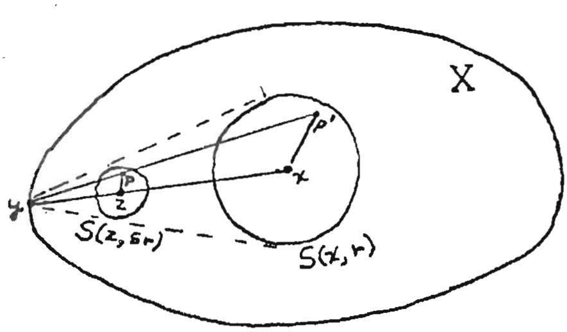

FIGURE 1

THE INTERIOR OF A CONVEX SET IS CONVEX

COROLLARY 2.7. The set of interior points of a convex set is convex.

COROLLARY 2.8. Every half-line issuing from an interior point of a bounded convex set X contains one and only one boundary point of X.

II. CONVEXITY AND ITS RELATION TO AFFINE GEOMETRY

Frequently convexity is regarded as a property of affine geometry, which is the study of properties invariant under the affine group. The affine group consists of the transformations A which carry a point x into the point

$$A(x) = T(x) + u,$$

where u is a fixed vector and T a non-singular linear transformation. When considering properties preserved under the affine group of transformations, vector spaces are usually referred to as <u>affine</u> spaces.

A non-singular affine transformation carries segments into segments, so that a convex set is transformed into another convex set; that is, convexity is invariant under the affine group.

DEFINITION 2.9. The m + 1 points $x_1, \ldots, x_{m+1}$ are called <u>affinely independent</u> if the m vectors $x_2 - x_1, \ldots, x_{m+1} - x_1$ are linearly independent; i.e. if

$$a_2(x_2 - x_1) + \ldots + a_{m+1}(x_{m+1} - x_1) = 0,$$

then $a_2 = \dots = a_{m+1} = 0$.

Or equivalently, if

$$a_1x_1 + a_2x_2 + \dots + a_{m+1}x_{m+1} = 0,$$
$$a_1 + a_2 + \dots + a_{m+1} = 0,$$

then $a_1 = a_2 = \dots = a_{m+1} = 0$.

A non-singular affine transformation carries affine independent points into affine independent points, for

$$T(x_i - x_1) = A(x_i) - A(x_1).$$

DEFINITION 2.10. Given m + 1 points $x_1, \dots, x_{m+1}$ in R^n and m + 1 real numbers $a_1, \dots, a_{m+1}$ such that $a_1 + \dots + a_{m+1} = 1$, a <u>centroid</u> of the points $x_1, \dots, x_{m+1}$ is a point x of the form

$$x = a_1x_1 + \dots + a_{m+1}x_{m+1}.$$

THEOREM 2.11. The m + 1 points $x_1, \dots, x_{m+1}$ are affinely independent if and only if every point x in $R^m \subset R^n$ has a unique representation as a centroid of $x_1, \dots, x_{m+1}$.

The numbers $a_1, \dots, a_{m+1}$ in Definition 2.10 are called the <u>barycentric coordinates</u> relative to the basis $x_1, \dots, x_{m+1}$. This term was introduced by Möbius (see [6, p. 204]). It has the following basis. A point of R^n to which a real number m, the mass or weight of the point, is assigned is called a mass point. If mass points x_i with weights m_i $(i = 1, \dots, r+1)$ are given and if $a_i = m_i/(m_1 + \dots + m_{r+1})$, then the point $x = a_1x_1 + \dots + a_{r+1}x_{r+1}$ is, by definition, the center of gravity or centroid of this mass distribution. The numbers m_i may or may not be positive. They are arbitrary real numbers such that $m_1 + \dots + m_{r+1} \neq 0$.

It is an interesting fact that the centroid is invariant under an affine transformation, that is,

THEOREM 2.12. An affine transformation carries centroids to centroids with the same weights.

The theorem says this: if T is an affine mapping, c the centroid of points x_i with weights a_i, then T(c) is the centroid of the mass points $T(x_i)$ with the same weights a_i. (In other words, the barycentric coordinates of a point are unchanged under an affine transformation.) For a more detailed discussion of affine geometry and proofs of the above results, see Birkhoff-MacLane [7, pp. 287-294].

III. CONVEX COVERS; SIMPLEXES; CONVEX POLYTOPES

The most basic intersection property of convex sets is the following.

THEOREM 2.13. The intersection of any collection of convex sets is a convex set, although it may be empty.

Given any set X, there can be associated with X a convex set C(X) called the <u>convex cover</u>, or <u>convex hull</u>, of X. (Since the term "hull" is somewhat misleading, the term "cover" will be used throughout the subsequent discussion.)

DEFINITION 2.14. The convex cover C(X) of a set X in

R^n is the intersection of all convex sets in R^n containing X. Alternatively, it is the smallest convex set that contains X.

The following is trivial but fundamental.

THEOREM 2.15. Let X be a subset of R^n. Then $X = C(X)$ if and only if X is convex.

If $X_1, \ldots, X_k$ is a finite collection of sets, the convex cover of their union is denoted by $C(X_1, \ldots, X_k)$.

DEFINITION 2.16. The <u>diameter</u> of a set X, denoted by diam X, is the least upper bound of the distances between two arbitrary points of the set X.

The following is important for later considerations, and so a proof is given here.

THEOREM 2.17. The diameter of the convex cover C(X) of a set X is equal to the diameter of X.

PROOF. Since $X \subset C(X)$, diam $X \leq$ diam $C(X)$. It remains to show that diam $C(X) \leq$ diam X. It suffices to show that if $d(x,y) < r$ for any pair of points x, y of X, then $d(x^*,y^*) < r$ for any pair of points x^*, y^* of C(X). Since $d(x,y) < r$ for all $y \in X$, $X \subset S(x,r)$. Since $S(x,r)$ is convex, it follows that $C(X) \subset S(x,r)$. Thus $x^* \in S(x,r)$ for all $x \in X$. Hence $x \in S(x^*,r)$, i.e. $X \subset S(x^*,r)$. Hence $C(X) \subset S(x^*,r)$. This implies that $d(x^*,y^*) < r$ and diam $C(X) \leq$ diam X. This proves the theorem.

DEFINITION 2.18. Let X and Y be subsets of R^n. Then, for some fixed $y \in R^n$, the set $X + y = \{x + y : x \in X\}$ is called a translate of X and, for some real number a, the set $aX = \{ax : x \in X\}$ is called a scalar multiple of X. Also, $X + Y = \{x + y : x \in X, y \in Y\}$.

DEFINITION 2.19. A flat, a linear manifold, or an affine subspace, is a translate of a linear subspace of R^n. The dimension of a flat is the dimension of the corresponding linear subspace.

For example, a flat of dimension 1 is called a line and a plane is a flat of dimension 2.

DEFINITION 2.20. The convex cover S of a finite set of k + 1 points $x_1, \ldots, x_{k+1}$ in R^n is called a k-dimensional simplex, or k-simplex, if the flat of minimal dimension containing S has dimension k. The points x_i $(i = 1, 2, \ldots, k+1)$ are called vertices. The k-simplex is regular if each two of its vertices determine the same distance.

Segments are 1-simplexes. Triangles are 2-simplexes, while the flat of minimal dimension containing a triangle is a plane (of dimension 2). In particular, an equilateral triangle is a regular 2-simplex. Tetrahedrons in space are 3-simplexes. The following theorem is of interest and follows immediately from Theorem 2.17.

THEOREM 2.21. The diameter of a simplex is equal to the maximum of the distances between its vertices.

DEFINITION 2.22. A convex polytope is a set which is the convex cover of a finite number of points.

THEOREM 2.23. The convex cover of the points x_1, x_2, ... , x_k is identical with the set of all convex combinations of x_1, ... , x_k.

In other words, the convex cover of the points x_1, x_2, ... , x_k consists of the centroids of all possible non-negative weights located at the points x_1, ... , x_k. In particular, if the points x_1, ... , x_k are affinely independent, that is, are vertices of a (k-1)-simplex, then the convex cover of the set consists of all the points of $R^{k-1} \subset R^n$ whose barycentric coordinates with respect to the basis are non-negative.

PROOF. Denote the set of all convex combinations of x_1, ... , x_k by K. The set K is convex. For suppose $x \in K$ and $y \in K$ where

$$x = a_1x_1 + \dots + a_kx_k,$$
$$y = b_1x_1 + \dots + b_kx_k.$$

Let $z \in xy$, i.e.,

$$z = sx + (1 - s)y \quad (0 \leqslant s \leqslant 1).$$

Then

$$\begin{aligned} z &= sa_1x_1 + \dots + sa_kx_k + (1 - s)b_1x_1 + \dots + (1 - s)b_kx_k \\ &= (sa_1 + (1 - s)b_1)x_1 + \dots + (sa_k + (1 - s)b_k)x_k. \end{aligned}$$

Since the coefficients $sa_i + (1 - s)b_i$ are non-negative and their sum is 1, i.e., z is a convex combination of x_1, ... ,

x_k, it follows that $z \in K$. Thus K is convex. Since K contains each x_i $(i = 1, \ldots, k)$, it follows that $C(x_1, \ldots, x_k) \subset K$. Furthermore, any convex set containing $x_1, \ldots, x_k$ also contains K by Theorem 2.4, page 7. Thus $K \subset C(x_1, \ldots, x_k)$, completing the proof.

THEOREM 2.24. Let X be a subset of R^n. Then the set of all finite convex combinations of points of X coincides with the convex cover $C(X)$.

PROOF. As in the proof of the previous theorem, denote the set of all finite convex combinations of points of X by K. Then for any point x of K,

$$x = a_1x_1 + \ldots + a_kx_k$$

for some positive integer k, where $x_1, \ldots, x_k$ are points of X and $a_1, \ldots, a_k$ are real numbers such that

$$a_1 + \ldots + a_k = 1 \ (a_i \geq 0,\ i = 1, \ldots, k).$$

Thus, by Theorem 2.23, $x \in C(x_1, \ldots, x_k)$. But $x_1, \ldots, x_k \in K$, and hence $C(x_1, \ldots, x_k) \subset C(X)$, so $x \in C(X)$. It follows that $K \subset C(X)$. Now it can be verified, as in the proof of the previous theorem, that K is convex. Clearly $X \subset K$. Thus, since K is convex, it follows that $C(X) \subset K$. Hence $K = C(X)$. This completes the proof.

The set of points in R^n whose barycentric coordinates are positive is a convex open set. Thus the following definition.

DEFINITION 2.25. The interior of the k-simplex (relative to the flat of minimal dimension containing it) is the set of all points x of the form

$$x = a_1x_1 + \dots + a_{k+1}x_{k+1},$$

where $a_1, \dots, a_{k+1}$ are real numbers such that

$$a_1 + \dots + a_{k+1} = 1 \ (a_i > 0, \ i = 1, \dots, k+1)$$

and $x_1, \dots, x_{k+1}$ are its vertices.

IV. SUPPORT HYPERPLANES AND SEPARATION THEOREMS

The existence of supporting hyperplanes at certain points of a convex set and separating hyperplanes for certain pairs of convex sets are fundamental results in the theory of convexity. Separating hyperplanes, in particular, will play a fundamental role in the results of this paper.

DEFINITION 2.26. A hyperplane is an (n-1)-dimensional flat of R^n.

Equivalently, a hyperplane is the set of points $x = (x_1, \dots, x_n)$ which satisfy an equation of the form

$$a_1x_1 + \dots + a_nx_n = b,$$

where not all the a_i are zero and b is some real number. Using the inner product notation, this means there exists a nonzero vector $a = (a_1, \dots, a_n)$ and a real number b such that the given hyperplane consists of all points x for which $a \cdot x = b$. For example, the hyperplanes in R^2 are the lines, while in R^3 they correspond to the planes.

DEFINITION 2.27. A hyperplane divides the space R^n into two open halfspaces consisting of the points x for which $a \cdot x < b$ and those for which $a \cdot x > b$. Similarly, each hyperplane determines two closed halfspaces for which $a \cdot x \leq b$ and $a \cdot x \geq b$. The hyperplane is said to bound the halfspaces.

DEFINITION 2.28. The hyperplane $a \cdot x = b$ separates two sets Y and Z in R^n if either $a \cdot y \leq b$, $a \cdot z \geq b$ or $a \cdot y \geq b$, $a \cdot z \leq b$ holds when $y \in Y$, $z \in Z$. The hyperplane $a \cdot x = b$ strictly separates Y and Z if either $a \cdot y < b$, $a \cdot z > b$ or $a \cdot y > b$, $a \cdot z < b$ when $y \in Y$, $z \in Z$.

DEFINITION 2.29. A hyperplane $a \cdot x = b$ is said to cut the convex set Y if and only if there exist points y_1 and y_2 in Y such that $a \cdot y_1 < b$ and $a \cdot y_2 > b$.

DEFINITION 2.30. The dimension of a convex set X is the largest integer m such that X contains m + 1 points $x_1, \ldots, x_{m+1}$ which are affinely independent.

In particular, every point x of X is of the form

$$x = a_1 x_1 + \ldots + a_{m+1} x_{m+1} \quad (a_i \geq 0,\ a_1 + \ldots + a_{m+1} = 1).$$

The flat (of dimension m) spanned by $x_1, \ldots, x_{m+1}$, i.e., the set of points x of the form

$$x = a_1 x_1 + \ldots + a_{m+1} x_{m+1} \quad (a_1 + \ldots + a_{m+1} = 1),$$

is denoted by L(X) and is said to be the flat carried by X, the flat spanned by X, or the minimal flat containing X.

DEFINITION 2.31. The relative interior and relative boundary of a set X are defined to mean the interior and boundary of X relative to L(X).

Barycentric coordinates are used to prove the following result.

THEOREM 2.32. The relative interior of a convex set is nonempty.

PROOF. Let $x_1, \ldots , x_{m+1}$ be m + 1 points of X which form a basis of L(X), i.e., $x \in L(X)$ if and only if

$$x = a_1x_1 + \ldots + a_{m+1}x_{m+1} \quad (a_1 + \ldots + a_{m+1} = 1).$$

Consider the point $x_0 = (x_1 + \ldots + x_{m+1})/(m+1)$. Then clearly $x_0 \in L(X)$, and since $x_i \in X$ $(i = 1, \ldots , m+1)$, it follows that $x_0 \in X$ by Theorem 2.4, page 7. Since each a_i $(i = 1, \ldots , m+1)$ depends continuously upon the coordinates of x, there exists a positive number d such that if $x \in S(x_0,d) \cap L(X)$, then each a_i is positive. Hence each of these points x belongs to X. Thus x_0 is a point of the relative interior of X, completing the proof.

THEOREM 2.33. The hyperplane H cuts the convex set X if and only if the following two conditions hold:

(i) $L(X) \not\subset H$;

(ii) H intersects the relative interior of X.

COROLLARY 2.34. If a hyperplane cuts X, it also cuts the relative interior of X.

For a proof of Theorem 2.33 and the corollary, see Eggleston [2, p. 17].

The following <u>separation</u> <u>theorem</u> is basic to the subsequent considerations of this paper.

THEOREM 2.35. (Separation Theorem) Suppose X and Y are two convex sets. Also suppose $X \neq \emptyset$, int $Y \neq \emptyset$, and that $X \cap \text{int } Y = \emptyset$. Then there exists a hyperplane H which separates X and Y. (For a proof, see [2], [4].)

DEFINITION 2.36. A hyperplane that intersects the closure of a convex set X and does not cut X is said to be a supporting hyperplane of X.

Planes of support play an important role in the theory of convex sets. The next two theorems concerning supporting hyperplanes give a characterization of convexity.

THEOREM 2.37. Through every point on the boundary of a convex set X there passes at least one support hyperplane of X.

THEOREM 2.38. If the closed set X has a nonempty interior and if through every point of its boundary there passes a supporting hyperplane to X, then X is convex.

Proofs of the above theorems and related results may be found in Eggleston [2] and Valentine [4]. Also, see Yaglom-Boltyanskii [5] for a treatment of the plane case.

This characterization of convexity, when taken as a definition, forms the basis of the duality theory of convex sets. Although it has not been seriously pursued in the preparation of this little tract, the duality theory provides efficient machinery in the study of convexity. In fact, Valentine [4] says: "Always look at the dual situation . . . for it may save you some embarrassment. The theory . . . may be intuitively simpler when viewed in the dual situation." In [4], the duality theory and the dual cone are employed as an approach to Helly's theorem.

There is no exact duality in convexity as in the case of projective geometry, and so there is a choice of dual spaces available, a "dual space" to R^n being a space in which the hyperplanes or halfspaces of R^n are represented by points or, possibly, halflines. In fact, "duality" is simply a correspondence between points, on the one hand, and hyperplanes on the other. While correspondence is a more modest term, the use of the term duality gives a sort of "dual feeling" and is somewhat more natural and geometric. The idea of duality is important for two main reasons: (1) it often suggests alternative proofs of known results, and (2) it often suggests new results which are "dual" to known results.

The following theorem, which gives an alternative definition of a convex polytope (see Definition 2.22, page 14), illustrates the scope of duality in Euclidean space.

THEOREM 2.39. A bounded nonempty subset of R^n is the intersection of a finite number of closed halfspaces of R^n if and only if it is a convex polytope.

The dual properties are those of being the intersection of a finite number of closed halfspaces, on the one hand, or of being the convex cover of a finite set of points on the other. This dual aspect of Euclidean space is investigated further in Chapter III.

THEOREM 2.40. If X is a closed convex set and Y is a convex body which does not intersect X, then there exists a hyperplane strictly separating X from Y.

PROOF. Let x_1 and y_1 be points of X and Y, respectively, such that

$$d(x_1,y_1) = \inf\{ d(x,y) : x \in X, \ y \in Y\}.$$

Then $d(x_1,y_1) > 0$. (It can be shown that such a pair of points x_1, y_1 always exists, and also that $d(x_1,y_1) > 0$.) Let H_1 and H_2 be the hyperplanes through x_1 and y_1 perpendicular to the segment x_1y_1. Then the hyperplane H_1 through x_1 is a supporting hyperplane of X at x_1 and the hyperplane H_2 through y_1 supports Y at y_1. For suppose there is a point $q \in X$ on the same side of H_1 as y_1, that is, q and y_1 are in the same open halfspace bounded by H_1. (Refer to Figure 2.) In the 2-flat spanned by x_1, y_1, and q drop a perpendicular from y_1 to x_1q with foot p. If $p \in x_1q$, then $y_1p < y_1x_1$, contrary to assumption. If $q \in x_1 p$, then $y_1 q < y_1 x_1$, also contrary to assumption. Therefore all of X lies on one side of H_1; hence H_1 is a supporting hyperplane of X. Similarly H_2 supports Y, and any hyperplane between H_1 and H_2 strictly separates X from Y. This completes the proof.

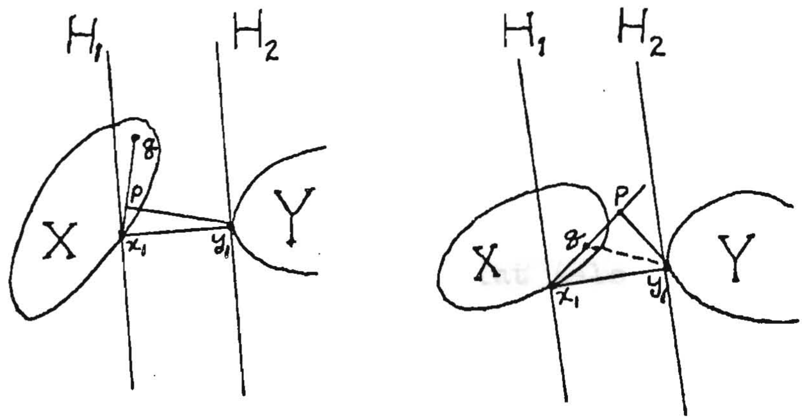

FIGURE 2

THE EXISTENCE OF A SEPARATING HYPERPLANE

V. PROJECTIONS AND MAPPINGS

Many theorems proved for convex sets in R^1 and R^2 are extended to R^n by induction on the dimension, so that it is necessary to relate the property in R^n to that in R^{n-1}. There are essentially two ways of doing this: (1) projection of a convex set from a point, and (2) projection of a convex set parallel to a fixed direction, that is, by orthogonal projection. The first mapping is described as follows. Let X be a convex set, and let O be a point not belonging to X.

The union of all halflines joining O with points of X is called the cone subtended by X at the vertex O. If H is a hyperplane not containing O and intersecting every halfline joining O to any point x X in a single unique point T(x), then a mapping of the set X into H is defined. This mapping is called the projection of X <u>onto</u> H <u>from</u> O. The second type of mapping is orthogonal projection. Given a convex set X, the set of all lines parallel to a given line and intersecting the convex set X is a convex set. The intersection of this convex set of lines with a flat (also convex) perpendicular to the given line is called the orthogonal projection of the given set X onto the flat. The orthogonal projection of the set X onto the flat is convex since it is the intersection of two convex sets.

Sometimes it is convenient to reduce a problem concerning a closed bounded convex set to one on a closed spherical neighborhood or closed n-ball. This is done by "central projection" as follows.

THEOREM 2.41. All n-dimensional convex bodies are homeomorphic to a closed n-ball.

PROOF. Let X be an n-dimensional convex body in R^n, and let B^n denote a closed n-ball whose center is any interior point x_0 of the convex set X. Let x by any point in X different from x_0. See Figure 3 for the case n = 2. Denote by p(x) the boundary point of X lying on the ray x_0x, by q(x) the point in which the ray x_0x intersects the boundary of B^n (see Corollary 2.8, page 10), and by $f(x)$ the point which divides the segment $x_0q(x)$ in the same ratio as the point x divides the segment $x_0p(x)$. Then f is a homeomorphism such that $f(X) = B^n$ with center $f(x_0) = x_0$; that is,

$$f(x) = (x - x_0)/(|p(x) - x_0|) + x_0 \text{ if } x \neq x_0,$$

$$f(x_0) = x_0.$$

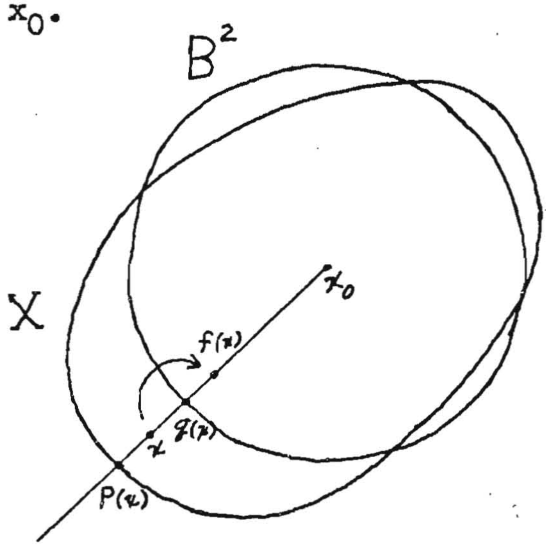

FIGURE 3

A HOMEOMORPHISM

DEFINITION 2.42. A <u>positive</u> <u>homothety</u> is a transformation which, for some fixed $y \in R^n$ and some real number $a > 0$, sends $x \in R^n$ into $y + ax$. The image of a set X under a positive homothety is called a <u>homothet</u> of X.

CHAPTER III

THE THEOREMS OF HELLY, CARATHEODORY, AND RADON

I. HELLY'S THEOREM

Prologue. Eduard Helly was born in Vienna on June 1, 1884. He received the Ph.D. degree in 1907 at the University of Vienna. In addition to his famous theorem on the intersection of convex sets, which he discovered in 1913, he contributed a number of other important results in mathematics during the years to follow. In 1938 Helly emigrated to America, with his wife and seven-year-old son, where he was on the staff of two Eastern colleges and the Illinois Institute of Technology. He died in Chicago in 1943. A more detailed account of Helly's life, obtained directly from his wife by the authors, is included in Danzer-Grünbaum-Klee [1, p. 101]. Helly's theorem is formulated as follows.

THEOREM 3.1. (Helly's theorem) Let F be a family of at least n+1 convex sets in affine n-space R^n, and suppose F is finite or each member of F is compact. Then if each n+1 members of F have a common point, there is a point common to all the members of F.

A vector space satisfying Helly's theorem is essentially one whose dimension is finite (see [2, p. 33]). In particular, Helly's theorem on the intersection of convex sets is one of the most striking properties of Euclidean n-space. As an illustration of Helly's theorem, consider three convex sets in R^2 which have a common point (see, e.g., Figure 4). Helly's theorem says that if a convex set in R^2

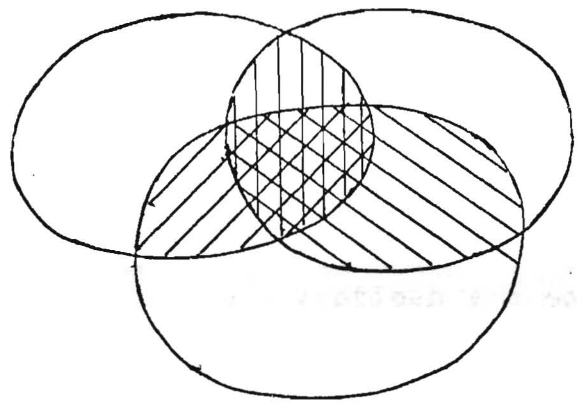

FIGURE 4

AN ILLUSTRATION OF HELLY'S THEOREM

intersects each of the three shaded areas, which are the pairwise intersections of the sets, then it must intersect the supershaded area, which is the intersection of all three of the given sets. The theorem of Helly is closely related to the theorems of Caratheodory and Radon on convex covers.

THEOREM 3.2. (Caratheodory's theorem) Let X be a subset of R^n. Then each point of C(X), the convex cover of X, is a convex combination of n+1 (or fewer) points of X.

This theorem was first published in 1907. The theorem of Caratheodory and an extension of it, where it is assumed that X in R^n has at most n components (that is, separated pieces), are proved in this chapter. A proof of Helly's theorem by means of Caratheodory's theorem is also given. Caratheodory's theorem is described by saying that the convex cover of a given set X is the union of an aggregate of simplexes whose vertices are among <u>the points of X</u>. Let X be a subset of R^2 consisting of, say, three components (as, for example, in Figure 5). Caratheodory's theorem guarantees that each point of the convex cover C(X) of X either lies inside a triangle (2-simplex) whose vertices are points of X, is on a segment (1-simplex) whose endpoints are points of X, or is itself a point (0-simplex) of X.

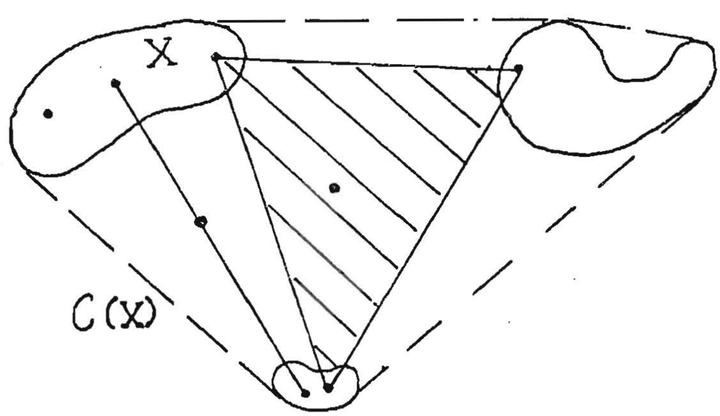

FIGURE 5

AN ILLUSTRATION OF CARATHEODORY'S THEOREM

THEOREM 3.3. (Radon's theorem) Each set of n+2 or more points in R^n can be expressed as the union of two disjoint sets whose convex covers have a common point.

A proof of Helly's theorem, based on Radon's theorem, was first published by Radon in 1921. Radon's proof of Helly's theorem is also included in this chapter. As an example of Radon's theorem, let X be any subset of R^2 consisting of four points (see Figure 6). Radon's theorem says this: either one of the points lies in the triangle determined by the other three, or else the segment joining some pair of points intersects the segment determined by the other pair of points.

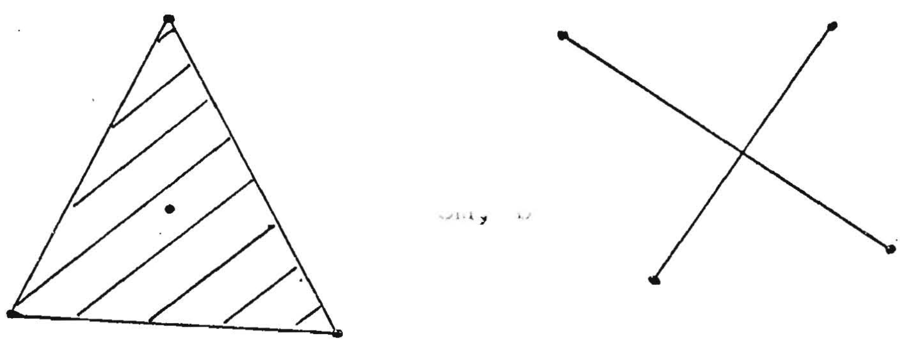

FIGURE 6

AN ILLUSTRATION OF RADON'S THEOREM

Each of the three theorems of Helly, Caratheodory, and Radon can be derived from each of the others (see [1,p. 109]). Indeed, this is the most astounding fact discovered in this investigation. It thus appears that these three results on the intersection of convex sets and the representation of convex covers are the manifestation of some underlying property of Euclidean space. In particular, this relationship is closely connected with the concept of duality in the theory of convexity. The "dual" aspects are the intersection of convex sets, on the one hand, and the representation of the convex covers of sets, on the other.

Except for the derivation of Radon's theorem from the other two theorems, the equivalence of the three theorems is established in this chapter. The theorems of Helly and Radon are also both proved independently and "directly". Some special cases of Helly's theorem for the line and plane are first considered, including several different interesting proofs for the plane case.

<u>Helly's theorem in the line and plane</u>.

THEOREM 3.4. (Helly's theorem for the line) If each pair of n segments of a line have a common point, then all n segments have a common point.

PROOF. Designate the left endpoints of the given segments by a_1, a_2, ... , a_n and the right endpoints by b_1, b_2, b_3, ... , b_n. Since the segments a_ib_i and a_jb_j ($i = 1, \ldots, n$; $j = 1, \ldots, n$) intersect, it follows that the left endpoint a_i of the first segment cannot be to the right of the endpoint b_j of the second segment; in other words, none of the left endpoints lies to the right of any of the right endpoints. Let a denote the left endpoint lying furthest to the right, b denote the right endpoint lying furthest to the left. If a coincides with b, then a = b is the only point belonging to all the segments a_1b_1, a_2b_2, ... , a_nb_n. If a lies to the left of b, then the entire segment ab is contained in all the given segments. This completes the proof.

LEMMA 3.5. Let four convex figures be given in the plane, each three of which have a common point. Then all four figures have a common point.

PROOF. A relaxed form of Radon's theorem is used. Denote the convex figures by C_1, C_2, C_3, C_4. Let a_4 be a common point of C_1, C_2, C_3, let a_3 be a common point of C_1, C_2, C_4, and so forth. Since a_1, a_2, a_3 belong to C_4, the triangle $a_1a_2a_3$ is contained in C_4. Similarly, the triangle $a_1a_2a_4$ is contained in C_3, $a_1a_3a_4$ in C_2, and $a_2a_3a_4$ in C_1. See Figure 7. According to Radon's theorem, two cases can arise: (1) either one of the points a_1, a_2, a_3, a_4 lies inside (or on one side) of the triangle formed by the other three, or (2) none of the points lies in the triangle formed by the other three, that is, the four points are vertices of a convex quadrilateral. Suppose, for example, in the first case, that a_1 lies inside the triangle $a_2a_3a_4$. Then a_1 belongs to all four figures. The argument remains valid even if the triangle is degenerate. Suppose the second case occurs. Then the intersection of the diagonals of the quadrilateral belongs to all four triangles under consideration, and hence to all four figures C_1, C_2, C_3, C_4. The other cases are all similar. This completes the proof.

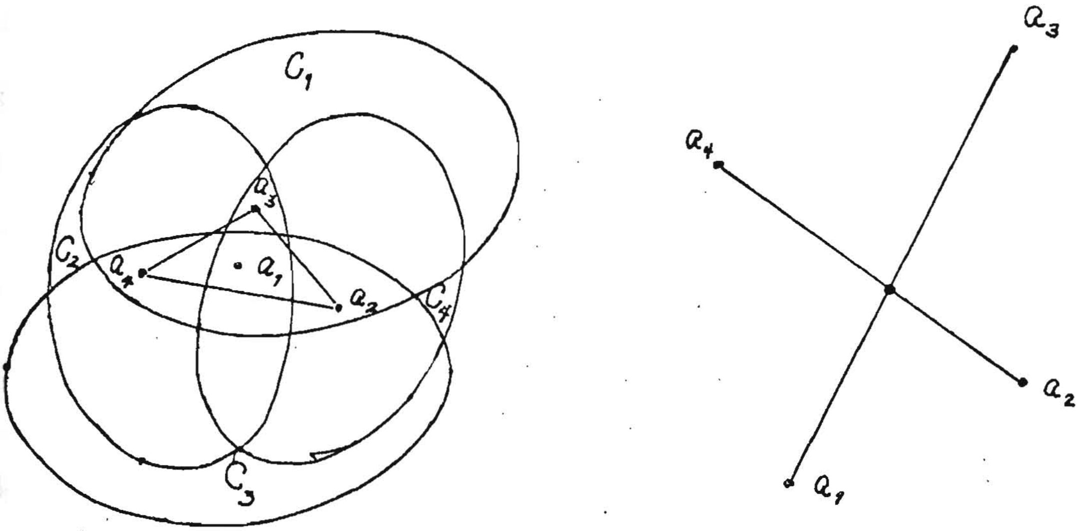

FIGURE 7

THE FOUR FIGURES HAVE A COMMON POINT

(IF EACH THREE HAVE A COMMON POINT)

THEOREM 3.6. (Helly's theorem for the plane) Let n convex figures be given in the plane, each three of which have a common point. Then all n figures have a common point.

PROOF. The proof is by induction. If the number of figures is three or four, then the theorem is true. Assume inductively that it is true for k figures. Let C_1, C_2, C_3, ... , C_{k+1} be k+1 convex figures, each three of which have a common point. Denote the intersection of the figures C_k and C_{k+1} by C_k^*. Then C_1, ... , C_{k-1}, C_k^* are k convex figures, each three of which have a common point. For by hypothesis there exists a common point for each three of the figures distinct from C_k^*. Further, by Lemma 3.5, there exists a common point of the figures C_i, C_j, C_k^* (that is, a common point of the figures C_i, C_j, C_k, C_{k+1}, since by hypothesis each three have a common point). In other words, each three of the figures $C_1, \ldots, C_{k-1}, C_k^*$ have a common point. By the inductive assumption, there is a point belonging to these k figures, and hence to each of the figures $C_1, \ldots, C_k, C_{k+1}$. This completes the proof.

In general, it is not true that an infinite number of unbounded convex figures have a nonempty intersection if each three have a common point. Consider, for example, an "upper" halfplane bounded below by some horizontal line. If halfplanes situated "higher" are adjoined, each three have a nonempty intersection. Adjoining more and more such halfplanes, the intersection moves higher and higher and gradually "slips away to infinity" with the continual adjunction of higher halfplanes. See Figure 8.

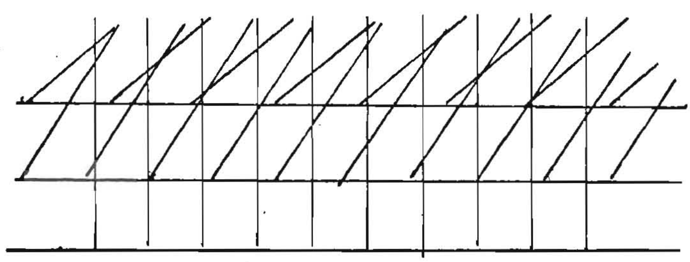

FIGURE 8

THE INTERSECTION OF THE HALFPLANES

"SLIPS AWAY TO INFINITY"

On the other hand, this situation cannot occur if bounded figures are considered, for the intersection must then remain in a bounded part of the plane (in fact, the boundedness of at least one of the given figures suffices). The following is true.

COROLLARY 3.7. If a finite or an infinite number of closed and bounded convex figures in the plane are given such that each three have a common point, then they all have a common point.

The preceding discussion is suggestive as to why Helly's theorem deals with only two types of families: those which are finite and those whose members are closed and bounded. This matter is investigated later.

A different proof of Helly's theorem in the plane is now given. This proof is more readily generalized to prove Helly's theorem for convex sets in R^n. The theorem is proved here only for four convex figures. The proof for any finite number of figures then proceeds by induction, as in the proof of Theorem 3.6.

THEOREM 3.8. Let four <u>bounded</u> convex figures of the plane be given, each three of which have a common point. Then they all have a common point.

PROOF. As before, denote the given figures by C_1, C_2, C_3, C_4. Let C denote the convex figure which is the intersection of C_1, C_2, C_3. It is evident, after a moment's reflection, that the four given figures have a common point if and only if C and C_4 have a common point. It remains, therefore, to prove that C and C_4 have a common point. Assume that they do not. Since C and C_4 are then two disjoint closed and bounded convex figures, there exists a line L which strictly separates the two figures (by Theorem 2.40, page 22). Assume that the line L is horizontal and that C lies above L and C below it, as illustrated in Figure 9.

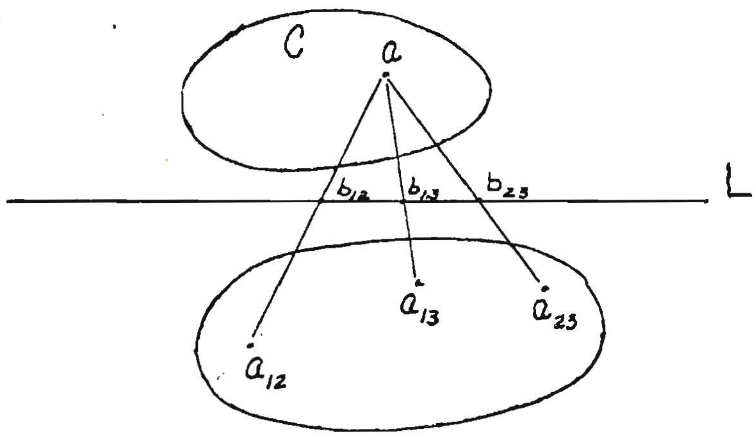

FIGURE 9

A CONTRADICTION TO THE SEPARATION ASSUMPTION

Since each three of the given four figures intersect, there exists a point a_{12} belonging to C_1, C_2, C_4, a point a_{13} belonging to C_1, C_3, C_4 and a point a_{23} to C_2, C_3, C_4. All three points lie below the line L (since they all belong to C_4). Let a be any point of C (that is, the intersection of C_1, C_2, C_3). Since the figures C_1, C_2, C_3 are convex, the segment $a_{12}a$ lies in the intersection of the figures C_1 and C_2, the segment $a_{13}a$ in the intersection of C_1 and C_3, and $a_{23}a$ in the intersection of C_2 and C_3. Denote by b_{12}, b_{13}, b_{23} the intersections, respectively, of the three segments with the line L. In particular, these intersection points belong to the given figures. Thus each two of the figures C_1, C_2, C_3 have a common point on the line L. Furthermore, each of the three figures has a segment in common with L. Since each two of the three segments have a common point, it follows that there is a point b of the line L which is common to all three segments (by Helly's theorem for the line). In particular, the point b belongs to all three of the figures C_1, C_2, C_3, hence to C. This is a contradiction to the assumption that L has no point in common with C, the intersection of the figures C_1, C_2, C_3. Hence C and C_4 intersect, so that C_1, C_2, C_3, C_4 have a point in common. This concludes the proof.

It should be noted that the figures were assumed to be bounded in the above theorem. This was necessary in order to apply the separation theorem, which is valid only for bounded figures. However, if the number of figures is finite, the proof is still valid; for a closed disk of sufficiently large radius can be taken such that it at least partially covers the intersections of each three of the given convex figures in the theorem. Since the disk is convex, all such intersections are convex, and the problem reduces at once to a finite number of bounded convex figures, each three of which have a common point. The separation theorem can then be applied. This situation is discussed later.

THEOREM 3.9. (Helly's theorem in space) Let n convex bodies be given, each four of which have a common point. Then all n bodies possess a common point.

The following is also true: If infinitely many bounded convex bodies are given in space, each four of which have a common point, then all the bodies possess a common point. The proof is essentially the same as that of Theorem 3.8, except that a separating plane is used instead of a line. Since Helly's theorem will be proved in the general n-dimensional case using this same approach (in Helly's proof), the theorem is only stated here as a further illustration of Helly's theorem.

DEFINITION 3.10. Suppose n convex sets $C_1, C_2, \ldots, C_n$ are given in R^n. The distance from a point p to the set of convex sets $C_1, C_2, \ldots, C_n$ is the greatest of the distances from p to the individual sets $C_1, C_2, \ldots, C_n$.

THEOREM 3.11. Let p be a point of the plane such that the distance d from p to the set of convex figures $C_1, \ldots, C_n$ is a minimum. Then

(i) either there are three of the given figures C_i, C_j, C_k for which the points a_i, a_j, a_k nearest to p are at a distance d from p and form a triangle containing the point p within itself; or

(ii) there are two figures C_i, C_j for which the points a_i, a_j nearest to p and at distance d from it are the ends of a segment containing the point p.

See Yaglom-Boltyanskii [5, pp. 165-167] for another interesting elementary proof of Helly's theorem in the plane using Theorem 3.11, together with a proof of this theorem. The theorem is stated here, not only as a curiosity, but as indications of things to come, since essentially the same approach is used later in proving Helly's theorem in R^n by means of Caratheodory's. Actually, Theorem 3.11 is an elementary variant of Caratheodory's theorem in the plane. Also, another interesting proof of Helly's theorem for the plane is given in Hadwiger-Debrunner [3, p. 60], using a somewhat more general form of Radon's theorem than used in Lemma 3.5, page 31.

<u>Helly's theorem in Euclidean n-space</u>. Helly's theorem deals with two types of families: those which are finite and those whose members are all compact. For a family of compact convex sets, it is sufficient to prove the result for finite families, for then the finite intersection property implies

that the intersection of such a family is nonempty. For let F be a family of compact convex sets, each finite subfamily of which has a nonempty intersection, and suppose further that the members of F have an empty intersection. Select $F_1 \in F$. Since the members of F have an empty intersection, for $x \in F_1$ there exists a member $F_x \in F$ which does not contain x. Since F_x is closed, there exists a neighborhood $N(x)$ of x such that $N \cap F_x = \emptyset$. By the Heine-Borel property which defines compactness, a finite covering $N(x_1)$, $N(x_2)$, ... , $N(x_k)$ of F_1 is thus obtained such that members $F_{x_i} \in F$ exist such that $N(x_i) \cap F_{x_i} = \emptyset$ $(i = 1, \ldots, k)$. Then

$$F_1 \cap F_{x_1} \cap F_{x_2} \cap \ldots \cap F_{x_k} = \emptyset,$$

a contradiction, since every finite subfamily of F has a nonempty intersection. The following theorem enables one to work with bounded closed sets.

THEOREM 3.12. Suppose $K = \{K_1, \ldots, K_N\}$ is a finite family of N convex sets (in some linear space), each $n+1$ of which have a common point. Then there exist N convex (compact) polyhedra P_i $(i = 1, \ldots, N)$ such that $P_i \subset K_i$ $(i = 1, \ldots, N)$; and such that every $n+1$ of them have a nonempty intersection.

PROOF. Consider all possible ways of choosing $n+1$ members of K, and for each such choice select a single point in the intersection of the $n+1$ sets chosen. Let J be the finite set of all points so selected. For each $K_i \in K$ let P_i be the convex cover of $K_i \cap J$. Then each P_i is a convex polyhedron and each n+1 of the sets P have a common point. Also, any point common to all the sets P_i must lie in the intersection of the family F. Thus, for a finite family of convex sets, Helly's theorem may be reduced to the case of a finite family of compact convex polyhedra. (Compare this argument with that on page 36.) In particular, the separation theorem can then be applied to the compact sets thus obtained.

The literature contains many proofs of Helly's theorem. Three proofs are presented in this section: (1) Helly's own proof; (2) Radon's proof (using Radon's theorem); and (3) a proof by means of Caratheodory's theorem. Each approach adds further illumination, and in many cases these different characterizations lead to different generalizations and results.

Helly's own proof depends on the separation theorem for convex sets in R^n and proceeds by induction on the dimension of the space. Among the many proofs, his is the most geometric and intuitive. Refer to Figure 9, page 35, for the case n = 2.

PROOF (1). (Helly's) The theorem is obvious for R^0. (It is also true for R^1 and R^2, but this fact is not needed in the present argument.) Assume inductively that it is true for R^{n-1}. Let F be a finite family of at least n+1 compact convex sets, each n+1 of which have a common point. Suppose the intersection $\bigcap F$ is empty. Then there are a subfamily F' of F and a member A of F' such that $\bigcap F' = \emptyset$, but such that $\bigcap(F' \setminus \{A\}) = M \neq \emptyset$. Since A and M are disjoint nonempty compact convex subsets of R^n, the separation theorem guarantees the existence of a hyperplane H in R^n such that A lies in one of the open halfspaces determined by H and M lies in the other. Let J denote the intersection of some n members of $F' \setminus \{A\}$. Obviously $J \supset M$. Since each n+1 members of F have a common point, J must intersect A. Since J is convex, in extending across H from M to A it must intersect H, and thus there is a common point for each n sets of the form $G \cap H$ with $G \in F' \setminus \{A\}$. From the induction hypothesis as applied to the (n-1)-dimensional space H it follows that $M \cap H$ is nonempty, a contradiction, thus completing the proof.

Helly's proof above can be found in [1, pp. 106-107], [4, pp. 70-71]. According to [1, p. 106], essentially the same proof was given by König (see page 1). The next proof is due to Radon. It is a very elegant algebraic proof and uses Radon's theorem stated above (see page 28).

PROOF (2). (Radon's) Let F_i $(i = 1, \ldots , r)$ be r members of the given family F of convex sets, where $r \geqslant n+2$.

The proof is by induction on r. Suppose every r-1 members of $\{F_i\ (i = 1, \ldots, r)\}$ have a point in common. It remains to prove $\bigcap_{i=1}^{r} F_i \neq \emptyset$. By the inductive hypothesis applied to the subfamily which consists of the whole family except F_j, there is a point $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$ which belongs to F_i if $i \neq j$. The equations

$$\sum_{j=1}^{r} a_j x_k^{(j)} = 0 \quad (k = 1, \ldots, n),$$

$$\sum_{j=1}^{r} a_j = 0,$$

form a set of n+1 equations in the r unknowns $a_1, \ldots, a_r$. Since $r > n+1$, these equations have non-trivial sets of solutions. For one such solution denote by $a_{i_1}, \ldots, a_{i_k}$ those a that are non-negative and by $a_{h_1}, \ldots, a_{h_{r-k}}$ those that are negative. Define the point $y = (y_1, \ldots, y_n)$ by

$$y_k = (\sum_r a_{i_r} x_k^{(i_r)})/(\sum_r a_{i_r}).$$

Since $x^{(i_r)} \in F_i$ if $i \neq i_r$, it follows that $y \in F_i$ by Theorem 2.4, page 7, provided $i \neq i_1, \ldots, i_k$, i.e., y belongs to $F_{h_1}, \ldots, F_{h_{r-k}}$. But according to the system of equations above,

$$y_k = (\sum_s (-a_{h_s}) x_k^{(h_s)})/(\sum_s -a_{h_s}),$$

and thus y also belongs to $F_{i_1}, \ldots, F_{i_k}$. Thus y is a point common to all the sets F_j, $j = 1, \ldots, r$. Hence, by induction, the members of each finite subset of the family F have a point in common. By compactness, this implies that all members of F have a point in common. This completes the proof.

PROOF (3). (using Caratheodory's theorem) The proof is by contradiction. Suppose there is a collection of N convex sets satisfying the conditions of Helly's theorem, but that there is no point common to all the members. Suppose, also, that the sets $C_1, \ldots, C_N$ are compact. Let x be any point of R^n. There exists a point x_0 such that $f(x) = \max_{1 \leq r \leq N} d(x, C_r)$ attains its least value at $x = x_0$. Also $f(x_0) > 0$. Among the C_r there are some, say $C_1, \ldots, C_k$, which are such that $f(x_0) = d(x_0, C_r)$, $(r = 1, \ldots, k)$. Suppose that $x_r \in C_r$ and $|x_0 - x_r| = f(x_0)$, $r = 1, \ldots, k$. (See Figure 10 for the case $n = 2$.) The points x_r exist because each C_r is compact; each x_r is unique because each C_r is convex. Then $x_0 \in C(x_1, \ldots, x_k)$; otherwise, $f(x_0)$ could

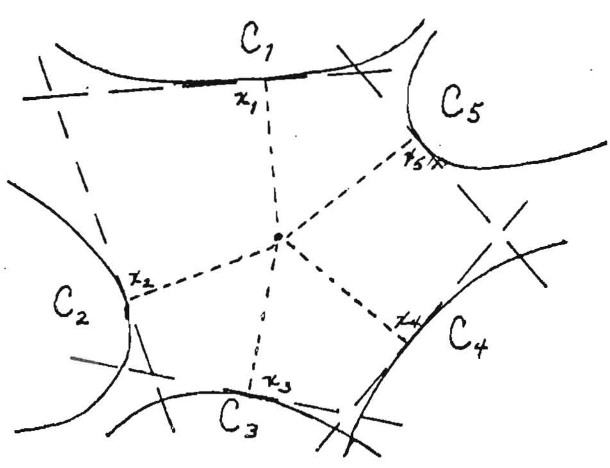

FIGURE 10

SUPPORTING HYPERPLANES AT NEAREST POINTS be reduced by moving x_0 towards $C(x_1, \ldots, x_k)$. By Caratheodory's theorem there is a subset of the points $x_1, x_2, \ldots, x_k$ consisting of at most $n + 1$ members, say $x_1, x_2, \ldots, x_s$, such that $x_0 \in C(x_1, \ldots, x_s)$. Then

$$x_0 = \sum_{r=1}^{s} a_r x_r \quad (a_i \geqslant 0,\ a_1 + \ldots + a_s = 1).$$

The hyperplane through x_r perpendicular to x_0x_r supports C_r at x_r. Thus if $y \in C_r$, then

$$(y - x_0)\cdot(x_r - x_0) \geqslant (|x_0 - x_r|)^2 > 0 \quad (r = 1, \ldots, k),$$

since none of the x_r coincide with x_0, $r = 1, \ldots, k$. Thus if $y \in C_r$, $r = 1, \ldots, s$ (such a point exists since $s \leqslant n+1$ by the hypothesis of Helly's theorem),

$$\begin{aligned} 0 &= (y - x_0)\cdot(x_0 - x_0) \\ &= (y - x_0)\cdot\Big(\sum_{r=1}^{s} a_r(x_r - x_0)\Big) \\ &= \sum_{r=1}^{s} a_r (y - x_0)\cdot(x_r - x_0) \\ &> 0, \end{aligned}$$

since all $a_r \geqslant 0$ and at least one $a_r > 0$. This contradiction establishes Helly's theorem in the case when the sets are compact.

The literature contains many different approaches to Helly's theorem. Eggleston [2] employs Caratheodory's theorem, which is closely connected to the approach by means of the dual cone. Valentine [4] employs Caratheodory's theorem and the duality theory of convex cones. Hadwiger has obtained Helly's theorem and other results by an application of the Euler-Poincare' characterization (see, e.g., Hadwiger-Debrunner [3]). Levi has developed an axiomatic approach based on Radon's theorem, and additional proofs of Helly's theorem have been given by Krasnoselskii, R. Rado and many others (see [1, p. 109]). Helly has also proved a topological generalization by means of combinatorial topology; this approach has lead to many interesting problems, but it remains to be fully exploited. Finally, for a complete bibliography of these various results, consult the compendium Danzer-Grünbaum-Klee [1].

II. THE THEOREMS OF CARATHEODORY AND RADON

In this section Radon's theorem is proved. The theorem of Caratheodory is deduced from the theorems of Helly and Radon, and an extension of this theorem is also proved.

Proof of Radon's theorem. In terms of a basis in R^n, the set of n + 1 equations corresponding to

$$a_1x_1 + a_2x_2 + \dots + a_rx_r = 0$$
$$a_1 + a_2 + \dots + a_r = 0$$

has a nontrivial solution $a_1^*, a_2^*, \dots, a_r^*$, since $r \geq n+2$. At least two of the numbers $a_1^*, a_2^*, \dots, a_r^*$ must have opposite signs. Without loss of generality, suppose $a_i^* \geq 0$ if $i = 1, \dots, j$ and $a_k^* < 0$ if $k = j+1, \dots, s$. Also

$$\sum_{i=1}^{j} a_i^* > 0, \quad \sum_{k=j+1}^{r} a_k^* < 0.$$

Then, if $t = \sum_{i=1}^{j} a_i^*$,

$$x = \sum_{i=1}^{j} (a_i^*/t)x_i = \sum_{k=j+1}^{r} (a_k^*/-t)x_k.$$

This completes the proof.

<u>Deduction of Caratheodory's theorem from Radon's.</u> Consider a set $X \subset R^n$ and a point $x \in C(X)$. According to Theorem 2.24, page 16, there is some integer k such that

$$x = a_1x_1 + \ldots + a_kx_k, \; a_i \geqslant 0, \; a_1 + \ldots + a_k = 1, \; x_i \in X.$$

It remains to show that an expression can be found for x with $k \leqslant n+1$. Let k be the smallest integer such that x can be represented as above and suppose $k \geqslant n+2$. Then by Radon's theorem there exist numbers $b_1, \ldots, b_k$ not all zero such that

$$b_1x_1 + \ldots + b_kx_k = 0, \; b_1 + \ldots + b_k = 0.$$

Let $V = \{i : b_i < 0\}$ and let $j \in V$ be such that $r_j = a_j/b_j \geqslant a_i/b_i$ for all $i \in V$. Then

$$x = (a_1 + r_jb_1)x_1 + \ldots + (a_k + r_jb_k)x_k,$$
$$a_i + r_jb_i \geqslant 0 \quad (i = 1, \ldots, k),$$
$$(a_1 + r_jb_1) + \ldots + (a_k + r_jb_k) = 1.$$

Since the coefficient of x_j is zero, it follows that x is a convex combination of k - 1 points of X. This contradicts the choice of k and establishes the theorem of Caratheodory.

<u>Deduction of Caratheodory's theorem from Helly's.</u> The case of Caratheodory's theorem is proved in which the given

set X is compact. Let $y \in C(X)$. It can be assumed that y is an interior point of C(X); otherwise, C(X) is supported at y by a hyperplane, say H, and $y \in C(X \cap H)$. If y is an interior point of $C(X \cap H)$, then the argument can proceed with $X \cap H$ instead of with X. Otherwise, the process is repeated until a flat Q is reached such that y is an interior point of $C(X \cap Q)$ relative to Q. Therefore, it is assumed that y is an interior point of C(X). If $y \in X$, there is nothing to prove. If $y \notin X$, then for each point $x \in X$ denote by T(x), W(x) the closed half-spaces bounded by the hyperplane through x perpendicular to xy (see, e.g., Figure 11). W(x) is the half-space which contains y. Now the set $\bigcap_{x \in X} T(x)$ is empty. For suppose it contained a point z. Let H be the hyperplane

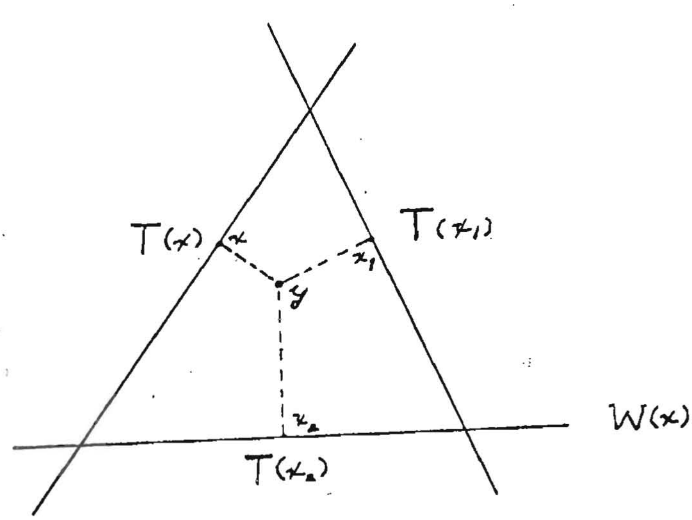

FIGURE 11

THE INTERSECTION OF THE HALF-SPACES IS EMPTY

hrough y perpendicular to zy. There is a point, say x_0, of
separated from z by H. Then the hyperplane perpendicular
o x_0y through x_0 does not separate y from z, a contradiction,
ince $z \in T(x_0)$ and $y \in W(x_0)$. Since each T(x) is convex it
ollows from the converse of Helly's theorem that there are
points, $s \leq n+1$, $x_1, \ldots, x_s$ of X such that $\bigcap_{i=1}^{s} T(x_i) = \emptyset$.
ut this implies that $y \in C(x_1, \ldots, x_s)$; otherwise, there is
hyperplane Q strictly separating y from $C(x_1, \ldots, x_s)$.
et the halfline that terminates at y, which is perpendicular
o Q, and which meets Q be L. Then L meets every $T(x_i)$ and
ll of L except a finite segment is contained in $T(x_i)$. But
his implies that $\bigcap_{i=1}^{s} T(x_i) \neq \emptyset$. This last relation is false,
nd hence $y \in C(x_1, \ldots, x_s)$ and Caratheodory's theorem is
roved.

<u>An extension of Caratheodory's theorem.</u> An extension
f Caratheodory's theorem is proved here in which it is
ssumed that X in R^n has at most n components.

DEFINITION 3.13. A convex set C in R^n which has at
east two points is called a convex cone with vertex y if for
ach $a \geq 0$ and for each $x \in C$, $x \neq y$, then $(1 - a)y + ax \in C$.
cone which is a proper subset of a line is a ray.

THEOREM 3.14. Let X be a subset of R^n with at most
components, and let $y \in C(X)$. Then there is a set of s
oints $x_1, \ldots, x_s$ all belonging to X with $s \leq n$ such that

is a point of the simplex whose vertices are $x_1, \ldots, x_s$.

PROOF. Let y be any point of C(X). By Caratheodory's theorem, there are s points of X with $s \leq n+1$, $y \in C(x_1, \ldots x_s)$. If $s < n+1$, the theorem is true. Suppose then that $s = n+1$. Then

$$y = b_1x_1 + \ldots + b_{n+1}x_{n+1} \quad (b_i \geq 0,\ b_1 + \ldots + b_{n+1} = 1),$$

where if any $b_i = 0$, the theorem is true, so suppose $b_i > 0$ ($i = 1, \ldots, n+1$). Let x_i' be the reflection of x_i in y, i.e., $x_i' = 2y - x$ $(1 \leq i \leq n+1)$. Let C_j be the cone subtended at y by the simplex whose vertices are the n points $x_1', \ldots, x_{j-1}', x_{j+1}', \ldots, x_{n+1}'$ (see Figure 12 for the case n = 2).

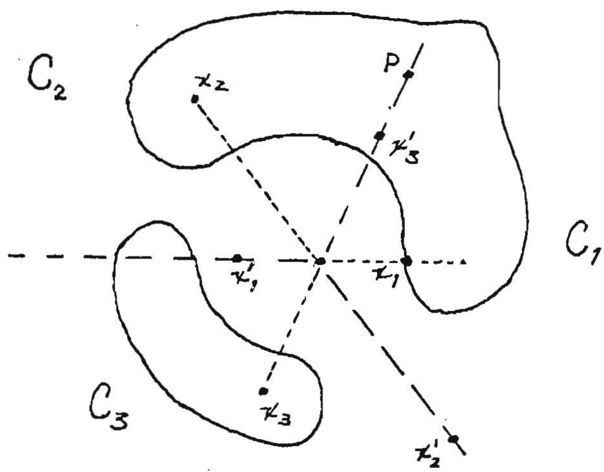

FIGURE 12

CONES SUBTENDED AT A POINT BY SIMPLEXES

Consider for definiteness the cone C_1. By definition, its points z are of the form

$$z = sy + t(a_2x_2' + \ldots + a_{n+1}x_{n+1}'),$$

where $a_i \geqslant 0$, $a_2 + \dots + a_{n+1} = 1$, $s + t = 1$, $t \geqslant 0$.

Take y to be the origin. Then

$$z = a_2'x_2' + \dots + a_{n+1}'x_{n+1}' \quad (a_i' \geqslant 0,\ 2 \leqslant i \leqslant n+1),$$

and similarly for the sets C_j, $2 \leqslant j \leqslant n+1$. It follows that

$$\begin{aligned} x_j' &= -x_j \\ &= (1/b_j)(b_1x_1 + \dots + b_{j-1}x_{j-1} + b_{j+1}x_{j+1} + \dots + b_{n+1}x_{n+1}). \end{aligned}$$

Thus x_j is an interior point of C_j. Since there are n+1 cones each containing points of X, and since X has at most n components, there is at least one point of X in the boundary of one of the cones C_j. Suppose, for example, that $p \in X \cap \mathrm{Bd}\ C_1$. Boundary points of C_1 are points z of the form above in which at least one of the numbers a_2', ... , a_{n+1}' is zero. Suppose then that $a_2' = 0$, say

$$p = c_3x_3' + \dots + c_{n+1}x_{n+1}' \quad (c_i \geqslant 0,\ i = 3, \dots, n+1).$$

$$\begin{aligned} \text{Then } y &= 0 \\ &= (p + c_3x_3 + \dots + c_{n+1}x_{n+1})/(1 + c_3 + \dots + c_{n+1}). \end{aligned}$$

Thus y is a point of the simplex whose n vertices p, x_3, x_4, ... , x_{n+1} all belong to X. The other possible cases are all similar and the proof of the theorem is complete.

The least number of points in Caratheodory's theorem cannot be reduced any further by imposing even more severe conditions on the connectivity of X. For example, if x_1, x_2, ... , x_{n+1} are the vertices of a non-degenerate simplex in R^n and X is the n segments x_1x_i, $2 \leqslant i \leqslant n+1$, then X is connected, but there are points of C(X) that are not contained in any simplex which has s vertices all belonging to X where $s \leqslant n-1$.

CHAPTER IV

APPLICATIONS OF HELLY'S THEOREM

A number of interesting results which can be proved with the aid of Helly's theorem are given in this chapter.

I. KIRCHBERGER'S THEOREM

In this section an example is given of the use of Helly's theorem to prove a result due to Kirchberger.

THEOREM 4.1. (Kirchberger's theorem) Let X and Y be two finite subsets of R^n. If for every subset S consisting of n+2 points selected from the union $X \cup Y$ it is possible to find a hyperplane that strictly separates $S \cap X$ from $S \cap Y$, then there is a hyperplane that strictly separates X from Y.

This theorem was formally established by Kirchberger in 1902. The following proof was given by Rademacher and Schoenberg in 1950. It is of interest to note that the original proof, which did not employ Helly's theorem, is nearly twenty-four pages long.

PROOF. For each $x = (x_1, \ldots, x_n) \in X$ and $y = (y_1, \ldots, y_n) \in Y$, define the open halfspaces J_x and Q_y in R^{n+1} by

$$(a_1, \ldots, a_n, b) \in J_x \text{ if } a_1x_1 + \ldots + a_nx_n + b > 0,$$

$$(a_1, \ldots, a_n, b) \in Q_y \text{ if } a_1y_1 + \ldots + a_ny_n + b < 0.$$

By hypothesis, each n+2 of the halfspaces $\{J_x : x \in X\} \cup \{Q_y : y \in Y\}$ have a point in common, and hence by Helly's theorem there is a point common to all of them. This point is, say, $(a_1', \ldots, a_n', b')$, and the sets X and Y are strictly separated by the hyperplane $a_1'z_1 + \ldots + a_n'z_n + b' = 0$, where $z = (z_1, \ldots, z_n) \in R^n$. This completes the proof.

The number n + 2 used in the theorem cannot be reduced. Consider, for example, the case in which X is the vertices of a regular simplex and Y is the centroid of the simplex. Then every subset S consisting of n + 1 points of $X \cup Y$ is such that $S \cap X$ can be separated from $S \cap Y$, but X cannot be separated from Y. In R^2, for instance, X would be the vertices of an equilateral triangle and Y would be the centroid, that is, the "center", of the triangle.

II. ESTIMATES OF "CENTEREDNESS"

The applications presented in this section indicate that for an arbitrary set there are points which behave approximately like a center of the set.

DEFINITION 4.2. Let S be a set of points. If for each point of the set S there is a point of S such that the segment joining the two points is always bisected by the same point O, then O is called the _center of symmetry_ (or simply the _center_) of S.

It follows from this definition that the center of a convex set is a point which besects every chord passing through it. Since every set does not have a center of symmetry, the following estimates are presented to show to what degree certain arbitrary sets are "centered".

THEOREM 4.3. Let n points be given in the plane. Then there exists a point O in the plane such that on each side of any line L through the point O there are at least $n/3$ of the given points (including points on the line itself).

PROOF. Let $P_1, P_2, \ldots, P_n$ be the given points. Consider all closed halfplanes which contain more than two-thirds of the given points (including points on the boundary lines of the halfplanes). It is then asserted that each three of these halfplanes have a common point. For suppose H_1, H_2, H_3 are any three such halfplanes. Let H' be the complement of H. Then $(H_1 \cap H_2 \cap H_3)' = H_1' \cup H_2' \cup H_3'$. Since each H_i' contains less than $n/3$ of the given points, it follows that $H_1' \cup H_2' \cup H_3'$ contains less than n points (of the set) and so $H_1 \cap H_2 \cap H_3$ contains at least one point of the set. This proves the assertion. Helly's theorem then implies that there is a point O common to all of the halfplanes under consideration. The point O is now shown to be the desired point. Let L be any line through O and assume it has been given a definite direction. Let P be either of the half-planes into which L divides the plane, say the half-plane to the left of L. There are at least n/3 of the given points in the halfplane P (including the line L). For suppose the contrary. Then there are more than 2n/3 of the n given points to the right of L, not counting those that belong to L itself. Consider the line M parallel to and to the right of L and sufficiently close to L so that none of the given points lies between L and M. Then there are more than 2n/3 points to the right of M. Hence the halfplane to the right of M must contain the point O, a contradiction, since O lies on the line L and is to the left of M. Thus the halfplane P contains at least n/3 of the given points.

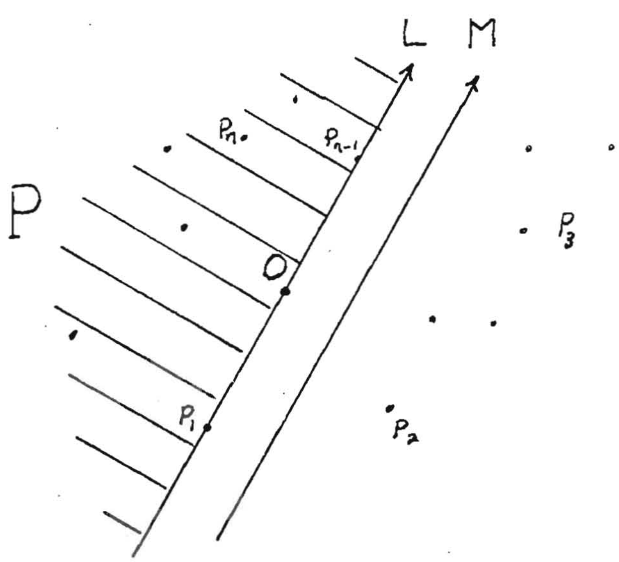

FIGURE 13

EACH HALFPLANE CONTAINS AT LEAST

ONE-THIRD OF GIVEN POINTS

THEOREM 4.4. Let a bounded curve K of length L be given in the plane (consisting possibly of separated pieces). Then there is a point of the plane such that each line through the point divides the curve K in two parts, each having a length of not less than $L/3$.

THEOREM 4.5. Let C be a plane bounded figure (consisting possibly of separated pieces) with area S. Then there is a point in the plane such that every line through the point divides the figure into two parts, each having an area of not less than $S/3$.

The estimates in the preceding theorems cannot be improved. For example, three non-collinear points constitute a set of points admitting no point O such that on each side of any line through O there are more than $n/3$ of the three points; three non-intersecting small circles about the vertices of a triangle furnish an example of a curve K for which the estimates cannot be improved.

THEOREM 4.6. Let S be any finite set of points in space. Then there is a point O such that every closed halfspace bounded by a plane through O contains at least $n/4$ points of S.

PROOF. Let $P_1, P_2, \ldots, P_n$ be the given points. Consider the set of all closed halfspaces which contain more than $3n/4$ points of S. It is asserted that each four of these halfspaces have a common point. Helly's theorem can then be applied to conclude the existence of a point O common to all such halfspaces. To prove the assertion, let H_1, H_2, H_3, H_4 be any four closed halfspaces each containing more than $3n/4$ points of S. Let H' be the complement of H. It is known that $(H_1 \cap H_2 \cap H_3 \cap H_4)' = H_1' \cup H_2' \cup H_3' \cup H_4'$. Since the H_i' each contain less than $n/4$ points of S, it follows that $H_1' \cup H_2' \cup H_3' \cup H_4'$ contains less than n points of S and hence $H_1 \cap H_2 \cap H_3 \cap H_4$ contains at least one point of S. This proves the assertion. It remains to show that the point O is the desired point. Suppose the contrary. Then there is a plane P through O which bounds a closed halfspace containing less than $n/4$ points of S. The opposite open halfspace H will contain more than $3n/4$ points of S. Let P' be the plane parallel to P passing through the points of $S \cap H$ closest to P. The closed halfspace H' bounded by P' and lying in H contains more than $3n/4$ points of S, but H' does not contain O. This contradiction completes the proof.

THEOREM 4.7. Let S be a bounded set of points in space having volume V. Then there is a point O such that every closed halfspace through O intersects S in a set of volume at least $V/4$.

THEOREM 4.8. Inside every bounded convex figure C there exists a point O such that every chord AB of C which passes through O is divided into two segments AO and BO, each of whose length is not less than 1/3 the length of the segment AB.

PROOF. Let C be a given convex figure and let A be any boundary point of C. Consider all possible chords of the figure C through A and lay off on any chord AB the segment AD whose length is 2/3 the length of AB. Refer to Figure 14. Now all of the points D thus obtained form the boundary of a certain figure C_A which is similar to the figure C and which lies in a position similar to the figure C. The point A is the center of similitude and 2/3 is the ratio of similarity. Let O be the point whose existence is to be proved, and let AB be a chord of the figure through this point. Since by definition of the point O the inequality $AO \leq 2/3AB$ must hold, the point O must belong to the figure C_A. The assertion in the theorem is equivalent to the assertion that there exists a point O belonging to every figure C_A whose center of similitude A lies on the boundary of C and whose ratio of similarity is 2/3. Since all the figures C_A are convex (that is, since all figures similar to a convex figure are convex), it is sufficient, by Helly's theorem, to show that any three of the figures under consideration have a common point. Let C_A, C_B, and C_C be three such figures similar to the figure C whose centers of similitude are the boundary points A, B, and C respectively. Draw the three chords AB, BC, and AC of the figure C. Let M, N, and P be the midpoints of the triangle ABC and let Q be the intersection of the medians AM, BN, and CP of this triangle. The point M belongs to the figure C. According to a well-known property of the medians of a triangle, the segment AQ equals 2/3 AM; hence the point Q belongs to the figure C_A. Similarly, Q also belongs to the figures C_B and C_C. Thus C_A, C_B, and C_C possess a common point. The proof is complete.

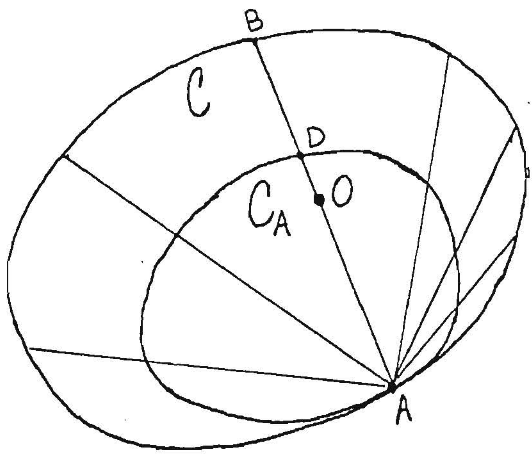

FIGURE 14

A FIGURE SIMILAR TO A CONVEX FIGURE

(WITH CENTER OF SIMILITUDE A)

The estimate in the preceding theorem cannot be improved. Within a triangle there is no point O such that both segments of each chord through O are greater than 1/3 of the entire chord. For let O be an interior point of the triangle ABC (in Figure 15) and let D, E and F be the respective intersections of OA, BO, and CO with the sides BC, AC, and AB of the triangle. If DO >1/3DA, then O lies in the interior (not the boundary) of the triangle cut off from ABC by a line parallel to BC and passing through the intersection M of the medians. If EO >1/3EB then O is similarly situated within the triangle cut off from ABC by a parallel to AC through M. If FO >1/3FC, then O lies within the triangle cut off from ABC by a parallel to AB through M. These three triangles have no common interior point; hence no such point O of triangle ABC can exist.

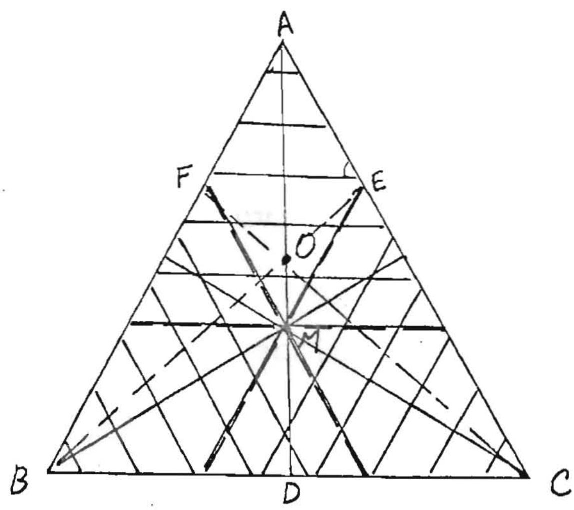

FIGURE 15

THERE EXISTS NO POINT O WITHIN A TRIANGLE SUCH THAT BOTH SEGMENTS OF ALL CHORDS THROUGH O ARE GREATER THAN 1/3 THE LENGTH OF THE CHORD

The next result is a generalization of Theorem 4.8.

The proof is due apparently to Yaglom-Boltyanskii [5]. Radon also proved that the centroid of a convex body has the stated property (see [1, p. 115]).

THEOREM 4.9. If C is a convex body in R^n, there exists a point $z \in C$ such that for each chord uv of C which passes through z, $(|z-u|)/(|v-u|) \leq n/(n+1)$.

PROOF. For each point $x \in C$, let $C_x = x + n(n+1)^{-1}(C-x)$. It is claimed that $\bigcap_{x\in C} C_x \neq \emptyset$. By Helly's theorem, it suffices to show that if $x_1, \ldots, x_{n+1}$ are any $n+1$ points of C, then $\bigcap_{i=1}^{n+1} C_{x_i}$ includes the point $y = (n+1)^{-1}\sum_{i=1}^{n+1} x_i$. This is true, since for each j it is true that

$$y = x_j + \frac{n}{n+1}\left(\frac{1}{n}\sum_{i\neq j} x_i - x_j\right) \in x_j + \frac{n}{n+1}(C-x_j).$$

Consider an arbitrary chord uv passing through the point $z \in \bigcap_{x\in C} C_x$. Then $z \in u + \frac{n}{n+1}(uv - u)$, whence $z = u + \frac{n}{n+1}s(v-u)$ for some s, $0 \leq s \leq 1$. Thus

$$|z-u|/|v-u| = sn/(n+1) \leq n/(n+1),$$

completing the proof.

III. TRANSLATION AND COVERING PROBLEMS

The following translation problem brings out the relation between covering and intersection properties of convex sets.

THEOREM 4.10. Let $K = \{K_a : a \in A\}$ be a collection of compact convex sets in R^n containing at least $n + 1$ members, et C be a compact convex set. If for each set of n+1 mem-ers of K there exists a translate of C which intersects (is ontained in; contains) all n+1 of them, then there exists uch a translate of C which intersects (is contained in; ontains) all the members of K.

PROOF. For each $K_a \in K$, let $K_a' = \{x \in R^n : (C+x) r K_a\}$, here r means "intersects" or "is contained in" or "contains". hen K_a' is convex. For instance, if r means "contains", then f $K_a \subset C + x_1$, $K_a \subset C + x_2$, it follows that $K_a \subset C + tx_1 + (1-t)x_2$, ≥ 0, for if $x = y + x_1 = z + x_2$, $x \in K_a$, $y \in C$, $z \in C$, then $= ty + (1-t)z + tx_1 + (1-t)x_2$, so that $x \in C + tx_1 + (1-t)x_2$. ence $tx_1 + (1-t)x_2 \in K_a'$ and K_a' is convex. It is also compact. y hypothesis, every n+1 of the members of $\{K_a' : a \in A\}$ have point in common. Hence there exists a point q common to ll K_a' by Helly's theorem, and C+q is the desired translate f C.

Theorem 4.10 is a generalization of Helly's theorem, or the latter results when C consists of a single point and means "intersects" or "is contained in". The theorem is specially useful for various covering problems when the amily K consists of one-pointed sets, as the following orollaries show.

COROLLARY 4.11. Let n points be given in the plane uch that each three of them can be enclosed in a circle of adius 1. Then all n points can be enclosed in a circle of adius 1.

Although Corollary 4.11 follows readily from heorem 4.10, a "direct" proof is given here for simplicity.

PROOF. It is necessary to show there exists a point of the plane (the center of the desired circle) whose istance from all the points is not greater than 1; or quivalently, that there exists a point 0 of the plane which elongs to all the circles of radius 1 about the given points. ccording to Helly's theorem, it suffices to show that any hree of the circles (about any three of the given points) ntersect. By virtue of the hypothesis, any three points an be enclosed in a circle of radius 1. The center of this ircle is a point belonging to the three unit circles about he three points, say A, B, D(since it is at most at distance from each of the three points).

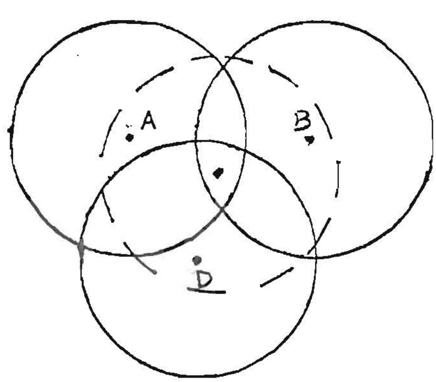

FIGURE 16

THE CIR CLES ABOUT THE THREE GIVEN POINTS INTERSECT

The next result is of some interest in itself.

COROLLARY 4.12. A class of convex set in R^n is such hat to every subclass of n+1 members of the class there orresponds a point whose distance from each of the n+1 conex sets is less than or equal to a fixed positive number d. hen there is a point whose distance from each convex set of he whole class is less than or equal to d.

COROLLARY 4.13. If X is a subset of R^n and each n+1 r fewer points of X can be covered by some translate of the onvex body C in R^n, then X lies in some translate of C.

THEOREM 4.14. Suppose a convex set in R^n is covered y a finite family of open or closed halfspaces. Then it is overed by some n+1 or fewer of these halfspaces.

This theorem illustrates the use of Helly's theorem n the contrapositive.

PROOF. Let $H = \{H_1, H_2, \ldots, H_n\}$ be the family of alfspaces covering a convex set C. Let H_i' denote the compement of H_i relative to C. Then $H' = \{H_1', H_2', \ldots, H_n'\}$ is finite family of convex sets whose intersection is empty, o by Helly's theorem there are n+1 or fewer sets in this amily whose intersection is empty. This completes the proof.

IV. A CHARACTERIZATION OF STARSHAPEDNESS

DEFINITION 4.15. A set S is said to be <u>starshaped</u>

vith respect to the point P of S, starshaped from P, if and only if for each point Q of S the entire segment PQ lies in S.

DEFINITION 4.16. If x and y are points of a set $S \subset R^n$, y is said to be visible from x (in S) provided $xy \subset S$.

DEFINITION 4.17. Polygons having the property that all segments which join a given interior point with all boundary points of the polygon also lie in the polygon are called star-shaped polygons. (See Figure 17.)

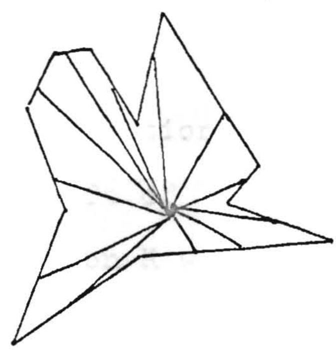

FIGURE 17

A STAR-SHAPED POLYGON

The following theorem is a characterization of star-shapedness for polygonal domains.

THEOREM 4.18. (Krasnosel'skii's theorem) If a domain in R^2 is bounded by a simple closed polygon, and if for each three sides of the polygon there is an interior point from which these three sides are visible, then there is some point of the domain from which all the sides are visible.

The following picturesque form is given by Yaglom-Boltyanskii [5]: If for each three paintings in a gallery one can find a place from which all three can be viewed, then there must be a place in the gallery from which all its pictures are visible.

PROOF. Let K be the given polygon. Let K be given a positive orientation on its boundary, each side of K being directed so that, when traversing the boundary in this sense, in the neighborhood of each side interior points of K lie to the left. Let AB be a side of the polygon lying on the line L. Assume L has the same direction as the side AB and denote by H the halfplane to the left of L. H will be called the "left halfplane of the polygon K with respect to the side AB". It remains to prove that there is a point O belonging to the left halfplanes with respect to all the sides of the polygon K and that this point satisfies the conditions of the theorem. According to Helly's theorem, it suffices to show that any three of the left halfplanes have a common point. Let H_1, H_2, H_3 be the three left halfplanes with respect to any three sides of the polygon K and let A_1, A_2, A_3 be any points on the corresponding sides. By hypothesis, there is an interior point M of K from which these three sides are visible, that is, the segments A_1M, A_2M, A_3M, in particular, lie inside the polygon K. It follows that when the three sides are traversed, the point M lies on the same side as the the interior points of the polygon K, that is, to the left. Thus M lies in all three halfplanes H_1, H_2, H_3. As an illustration see Figure 18. Let O be a common point of all the left halfplanes of the polygon K with respect to its sides (by Helly's theorem). It remains to show that O is the desired point. First it is shown that O lies inside K. Refer to Figure 19. Assume that O lies outside the polygon K and that X is the boundary point of the polygon K which is nearest to the point O (or one of several nearest boundary points). Then the segment OX, except for the point X, lies entirely outside K. If the point X were a vertex of the polygon K, then a point sufficiently close to X could be chosen on one of the two sides through this vertex such that it would not be a vertex of the polygon and such that, except for the point itself, the entire segment connecting it with O would lie outside the polygon. In other words, if O lies outside K, then a point X of K can be found which is not a vertex and such that except for X the segment OX lies outside K. In particular, if AB is the side on which X lies, then O is on the same side of AB as the exterior points of the polygon, that is, to the right, a contradiction. Hence O lies within K. It remains to show that the entire segment OC lies within K if C is any point of the polygon K. Assume the

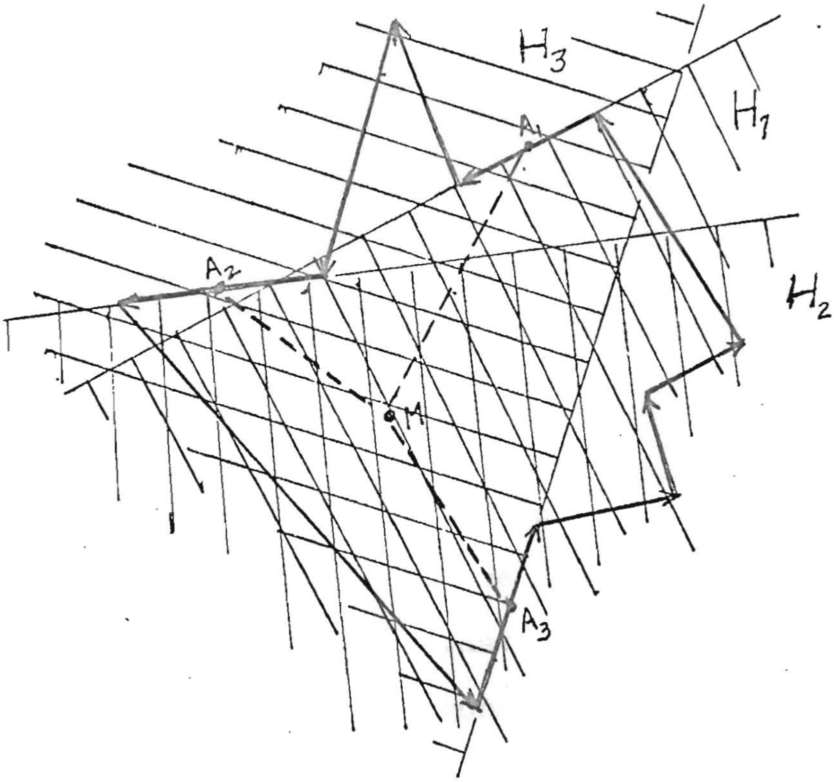

FIGURE 18

EACH THREE LEFT HALFPLANES HAVE A COMMON POINT

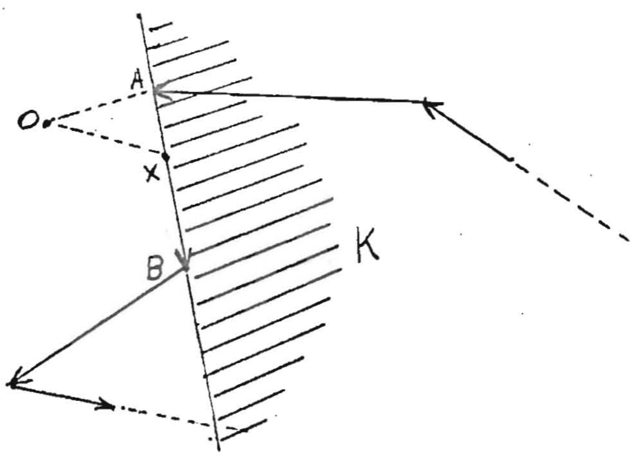

FIGURE 19

THE POINT O LIES INSIDE THE POLYGON K

contrary and let M be a point of the segment OC which lies exterior to the polygon K (see Figure 20). Let P be the point of intersection of the segment CM with the boundary of K nearest to M. If P is not a vertex of K and AB is the side on which P lies, then the point O lies on the same side of AB as M, that is, on the same side as the exterior points of the polygon, thus in the right halfplane with respect to AB, a contradiction that O belongs to all the left halfplanes of K. If P is a vertex of K, then on each of the sides through P a point P' sufficiently close to P can be found which is not a vertex of K and such that on the segment OP' there is a point M' outside the polygon K. The above argument is then repeated. It follows that all sides of the polygon K are visible from O. This completes the proof.

The preceding theorem holds not only for polygons, but for any plane figure. Since there are then infinitely many left halfplanes, and because of the unboundedness of the halfplanes, Helly's theorem cannot be applied immediately; however, it is nevertheless still valid as follows. For each left halfplane a square is chosen that has one side on the boundary line, contains the "side" (which may be degenerate) of the figure lying on the boundary line, and in addition is so large that it entirely contains the part of the figure that lies in the left halfplane. In this way, the problem is reduced to bounded "left squares" and Helly's theorem is then valid.

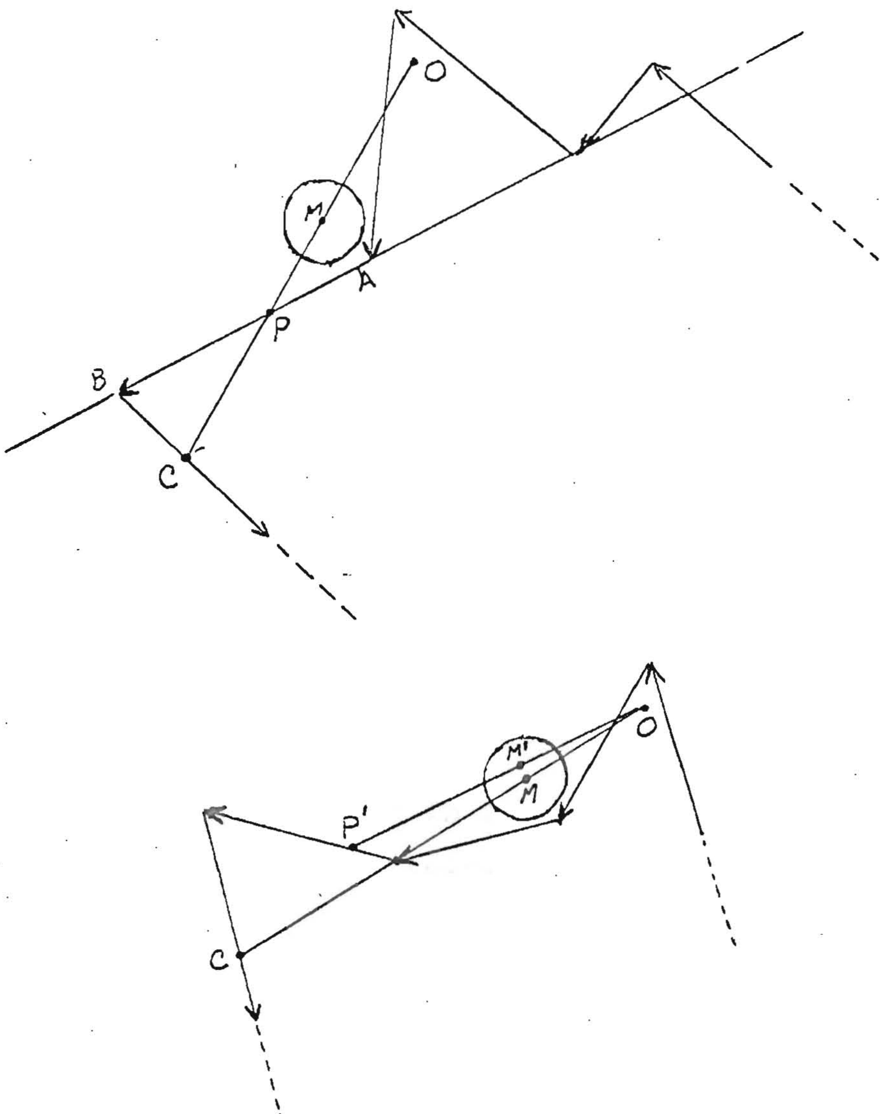

FIGURE 20

THE POLYGON IS STAR-SHAPED WITH RESPECT TO THE POINT O

A rigorous proof of the general n-dimensional case of heorem 4.18 is now given.

THEOREM 4.19. Let S be an infinite compact subset of ^{n}, and suppose that for each n+1 points of S there is a point rom which all n+1 are visible. Then S is starshaped (with espect to some point).

PROOF. For each $x \in S$, let $V_x = \{y : [xy] \subset S\}$. The ypothesis is that each n+1 of the sets V_x have a common oint, and it remains to prove that $\bigcap_{x \in S} V_x \neq \emptyset$. By Helly's heorem, there exists a point $y \in \bigcap_{x \in S} C(V_x)$, and it will be roved that $y \in \bigcap_{x \in S} V_x$. Suppose the contrary. Then there xist $x \in S$, $u \in [yx) \sim S$, and there exists $x' \in S \cap [ux]$ with $ux') \cap S = \emptyset$. Further, there exist $w \in (ux')$ such that $w - x'| = (1/2)d(u,S)$, and $v \in [uw]$ and $x_0 \in S$ such that $x_0 - v| = d([uw],S)$. Since x_0 is a point o S nearest to v, t is evident that V_{x_0} lies in a closed halfspace Q which isses v and is bounded by the hyperplane through x_0 perpendicular to $[vx_0]$. But then $y \in C(V_{x_0}) \subset Q$ and $\measuredangle yx_0v \geqslant \pi/2$, hence $\measuredangle x_0vy < \pi/2$. Since $d(v,S) \leqslant d(w,S) < d(u,S)$, it follows hat $u \neq v$ and hence some point of $[uv)$ is closer to x_0 than . This contradicts the choice of v and completes the proof.

V. APPLICATIONS TO APPROXIMATION THEORY OF POLYNOMIALS

Some important results in the field of analysis related to the approximation of functions by polynomials have been obtained with the aid of Helly's theorem. This section is devoted to a general investigation of the way in which this part of mathematics has been approached by means of Helly's theorem.

DEFINITION 4.20. A <u>common transversal</u> for a family of sets is a line which intersects every set in the family.

The following theorem due to Santalo' is a consequence of Helly's theorem and has important applications in analysis.

THEOREM 4.21. Let S be a finite collection of parallel line segments in the plane. If for every three members of S there is a line which intersects all three, then there is a line which intersects all members of S.

PROOF. Choose a coordinate system with the y-axis parallel to the given segments. Let $P_i(x_i, y_i')$ and $Q_i(x_i, y_i'')$ be the endpoints of the ith segment where $y_i' < y_i''$. If the line $y = mx + b$ intersects the ith segment, then

$$y_i' \leq mx_i + b \leq y_i''.$$

Since every line L not parallel to the y-axis is completely determined by its slope and y-intercept, to each point (m,b) there corresponds a unique line L, and conversely. Consider the equations $b = -x_i m + y_i$ and $b = -x_i m + y_i$. These represent parallel lines in the mb-plane, since they have the same slope $-x_i$. It follows from the above inequalities that the line L intersects the ith segment if and only if the point (m,b) corresponding to L lies in the strip S_i bounded by these parallel lines. Figure 21 illustrates this situation. By hypothesis each three segments are intersected by a line. Therefore each three of the corresponding strips contains a common point. By Helly's theorem there is a point (m_1, b_1) common to all the strips. The line $y = m_1x + b_1$ intersects all the segments. This completes the proof.

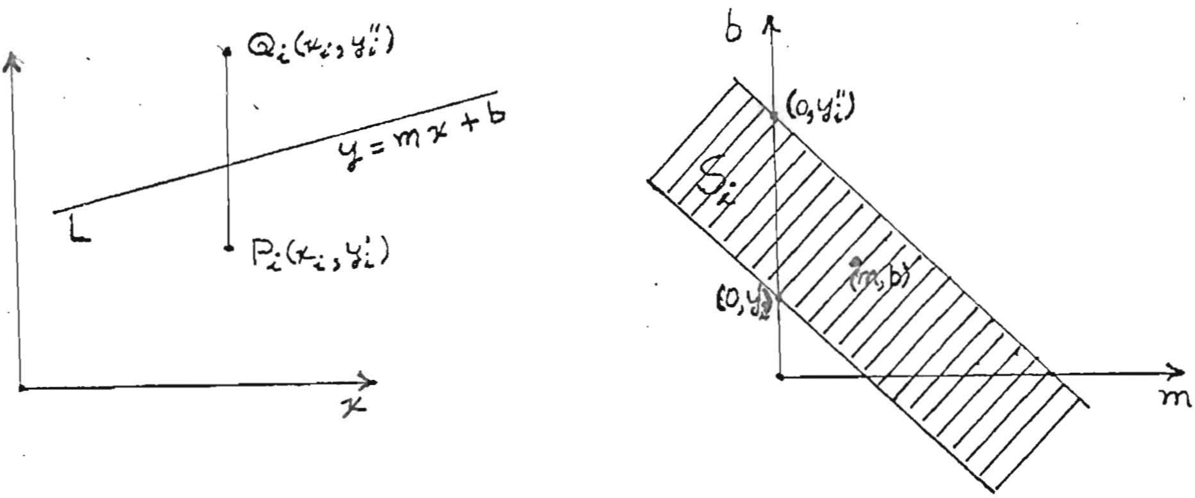

FIGURE 21

THERE IS A POINT COMMON TO ALL THE STRIPS

(SINCE EACH THREE HAVE A COMMON POINT)

The above theorem is suggestive of how Helly's theorem is applied to the theory of approximation of functions. To illustrate the use of the theorem consider a continuous function f defined on some interval. It is said that the line $y = mx + b$ approximates the function f on the set S (contained in the interval) with exactness within $\epsilon > 0$ if and only if $|f(x) - (mx+b)| \leq \epsilon$ for every x in S. The problem is finding a line that best approximates the function f on the given interval. It is sufficient to seek the lines which approximate the function, within given exactness, at all possible triples of points of the interval, and it then follows from the theorem that there is a line which approximates f within ϵ on any finite number of points of the interval. Together with the continuity of f this implies the existence of a line which approximates f within ϵ on the whole interval.

Using Helly's theorem for space, the following theorem can also be proved.

THEOREM 4.22. Let S be a finite collection of parallel line segments in the plane. If for every four members of S there is a parabola which intersects all four, then there is a parabola which intersects all members of S.

By analogy with the preceding discussion, it is sufficient to seek the parabolas which best approximate the function f at all possible quadruples of points of the interval. In general, the straight-line transversals in Theorem 4.21 can be replaced by nth-degree polynomial curves as follows.

THEOREM 4.23. Let S be a finite collection of parallel line segments in the plane containing at least n+2 members. Suppose every n+2 segments in S are intersected simultaneously by some polynomial of degree n, $y = a_0x^n + a_1x^{n-1} + \ldots + a_n$. Then all the segments of S are intersected by such a polynomial.

The proof of this theorem corresponds to that of Theorem 4.21, except that (m,b) is replaced by $(a_0, \ldots ,a_n)$. Whereas in the former proof a duality between points in a plane and lines in a plane was used, the proof of this theorem utilizes a duality between points in R^{n+1} and nth-degree polynomial curves in the plane. Similarly, Helly's theorem in R^n yields the following "fitting theorem" of Karlin-Shapley (see [1]).

THEOREM 4.24. Suppose $f_1, \ldots , f_n$ are real-valued functions on a linear space L; $x_1, \ldots , x_m$ are points of L; $a_1, \ldots , a_m$ real numbers; and $e_1, \ldots , e_m$ are real non-negative numbers. Then the existence of a linear combination of the f_i's which fits each point (x_i,a_i) within e_i (i.e., $|f(x_i) - a_i|\leq e_i$) is guaranteed by the existence of such a fitting for each n+1 points (x_i,a_i).

More general results relating to the approximation of a function by polynomials of arbitrary degree have been obtained in a similar way by the Russian mathematician L.G.

Snirel'man with aid of Helly's theorem. The well-known Russian mathematician P. L. Tschebyscheff obtained the same results with an entirely different method. Together with further generalizations, similar results have been given by Rademacher and Schoenberg in particular, using Helly's theorem. Theorem 4.23 is apparently due to Rademacher and Schoenberg (see [4, p. 80]).

VI. JUNG'S THEOREM

The following is the plane case of Jung's theorem.

THEOREM 4.25. (Jung's theorem for the plane) Every point set of diameter 1 can be enclosed in a circle of radius $1/\sqrt{3}$.

The approximation given in Jung's theorem cannot be improved. An equilateral triangle is an example of a figure of diameter 1 which cannot be enclosed in a circle of radius less than $1/\sqrt{3}$. It is of interest to know that this theorem is used in certain number theory problems (see [5, p. 18]). The theorem is illustrated as follows: If there is a spot of diameter d on a table cloth, then it is certain that it can be covered with a circular napkin of radius $d/\sqrt{3}$.

PROOF. Let S be a point set of diameter 1. According to Corollary 4.11, page 62, it is sufficient to show that any three of the given points of S can be enclosed in a circle of radius $1/\sqrt{3}$. No side of a triangle ABC formed from any three of the points of S is greater than 1. If this triangle is obtuse or right-angled, then it is completely enclosed by the circle that is constructed on the largest side as diameter. The radius of this circle is not greater than 1/2, and is therefore smaller than $1/\sqrt{3}$. If the triangle ABC is acute-angled, then the radius of the circumscribing circle can likewise not be greater than $1/\sqrt{3}$, for at least one of the angles of this triangle, say angle A, is not less than $60°$. Hence the side BC, as chord of an arc not less than $120°$ but less than $180°$, is not smaller than $r\sqrt{3}$, where r is the radius of the circumcircle of the triangle ABC (the chord of an arc of $120°$ has length $r\sqrt{3}$). Hence $BC \geqslant r\sqrt{3}$, and since $BC \leqslant 1$, it follows that $r\sqrt{3} \leqslant 1$, and thus $r \leqslant 1/\sqrt{3}$. This completes the proof.

It is also true that every figure of diameter 1 can be covered by a regular hexagon inscribed in a circle of radius $1/\sqrt{3}$, but that even this hexagon is not the smallest figure possessing this property. The following problem, the solution of which is unknown, is closely related to Jung's theorem: Find a figure of least area which covers every plane figure of diameter 1. It has been proved that such a figure exists (see, e.g., [5, pp. 100-104]).

Here is the n-dimensional version of Jung's theorem.

THEOREM 4.26 (Jung's theorem) If X is a set in R^n with diam $X \leq 2$, then X lies in a Euclidean cell of radius $[2n/(n+1)]^{\frac{1}{2}}$. If X does not lie in any smaller cell, then cl X, the closure of X, contains the vertices of a regular n-simplex of edge-length 2.

PROOF. By Helly's theorem, this theorem can be reduced to the case of sets of cardinality $\leq n+1$. For consider $X \subset R^n$ with card $X \geq n+1$, and for each $x \in X$ the cell $B_x = \{y : |y-x| \leq [2n/(n+1)]^{\frac{1}{2}}$. If this theorem is known for sets of cardinality $\leq n+1$, then each $n+1$ of the sets B_x have a common point, so that $\bigcap_{x \in X} B_x$ is nonempty by Helly's theorem and the desired conclusion follows. Therefore suppose $X \subset R^n$ with card $X \leq n+1$. Let y denote the center of the smallest Euclidean cell B containing X and let $r = r(X)$ be its radius. Let $\{z_0, \ldots , z_m\} = \{x \in X : |y-x| = r\}$, where $m \leq n$. It is verified that $y \in C(z_0, \ldots ,z_m)$, and it is assumed without loss of generality that $y = 0$, whence

$$\sum_0^m a_i z_i = 0 \text{ with } \sum_0^m a_i = 1 \text{ and } a_i \geq 0.$$

For each i and j, let $d_{ij} = |z_i - z_j| \leq 2$, whence

$$d_{ij}^2 = 2r^2 - 2(z_i, z_j).$$

For each j,

$$1 - a_j = \sum_{i \neq j} a_i \geq \sum_0^m a_i d_{ij}^2/4$$

$$= r^2/2 - (\sum_0^m a_i z_i, z_j)/2 = r^2/2.$$

Summing on j (from 0 to $m \leq n$) leads to the conclusion that

$\geqslant (m+1)r^2/2$, whence $r^2 \leqslant 2m/(m+1) \leqslant 2n/(n+1)$. Further, quality implies that $m = n$ and $d \quad = 2$ for all $i \neq j$, so he proof is complete.

The proof of this theorem shows how the theorems of elly and Caratheodory can sometimes substitute for each ther in applications. The assumption that card X n+1 justified by Helly's theorem) was made only to insure that he point y C(X) could be expressed as a convex combination f n+1 or fewer points of X. On the other hand, this is also nsured by Caratheodory's theorem, so that the above proof ould also be based on the latter. Caratheodory's theorem as employed by Eggleston.

VII. BLASCHKE'S THEOREM

The theorem of Blaschke is of interest in itself, aving implications far beyond the scope of this thesis. he purpose here is to illustrate the use of Helly's theorem n proving this theorem. The present discussion is restricted o the plane case of the theorem; there is an analogue of laschke's theorem for general n-space, but only the analogous heorem for 3-space is stated here.

DEFINITION 4.27. The smallest distance between arallel supporting lines of a bounded convex figure is called he <u>width</u> of this convex figure.

THEOREM 4.28. (Blaschke's theorem for the plane) Every bounded convex figure of width 1 contains a circle of radius 1/3.

PROOF. Let C be a convex figure satisfying the hypothesis of the theorem, and let O be the point whose existence is asserted in Theorem 4.8, page 57. Then O is the center of a circle of radius 1/3 which is entirely enclosed by C. It suffices to show that the point O has a distance at least 1/3 from each of the boundary points of C. Refer to Figure 22. Consider a supporting line L through any boundary point B of the figure C. Let B_1 designate the point of intersection of the figure C with the supporting line L_1 of C parallel to L. The distance between L and L_1 cannot be less than the width 1 of C.

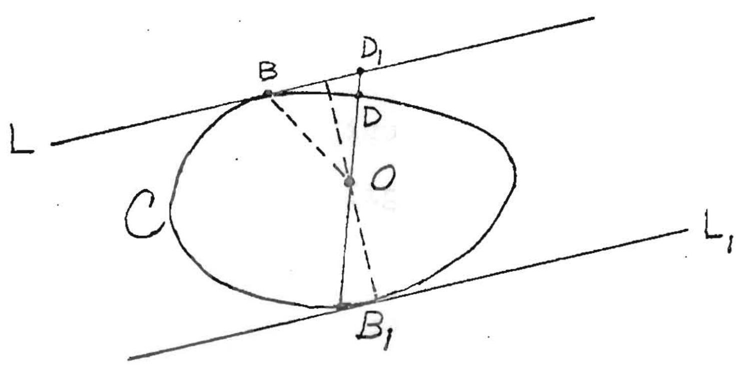

FIGURE 22

THE DISTANCE OF THE POINT FROM THE SUPPORTING LINE IS AT LEAST ONE-THIRD THE DISTANCE BETWEEN THE TWO PARALLEL SUPPORTING LINES

Let D and D_1 be the intersection points of the line B_1O with the boundary of the figure C and the line L, respectively. Since $DO \geqslant 1/3 DB_1$ (by definition of O), it follows that

$$D_1O = DO + DD_1 \geqslant 1/3(DB_1 + 3DD_1) \geqslant 1/3 D_1B_1.$$

Hence the distance of the point O from the line L is at least 1/3 the distance between the lines L and L_1, that is, at least 1/3. It follows that the distance between the points O and B is at least 1/3. This completes the proof.

The approximation in this theorem cannot be improved. An equilateral triangle of altitude 1 is an example of a convex figure of width 1 in which no circle can be drawn with radius greater than 1/3. For another interesting proof of Blaschke's theorem in the plane, see Yaglom-Boltyanskii [5, pp. 123-125]. Helly's theorem is applied in the proof to obtain the point O which is chosen as the center of the desired circle.

The question of the figure of greatest area which can be enclosed in every convex figure of width 1 is unsolved, although the existence of such a figure has been established.

THEOREM 4.29. (Blaschke's theorem for space) Inside each convex body of width 1 in 3-space a sphere of diameter $1/\sqrt{3}$ can be placed.

VIII. APPLICATIONS OF CARATHEODORY'S THEOREM

The theorem of Caratheodory has some interesting pplications. An especially useful consequence of aratheodory's theorem is the following.

THEOREM 4.30. The convex cover of a compact set is ompact.

PROOF. Let X be a compact subset of R^n. Define the ompact subset B of R^{n+1} by

$$B = \{b = (b_0, \ldots ,b_n) \in R^{n+1}: \sum_0^n b_i = 1 \text{ and } b_i \geq 0\}.$$

'or each point

$$(b,x) = ((b_0, \ldots ,b_n),(x_0, \ldots ,x_n)) \in B \times X^{n+1}$$

efine the mapping f by

$$f(b,x) = \sum_0^n b_i x_i.$$

ince f is continuous and $B \times X^{n+1}$ is compact, the set $(B \times X^{n+1})$ is compact. By Caratheodory's theorem,

$$f(B \times X^{n+1}) = C(X).$$

his completes the proof.

The following theorem is due to Steinitz. The proof y means of Caratheodory's theorem is due to Valentine and rünbaum.

THEOREM 4.31. (Steinitz's theorem) If a point y is nterior to the convex cover of a set $X \subset R^n$, then y is interor to the convex cover of some set of 2n or fewer points of .

PROOF. Assume without loss of generality that y is the origin 0. With $0 \in \text{int } C(X)$, there is a finite subset Y of $C(X)$ such that $0 \in \text{int } C(Y)$ and the existence of a finite set $V \subset X$ with $0 \in \text{int } C(V)$ is concluded. Let J denote the set of all linear combinations of $n-1$ (or fewer) points of V. Since $J \neq R^n$, there exists a line L through 0 such that $L \cap J = \{0\}$. Let w_1 and w_2 be the two points of L which are boundary points of $C(V)$ and let H_i denote a hyperplane which supports $C(V)$ at w_i. Clearly $0 \in (w_1 w_2)$ and $w_i \in C(V \cap H_i)$. By Caratheodory's theorem and the choice of L, w_i can be expressed as a convex combination of some n points $v_1^i, \ldots, v_n^i$ of $V \cap H_i$ but cannot be expressed as a linear combination of fewer than n points of V. It follows that w_i is interior to the set $C(v_1^i, \ldots, v_n^i)$ relative to H_i, and since $0 \in (w_1 w_2)$, then $0 \in \text{int } C(v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2)$. This completes the proof.

DEFINITION 4.32. A set Y in R^n is said to be <u>convexly independent</u> provided no point of Y is a convex combination of other points of Y.

For $n = 3$, this definition requires that Y should consist of one or two points or is the set of all vertices of a convex polygon or a convex polytope. The following theorem follows at once from Caratheodory's theorem.

THEOREM 4.33. If a set in R^n is such that each $n+2$ of its points are convexly independent, then the entire set is convexly independent.

DEFINITION 4.34. Given a subset X of R^n and an integer j between 0 and n, the <u>j-interior</u> $\mathrm{int}_j\, X$ is the set of all points y such that, for some j-dimensional flat $F \subset R^n$, y is interior to $X \cap F$ relative to F.

Then $\mathrm{int}_0\, X = X$ and $\mathrm{int}_n\, X = \mathrm{int}\, X$. The following theorem of Bonnice and Klee on the generation of affine hulls (covers) is of interest.

THEOREM 4.35. If $X \subset R$ and $y \in \mathrm{int}_j\, C(X)$, then $y \in \mathrm{int}_j\, C(Y)$ for some set Y consisting of at most $\max\{2j, n+1\}$ points of X.

For a positive integer j and a set X in R^n, let $H_j(X)$ denote the set of all convex combinations of j or fewer elements of X. Then $C(X) = \bigcup_{j=1}^{\infty} H_j(X)$. On the other hand, the convex cover $C(X)$ can also be generated by iteration of the operation H_j for fixed $j > 1$; that is, $C(X) = \bigcup_{i=1}^{\infty} H_j^i(X)$, where $H_j^1(X) = H_j(X)$ and (for $i > 1$) $H_j^i(X) = H_j(H_j^{i-1}(X))$. The question is asked how many times the operation H_j must be iterated to produce the convex cover of a set in R^n. The question is trivial (<u>modulo</u> Caratheodory's theorem) in view of the following fact: $H_j(H_k(X)) = H_{jk}(X)$. As then noted by Bonnice and Klee: if $X \subset R^n$ and $j_1 j_2 \dots j_n \geq n+1$, then $H_{j_1}(H_{j_2} \dots (H_{j_n}(X))\dots) = C(X)$; conversely, if X is the set of all vertices of an n-simplex and $j_1 j_2 \dots j_n \leq n$, then $H_{j_1}(H_{j_2} \dots (H_{j_n}(X))\dots) \neq C(X)$.

CHAPTER V

HELLY-TYPE THEOREMS

In this chapter some generalizations and variants of elly's theorem are presented to shed some additional light n the subject. The generalizations usually involve attempts o find theorems with assumptions of Helly type, that is, of he type indicated below, so that the intersections of given amilies of convex sets are nonempty, and from which Helly's heorem follows as a special case.

Specifically, the contrapositive of Helly's theorem tates that if a family of convex sets is finite or is nfinite and its members are compact, and if the intersection f all the members is empty, then there is a subfamily of + 1 or fewer members whose intersection is empty. Regarding elly's theorem as saying something about the "structure" of ertain families of convex sets (namely, those which are inite or whose members are compact), attempts are then made o arrange the structure of <u>every</u> family of convex sets in R^n or which the intersection is empty, having Helly's theorem s a consequence. The point common to all the members of the amily of convex sets in the conclusion of Helly's theorem ay be regarded (for $j = 0$) in any of the following six ways: 1) as a j-dimensional convex set contained in each member;

(2) as a j-dimensional convex set which intersects each member; (3) as a j-dimensional flat contained in each member; (4) as a j-dimensional flat which intersects each member; (5) as a (j+1)-pointed set which is contained in each member; and (6) as a (j+1)-pointed set which intersects each member of the family. The question is asked: What condition on a certain family would assure the existence of such sets for other values of j? The generalizations of Helly's theorem are results of questions of this sort, and some of the answers are given in this chapter.

The following theorem, due to de Santis, is a typical generalization of Helly's theorem.

THEOREM 5.1. If every $k + 1$ members of a finite family F of convex sets in R^n contain a common flat of dimension $n - k$, then all the members of F contain a common flat of dimension $n - k$, provided F contains at least $k + 1$ members.

For $k = n$, this becomes Helly's theorem. The proof of Theorem 5.1 by means of duality can be found in Valentine [4]. This theorem, together with other general theorems of Helly type, is developed there.

A generalization of Helly's theorem has been developed in the theory of games. Helly also has a topological theorem in R^2 concerning simply connected compact sets. The following theorem, established by Molnar, is an improvement. For a discussion of these and numerous other general results, see Danzer-Grünbaum-Klee [1].

THEOREM 5.2. A family of at least three simply connected compact sets in R^2 has nonempty simply connected intersection provided each two of its members have connected intersection and each three have nonempty intersection.

DEFINITION 5.3. A j-transversal of a family of sets in R^n is a j-dimensional flat which intersects each member of the family.

Helly's theorem deals with 0-transversals. Some problems dealing with 1-transversals (lines) are included in this chapter. They are called <u>common transversals</u> (see Definition 4.20, page 72). Included also are some theorems on common transversals for infinite families of sets which have no counterpart for finite families.

There are two general approaches. One, while restricting the relative positions and the distribution of the convex sets, allows the sets themselves to be quite general. Examples of this type will be given where the sets are said to be "totally separable" or "sufficiently sparsely distributed", described in terms of the "viewing angle". The other approach, while weakening the condition on the relative positions of the sets, places restrictions on their shape or assumes they are disjoint and congruent. The virtual necessity of these assumptions is shown by various examples in Hadwiger-Debrunner 3 .

The remaining material in this chapter will be restricted to a group of theorems of Helly type generally referring to ovals, which are closed and bounded convex sets in the plane. Many of them are proved with the aid of Helly's theorem for the line or plane, after a suitable transformation, and the proofs of these are included here, mainly to illustrate the tremendous utility of the theorem of Helly. Otherwise, the proofs are omitted. Their proofs can be found in 3 , along with many other theorems of Helly type, in particular the plane cases of Helly's theorem, Radon's theorem, Kirchberger's theorem, and Krasnosel'skii's theorem.

THEOREM 5.4. If each two rectangles of a family of parallel rectangles, that is, with sides parallel to the coordinate axes, have a common point, then all the rectangles of the family have a common point.

PROOF. This theorem follows from Helly's theorem for the plane if it can be shown that each three of the rectangles have a common point. Let R_1, R_2, R_3 be any three of the rectangles. Choose a Cartesian system so that the axes are parallel to the sides of the rectangles and let $P_i(x_i,y_i)$ be a point that is in all three of the rectangles except perhaps R_i. Then P_i and P_j points of R_k and R_k contains the entire rectangle whose sides are parallel to the axes and whose diagonal is the segment P_iP_j; that is, R_k includes all points $P(x,y)$ for which x lies in the interval (x_i,x_j) and y in the interval (y_i,y_j). If the indices are chosen so that $x_1 \leq x_2 \leq x_3$ and $y_i \leq y_j \leq y_k$, then the point $P(x_2,y_j)$ satisfies these conditions for all three of the rectangles and hence belongs to all of them.

COROLLARY 5.5. If each two segments of a family of segments in the line have a common point, then all the segments of the family have a common point. (Helly's theorem for the line)

Some theorems of Helly type are given next which involve families of circular arcs which lie on the same circle. The theorems are closely related to the above corollary and are useful for applications.

THEOREM 5.6. If a family of circular arcs, all smaller than a semicircle, is such that each three of the arcs have a common point, then all the arcs of the family have a common point.

PROOF. This theorem can be reduced to Helly's theorem in the plane. A family of circular arcs, each smaller than a semicircle, has a common point if and only if this is true of the corresponding segments of the disk. For this it suffices, by virtue of Helly's theorem, that each three should have a common point.

The condition on the size of the arcs cannot be weakened, for it is false for semicircles; nor can the number three be replaced by two. For example, four points evenly distributed on a circle determine four semicircles for which the theorem is false. However, the following is known.

THEOREM 5.7. If a family of circular arcs, all smaller than one-third of a circle, is such that each two of the arcs have a common point, then all the arcs of the family have a common point.

PROOF. This theorem follows from Helly's theorem for the line (Corollary 5.5). If each of the circular arcs is smaller than one-third of a circle while each two of them have a common point, then they leave some point of the circle uncovered, for example, the point antipodal (that is, diametrically opposite) to the midpoint of one of the arcs. The circle can be cut at this point and unrolled on a line so that each of the arcs turns into a segment of the line. The desired result then follows by applying Helly's theorem.

All assumptions on the size of the arcs are dropped for the next result.

THEOREM 5.8. If a family of circular arcs is such that each two of the arcs have a common point, then there is an antipodal pair of points such that each arc of the family includes at least one point of the pair. In other words, there is a diameter of the circle that intersects all the arcs.

PROOF. Let $L(a)$ be the directed line through the center Z of the circle, making an angle a with a fixed direction. Projecting the given pairwise intersecting arcs orthogonally onto $L(a)$, the resulting segments have the same property. Thus the intersection $D(a)$ of all of these segments is a point or a segment. By Helly's theorem for the line, the intersection is not empty. The set $D(a)$ includes the center Z for at least one angle a_0. To see this, note that the position of $D(a)$ relative to Z in $L(a)$ is exactly antipodal to the position of $D(a + \pi)$ relative to Z in $L(a + \pi)$. (Recall that these are directed lines.) Since the orthogonal projection of each arc on $L(a)$ varies continuously with a, so does the set $D(a)$, and thus rotation through an angle of π must yield at least one a_0 for which $Z \in D(a_0)$. The line $L(a_0 + \pi/2)$ is then a diametral line that intersects all the arcs.

THEOREM 5.9. If a family of ovals is such that each two of its members have a common point, then through each point of the plane there is a line that intersects all the ovals of the family.

PROOF. If the pairwise intersecting ovals are mapped into a circle by central projection, they give rise to arcs hat satisfy Theorem 5.8. Then every oval in the family is ntersected by the line determined by the two antipodal oints specified in Theorem 5.8.

THEOREM 5.10. If a family of ovals is such that each wo of its members have a common point, then for each line n the plane there is a parallel line that intersects all the vals of the family.

PROOF. By orthogonal projection of the ovals into a ine, a family of segments satisfying Helly's theorem is enerated. All the ovals of the family are intersected by the erpendicular line that passes through a point common to all hese segments.

Theorems 5.9 and 5.10 are the plane cases of more eneral theorems of A. Horn, answering the question as to hat can replace the conclusion of Helly's theorem when the umber three is replaced by two in its hypothesis. The idea s to relax the intersection condition on the class of sets n R^n. The following is Horn's extensions of Helly's theorem n the general setting. The modified form requires only that very subclass of k members, $1 \leq k \leq n$, have a common point. proof is given in Eggleston [2, pp. 43-44].

THEOREM 5.11. A finite collection of compact convex ets in R^n has the property that every k of the sets have a oint in common, $1 \leq k \leq n$. Then, given any (n-k)-dimensional inear space M, there can be found an (n-k+1)-dimensional linear space N such $N \supset M$ and N intersects each of the convex sets of the given collection.

The question is asked as to whether points can be replaced by lines in the conclusion of Helly's theorem in the sense that the following form is correct: If each h members of a family of ovals are intersected by a line, then there is a line that intersects all the ovals of the family. The answer is negative, as to the existence of such a Helly "stabbing number" h, for L. A. Santalo' has proved that for each natural number $n > 2$ it is possible to construct a family of n ovals so that each n-1 members of the family admit a common transversal, but not all of them. For a verification of this, see Hadwiger-Debrunner [3, pp. 8-9]. As indicated before, theorems of this sort are established by placing certain conditions on the shape and positions of the ovals. The following theorems are typical.

THEOREM 5.12. If each three rectangles of a family of parallel rectangles are intersected by an ascending line, then there is an ascending line that intersects all the rectangles of the family.

PROOF. The conclusion follows at once if among the parallel rectangles of the family there are two that have a unique ascending transversal in common, for then this line must intersect every other rectangle of the family. Therefore assume that each three rectangles of the family admit a common ascending transversal that is not parallel to the x-axis. But then the same is true for any finite number of rectangles in the family. To see this, lay out two lines parallel to the x-axis and associate with each transversal a point of an auxiliarly (or "dual") plane, the coordinates of this point being the x coordinates of the intersection of the transversal with the two parallel lines. The set of all ascending transversals of a rectangle of the family is thus associated with a convex, closed, but unbounded point set in the auxiliarly plane. By hypothesis, each three of these sets have a common point. For any finite number of these convex sets, the intersections with a sufficiently large disk are ovals that, according to Helly's theorem in the plane, have a common point. The line associated with this point intersects the corresponding finite number of rectangles. In order to carry out the proof for infinite sets of rectangles also, without using a stronger variant of the plane case of Helly's theorem, it is required from the above proof only the fact that each four rectangles of the family have a common transversal. With each line forming an angle a with the two parallels, associate a point on a circle having angular coordinate a. The set of all ascending lines that intersect two given rectangles of the family is thus associated with a circular arc that is smaller than one-third of a circle.

carried out for all pairs of rectangles from the family, this mapping produces a family of arcs that intersect pairwise because each four of the rectangles admit a common ascending transversal. There is a point common to all these arcs by Theorem 5.7, page 90, and each two rectangles of the family admit a common ascending transversal parallel to the line L that corresponds to this point. Then under projection parallel to this line, the family of rectangles is carried into a family of segments that have a common point by Helly's Theorem for the line. But then the line through this point parallel to L intersects all the rectangles of the family. This completes the proof.

Klee posed the question as to whether there is a Helly stabbing number when the ovals are pairwise disjoint. Again, the answer is negative, as shown by the construction of a rosette of circular segments (see, e.g., [3, p. 10]). This same rosette is used to demonstrate the non-existence of various other questions that are considered. The next two propositions, however, show to what extent the existence of a common transversal can be deduced from Helly type assumptions with certain supplementary conditions.

THEOREM 5.13. If each four members of a family of homothetic ovals admit a common transversal, then there are four lines, parallel or orthogonal in pairs, such that each of the ovals is intersected by at least one of the lines.

PROOF. Let P be a point of a circle. To each line L in the plane, lay a parallel through P; let its second point of intersection with the circle be the image of the line L. Under this mapping, the set of all lines that intersect two fixed ovals goes into an arc. Carrying this out for all pairs of ovals from a family in which each four ovals have a common transversal, a family of pairwise intersecting arcs is obtained. There are two orthogonal directions corresponding to the antipodal pair of points that intersects all the arcs by Theorem 5.8. Hence, if each four ovals of a family admit a common transversal, then there exist two orthogonal directions such that each two ovals of the family admit a common transversal in one of these directions. If the ovals of this family are mutually homothetic, and if four lines are laid out in two orthogonal directions so that they form a rectangle circumscribed to a given oval of the family, then each of the family's (homothetic) ovals that is not smaller than the given one must be intersected by one of these four lines. Thus if there is a smallest oval of the family, the lines circumscribed to it meet all ovals of the family. If there is no smallest oval in the family, the desired result is obtained from some supplementary considerations on the limiting behavior of the size and position of the ovals in question.

If the ovals are not only homothetic but are mutually congruent, it can be verified that some three of these four lines intersect all the ovals, completing the proof.

DEFINITION 5.14. A system of ovals is said to be totally separable if there exists a direction such that each line in this direction intersects at most one oval of the system. (Pairwise disjoint parallel strips can then be formed in the plane in such a way that each strip contains exactly one oval from the system.)

THEOREM 5.15. If each three members of a totally separable system of ovals admit a common transversal, then there is a transversal common to all of the ovals of the system.

PROOF. Let a line in the separating direction be chosen as the x-axis. Every other line in the plane forms an angle ϕ (measured counterclockwise) with the x-axis for which $0 \leq \phi < \pi$. The set of all lines that intersect two ovals of the system, say A and B, corresponds on a ϕ-axis to an interval of angles between 0 and π which is denoted by (AB), and similarly for other pairs of sets. It is then claimed that each two of these intervals have a common point. Assuming this, there follows from Helly's theorem for the line the existence of an angle ϕ_0 such that each two ovals of the system admit a common transversal in the direction ϕ_0. In other words, the parallel projections of the ovals in this direction form a system of pairwise intersecting segments on the x-axis. Then all the ovals of the system are intersected by the projecting line through a point common to all the segments (by Helly's theorem for the line). It remains to show that each two intervals of angles have a common point. For such pairs of intervals as (AB) and (BC) this is assured by the assumption of a common transversal for A, B, and C. If two intervals, say (AB) and (CD) should have no common point, then a contradiction would result as follows. Each of the intervals (AC), (AD), (BC), and (BD) would have points in common with both (AB) and (CD), so that the following situation arises for an angle ϕ_1 between (AB) and (CD): the ovals A and B, and also C and D, are separable by lines of direction ϕ_1, from which follows the separability of an additional pair by means of each of these two separating lines, but the pairs A and C, A and D, B and C, and B and D are not separable in this way. This is a contradiction, which establishes the proof.

A corollary of Theorem 5.15 is the theorem due to L. A. Santalo' (Theorem 4.21, page 72), according to which there is a transversal common to all the members of a family of parallel segments if each three segments from the family admit a common transversal.

Another interesting question arises as to what peculiar properties of a system of ovals lead to its total separability. The ovals are then said to satisfy a transversality condition and are said to be "sufficiently sparsely distributed" in the plane. This is described in terms of the size of the viewing angle (see Figure 23).

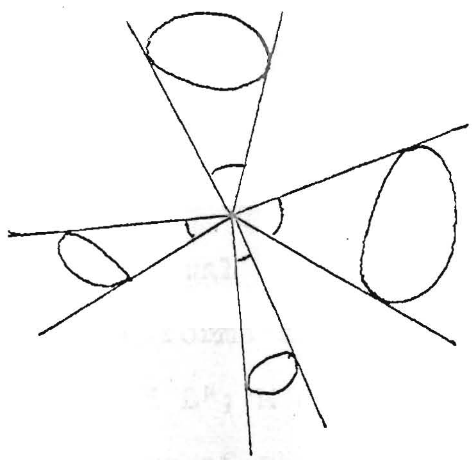

FIGURE 23

THE VIEWING ANGLE OF A SYSTEM OF OVALS

THEOREM 5.16. If the ovals of a system are so sparsely distributed that from each point in the plane at most one of the ovals subtends an angle of $\pi/3$ or more, and if each three of the ovals of the system admit a common transversal, then the system is totally separable, and there is a transversal common to all of the ovals of the system.

PROOF. The assertion is first proved for a system of four ovals C_i $(i = 1, 2, 3, 4)$. Let L be a line that intersects C_1, C_2, and C_3. By M' and M'' denote two lines that orm an angle of $\pi/3$ with each other as well as with L. No ine parallel to M' or M'' can intersect more than one of the vals C_1, C_2, and C_3, since otherwise more than one of the vals would subtend an angle of at least $\pi/3$ at the point here this line intersects L. The same argument shows also hat either no parallel to M' or no parallel to M'' can intersect more than one of the ovals C_1, C_2, C_3, C_4; otherwise, a arallel L' to M' would intersect C_i and C_4 and a parallel L'' o M'' would intersect C_k and C_4, where i and k are among the umbers 1, 2, 3; a transversal M of C_i, C_k, and C_4, which must xist by hypothesis, then forms a nonobtuse angle $\pi/3$ with ne of the lines L, L' and L''; M and this line would then oth intersect the same two of the four ovals C_1, C_2, C_3, and nd C_4, which is impossible because of the condition on subended angles. Thus the four ovals are totally separable ither by parallels to M' or by parallels to M'', and according to Theorem 5.15 must admit a common transversal. It emains to prove the assertion for a system of more than four vals. According to what has been proved already, it may be ssumed that for each four ovals of the system there is a ommon transversal. Let P be a point of a circle. With each ine L that intersects two ovals of the system associate a arallel through P and regard its second point of intersection ith the circle as the image of the line L. In this way the et of all transversals common to two ovals is carried onto an arc; effecting this construction for all pairs of ovals of the system, a family of arcs is obtained, each smaller than one-third of a circle by the condition on subtended angles, and each two intersecting by the existence of a transversal for each four ovals. Thus all the arcs have a common point Q by Theorem 5.7, and the antipodal point Q* is not in any of the arcs. Hence the line determined by the points P and Q* yields a direction not corresponding to a transversal of any two ovals. The system is totally separable by lines in this direction, and from Theorem 5.15 there follows the existence of a line that intersects all the ovals of the system. This establishes the theorem.

COROLLARY 5.17. If a family of disks in the plane is so sparsely distributed that even the disks with the same centers but doubled radii are all disjoint, and if each three disks of the family have a common transversal, then there is a transversal common to all of them.

PROOF. The set of all points at which a circle subtends an angle of at least $\pi/3$ is a concentric disk having twice the radius. Thus the hypothesis that the disks with doubled radii are disjoint implies that at no point of the plane does more than one of the disks subtend an angle $\geq \pi/3$. Consequently the result is a corollary of Theorem 5.16.

The next few theorems are some examples of covering and intersection problems. Jung's theorem (see page 76) on the circumcircle of a set is typical of this sort of problem.

DEFINITION 5.18. A set of lines is called <u>bounded</u> if it includes no parallel lines and the set of all intersection points of pairs of lines from the set is bounded.

According to this definition, a single line would constitute a bounded set of lines (since the empty set is bounded).

DEFINITION 5.19. The <u>intersection radius</u> of a bounded set of lines is the radius of a smallest closed disk that intersects all lines of the set.

DEFINITION 5.20. The <u>diameter</u> of a set of lines is the diameter of the set of all intersection points of the various pairs of lines involved.

THEOREM 5.21. If each three lines of a bounded set of lines are intersected by some disk of radius R, then some such disk intersects all lines of the set.

PROOF. This theorem is a special case of Theorem 4.10, page 61; for the lines can be replaced by sufficiently long segments.

THEOREM 5.22. The intersection radius of a set of lines of diameter $D = 1$ is $r \leq 1/2\sqrt{3}$. (Dual to the plane case of Jung's theorem)

PROOF. By Theorem 5.21, it suffices to prove the assertion for a set of diameter 1 consisting of three lines.

These lines form a triangle of perimeter $P \leq 3$ that is circumscribed about the smallest intersecting circle. Since the equilateral triangle of perimeter $6r\sqrt{3}$ has the smallest perimeter of any triangle that can be circumscribed about a circle of radius r (see e.g. [9]), it follows that $6r\sqrt{3} \leq P \leq 3$ and hence $r \leq \frac{1}{2}\sqrt{3}$. This completes the proof.

The more the various ovals of a system are drawn together, the less the possibility that the members of the system can be separated by a line.

DEFINITION 5.23. A system of ovals is said to be <u>separable</u> if there is a line that intersects none of the ovals, but such that each of the two open halfplanes determined by the line contains an oval of the system.

DEFINITION 5.24. If an oval has interior points, it is said to be <u>proper</u>, otherwise to be <u>degenerate</u>.

The next result is a good illustration of the close connection among various groups of theorems and methods of proof in convexity and combinatorial geometry, especially since it is of Helly type. Hadwiger-Debrunner [3, p. 18] state it picturesquely as follows: If each two members of a system of congruent disks can be pierced by a needle, then three needles suffice to pierce all the disks of the system.

LEMMA 5.25. A point set of diameter D=1 can be covered by an equilateral triangle of side s = 3.

PROOF. Let S be an equilateral triangle that is circumscribed about the set, so that each of its sides includes a point of the set, and let S* be another such triangle that is obtained by reflecting S in a point and then translating and magnifying, or contracting, if necessary, to obtain a second circumscribed equilateral triangle. Then either S or S* has sides of length $s \leq \sqrt{3}$. To see this, consider an arbitrary point that is common to S and S*, and consider the perpendiculars from this point to the sides of the triangles. By a theorem from plane geometry, the sum of the three perpendiculars from any point in an equlateral triangle is equal to the altitude of that triangle. Since the set is of diameter ≤ 1, the sum of a perpendicular to S and corresponding perpendicular to S* must be ≤ 1, so that one of the triangles has altitude $\leq 3/2$ and side of length $\leq \sqrt{3}$.

LEMMA 5.26. A point set of diameter $D = 1$ can be covered by a regular hexagon of side $s = 1/\sqrt{3}$.

PROOF. In addition to the proof of the preceding Lemma, it is verified that the length of the side of the circumscribed equilateral triangle S varies continuously with the directions of the sides and becomes that of S* after a rotation through the angle π. Thus for some position of S, S and S* are congruent and their intersection, which contains the given set of diameter 1, is a (possibly degenerate) centrally symmetric hexagon in which the distance between parallel sides is ≤ 1. This hexagon is contained in a regular exagon that has the same center of symmetry and same directons for its sides, and in which the distance between parallel ides is equal to 1. The regular hexagon has sides of length $/\sqrt{3}$ and contains the given set.

The above proofs follow conventional lines. For many ther interesting problems of this sort, see Kazarinoff [9] nd Yaglom-Boltyanskii [5], also Lyusternik [8].

The following theorem can now be justified.

THEOREM 5.27. If a system of congruent disks is such hat each two of its members have a common point, then there xists three points such that each disk of the system covers t least one of the three points.

PROOF. A point set of diameter $D \leq 2$ is formed by the enters of the disks of radius $R = 1$ that intersect pairwise. t follows from Lemma 5.26 that this set can be covered by regular hexagon having sides of length $2/\sqrt{3}$. In this exagon there are three points, the midpoints of three diagnals, at a mutual distance of 1 such that all points of the exagon, and in particular the centers of the given disks, re at distance ≤ 1 from at least one of these points. ccordingly, each of the given disks includes at least one f the three points. This completes the proof.

That the "piercing number" $n = 3$ cannot be reduced is llustrated in [3, p. 19] by a group of 9 disks arranged in uch a way that 2 needles would not pierce all of them.

The following two theorems are characteristic of the ubject.

THEOREM 5.28. If a point set A on a circle consists f at least three points, and each three points of A lie in ome closed semicircle, then the following alternatives arise: ither A is a four-pointed set formed from two antipodal airs, or A itself lies entirely in a semicircle.

THEOREM 5.29. If in a family of ovals that are all omothetic to a parallelogram A, each two have a nonempty ntersection, then they all have a nonempty intersection. he assertion is no longer true when A is a proper oval that s not a parallelogram.

THEOREM 5.30. If a family of ovals all homothetic to parallelogram A is such that for each line there is a arallel line intersecting all the ovals of the family, then he ovals have a nonempty intersection. The assertion is no onger true when A is a proper oval that is not a parallel-gram.

The next result appears to be some quirk of the magination.

THEOREM 5.31. If each three ovals of an infinite amily of pairwise disjoint congruent proper ovals are inter-ected by some line, then there is a line that intersects ll of them.

The theorem is no longer true if any one of the four special conditions (proper, congruent, disjoint, infinite) is omitted. For example, the four segments shown in Figure 24 have the property that each three can be intersected by a line; however, no line intersects all four.

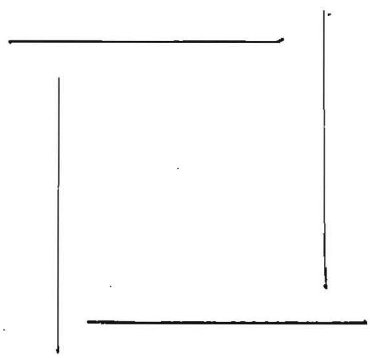

FIGURE 24

THE OVALS ARE NOT PROPER

If one imagines very small rectangles in place of the segments, each containing a countably infinite number of disjoint segments of the same length, then again each three of these segments will be intersected by a line, but not all of them. The ovals are not proper. For examples showing the necessity of the other three conditions and a proof of the theorem, the reader is referred to Hadwiger-Debrunner [3].

Hadwiger-Debrunner [3] have generalized Helly's theorem in a form so that one can decide when a given collection of convex sets can be partitioned into subcollections, each of which has a nonempty intersection. The following theorem is stated. Its proof can be found in [3, p. 83].

THEOREM 5.32. If the p-pointed subsets of an infinite set A are divided into k classes, then A contains an infinite subcollection all of whose p-pointed subsets belong to one and the same class.

The following propositions are closely related to Helly's theorem.

THEOREM 5.33. If each line meets only finitely many ovals in a given infinite family of ovals, then there is an infinite subfamily consisting of mutually disjoint ovals.

PROOF. The pairs of ovals from the family are divided into two classes according to whether the two ovals of the pair have an empty or nonempty intersection. By Theorem 5.32 the family of ovals contains an infinite subfamily such that all its pairs belong to the same class. If there were no infinite subfamily consisting of pairwise disjoint ovals, then there would be an infinite subfamily whose ovals are pairwise intersecting. If the ovals of the subfamily are projected orthogonally onto a line T, the resulting segments intersect pairwise and hence by Helly's theorem have a common point P. The line L that is perpendicular to T at P intersects all the ovals of the subfamily, contradicting the hypothesis.

THEOREM 5.34. If an infinite family of mutually parallel rectangles does not include infinitely many that are pairwise disjoint, then some infinite subfamily has a nonempty intersection.

PROOF. The pairs of rectangles are divided into two classes according to whether the two rectangles of the pair have an empty or nonempty intersection. By theorem 5.32 the family of rectangles has an infinite subfamily whose pairs all belong to the same class. By hypothesis this can only be the second class and then the desired conclusion is obtained with the aid of Theorem 5.4.

THEOREM 5.35. If an infinite family of ovals is such that each of its infinite subfamilies includes three ovals with a nonempty intersection, then some infinite subfamily has a nonempty intersection.

PROOF. The triples of ovals are divided into two classes according to whether the three ovals have an empty or nonempty intersection. By Theorem 5.32 the family has an infinite subfamily whose triples all belong to the same class. By hypothesis this cannot be the first class; hence, by Helly's theorem for the plane, the ovals of the subfamily have a nonempty intersection.

CHAPTER VI

SUMMARY AND CONCLUSIONS

I. SUMMARY

The following fundamental results of this paper are re-emphasized here: (1) affine independence; (2) the existence of supporting and separating hyperplanes; (3) duality; (4) convex covers and simplexes; and, finally, (5) the interdependence of the theorems of Helly, Caratheodory, and Radon. All these results are inter-related and were necessary for the normal development of this paper.

Helly's theorem has been applied in many different parts of mathematics, and various applications were presented in this report. In particular, the applications of this theorem to estimates of "centeredness" and to the approximation theory of polynomials are of interest. The theorems of Kirchberger, Jung, and Blaschke can be proved with the aid of Helly's theorem. Using Helly's theorem, a characterization of starshapedness results. The translation problem proved in conjunction with the applications of Helly's theorem, while being especially useful for sets consisting of one-pointed sets, brings out the "dual" relation between covering and intersection properties of convex sets.

The Helly-type theorems presented in the last chapter, consisting mainly of transversal and covering problems, are of interest. In addition to illustrating the relation between the covering and intersection properties of convex sets, which was mentioned above, the proofs of these theorems indicate the principal methods and techniques used in the theory; the various mappings, projections, and "dual" spaces employed are typical.

II. CONCLUSIONS

The various applications of Helly's theorem, along with the numerous Helly-type theorems, all show that not only is Helly's theorem one of the most interesting theorems, but it is one of the most important tools in the study of convexity.

The inter-dependence of the theorems of Helly, Caratheodory, and Radon seems to lie at the core of the whole matter. Ultimately, and particularly in the light of the proof of Radon's theorem, the problem appears to be reducible to the idea of (affine) independence and the concept of dimension. Studying the inter-relationship between these three theorems, one arrives at the conclusion or conjecture that any further investigation of this underlying problem would lead inevitably into a study of dimension theory and associated concepts of combinatorial topology. The present investigation was leading naturally in this direction.

BIBLIOGRAPHY

BIBLIOGRAPHY

A. PRIMARY SOURCES

1. Danzer, Ludwig, Branko Grünbaum, and Victor Klee. "Helly's Theorem and its Relatives," Convexity, pp. 100-181. Proceedings of the Seventh Symposium in Pure Mathematics of the American Mathematical Society, Vol. VII. Providence, Rhode Island: American Mathematical Society, 1963.
   A compendium by three prominent mathematicians in the field, containing some important applications of Helly's theorem and a complete and extended bibliography.

2. Eggleston, H. G. Convexity. Cambridge, Eng.: Cambridge University Press, 1958. pp. 1-44.
   A valuable source on the theory of convex sets in n-dimensional real Euclidean space; an approach to Helly's theorem via Caratheodory's.

3. Hadwiger, Hugo, and Hans Debrunner. Combinatorial Geometry in the Plane. Trans. Victor Klee. New York: Holt, Rinehart and Winston, Inc., 1964.
   An extensive collection of special problems, including numerous Helly-type theorems, representing the methods of plane combinatorial geometry, which is largely associated with the notion of convexity.

4. Valentine, F. A. Convex Sets. New York: McGraw-Hill Book Company, Inc., 1964. pp. 1-90.
   A rigorous development of the theory of convexity in an n-dimensional topological linear space, with an approach to Helly's theorem via the duality theory; a standard text in the field of convexity.

5. Yaglom, I. M., and V. G. Boltyanskii. Convex Figures. Trans. P. J. Kelly and L. F. Walton. New York: Holt, Rinehart and Winston, Inc., 1961.
   An elementary treatment of convex figures in the plane; predominantly intuitive, and presents some interesting applications of Helly's theorem.

B. SECONDARY SOURCES

6. Aleksandrov, P. S. Combinatorial Topology, Vol. I. Trans. Horace Komm. New York: Graylock Press, 1956. pp. 202-215.
An introductory but rigorous treatment of simplexes; also, including an introduction to dimension theory and complexes and triangulations.

7. Birkhoff, Garrett, and Saunders MacLane. A Survey of Modern Algebra. Revised edition. New York: The Macmillan Company, 1953. pp. 251-294.
A clear and comprehensive discussion of the full linear and affine groups in an n-dimensional setting; the notion of affine space is introduced, and the concepts of affine independence and barycentric coordinates are rigorously developed.

8. Kazarinoff, N. D. Geometric Inequalities. New York: Random House, Inc., 1961.
An excellent introduction to elementary inequalities, treating various geometric problems involving maxima and minima and isoperimetric theorems, and some problems concerning the reflection principle.

9. Lyusternik, L. A. Convex Figures and Polyhedra. Trans. T. J. Smith. New York: Dover Publications, Inc., 1963.
An elementary text on basic concepts of convex figures; a thorough discussion of linear systems of convex bodies and covering and displacement problems.