A NON-EUCLIDEAN AND EUCLIDEAN DEVELOPMENT OF SPHERICAL TRIGONOMETRY

515

A Thesis

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by

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PREFACE

The primary purpose of this thesis is to develop spherical trigonometry from both the Euclidean and Non-Euclidean sets of postulates and to show the results equivalent in the case of right triangles. The first five chapters develop spherical trigonometry using the familiar postulates and theorems of Euclidean Geometry. Any student with a basic knowledge of plane trigonometry should be able to follow the development of the first four chapters and to appreciate the applications illustrated in Chapter V. Chapter VI develops spherical trigonometry for right triangles using a set of postulates of Riemannian Geometry as a basis. At least a basic knowledge of Riemannian(Elliptic)Geometry is needed by any person desiring to read Chapter VI.

Only with the guidance and help of Dr. Donald L. Bruyr has this work been possible. His helpful suggestions and his just criticisms have added immensely to whatever degree of quality this work possesses. Few students are so fortunate as to work under an instructor who willingly sacrifices his vacation to help a student. Thank you very much Dr. Bruyr.

Dr. Marion P. Emerson is to be thanked for the time saving suggestion of using special footnotes and for his help in selecting this topic. Dr. John M. Burger is to be thanked for reading the final draft and for the suggestions he made to improve the final draft.

Special thanks must be expressed to my wife Nancy and my little boy Daniel. Nancy spent countless hours typing, editing, and retyping this manuscript. Without her efforts, this work would have been much more difficult. Daniel has had to spend many hours entertaining himself while his mother and father have been busy working on this manuscript. The last six months have been indeed trying on our family routine but as the end seems near, I am sure we will be able to look back on this work with a feeling of accomplishment.

T.R.H.

July 26, 1967
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CHAPTER I

THE PROBLEM, THEOREMS AND DEFINITIONS TO BE USED

The study of plane trigonometry as a college student developed a strong desire to study spherical trigonometry. At the suggestion of an advisor, the decision was made to make this work into the development and comparison of the trigonometries of the sphere and of Riemannian Geometry.

I. PLAN OF THE PAPER

The remainder of this chapter will include a brief statement of the history and uses of spherical trigonometry, a statement of some of the theorems from spherical geometry needed in the following chapters, definitions and terminology of spherical geometry, and a statement of the formulas of spherical trigonometry to be developed in Chapters II, III, and IV.

Chapter II will develop the formulas for right spherical triangles and will contain several examples worked out. Chapter III will consider some of the possible methods of solving oblique spherical triangles and Chapter IV will contain more advanced methods. Both chapters will include examples of solutions. Chapter V will be devoted to discussing some of the applications of spherical trigonometry.

Chapter VI will develop the trigonometry of Riemannian Geometry and show it equivalent to that of spherical trigonometry.

Footnotes. Footnotes used in this work will be of the type used in many of the articles in mathematical periodicals. Two groups of numerals, separated by a semicolon, enclosed in brackets will indicate both the source and the page number or numbers of the source referred to. The first number, to the left of the semicolon, will indicate the reference source listed in the references at the end of this work. The number or numbers following the semicolon will give the page number or numbers of the source being referred to.

II. SPHERICAL TRIGONOMETRY

Trigonometry is one of the oldest branches of mathematics known to the world at large, not by name but by its applications to the measuring of heights, depths, distances, areas, and the like. In early centuries, it was the custom to tell the seasons of the year and the time of day from the shadows cast by a tree or some other upright object. The Greeks called the upright object used for this purpose a gnomon("inspector"), and the shadow was called an umbra. (Figure 1)The winter soltice occurs when the umbra(AC)is longest and the sun(S)is farthest south; the summer soltice occurs when the umbra is shortest.

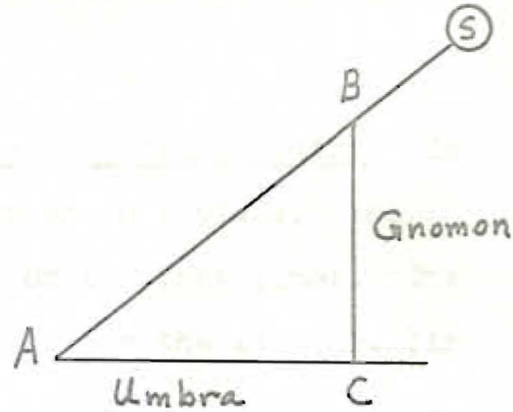


Figure 1

"Trigonometry" comes from the Greek tri("three")+ gonnia("angle")+ metron("measure")or metrein("to measure"). It is in this three-angle-measure that is found one of the first traces of trigonometry [15;229]. Only a century ago sundials were in frequent use in our country and are still in general use in some oriental countries. Evidence of the use of trigonometry as a means of measurement in India, Iraq, Babylon, Egypt, and China date nearly 4000 years ago.

The Greeks, at one time the most scholarly of the ancient peoples of the Mediterranean region, developed trigonometry as a science. Hipparchus(about 140 B. C.)and Ptolemy (300 years later)studied both plane and spherical trigonometry. They knew a number of the present formulas, including $\sin^2 x + \cos^2 x = 1$, although using different names and symbols(chords instead of sines). Menelaus, who lived about 100 A. D., both in Alexandria and in Rome, studied spherical trigonometry scientifically. Thereafter, his work in both plane and spherical trigonometry was well known, even in the Far East [15;230].

Comparison of plane and spherical trigonometry. In plane trigonometry, triangles are drawn in a plane. The sides of the triangles are segments of straight lines. The shortest distance between two points is on the straight line containing the two points. Distance between two points is measured in linear units. The sum of the measures of the angles of a plane triangle is always 180°. Through any point in a plane, there exists a line in the plane parallel to a given line in the plane. The geometry of plane trigonometry is Euclidean.

In spherical trigonometry, triangles are drawn on a sphere. The sides of the spherical triangles are arcs of great circles. The shortest distance between two points is on a great circle containing the two points. Distance between two points is measured in angular units. The sum of the measures of the angles of a spherical triangle is always greater than 180°. Through a given point on a sphere, there does not exist a great circle parallel to another great circle on the sphere. The geometry of spherical trigonometry is non-Euclidean [4;5].

Uses of spherical trigonometry. Spherical trigonometry is used by engineers and geologists, who deal with surveying, geodesy, and astronomy; to physicists, chemists, mineralogists, and metallurgists, in their study of crystallography; and to Navy and Aviation officers, in the solution of navigation problems [4;vii].

Surveying, when dealing with small enough regions of the earth, may be considered plane as a first approximation. Geodesy deals with larger regions for which the curvature of the earth must be taken into account, in which spherical trigonometry must be used to obtain satisfactory approximations. Here, the earth is considered to be spherical.

In astronomy, the earth is considered as the center of the celestial sphere. The line of sight of an observer is the radius of the celestial sphere. Distances between heavenly bodies are measured in angular measure. Thus, the distance between two heavenly bodies is the side of a spherical triangle on an immense sphere [15;5-6].

Some of the uses of spherical trigonometry will be examined in much fuller detail in chapter five. Included will be numerical examples illustrating the applications.

III. RESULTS FROM THE GEOMETRY OF THE SPHERE

A sphere is a closed surface which is the set of all points in space at a given distance from a given point called its center. The line segment from the center to any point on the sphere is a radius; a line segment whose endpoints are the points of intersection of a line passing through the center of the sphere and the sphere is called a diameter.

The intersection of a plane and a sphere, which is not empty and is not a single point, is a circle. If the plane passes through the center of the sphere, the intersection is a great circle; if not, the intersection is a small circle(or lesser circle). A great circle has the same center and radius as the sphere. In spherical trigonometry, all arcs are arcs of great circles. Thus, any two points on a sphere which are not endpoints of the same diameter determine a unique arc, or great circle.

The poles of a great circle of a sphere are the endpoints P and P' of the diameter of the sphere which is perpendicular to the plane of the great circle. A given point on a sphere is a pole of exactly one great circle.

A spherical angle is the angle formed by two arcs of intersecting great circles drawn from the same point on a sphere. The point of intersection is called the vertex of the angle, and the arcs the sides of the angle. The measure of a spherical angle ABC is defined to be

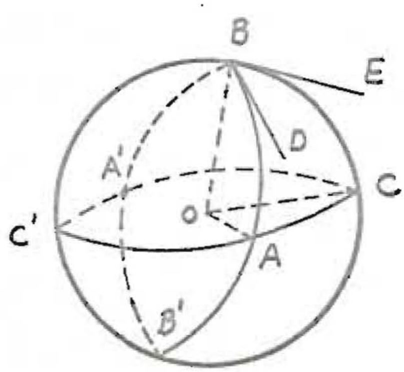


Figure 2

the measure of the plane angle DBE formed by the tangents BD and BE to the arcs AB and BC at their point of intersection, B.(Figure 2)This is numerically equal to the measure of the dihedral angle formed by the intersection of the planes of the great circles BAB' and BCB'. If the planes are perpendicular, angle ABC is a right spherical angle.

A spherical triangle can be described in two equivalent ways. First, a spherical triangle is formed by three arcs of great circles. Secondly, let A, B, and C be three points of a sphere not all on the same great circle of the sphere. Then, the three arcs of great circles connecting A, B, and C form spherical triangle ABC.(Figure 2)

The arcs are the sides of the triangle, the spherical angles formed by the intersection of the arcs are the angles of the triangle, and the points of intersection of the arcs are the vertices of the triangle.

The planes of the sides of the spherical triangle form the trihedral angle O-ABC.(Figure 2)Each side of a spherical triangle is measured by the central angle of the sphere which intercepts it. Thus, side AC has the measure of central angle AOC, or of the corresponding face angle of the trihedral angle.

A common notation for indicating the sides and angles of a spherical triangle ABC is to let A, B, and C represent the angles at A, B, and C, respectively, and to let a, b, and c represent the corresponding opposite sides. It is also a common agreement, although not essential, to consider only spherical triangles whose sides and angles are each less than 180° [6;177-178], [9;213-215].

Definitions

The following definitions will be useful in the work that follows.

Convex spherical triangles. Let ABA', ACA', and BCB' be any three planes having only the center of the sphere in common. (Figure 2) These three planes divide the sphere into eight spherical triangles. The surface of the hemisphere A-BCB'C' is divided into four spherical triangles: ABC, AB'C, AB'C', and ABC'. Each part of each one of the last three triangles is equal to or is a supplement of the parts of triangle ABC. Each one of the four triangles in the hemisphere A'-BCB'C' is symmetric to one of the triangles in hemisphere A-BCB'C'. Each of these eight spherical triangles has only parts that are each less than 180°. Such triangles are convex. In the work that follows only convex spherical triangles will be considered. Any reference to a spherical triangle shall mean a convex spherical triangle unless otherwise stated. Frequently, when no confusion can arise from it, triangle will mean spherical triangle.

Any spherical triangle which has a side greater than 180°, as ABCA'C', will be made up of three or five convex triangles. Such a triangle is called a reentrant triangle. A reentrant triangle has one or two reentrant angles. If either side of a reentrant angle is extended through the vertex, it enters the triangle. The parts of a reentrant triangle are easily determined from the parts of the convex triangle, as ABC [14;152-153].

Lune. A lune is the part of a sphere enclosed by two great semicircles. The angle of a lune is the angle formed by the semicircles [14;156].

Spherical excess. The spherical excess of a spherical triangle is the amount by which the sum of the three angles exceeds 180° or π radians [14;191].

Types Of Spherical Triangles

A spherical triangle, like a plane triangle, may be right, obtuse, or acute. It may also be classified equilateral, equiangular, scalene, or isosceles [15;188].

Right triangle. A spherical triangle is right if it has at least one angle equal to 90°.

Bi-rectangular triangle. A spherical triangle is bi-rectangular if it has two right angles.

Tri-rectangular triangle. A spherical triangle is tri-rectangular if it has three right angles.

Isosceles triangle. A spherical triangle is isosceles if two of its sides are equal.

Oblique triangle. A spherical triangle is oblique if it is not a right triangle [8;126].

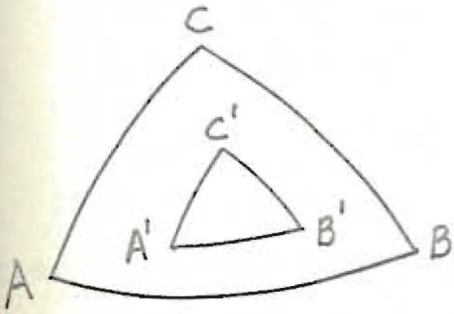
Quadrantal triangle. A spherical triangle is quadrantal if at least one side is equal to a quadrant, or 90° [7;76].

Bi-quadrantal triangle. A spherical triangle is bi-quadrantal if two sides are each equal to a quadrant, or 90°.

Tri-quadrantal triangle. A spherical triangle is tri-quadrantal if all three sides are each equal to a quadrant, or 90° [3;126].

Polar triangles. If ABC is a spherical triangle, it is possible to construct the three great circles that have A, B, and C, respectively, as poles. These three great circles will intersect to form eight spherical triangles, exactly one of them labeled A'B'C' (Figure 3) and having the following properties:

1. A, B, and C are, respectively, poles of the great circles on which B'C', A'C', and A'B' lie.
2. A and A' are on the same side of the great circle containing B'C'; B and B' are on the same side of the great circle containing A'C'; C and C' are on the same side of the great circle containing A'B'.



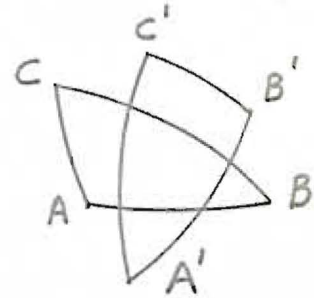


Figure 3

Triangle A'B'C' is called the polar triangle of ABC. It is also true that ABC is the polar triangle of A'B'C' [6;179].

<u>Colunar triangles</u>. Associated with any spherical triangle ABC are three colunar triangles A'BC, AB'C, and ABC' formed by extending pairs of sides of triangle ABC to meet in points A', B', and C' to form the lunes AA', BB', and CC', respectively. (Figure 4)

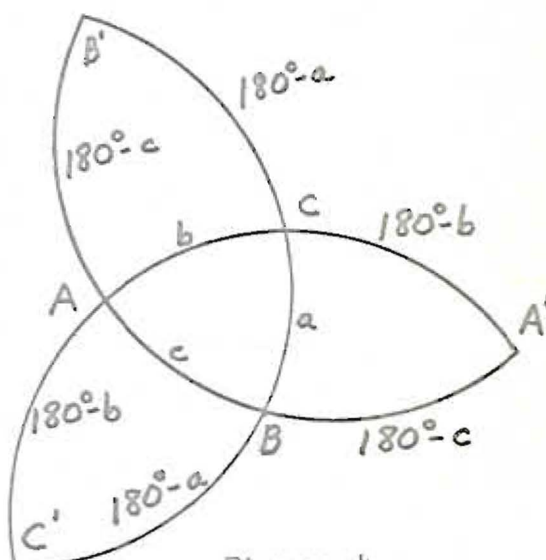


Figure 4

The relationship between the parts of triangle ABC and the parts of one of the colunar triangles is easily seen. Take lune BB', triangles ABC and AB'C, for example. In lune BB', angle B = angle B' and the sides of the lune each equal 180°.

Thus,

$$AB' = 180 - c \; ; \quad B'C = 180^\circ - a \; ;$$

$$b = b \; ; \quad \angle B'AC = 180^\circ - \angle BAC \; ;$$

$$\angle ACB' = 180^\circ - \angle ACB \; ; \; B = B' \; .$$

If any information is known of a spherical(colunar) triangle, the corresponding information is also known for the colunar(original)triangle [1;634-635].

Theorems From Spherical Geometry

The following theorems from spherical geometry will be stated without proof:

I. Any side of a spherical triangle is less than the sum of the other two sides.

II. The sum of the sides of a spherical triangle is less than 360°.

III. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

IV. In any isosceles spherical triangle the angles opposite the equal sides are equal.

V. If two angles of a spherical triangle are equal, the sides opposite these equal angles are equal, and the triangle is isosceles.

VI. In any spherical triangle, if two angles are unequal, the opposite sides are unequal, the greater side being opposite the greater angle.

VII. If two sides of a spherical triangle are unequal, the opposite angles are unequal, the greater angle being opposite the greater side [9;215-216].

VIII. If A'B'C' is the polar triangle of ABC, then ABC is the polar triangle of A'B'C'.

IX. Each side of triangle ABC is the supplement of the corresponding angle of the polar triangle A'B'C'; each angle of ABC is the supplement of the corresponding side of A'B'C' [6;179].

X. The difference between any two angles of a spherical triangle is less than the supplement of the third angle [11;20].

IV. BASIC FORMULAS OF SPHERICAL TRIGONOMETRY

The following formulas, essential to the solution of spherical triangles, are developed in chapters two through five.

Formulas Of Right Triangles

$\cos c = \cos a \cos b$	(1)	$\cos B = \tan a \cot c$	(6)
$\sin a = \sin c \sin A$	(2)	$\sin a = \tan b \cot B$	(7)
$\cos A = \tan b \cot c$	(3)	$\cos B = \cos b \sin A$	(8)
$\sin b = \tan a \cot A$	(4)	$\cos A = \cos a \sin B$	(9)
$\sin b = \sin c \sin B$	(5)	$\cos c = \cot A \cot B$	(10)

Napier's Rules

I. The sine of any middle part is equal to the product of the tangents of the adjacent parts.

II. The sine of any middle part is equal to the product of the cosines of the opposite parts.

Rules of Species

Rule 1. An oblique angle and its opposite side are always of the same species.

Rule 2. If the hypotenuse is less than 90°, the two oblique angles(and therefore the two sides)are of the same species; if it is greater than 90°, the two angles(and therefore the two sides)are of different species.

Formulas Of Oblique Triangles

Law of sines.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{11}$$

Law of cosines for sides.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{12}$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \tag{13}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \tag{14}$$

Law of cosines for angles.

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \tag{15}$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \tag{16}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \tag{17}$$

Law of cosines in terms of haversines.

$$\text{hav } a = \text{hav}(b-c) + \sin b \sin c \text{ hav } A \tag{18}$$

$$\text{hav } b = \text{hav}(a-c) + \sin a \sin c \text{ hav } B \tag{19}$$

$$\text{hav } c = \text{hav}(a-b) + \sin a \sin b \text{ hav } C \tag{20}$$

Half angle formulas.

$$\begin{aligned} \tfrac{1}{2}(a+b+c) &= s \quad ; \quad \tfrac{1}{2}(b+c-a) = s - a \ ; \\ \tfrac{1}{2}(a+c-b) &= s - b; \quad \tfrac{1}{2}(a+b-c) = s - c \ . \end{aligned} \tag{21}$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \tag{22}$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \qquad (23)$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}} \qquad (24)$$

Formulas (25) through (30) are similar formulas for the sine, cosine, and tangent of one-half of the angles B and C.

Half side formulas.

$$\left.\begin{array}{l} \tfrac{1}{2}(A+B+C) = S \ ; \quad \tfrac{1}{2}(B+C-A) = S - A \ ; \\ \tfrac{1}{2}(A+C-B) = S - B \ ; \ \tfrac{1}{2}(A+B-C) = S - C \end{array}\right\} \qquad (31)$$

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \qquad (32)$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}} \qquad (33)$$

$$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B) \cos(S-C)}} \qquad (34)$$

Formulas (35) through (40) are similar formulas for the sine, cosine, and tangent of one-half of the sides b and c.

Gauss's Equations or Delambre's Analogies.

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C \qquad (41)$$

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C \qquad (42)$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C \qquad (43)$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C \qquad (44)$$

Napier's Analogies.

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C \qquad (45)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C \qquad (46)$$

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c \qquad (47)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c \qquad (48)$$

Spherical Excess

$$E = A+B+C-180^\circ \qquad (49)$$

Area of a spherical triangle.

$$H = \pi r^2 E/180 \qquad (50)$$

Girard's Theorem.

$$H = r^2 E \qquad (51)$$

Lhuillier's Theorem.

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)} \qquad (52)$$

Cagnoli's Theorem.

$$\sin \tfrac{1}{2}E = \frac{\sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \qquad (53)$$

Two sides and their included angle.

$$\tan \tfrac{1}{2}E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C} \qquad (54)$$

Several other formulas for a very specific application are also developed within this work but are not of a basic nature and for this reason are not included in this compilation.

CHAPTER II

RIGHT SPHERICAL TRIANGLES

The development of methods of solving spherical triangles will be initiated by first examining right spherical triangles in which the sides adjacent to the right angle are each less than 90°. The ten basic formulas commonly associated with solving right spherical triangles will be developed, first for the special case mentioned above, then secondly expanded to the general case of not restricting the sides of the triangle. Napier's rules of circular parts will be stated and shown to be an easy device to use in remembering these formulas. Rules for determining the quadrants (species)of the parts obtained in solving right spherical triangles will be stated and shown to be valid. Several examples of solutions of triangles will be given illustrating the use of Napier's rules.

I. FORMULAS FOR RIGHT SPHERICAL TRIANGLES

Let ABC be a right spherical triangle with angle C a right angle. Let a, b, and c represent the sides opposite the angles A, B, and C respectively.(Figure 5)Restrict a and b to measures less than 90°.

At any point E on OB, pass a plane through E perpendicular to OA cutting OC at F. Since plane DEF is perpendicular to OA, DE and DF are each perpendicular to OA. Plane DEF is perpendicular to plane AOC, since plane DEF is perpendicular OA, a line in plane AOC. Angle C, a right angle, makes plane BOC perpendicular to plane AOC. Thus, EF, the intersection of the two planes BOC and DEF both perpendicular to plane AOC, is perpendicular to plane AOC. Hence, EF is perpendicular to both OC and DF. This makes triangle DEF right-angled at F[6;180]. DE in plane AOB and perpendicular to OA, and DF in plane AOC and perpendicular to OA, implies that angle EDF and angle A have the same measure.

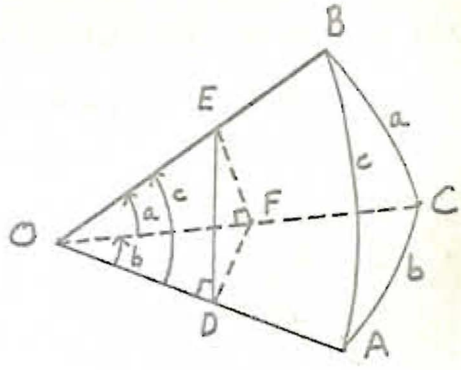


Figure 5

Angle AOC has the measure of side b, angle AOB has the measure of side c, and angle BOC has the measure of side a[9;217-218]. Triangles ODF, ODE, and OFE are right-angled at D, D, and F, respectively. These relationships and the following proportions give the desired results.

$$\frac{OD}{OE} = \frac{OF}{OE} \cdot \frac{OD}{OF} \quad \text{or} \quad \cos c = \cos a \cos b \quad (1)$$

$$\frac{EF}{OE} = \frac{DE}{OE} \cdot \frac{EF}{DE} \quad \text{or} \quad \sin a = \sin c \sin A \quad (2)$$

$$\frac{DF}{DE} = \frac{DF}{OD} \cdot \frac{OD}{DE} \quad \text{or} \quad \cos A = \tan b \cot c \quad (3)$$

$$\frac{DF}{OF} = \frac{EF}{OE} \cdot \frac{DF}{EF} \quad \text{or} \quad \sin b = \tan a \cot A \quad (4)$$

Since the labeling of right spherical triangle ABC was arbitrary with the exception of angle C, interchanging A and B and a and b gives:

$$\sin b = \sin c \sin B \quad (5)$$

$$\cos B = \tan a \cot c \quad (6)$$

$$\sin a = \tan b \cot B \quad (7)$$

from (2), (3), and (4) respectively.

Multiplying (2) and (6) yields:

$$\sin a \cos B = \sin c \sin A \tan a \cot c$$

$$\cos B = \frac{\sin c \sin A \sin a \cos c}{\sin a \cos a \sin c} = \frac{\cos c}{\cos a} \sin A,$$

but from (1) $\frac{\cos c}{\cos a} = \cos b$, therefore:

$$\cos B = \cos b \sin A \quad (8)$$

Interchanging A and B and a and b, (8) becomes

$$\cos A = \cos a \sin B \quad (9)$$

Multiplying (4) and (7) yields:

$$\sin a \sin b = \tan a \tan b \cot A \cot B$$

and dividing both sides by tan a tan b, this becomes

$$\cos a \cos b = \cot A \cot B$$

and using cos a cos b = cos c from (1) the final result is:

$$\cos c = \cot A \cot B \quad (10)$$ [9;218-219].

Equations (1) through (10) are the ten basic formulas used to solve right spherical triangles.

General proof of formulas (1) through (10). Formulas (1) through (10) were proved with the restriction that both a and b were less than 90°. Two other possibilities exist: (i) exactly one of sides a and b could be less than 90° and the other greater than 90°, or (ii) both a and b could be greater than 90°.

Now to examine the first possibility. Without loss of generality, assume that a is greater than 90° while b is less than 90°. Extend arcs BA and BC of right spherical triangle ABC through A and C respectively to meet in point B', cutting out the lune BB'. (Figure 6) With a greater than 90° this makes 180-a less than 90°. Formulas (1) through (10) can thus be applied to right spherical triangle AB'C.

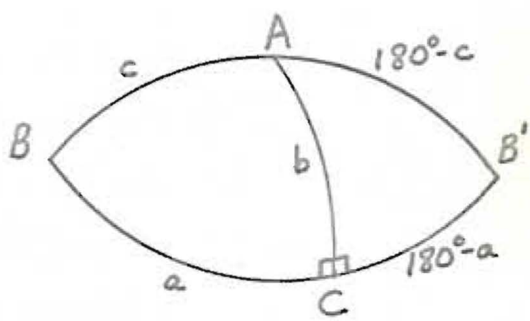


Figure 6

Next to proceed to show that formulas (1) through (10) are also valid in triangle ABC. Applying (1) to triangle AB'C gives:

$$\cos(180-c) = \cos(180-a)\cos b \quad \text{which is}$$

$$-\cos c = -\cos a \cos b \quad \text{or}$$

$$\cos c = \cos a \cos b.$$

This is formula (1) applied to triangle ABC. In a similar manner, formulas (2) through (10) will be shown to valid when applied to triangle ABC.

From (2), $\sin(180-a) = \sin(180-c)\sin(180-A)$

Therefore, $\sin a = \sin c \sin A$.

From (3), $\cos(180-A) = \tan b \cot(180-c)$

$-\cos A = \tan b(-\cot c)$

Therefore, $\cos A = \tan b \cot c$.

From (4), $\sin b = \tan(180-a)\cot(180-A)$

$\sin b = -\tan a(-\cot A)$

Therefore, $\sin b = \tan a \cot A$.

From (5), $\sin b = \sin(180-c)\sin B'$

Therefore, $\sin b = \sin c \sin B$.

From (6), $\cos B' = \tan(180-a)\cot(180-c)$

$\cos B = -\tan a(-\cot c)$

Therefore, $\cos B = \tan a \cot c$.

From (7), $\sin(180-a) = \tan b \cot B'$

Therefore, $\sin a = \tan b \cot B$.

From (8), $\cos B' = \cos b \sin(180-A)$

Therefore, $\cos B = \cos b \sin A$.

From (9), $\cos(180-A) = \cos(180-a)\sin B'$

$-\cos A = -\cos a \sin B$

Therefore, $\cos A = \cos a \sin B$.

From (10), $\cos(180-c) = \cot(180-A)\cot B'$

$-\cos c = -\cot A \cot B$

Therefore, $\cos c = \cot A \cot B$.

Consider the remaining possibility, that is, suppose right spherical triangle ABC has both a and b greater than 90°. Extend arcs CB and CA of right spherical triangle ABC through B and A respectively, meeting at C'. (Figure 7) With a and b each greater than 90°, 180-a and 180-b will each be less than 90°. Now to apply formulas (1) through (10) to right spherical triangle ABC'.

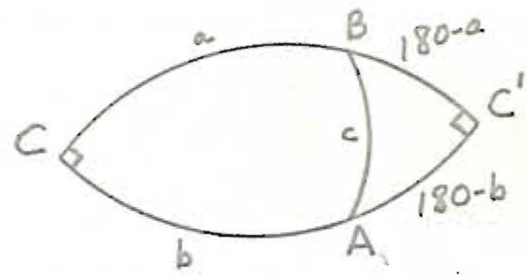


Figure 7

From (1), $\cos c = \cos(180-a)\cos(180-b)$

$\cos c = -\cos a(-\cos b)$

Therefore, $\cos c = \cos a \cos b$.

From (2), $\sin(180-a) = \sin c \sin(180-A)$

Therefore, $\sin a = \sin c \sin A$.

From (3), $\cos(180-A) = \tan(180-b) \cot c$

$-\cos A = -\tan b \cot c$

Therefore, $\cos A = \tan b \cot c$.

From (4), $\sin(180-b) = \tan(180-a) \cot(180-A)$

$\sin b = -\tan a(-\cot A)$

Therefore, $\sin b = \tan a \cot A$.

From (5), $\sin(180-b) = \sin c \sin(180-B)$

Therefore, $\sin b = \sin c \sin B$.

From (6), $\cos(180-B) = \tan(180-a) \cot c$

$$-\cos B = -\tan a \cot c$$

Therefore, $\cos B = \tan a \cot c$.

From (7), $\sin(180-a) = \tan(180-b) \cot(180-B)$

$$\sin a = -\tan b(-\cot B)$$

Therefore, $\sin a = \tan b \cot B$.

From (8), $\cos(180-B) = \cos(180-b) \sin(180-A)$

$$-\cos B = -\cos b \sin A$$

Therefore, $\cos b = \cos b \sin A$.

From (9), $\cos(180-A) = \cos(180-a) \sin(180-B)$

$$-\cos A = -\cos a \sin B$$

Therefore, $\cos A = \cos a \sin B$.

From (10), $\cos c = \cot(180-A) \cot(180-B)$

$$\cos c = -\cot A(-\cot B)$$

Therefore, $\cos c = \cot A \cot B$.

This completes the verification of formulas (1) through (10) for any right spherical triangle ABC with right angle at C [14;165-166].

II. NAPIER'S RULES OF CIRCULAR PARTS

John Napier(about 1614)discovered two simple rules for recalling the ten formulas just derived for solving right spherical triangles. If ABC is a right spherical triangle, right-angled at C, the parts of the triangle are C, a, B, c, a, b.(Figure 8)If C is omitted, for it is not involved in the ten formulas, and B, c, and A are replaced by co-B, co-c, and co-A respectively, the five circular parts of the right spherical triangle become a, co-B, co-c, co-A, and b.(Figure 9)Now co-B = 90-B, that is, co-B represents the co-function of B. The same is true for co-c and co-A. Notice that it is possible to think of these five parts as being arranged around a circle or around triangle ABC with the order preserved. For any circular part, usually referred to as a middle part, there are two parts adjacent to it and two other parts which are called opposite parts. Thus, with respect to the middle part co-B, co-c and a are the adjacent parts while co-A and b are the opposite parts.

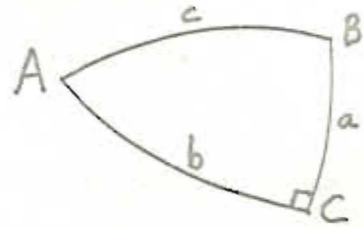


Figure 8

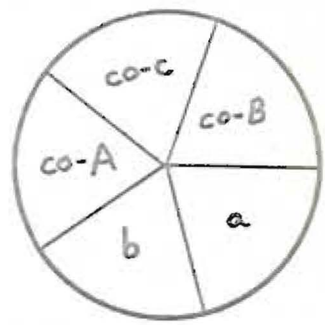


Figure 9

Using this terminology, Napier's rules of circular parts are:

I. The sine of any middle part is equal to the product of the tangents of the adjacent parts.

II. The sine of any middle part is equal to the product of the cosines of the opposite parts[9;219-220].

Using rule I and co-c as the middle part:

$$\sin(\text{co-}c) = \tan(\text{co-}A)\tan(\text{co-}B), \text{ or}$$

$$\cos c = \cot A \cot B.$$

This is formula (10). Using rule II and co-c as the middle part:

$$\sin(\text{co-}c) = \cos a \cos b, \text{ or}$$

$$\cos c = \cos a \cos b.$$

This is formula (1). If each rule were applied to each circular part all of formulas (1) through (10) would be produced [6;186].

A helpful device in remembering Napier's rules is to associate the a in tangent and adjacent and the o in cosine and opposite [3;137].

III. RULES OF QUADRANTS (SPECIES) FOR RIGHT SPHERICAL TRIANGLES

Two angular quantities, sides and/or angles, are said to be of the same species if their angular measures are both acute or both obtuse; and two angular quantities are of different species if one of their angular measures is acute and the other obtuse [6;181]. The following rules are useful in determining the species of parts of a right spherical triangle.

Rule 1. An oblique angle and its opposite side are always of the same species.

Rule 2. If the hypotenuse is less than 90°, the two oblique angles(and therefore the two sides)are of the same species; if it is greater than 90°, the two angles(and therefore the two sides)are of different species[3;138].

To verify rule 1, investigate the formula sin b = tan a cot A(4) and the formula sin a = tan b cot B(7). Sin b is always positive so that tan a and cot A must both have the same sign which is equivalent to saying a and A are of the same species. Using sin a = tan b cot B and a similar argument it is obvious that b and B must also be of the same species. This verifies rule 1.

To verify rule 2, recall that formula (10) relates c, A, and B by cos c = cot A cot B. First, if c is less than 90°, cos c is positive making cot A cot B positive. Thus cot A and cot B have the same sign making A and B of the same species. From rule 1, this makes a and b of the same species in this case. Secondly, if c is greater than 90°, again recalling cos c = cot A cot B, cos c is negative making cot A and cot B have opposite signs. Thus A and B are of different species, and again employing rule 1, it is obvious that a and b are of different species. This verifies rule 2.

IV. SOLUTION OF RIGHT SPHERICAL TRIANGLES

In order to solve a right spherical triangle it is necessary to know two parts of the triangle in addition to the right angle. As a result, there are six possible cases to be considered. Classifying them with the given parts, they are:

I. Given the hypotenuse and an angle.

II. Given the hypotenuse and a side.

III. Given the two angles.

IV. Given the two sides.

V. Given an angle and the adjacent side.

VI. Given an angle and the opposite side [3;139].

It is not necessary to refer to formulas (1) through (10) in the computation required. Instead, Napier's rules will give the desired relationships. Depending upon the parts given, any right spherical triangle will have one solution, two solutions, or no solution [6;182]. Each of these situations will be illustrated in examples to follow.

The following steps make the solution of right spherical triangles a logical and orderly process:

(a) Use Napier's rules to express each of the three unknown parts as a function of the two given parts.

(b) Arrange the functions in (a) for logarithmic computation.

(c) Compute using logarithms but being careful to keep track of negative factors.

(d) Use the rules of species to determine the species of each of the unknown parts if its species is not determined by the sign of the result.

(e) If a check is desired, use Napier's rules to express a relationship between the unknown parts, then carry this relationship through steps (b), (c), and (d) [3;139-140].

<u>Case I. Given the hypotenuse and an angle.</u> Let ABC be a right spherical triangle with $C = 90^\circ$, $B = 58^\circ 20'$, and $c = 85^\circ 40'$. (Figure 10) The unknown parts are a, b, and A. Using Napier's rules, $\sin b = \cos(\text{co-}B)\cos(\text{co-}c)$, $\sin(\text{co-}B) = \tan(\text{co-}c)\tan a$, and $\sin(\text{co-}c) = \tan(\text{co-}B)\tan(\text{co-}A)$ are the needed relationships between given parts and unknown parts. Rearranging and applying the co-function identities, these relationships become $\sin b = \sin B \sin c$, $\tan a = \cos B \tan c$, and $\cot A = \cos c \tan B$.

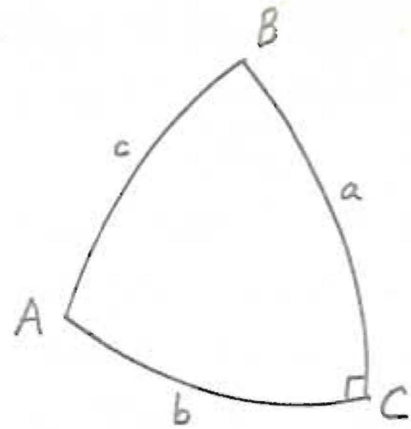


Figure 10

Substituting in the values of B and c, these equations become $\sin b = \sin 58^\circ 20' \sin 85^\circ 40'$, $\tan a = \cos 58^\circ 20' \tan 85^\circ 40'$, and $\cot A = \cos 85^\circ 40' \tan 58^\circ 20'$. Using four-place logarithms the computation is as follows:

log sin $58^\circ 20'$ = 9.9300 - 10
log sin $85^\circ 40'$ = 9.9988 - 10
log sin b = 9.9288 - 10
b = $58^\circ 5'$ or ~~$121^\circ 55'$~~.

log cos $58^\circ 20'$ = 9.7201 - 10
log tan $85^\circ 40'$ = 1.1205
log tan a = 0.8406
a = $81^\circ 47'$ or ~~$98^\circ 13'$~~.

log cos $85^\circ 40'$ = 8.8783 - 10
log tan $58^\circ 20'$ = 0.2098
log cot A = 9.0881 - 10
A = $83^\circ 1'$ or ~~$96^\circ 59'$~~.

The sine function is positive in both the first and second quadrants indicating b could be $58^\circ 5'$ or $121^\circ 55'$. The latter is eliminated by rule 1 of species which says an oblique angle and its opposite side are of the same species. B is acute, therefore b = $58^\circ 5'$.

The tangent and cotangent functions are each positive in the first quadrant and negative in the second quadrant and since all factors in computing tan a and cot A are positive, tan a and cot A are also positive. This makes a and A both acute, or a = $81^\circ 47'$ and A = $83^\circ 1'$.

To check the results, use Napier's rules and the unknown parts a, b, and A to get sin b = tan a tan(co-A) or sin b = tan a cot A. Substituting in the results for a, b, and A, this becomes sin $58^\circ 5'$ = tan $81^\circ 47'$ cot $83^\circ 1'$. Using four-place logarithms the computation is:

log tan $81^\circ 47'$ = 0.8406
log cot $83^\circ 1'$ = 9.0881 - 10
log tan a + log cot A = 9.9287 - 10
log sin $58^\circ 5'$ = 9.9288 - 10

A rounding off loss or gain could easily produce such a small difference in results. Thus, the conclusion is that the answers are correct.

Case II. Given the hypotenuse and a side. Let ABC be a right spherical triangle with $C = 90^\circ$, $b = 49^\circ 56'$, and $c = 112^\circ 51'$. (Figure 11) The unknown parts are A, B, and a. Using Napier's rules, $\sin(\text{co-}A) = \tan b \tan(\text{co-}c)$, $\sin b = \cos(\text{co-}c) \cos(\text{co-}B)$, and $\sin(\text{co-}c) = \cos a \cos b$. Solving these for the unknown parts gives $\cos A = \tan b \cot c$, $\sin B = \sin b / \sin c$, and $\cos a = \cos c / \cos b$. Substituting in the values of b and c and using four-place logarithms, the computation is as follows:

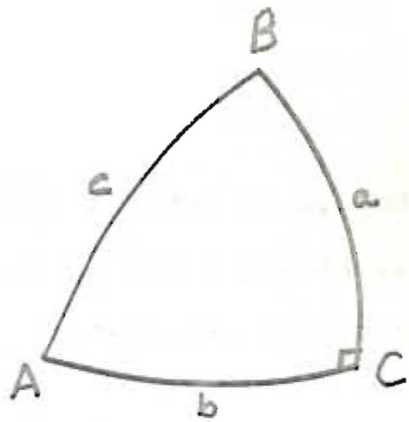


Figure 11

$\log \tan b = \log \tan 49^\circ 56' = \log \tan 49^\circ 56' = 0.0752$
$\log \cot c = \log \cot 112^\circ 51' = (n) \log \cot 67^\circ 09' = 9.6247 - 10$
$\log \cos A = 9.6999 - 10$

$A =$ ~~$59^\circ 56'$~~ or $120^\circ 4'$.

$\log \sin b = \log \sin 49^\circ 56' = \log \sin 49^\circ 56' = 19.8839 - 20$
$\log \sin c = \log \sin 112^\circ 51' = \log \sin 67^\circ 09' = 9.9645 - 10$
$\log \sin B = 9.9194 - 10$

$B = 56^\circ 10'$ or ~~$123^\circ 50'$~~.

$\log \cos c = \log \cos 112^\circ 51' = (n) \log \cos 67^\circ 09' = 19.5892 - 20$
$\log \cos b = \log \cos 49^\circ 56' = \log \cos 49^\circ 56' = \underline{9.8087 - 10}$
$\log \cos a = 9.7805 - 10$

$a = \sout{52^\circ 54'}$ or $127^\circ 6'$.

The (n) in the computation is a reminder that the factor following it is negative. In the computation of A and a, each process contained one negative factor meaning that cos A and cos a were negative, thus A and a must be in the second quadrant. The acute value for B is chosen since b is also acute.

The check requires a relationship between A, B, and a. Proceeding as before, sin(co-A) = cos(co-B) cos a yields cos $120^\circ 4'$ = sin $56^\circ 10'$ cos $127^\circ 6'$. Using four-place logarithms, the computation is:

$\log \sin 56^\circ 10' = \log \sin 56^\circ 10' = 9.9194 - 10$
$\log \cos 127^\circ 6' = (n) \log \cos 52^\circ 54' = \underline{9.7805 - 10}$
$\log \sin B + \log \cos a = 9.6999 - 10$

$\log \cos 120^\circ 4' = (n) \log \cos 59^\circ 56' = 9.6999 - 10$

This completes the check.

<u>Case III</u>. <u>Given the two angles</u>. Let ABC be a right spherical triangle with $C = 90^\circ$, $A = 48^\circ 20'$, and $B = 100^\circ 40'$. (Figure 12) The unknown parts are a, b, and c. As before, sin(co-A) = cos a cos(co-B), sin(co-B) = cos b

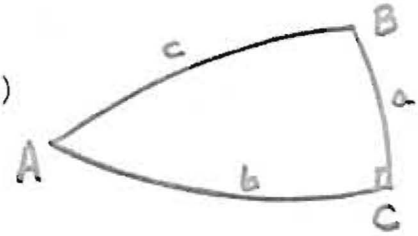


Figure 12

cos(co-A), and sin(co-c) = tan(co-A) tan(co-B). Then cos a = cos A/sin B, cos b = cos B/sin A, cos c = cot A cot B. The computation is:

$$\begin{aligned}
\log\cos A &= \log\cos 48^\circ20' = \log\cos 48^\circ20' = 19.8227 - 20\\
\log\sin B &= \log\sin 100^\circ40' = \log\sin 79^\circ20' = \underline{9.9924 - 10}\\
&\qquad \log\cos a = 9.8303 - 10
\end{aligned}$$

a = $47^\circ26'$ or ~~$132^\circ31'$~~.

$$\begin{aligned}
\log\cos B &= \log\cos 100^\circ40' = (n)\ \log\cos 79^\circ20' = 19.2674 - 20\\
\log\sin A &= \log\sin 48^\circ20' = \log\sin 48^\circ20' = \underline{9.8733 - 10}\\
&\qquad \log\cos b = 9.3941 - 10
\end{aligned}$$

b = ~~$75^\circ39'$~~ or $104^\circ21'$.

$$\begin{aligned}
\log\cot A &= \log\cot 48^\circ20' = \log\cot 48^\circ20' = 9.9494 - 10\\
\log\cot B &= \log\cot 100^\circ40' = (n)\ \log\cot 79^\circ20' = \underline{9.2750 - 10}\\
&\qquad \log\cos c = 9.2244 - 10
\end{aligned}$$

c = ~~$80^\circ21'$~~ or $99^\circ39'$.

The specific values of a, b, and c to accept are easily determined by the cosine function and the sign of the result as determined by the number of negative factors. Observe that the answers still obey the rules of species. A and a, as also B and b, are of the same species. The second rule of species states c should be obtuse since A and B are of different species and this also agrees with the results.

A check involves substituting the results into cos c = cos a cos b.

$$\begin{aligned}
\log\cos a &= \log\cos 47^\circ26' = \log\cos 47^\circ26' = 9.8303 - 10\\
\log\cos b &= \log\cos 104^\circ21' = (n)\ \log\cos 75^\circ39' = \underline{9.3941 - 10}\\
&\qquad \log\cos a + \log\cos b = 9.2244 - 10
\end{aligned}$$

log cos c = log cos $99^{\circ}39'$ = (n) log cos $80^{\circ}21'$ = 9.2244 - 10

The results agree in sign and magnitude, thus the check is complete.

<u>Case IV. Given the two sides.</u> Let ABC be a right spherical triangle with C = 90°, a = $98^{\circ}15'$, and b = $100^{\circ}33'$. (Figure 13) The unknown parts are A, B, and c. Expressing each of these in terms of a and b with the help of Napier's rules, the relationships are: sin b = tan a tan(co-A), sin a = tan b tan(co-B), and sin(co-c) = cos a cos b. These relationships become cot A = sin b cot a, cot B = sin a cot b, and cos c = cos a cos b. The computation is:

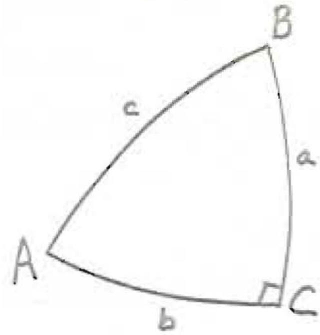


Figure 13

log sin b = log sin $100^{\circ}33'$ = log sin $79^{\circ}27'$ = 9.9926 - 10
log cot a = log cot $98^{\circ}15'$ = (n) log cot $81^{\circ}45'$ = 9.1614 - 10
log cot A = 9.1540 - 10

A = ~~$81^{\circ}53'$~~ or $98^{\circ}7'$.

log sin a = log sin $98^{\circ}15'$ = log sin $81^{\circ}45'$ = 9.9955 - 10
log cot b = log cot $100^{\circ}33'$ = (n) log cot $79^{\circ}27'$ = 9.2701 - 10
log cot B = 9.2656 - 10

B = ~~$79^{\circ}33'$~~ or $100^{\circ}27'$.

log cos a = log cos $98^{\circ}15'$ = (n) log cos $81^{\circ}45'$ = 9.1568 - 10
log cos b = log cos $100^{\circ}33'$ = (n) log cos $79^{\circ}27'$ = 9.2626 - 10
log cos c = 8.4194 - 10

c = $88^{\circ}30'$ or ~~$91^{\circ}30'$~~

It is not necessary to refer to the rules for species because the signs of the final results tell which answers to exclude. The check formula is sin(co-c) = tan(co-A) tan(co-B) or cos c = cot A cot B. The computation is:

$$\begin{aligned} \log \cot A = \log \cot\ 98^\circ 7' &= 9.1540 - 10 \\ \log \cot B = \log \cot 100^\circ 27' &= \underline{9.2656 - 10} \\ \log \cot A + \log \cot B &= 8.4196 - 10 \\ \log \cos 88^\circ 30' &= 8.4194 - 10 \end{aligned}$$

Both factors are negative. Cos c is positive.

The check agrees in sign and is satisfactory in magnitude.

<u>Case V. Given an angle and the adjacent side.</u> Let ABC be a right spherical triangle with C = 90°, A = 33°40', and b = 107°10'. (Figure 14) The unknown parts are a, B, and c. Thus, sin b = tan(co-A) tan a, sin(co-B) = cos b cos(co-A), and sin(co-A) = tan b tan(co-c). The formulas, changed into a more workable form, become tan a = sin b tan A, cos B = cos b sin A, and cot c = cos A cot b. The computation is:

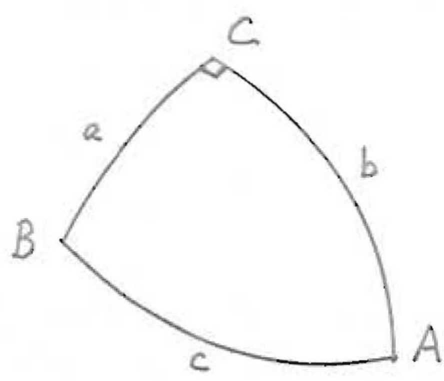


Figure 14

$$\begin{aligned} \log \sin b = \log \sin 107^\circ 10' = \log \sin 72^\circ 50' &= 9.9802 - 10 \\ \log \tan A = \log \tan\ 33^\circ 40' = \log \tan 33^\circ 40' &= \underline{9.8235 - 10} \\ \log \tan a &= 9.8037 - 10 \\ a &= 32^\circ 28' \text{ or } \underline{147^\circ 32'}. \end{aligned}$$

$\log \cos b = \log \cos 107^\circ 10' = (n) \log \cos 72^\circ 50' = 9.4700 - 10$
$\log \sin A = \log \sin 33^\circ 40' = \log \sin 33^\circ 40' = \underline{9.7438 - 10}$
$\log \cos B = 9.2138 - 10$

$B = \sout{80^\circ 35'}$ or $99^\circ 25'$.

$\log \cos A = \log \cos 33^\circ 40' = \log \cos 33^\circ 40' = 9.9203 - 10$
$\log \cot b = \log \cot 107^\circ 10' = (n) \log \cot 72^\circ 50' = \underline{9.4898 - 10}$
$\log \cot c = 9.4101 - 10$

$c = \sout{75^\circ 35'}$ or $104^\circ 25'$.

The signs of the factors makes it easy to determine the species for each unknown part. The check formula, again by Napier's rules, is sin(co-B) = tan a tan(co-c) or cos B = tan a cot c. The computation is:

$\log \tan a = \log \tan 32^\circ 28' = 9.8037 - 10$
$\log \cot c = \log \cot 104^\circ 25' = \underline{9.4101 - 10}$
$\log \tan a + \log \cot c = 9.2138 - 10$

$\log \cos B = \log \cos 99^\circ 25' = 9.2138 - 10.$

One factor is negative. Cos B is negative.

The check is completed.

<u>Case VI</u>. <u>The Ambiguous Case</u>. <u>Given an angle and the opposite side</u>. Let ABC be any right spherical triangle with $C = 90^\circ$ and in which an angle, say B, and the side opposite it, b, are known. (Figure 15) The formulas used to solve for a, c, and A are sin a = tan b tan(co-B), sin b = cos(co-c) cos(co-B),

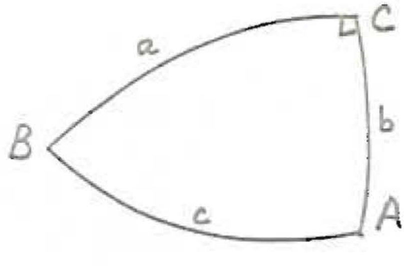


Figure 15

and sin(co-B) = cos b cos(co-A). In a more appropriate form these are sin a = tan b cot B, sin c = sin b/sin B, and sin A = cos B/cos b. In looking at each one of these formulas, since B and b by rule one of species must be of the same species, the right hand side of each one of these formulas must be positive. But the sin function is positive in both the first and second quadrants so that a, c, and A can each take on a value and its supplement from sin a, sin c, and sin A respectively.

This can be illustrated in another manner. If arcs BC and BA each be produced to meet in B', angle B = angle B'. (Figure 16)The two triangles, ABC and AB'C, each of which has the same given, B and b, have two sets of answers; a, c, and A in triangle ABC, and co-A, co-c, and co-a in triangle AB'C. Thus, it is possible to have two sets of answers when an angle and the side opposite are known [9;227]. An example of this follows.

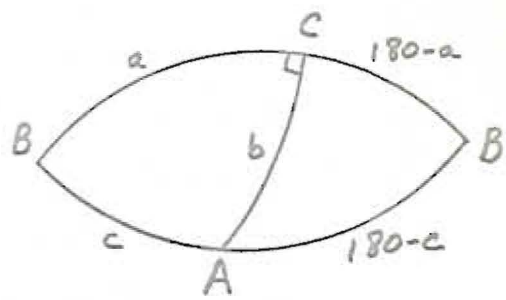


Figure 16

Let b = 114°20' and B = 96°15'. From the formulas derived above, substitutions and computation give:

$$\begin{aligned}
\log \tan b &= \log \tan 114^\circ 20' = (n) \log \tan 65^\circ 40' = 0.3447 \\
\log \cot B &= \log \cot 96^\circ 15' = (n) \log \cot 83^\circ 45' = \underline{9.0394 - 10} \\
&\qquad \log \sin a = 9.3841 - 10 \\
&\qquad a = 14^\circ 1' \text{ or } 165^\circ 59'.
\end{aligned}$$

$$\begin{aligned}
\log \sin b &= \log \sin 114^\circ 20' = \log \sin 65^\circ 40' = 19.9596 - 20 \\
\log \sin B &= \log \sin 96^\circ 15' = \log \sin 83^\circ 45' = \underline{9.9974 - 10} \\
&\qquad \log \sin c = 9.9622 - 10 \\
&\qquad c = 66^\circ 27' \text{ or } 113^\circ 33'.
\end{aligned}$$

$$\begin{aligned}
\log \cos B &= \log \cos 96^\circ 15' = (n) \log \cos 83^\circ 45' = 19.0368 - 20 \\
\log \cos b &= \log \cos 114^\circ 20' = (n) \log \cos 65^\circ 40' = \underline{9.6149 - 10} \\
&\qquad \log \sin A = 9.4219 - 10 \\
&\qquad A = 15^\circ 19' \text{ or } 164^\circ 41'.
\end{aligned}$$

The signs of the results do not give any indication of what to do with the results. It is necessary to employ the rules of species to group the answers correctly. Assuming $a = 14^\circ 1'$ is correct for triangle ABC, then by rule 1, $A = 15^\circ 19'$. By rule 2, the hypotenuse must be obtuse, thus $c = 113^\circ 33'$. The solution of triangle AB'C is: $a = 165^\circ 59'$; $c = 66^\circ 27'$; and $A = 164^\circ 41'$. These answers also satisfy the rules of species.

A check of the answers involves substituting into $\sin a = \sin c \sin A$.

$$\begin{aligned}
\log \sin c &= 9.9622 - 10 \\
\log \sin A &= \underline{9.4219 - 10} \\
\log \sin c + \log \sin A &= 9.3841 - 10 = \log \sin a
\end{aligned}$$

and the check is complete.

It may also occur that when given an angle and the side opposite it, no solution will exist. One obvious example of this would be if the angle and the side opposite it are of different species. To verify this, look at sin A = cos B/cos b, one of the relations developed above. If B and b are of different species, cos B and cos b are of opposite sign and sin A is negative. But this is impossible since sin A is positive when A is in the first or second quadrants. As a numerical example, consider B = 120, b = 45. Then sin A = cos 120/cos 45 = $(-\frac{1}{2})/(\sqrt{2}/2) = -\sqrt{2}/2$. Thus, A is greater than 180° and there is no solution since the study of spherical trigonometry is limited to spherical triangles in which each part is less than 180° [6;178].

A much less obvious contradiction can develop when an angle and the opposite side are the given parts. To see this, examine one of the formulas developed above, sin c = sin b/sin B. Recall $0 < \sin x \leq 1$ for $0^\circ < x < 180^\circ$ and that sin x = 1 when $x = 90^\circ$ and $\sin x \to 0$ as $x \to 0^\circ$ or $x \to 180^\circ$. Thus, for sin c to be less than or equal to one, the denominator of sin b/sin B must be greater than or equal to the numerator. But $|\sin B| \geq |\sin b|$ if and only if $|90-B| \leq |90-b|$. A similar argument can be used for A and a using sin c = sin a/sin A. It is possible to state this result as a theorem.

Theorem: In a right spherical triangle, an oblique angle always differs from a right angle by an amount less than or equal to the amount the side opposite it differs from a right angle.

To illustrate the preceding theorem, attempt to solve right spherical triangle ABC if $C = 90°$, $B = 45°$, and $b = 60°$. From $\sin c = \sin b/\sin B$, $\sin c = (\sqrt{3}/2)/(\sqrt{2}/2) = \sqrt{3}/\sqrt{2}$. But $\sqrt{3}/\sqrt{2}$ is greater than one. This is a contradiction since $\sin c \leq 1$ for all c, $0° < c < 180°$, therefore, there is no solution to this problem. The preceding examples used to illustrate cases I through VI also serve to substantiate this theorem.

V. SOLVING ISOSCELES SPHERICAL TRIANGLES

An isosceles spherical triangle(Figure 17)has two sides equal. It also has the angles opposite the equal sides equal. The bisector of the angle formed by the two equal sides will be perpendicular to and bisect the third side. That is, if $a = b$, then $A = B$, angle ACD = angle BCD = $\frac{1}{2}C$, $AD = BD = \frac{1}{2}c$, and CD is perpendicular to AB[9;229]. This gives a fairly easy method to solve an oblique spherical triangle if it is known to be isosceles, which two sides

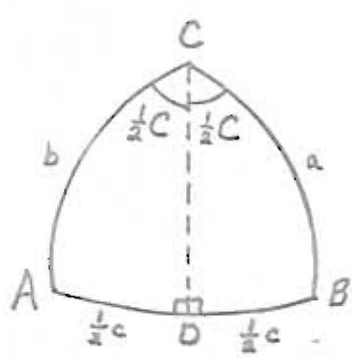


Figure 17

are equal, and the measure of the two equal sides (or angles) and the measure of one other part of the triangle. The procedure is to drop the perpendicular from the vertex formed by the two equal sides to the third side. Since this perpendicular bisects the third side and the vertex angle, the solution is either one of the congruent right spherical triangles formed.

Consider the following example. In Figure 17, let $a = b = 73^\circ 50'$ and $c = 110^\circ$. Then in right spherical triangle ACD, $b = 73^\circ 50'$ and $\frac{1}{2}c = 55^\circ$. By Napier's rules, $\cos A = \cot b \tan \frac{1}{2}c$ and $\sin \frac{1}{2}C = \sin \frac{1}{2}c / \sin b$. Using four-place logarithms, the computation is:

$$\begin{aligned} \log \cot b &= \log \cot 73^\circ 50' = 9.4622 - 10 \\ \log \tan \tfrac{1}{2}c &= \log \tan 55^\circ = \underline{0.1548} \\ \log \cos A &= 9.6170 - 10 \\ A &= 65^\circ 32' \text{ or } \sout{114^\circ 28'}. \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}c &= \log \sin 55^\circ = 19.9134 - 20 \\ \log \sin b &= \log \sin 73^\circ 50' = \underline{9.9825 - 10} \\ \log \sin \tfrac{1}{2}C &= 9.9309 - 10 \\ \tfrac{1}{2}C &= 58^\circ 31' \text{ or } \sout{121^\circ 29'}. \\ C &= 117^\circ 2'. \end{aligned}$$

The solution of isosceles spherical triangle ABC is $A = B = 65^\circ 32'$ and $C = 117^\circ 2'$. $A = 114^\circ 28'$ and $\frac{1}{2}C = 121^\circ 29'$ are eliminated by observing the signs of the factors in each computation. Also, C is less than 180°, thus $\frac{1}{2}C$ must be less than 90°. For a check use $\cos b = \cot A \cot \frac{1}{2}C$.

$$\log \cot A = \log \cot 65^\circ 32' = 9.6580 - 10$$
$$\log \cot \tfrac{1}{2}C = \log \cot 58^\circ 31' = \underline{9.7870 - 10}$$
$$\log \cot A + \log \cot \tfrac{1}{2}C = 9.4450 - 10$$

$$\log \cos b = \log \cos 73^\circ 50' = 9.4447 - 10.$$

The check is satisfactory considering the factor of ½ used in computing C increases the amount of error.

VI. SOLVING QUADRANTAL SPHERICAL TRIANGLES

A quadrantal spherical triangle is a triangle in which at least one side is equal to 90°. The polar triangle of a quadrantal triangle is a right-angled spherical triangle. This triangle can be solved using Napier's rules and the results converted into a solution of the original triangle. The following formulas relate the parts(A, B, C, a, b, c)of a spherical triangle ABC(Figure 18) and the parts(A', B', C', a', b', c')of its polar triangle A'B'C' [6;185, 179].(Figure 19)

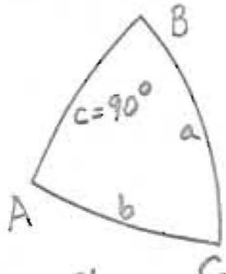


Figure 18

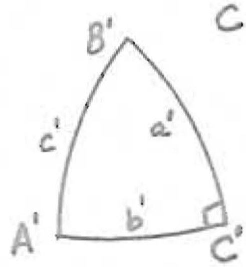


Figure 19

$a = 180-A'$; $a' = 180-A$;

$b = 180-B'$; $b' = 180-B$;

$c = 180-C'$; $c' = 180-C$.

As an illustration, let spherical triangle ABC have $c = 90^\circ$, $a = 151^\circ 30'$, and $B = 119^\circ 40'$. In polar triangle A'B'C', the above relationships yield:

$C' = 180-90 = 90^\circ$; $A' = 180-151^\circ 30' = 28^\circ 30'$; $b' = 60^\circ 20'$.

Using Napier's rules on right spherical triangle $A'B'C'$, $\tan a' = \sin b' \tan A'$, $\cos B' = \cos b' \sin A'$, and $\cot c' = \cos A' \cot b'$. Substituting and using four-place logarithms, the computation is:

$$\begin{aligned} \log \sin b' &= \log \sin 60^\circ 20' = 9.9390 - 10 \\ \log \tan A' &= \log \tan 28^\circ 30' = \underline{9.7348 - 10} \\ \log \tan a' &= 9.6738 - 10 \end{aligned}$$

$$a' = 25^\circ 16' \text{ or } \sout{154^\circ 44'}.$$

$$\begin{aligned} \log \cos b' &= \log \cos 60^\circ 20' = 9.6946 - 10 \\ \log \sin A' &= \log \sin 28^\circ 30' = \underline{9.6787 - 10} \\ \log \cos B' &= 9.3733 - 10 \end{aligned}$$

$$B' = 76^\circ 20' \text{ or } \sout{103^\circ 40'}.$$

$$\begin{aligned} \log \cos A' &= \log \cos 28^\circ 30' = 9.9439 - 10 \\ \log \cot b' &= \log \cot 60^\circ 20' = \underline{9.7556 - 10} \\ \log \cot c' &= 9.6995 - 10 \end{aligned}$$

$$c' = 63^\circ 24' \text{ or } \sout{116^\circ 36'}.$$

To check the values $a' = 25^\circ 16'$, $B' = 76^\circ 20'$, and $c' = 63^\circ 24'$, use the formula $\cos B' = \tan a' \cot c'$. Thus,

$$\begin{aligned} \log \tan a' &= \log \tan 25^\circ 16' = 9.6739 - 10 \\ \log \cot c' &= \log \cot 63^\circ 24' = \underline{9.6995 - 10} \\ \log \tan a' + \log \cot c' &= 9.3734 - 10 \end{aligned}$$

$$\log \cos B' = \log \cos 76^\circ 20' = 9.3733 - 10$$

and the check for triangle $A'B'C'$ is complete.

To solve the triangle ABC, use the following relationships:

$$A = 180 - a' = 180 - 25^\circ 16' = 154^\circ 44';$$

$$b = 180 - B' = 180 - 76^\circ 20' = 103^\circ 40';$$

$$C = 180 - c' = 180 - 63^\circ 24' = 116^\circ 36'.$$

It is possible, although usually not the most convenient way, to solve other oblique spherical triangles using auxiliary arcs and Napier's rules. A large part of the next chapter will be devoted to solving oblique spherical triangles by auxiliary arcs. Most of the next two chapters will deal with solving oblique spherical triangles by more elaborate techniques.

CHAPTER III

OBLIQUE SPHERICAL TRIANGLES

The methods of solving oblique spherical triangles are many and varied. This chapter will consider solutions by: (1) right triangles, (2) the law of sines, (3) the law of cosines, and (4) the law of cosines in terms of haversines. Several other methods will be considered in the next chapter.

Rules of species for oblique spherical triangles will be stated so that examples of solutions can be carried out. The proofs of the rules of species will be given in the next chapter, after the formulas essential for their proof have been developed.

Examples of solutions will not be numerous enough to illustrate each method of solution with each of the six possible cases in solving oblique spherical triangles. All of the possible cases will be considered by at least one method of solution.

<u>Solutions of oblique spherical triangles</u>. The six possible cases in solving oblique spherical triangles are:

I. Given the three sides.
II. Given the three angles.
III. Given the two sides and the included angle.
IV. Given two angles and the included side.
V. Given two sides and the angle opposite one of them.
VI. Given two angles and the side opposite one of them [6;189].

I. SOLUTION BY RIGHT TRIANGLES

It is possible to solve any oblique spherical triangle by passing an arc of a great circle through a proper vertex forming two right spherical triangles which can then be solved using Napier's rules. Labeling the parts of the right spherical triangles formed in a similar manner each time can help to make the solution easier to perform. The general procedure follows.

Through any vertex C of a spherical triangle ABC pass the arc p of the great circle that is perpendicular to side AB(Figure 20)or side AB extended(Figure 21)at D. The two

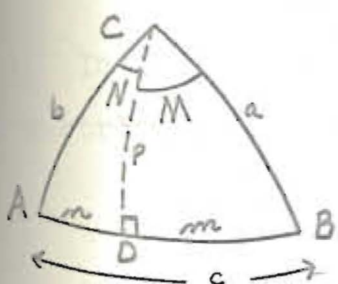


Figure 20

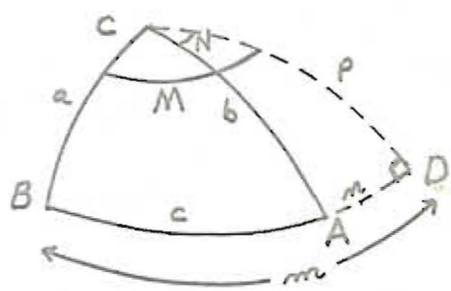


Figure 21

right triangles formed are ACD and BCD, both right-angled at D. Label the parts AD and BD of these right triangles n and m respectively, where the algebraic sum of m and n is c, that is, m + n = c. Label the angles opposite the parts m and n as M and N respectively. Thus, M and N can be considered as parts of angle C[3;168].

<u>A special formula for cases I and II.</u> Cases I and

II lend themselves to the development of a special formula that makes their solutions much easier. In each of Figures 20 and 21, using Napier's rules on parts b, p, and n of right triangle ACD, cos b = cos n cos p. Likewise, in right triangle BCD, cos a = cos p cos m. Upon solving each of these relationships for cos p, and then equating the results gives

$$\frac{\cos a}{\cos m} = \frac{\cos b}{\cos n} \quad [3;168].$$

Interchanging cos a and cos n, adding one to each side, and simplifying each side obtains

$$\frac{\cos n + \cos m}{\cos m} = \frac{\cos b + \cos a}{\cos a} .$$

Inverting each side, multiplying each side by two, then subtracting one from each side gives

$$\frac{2 \cos m}{\cos n + \cos m} - 1 = \frac{2 \cos a}{\cos b + \cos a} - 1.$$

Simplifying each side to a single fraction gives

$$\frac{\cos m - \cos n}{\cos m + \cos n} = \frac{\cos a - \cos b}{\cos a + \cos b} .$$

Using the trigonometric identities

$\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$ and

$\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$ on both sides

yields $\dfrac{-2 \sin \frac{1}{2}(m + n) \sin \frac{1}{2}(m - n)}{2 \cos \frac{1}{2}(m + n) \cos \frac{1}{2}(m - n)} = \dfrac{-2 \sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}$

Thus, $\tan \frac{1}{2}(m+n) \tan \frac{1}{2}(m-n) = \tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b)$, and since $m + n = c$,

$$\tan \tfrac{1}{2}(m-n) = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}c, \quad \text{(i)}$$

where $m = \tfrac{1}{2}c + \tfrac{1}{2}(m-n)$, and (ii)

$n = \tfrac{1}{2}c - \tfrac{1}{2}(m-n)$ [4;169]. (iii)

Formulas (i), (ii), and (iii) provide the means to solve any spherical triangle of case I or II. After the parts m and n are determined, the right triangles ABD and BCD can be solved by Napier's rules to find the remaining parts of triangle ABC. An example will serve to better illustrate.

<u>Case</u> <u>I</u>. <u>Given</u> <u>the</u> <u>three</u> <u>sides</u>. Let ABC be a spherical triangle in which $a = 49°20'$, $b = 38°40'$, and $c = 63°30'$. Let p be the arc CD perpendicular to side AB of triangle ABC. Call AD and BD n and m respectively. Recall that $m + n = c$ and that the sign of m and of n will indicate whether the side AB was extended or not. Using

$$\tan \tfrac{1}{2}(m-n) = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}C$$

of the last section, substituting in the proper values, and utilizing four-place logarithms the computation is:

$$\begin{aligned}
\log \tan \tfrac{1}{2}(a+b) &= \log \tan 44° &&= 9.9848 - 10\\
\log \tan \tfrac{1}{2}(a-b) &= \log \tan 5°20' &&= 8.9701 - 10\\
\log \cot \tfrac{1}{2}c &= \log \cot 31°45' &&= \underline{0.2084}\\
&\ \log \tan \tfrac{1}{2}(m-n) &&= 9.1633 - 10\\
&\ \tfrac{1}{2}(m-n) &&= 8°17'.
\end{aligned}$$

Thus, from $m = \tfrac{1}{2}c + \tfrac{1}{2}(m-n)$ and $n = \tfrac{1}{2}c - \tfrac{1}{2}(m-n)$, $m = 40°2'$ and $n = 23°28'$. Both m and n are positive making triangle ABC similar to Figure 20. The remainder of the solution is to solve right spherical triangles ACD and BCD for A, N, B, and M using Napier's rules on the known parts b, n, a, and m.

In triangle ACD, $\cos A = \cot b \tan n$ and $\sin N = \sin n/\sin b$. In triangle BCD, $\cos B = \tan m \cot a$ and $\sin M = \sin m/\sin a$. The computation is:

$$\begin{aligned} \log \cot b = \log \cot 38^\circ 40' &= 0.0968 \\ \log \tan n = \log \tan 23^\circ 28' &= \underline{9.6376 - 10} \\ \log \cos A &= 9.7344 - 10 \\ A &= 57^\circ 09' \text{ or } \sout{122^\circ 51'}. \end{aligned}$$

$$\begin{aligned} \log \sin n = \log \sin 23^\circ 28' &= 19.6001 - 20 \\ \log \sin b = \log \sin 38^\circ 40' &= \underline{9.7957 - 10} \\ \log \sin N &= 9.8044 - 10 \\ N &= 39^\circ 36' \text{ or } \sout{140^\circ 24'}. \end{aligned}$$

$$\begin{aligned} \log \tan m = \log \tan 40^\circ 02' &= 9.9243 - 10 \\ \log \cot a = \log \cot 49^\circ 20' &= \underline{9.9341 - 10} \\ \log \cos B &= 9.8584 - 10 \\ B &= 43^\circ 48' \text{ or } \sout{136^\circ 12'}. \end{aligned}$$

$$\begin{aligned} \log \sin m = \log \sin 40^\circ 02' &= 19.8084 - 20 \\ \log \sin a = \log \sin 49^\circ 20' &= \underline{9.8800 - 10} \\ \log \sin M &= 9.9284 - 10 \\ M &= 58^\circ \text{ or } \sout{122^\circ}. \end{aligned}$$

Using the second rule of species for right spherical triangles and noting that a, b, m, and n are acute, it is reasoned that, had p been computed, p also would have been acute. Therefore, A, B, M, and N must also be acute. Thus the solution of triangle ABC is that $A = 57^\circ 09'$, $B = 43^\circ 48'$, and $C = M + N = 97^{\circ}36'$.

Formulas involving all six parts of a spherical triangle are quite useful in checking solutions of triangles. These will be developed in the next chapter so that no check will be performed at this time.

Case II. Given the three angles. Let ABC be a spherical triangle in which A, B, and C are known. Perform the method of case I on polar triangle A'B'C' thus obtaining the solution of triangle ABC [3;170]. An example of this case will not be given for this method of solution.

Case III. Given two sides and the included angle. Let ABC be a spherical triangle in which the values of A, b, and c are known. From the ends of one of the sides(b is used as an example)pass the arc CD perpendicular to the opposite side or opposite side extended.(See Figure 20 or Figure 21). Thus, in right spherical triangle ACD, A and b are known so that N, p, and m can be found using Napier's rules, with A and p having the same species. Thus, $m = a - n$, and m is negative when a is less than m showing D is on side BC extended. Then, p and m are known in triangle BCD making the solution of B, a, and M possible by Napier's rules [3;171]. The example of the next case will partially help to illustrate the method described in this case.

Case IV. Given two angles and the included side. Let ABC be a spherical triangle in which A, b, and C are known. Through the vertex of one of the angles(C is used here)pass the arc perpendicular to the opposite side or opposite extended at D.(Figure 22)In right spherical triangle ACD, knowing A and b, Napier's rules can be used to solve for p, n, and N, taking p to have the same species as A and letting the signs of the factors determine n and N. Then M = C - N and a negative value of M indicates triangle BCD is exterior to triangle ABC. In

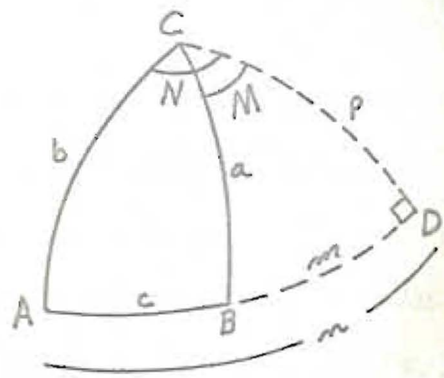


Figure 22

right triangle BCD p and M are known so that m, a, and CBD can be found [3;171-172]. An example follows.

Suppose A = 142°10', b = 55°40', and C = 105°30'. Then, in right spherical triangle ACD, sin p = sin A sin b, tan n = cos A tan b, and cot N = cos b cot A. The computation with four-place logarithms is:

log sin A = log sin 142°10' = log sin 37°50' = 9.7877 - 10
log sin b = log sin 55°40' = log sin 55°40' = 9.9169 - 10
log sin p = 9.7046 - 10

p = ~~30°26'~~ or 149°34'.

$$\log \cos A = \log \cos 142^\circ 10' = (n) \log \cos 37^\circ 50' = 9.8975 - 10$$
$$\log \tan b = \log \tan 55^\circ 40' = \log \tan 55^\circ 40' = 0.1656$$
$$\log \tan n = 0.0631$$
$$n = \sout{49^\circ 09'} \text{ or } 130^\circ 51'.$$

$$\log \cos b = \log \cos 55^\circ 40' = \log \cos 55^\circ 40' = 9.7513 - 10$$
$$\log \tan A = \log \tan 142^\circ 10' = (n) \log \tan 37^\circ 50' = 9.8902 - 10$$
$$\log \cot N = 9.6415 - 10$$
$$N = \sout{66^\circ 21'} \text{ or } 113^\circ 39'.$$

Notice that A greater than 90° implies $p = 149^\circ 34'$. The signs of the factors determine the species of n and N but also note that p and A each greater than 90° and b less than 90° implies that n and N must each be greater than 90°. This serves to double check the results obtained so far. From $M = C - N$, $C = 105^\circ 30'$, and $N = 113^\circ 39'$, the result that $M = -8^\circ 09'$ means that D is on BC extended and that triangle BCD is exterior to triangle ABC. Thus, in working with triangle BCD it must be noticed that m will be acute and negative, since M is acute and negative, and that angle B in triangle BCD is really the supplement of the angle B in triangle ABC that is being sought. The negative sign of M should be ignored when solving right triangle BCD so the proper quadrants will be obtained for a, m, and b.

Thus, using Napier's rules in triangle BCD,

$\cot a = \cos M \cot p$, $\tan m = \sin p \tan M$, and $\cos B = \cos p \sin M$.

The computation is:

log cos M = log cos $8^\circ 9'$ = log cos $8^\circ 9'$ = 9.9956 - 10
log cot p = log cot $149^\circ 34'$ = (n) log cot $30^\circ 26'$ = 0.2310
log cot a = 0.2266

a = ~~$30^\circ 41'$~~ or $149^\circ 19'$.

log sin p = log sin $149^\circ 34'$ = log sin $30^\circ 26'$ = 9.7046 - 10
log tan M = log tan $8^\circ 9'$ = log tan $8^\circ 9'$ = 9.1560 - 10
log tan m = 8.8606 - 10

m = ~~$4^\circ 9'$~~ or ~~$175^\circ 51'$~~.

m = $-4^\circ 9'$ since M is negative and acute.

log cos p = log cos $149^\circ 34'$ = (n) log cos $30^\circ 26'$ = 9.9356 - 10
log sin M = log sin $8^\circ 9'$ = log sin $8^\circ 9'$ = 9.1516 - 10
log cos CBD = 9.0872 - 10

CBD = ~~$82^\circ 59'$~~ or $97^\circ 01$.

The species of a and CBD are determined by the signs of the factors. Since triangle BCD is exterior to triangle ABC and knowing M is acute, the conclusion is that m = $-4^\circ 9'$. Finally, B = 180 - BCD and m + n = c, so that the completed solution to triangle ABC is:

a = $149^\circ 19'$; B = $82^\circ 59'$; and c = $126^\circ 42'$.

Case V. Given two sides and the angle opposite one of them. Let ABC be a spherical triangle in which A, b, and a are known. Through C, the vertex of the intersection of the two known sides, pass the arc p of the great circle perpendicular to third side or third side extended at D. (Figure 23)

Depending upon the values of a, b, and p there may be two solutions, exactly one solution, or no solution to triangle ABC. Crockett [3;129] gives the following summation of conditions possible and solutions obtainable:

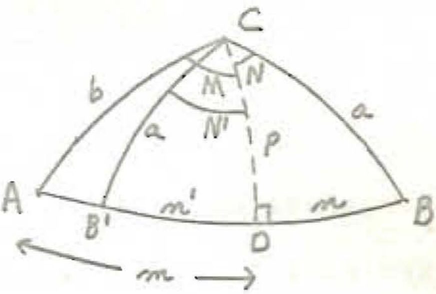


Figure 23

The conditions, therefore, for two solutions are:
A acute: $a > p$, $a < b$, $a < 180^\circ - b$.
A obtuse: $a < p$, $a > b$, $a > 180^\circ - b$.
Or, a must be intermediate in value between p and both b and $180^\circ - b$.

If a is intermediate in value only between p and either b or $180^\circ - b$, there will be one solution.

If a is not intermediate in value between p and either b or $180^\circ - b$, no solution will be possible, but if $p = a$, there will be one solution--a right triangle.

To illustrate let $A = 58^\circ 30'$, $a = 43^\circ 10'$, and $b = 48^\circ 20'$. In right spherical triangle ACD, using Napier's rules, $\sin p = \sin A \sin b$, $\cot M = \cos b \tan A$, and $\tan m = \cos A \tan b$. The computation is:

$$\begin{aligned} \log \sin A = \log \sin 58^\circ 30' &= 9.9308 - 10 \\ \log \sin b = \log \sin 48^\circ 20' &= \underline{9.8733 - 10} \\ \log \sin p &= 9.8041 - 10 \end{aligned}$$

$$p = 39^\circ 33' \text{ or } \cancel{140^\circ 27'}.$$

$$\begin{aligned} \log \cos b = \log \cos 48^\circ 20' &= 9.8227 - 10 \\ \log \tan A = \log \tan 58^\circ 30' &= \underline{0.2127} \\ \log \cot M &= 0.0354 \end{aligned}$$

$$M = 42^\circ 40' \text{ or } \cancel{137^\circ 20'}.$$

$$\begin{aligned}\log\cos A &= \log\cos 58^\circ 30' = 9.7181 - 10\\ \log\tan b &= \log\tan 48^\circ 20' = \underline{0.0506}\\ &\log\tan m = 9.7687 - 10\end{aligned}$$

$$m = 30^\circ 25' \text{ or } \sout{149^\circ 35'}.$$

The value of p is acute since A is given acute. A and p acute implies M must also be acute which in turn makes m acute as the signs of its factors also indicate.

Now to see how many solutions will be possible. The value of a is intermediate between the values of p and b as also is the value of a intermediate between the values of p and 180°-b. Thus, two solutions should be possible. (Figure 23)

Using Napier's rules and the values found above for a and p in right spherical triangle BCD(similar relationships hold for triangle B'CD)the needed relationships are: cos n = cos a/cos p, cos N = tan p cot a, and sin B = sin p/sin a. The computation is:

$$\begin{aligned}\log\cos a &= \log\cos 43^\circ 10' = 19.8629 - 20\\ \log\cos p &= \log\cos 39^\circ 33' = \underline{9.8871 - 10}\\ &\log\cos n = 9.9758 - 10\end{aligned}$$

$$n = 18^\circ 58' \text{ or } \sout{161^\circ 02'}.$$

$$\begin{aligned}\log\tan p &= \log\tan 39^\circ 33' = 9.9169 - 10\\ \log\cot a &= \log\cot 43^\circ 10' = \underline{0.0278}\\ &\log\cos N = 9.9447 - 10\end{aligned}$$

$$N = 28^\circ 19' \text{ or } \sout{151^\circ 41'}.$$

$$\begin{aligned} \log 0.42480 &= 9.62818 - 10 \\ \text{colog} \sin b = \text{colog} \sin 54^\circ 20' &= 0.09022 \\ \text{colog} \sin c = \text{colog} \sin 69^\circ 23' &= \underline{0.02874} \\ \log \text{hav } A &= 9.74714 - 10 \\ A &= 96^\circ 44'. \end{aligned}$$

Therefore, the solution to triangle ABC, without need
refer to any rules of species, is:

$A = 96^\circ 44'$; $B = 125^\circ 41'$; $c = 110^\circ 37'$.

<u>Case VI. Given two angles and the included side.</u> On
ges 54 and 55 of this chapter case V was discussed and found
be ambiguous as to the number of solutions possible. Con-
ions were given for determining the number of solutions
ler the method of solution of introducing an auxiliary arc.

Case VI is also an ambiguous case. The law of sines
kes an analysis of the number of solutions under cases V
VI relatively easy.

Let ABC be a spherical triangle in which A, B, and a
known. Solve for sin b by the law of sines obtaining

$$\sin b = \sin a \sin B / \sin A.$$

Depending upon the given parts there may be no solu-
on, exactly one solution, or two solutions. With the
rts as given there are the following possibilities:

1. If $\sin b > 1$, there is no solution.
2. If $\sin b = 1$, there is exactly one solution, with $b = 90^\circ$.

3. If sin b $<$ 1, there may be one, two, or no solutions. When sin b $<$ 1, an acute angle and its supplement are both possible solutions for b and must be checked by rules of species or other means for determining relationships between sides and angles of spherical triangles [6;197].

As an illustration let a = 72°, B = 118°, and A = 57°6'. e computation for sin b to determine the number of solutions :

$$\begin{aligned} \log \sin a &= \log \sin 72^\circ = 9.9782 - 10 \\ \log \sin B &= \log \sin 62^\circ = 9.9459 - 10 \\ \text{colog} \sin A &= \text{colog} \sin 57^\circ 6' = \underline{0.0759\qquad} \\ \log \sin b &= 10.0000 - 10 \end{aligned}$$

$$b = 90^\circ.$$

Since b = 90° there must be exactly one solution to iangle ABC. It is impossible to use the law of sines to lve for c and C since neither one of them is known. The ws of cosines and haversines are also not useful in this tuation. It would be possible to pass an arc through a rtex perpendicular to the opsite side as was done in the rst section of this chapter. is would be the only possility had b not been equal to °. But b = 90° implies that lar triangle A'B'C' is right-

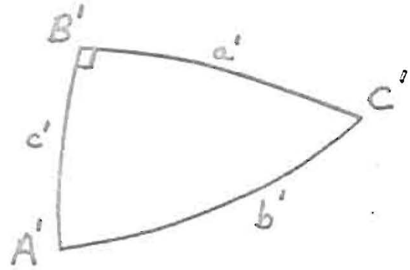


Figure 26

ngled at B'. Thus, using Napier's rules on right tri-
ngle A'B'C' in which $B' = 90^\circ$, $A' = 108^\circ$, $b' = 62^\circ$, and
$' = 122^\circ 54'$, the formulas needed to solve for C' and c'
re:

$$\cos C' = \tan a' \cot b'; \text{ and } \cos c' = \cos b'/\cos a'.$$

The computation is:

og tan a' = log tan $122^\circ 54'$	=	(n) log tan $57^\circ 6'$	=	0.1892
og cot b' = log cot 62°	=	log cot 62°	=	9.7257 - 10
		log cos C'	=	9.9149 - 10
		C'	=	~~$34^\circ 42'$~~ or $145^\circ 18'$.
og cos b' = log cos 62°	=	log cos 62°	=	19.6716 - 20
og cos a' = log cos $122^\circ 54'$	=	(n) log cos $57^\circ 6'$	=	9.7350 - 10
		log cos c'	=	9.9366 - 10
		c'	=	~~$30^\circ 13'$~~ or $149^\circ 47'$.

The odd number of negative factors determine the species
f C' and c'. Thus, in triangle ABC, the solution is:

$$c = 34^\circ 42'; \; C = 30^\circ 13'; \text{ and } b = 90^\circ.$$

The next chapter will be devoted to developing
ormulas for solving spherical triangles of all types and
o be able to give complete solutions of these triangles.
here will also be formulas that are easily solved by log-
rithms, one of the disadvantages of using the laws of co-
ines and haversines.

CHAPTER IV

SPECIAL METHODS OF SOLUTION

This chapter will be devoted to developing special rmulas for the solution of spherical triangles. Some of ese formulas will have as a purpose the solution of a rticular part of the triangle while others will be useful finding a complete solution of a spherical triangle unr a certain set of conditions. Formulas will be developed at are most useful in checking the results of a solution. e names of some of these formulas are: half angle formus; half side formulas; Gauss's Equations; Delambre's Analies; and Napier's Analogies. Spherical excess will be scussed. The theorems of Girard, Lhuillier, and Cagnoli, d a formula useful in computing spherical excess will be rived. The rules of species for oblique spherical trigles stated in the last chapter will be proven. A limited mber of examples will be considered so that the use of each pe of equation, formula, and theorem can be illustrated.

I. FORMULAS FOR HALF ANGLES

The formulas for half angles give the solution to lf of any angle of a spherical triangle when all three des of the triangle are known. Half of any angle can be

ound by the sine, cosine, or tangent functions. These
ormulas are particularly useful in a case I solution.

The derivation of the half angle formula for angle A
arts with formula (12) solved for cos A, that is

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad .$$

Then,

$$1 - \cos A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c} \quad .$$

Using the addition identity $\cos(x-y) = \cos x \cos y +$ n $x \sin y$ on the numerator of the right hand side of the st equation, the result is

$$1 - \cos A = \frac{\cos(b-c) - \cos a}{\sin b \sin c} \quad .$$

Using the product formula $\cos x - \cos y =$ $\sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$ on the numerator of the last equa- on, the result is

$$1 - \cos A = \frac{-2 \sin \frac{1}{2}(b-c+a) \sin \frac{1}{2}(b-c-a)}{\sin b \sin c} \quad .$$

But, $1 - \cos X = 2 \sin^2 \frac{1}{2}X$ and $\sin(-X) = -\sin X$, so e last equation becomes

$$\sin^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a+c-b)}{\sin b \sin c} \quad .$$

This result will be condensed after expressions for $s^2 \frac{1}{2}A$ and $\tan^2 \frac{1}{2}A$ have been developed.

Proceeding as above it is seen that

$$1 + \cos A = \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c} .$$

Using $\cos(x+y) = \cos x \cos y - \sin x \sin y$, the re-

lt is

$$1 + \cos A = \frac{\cos a - \cos(b+c)}{\sin b \sin c} \quad ; \text{ or}$$

$$2 \cos^2 \tfrac{1}{2}A = \frac{-2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(a-b-c)}{\sin b \sin c} \quad ; \text{ which is}$$

$$\cos^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c} .$$

But, $\tan^2 \frac{1}{2}A = \sin^2 \frac{1}{2}A/\cos^2 \frac{1}{2}A$, therefore

$$\tan^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a+c-b)}{\sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)} .$$

If the perimeter of spherical triangle ABC is given e value 2s, then the following relationships allow a use- l simplification in the results just obtained.

If $a+b+c = 2s$,

$$\left.\begin{aligned} \text{then} \quad \tfrac{1}{2}(a+b+c) &= s, \\ \tfrac{1}{2}(b+c-a) &= s - a, \\ \tfrac{1}{2}(a+c-b) &= s - b, \\ \text{and} \quad \tfrac{1}{2}(a+b-c) &= s - c. \end{aligned}\right\} \qquad (21)$$

If the appropriate formulas in group (21) are sub- ituted in the results obtained for $\sin^2 \frac{1}{2}A$, $\cos^2 \frac{1}{2}A$, and $n^2 \frac{1}{2}A$, the following equations are the results:

$$\sin^2 \tfrac{1}{2}A = \frac{\sin(s-c) \sin(s-b)}{\sin b \sin c} \quad ;$$

$$\cos^2 \tfrac{1}{2}A = \frac{\sin s \sin(s-a)}{\sin b \sin c} \quad ;$$

$$\tan^2 \tfrac{1}{2}A = \frac{\sin(s-c) \sin(s-b)}{\sin s \sin(s-a)} \quad .$$

Since $A < 180° \longrightarrow \frac{1}{2}A < 90°$, and thus $\cos \frac{1}{2}A$ is posi-
ve as is $\tan \frac{1}{2}A$. Taking the principal square root of each
de, the resulting half angle formulas for angle A are:

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}} \quad ; \qquad (22)$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \quad ; \qquad (23)$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}} \quad [15;208]. \qquad (24)$$

By symmetry, that is, interchanging A and B and a and
in formulas (22), (23), and (24), the half angle formulas
r angle B are:

$$\sin \tfrac{1}{2}B = \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin a \sin c}} \quad ; \qquad (25)$$

$$\cos \tfrac{1}{2}B = \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}} \quad ; \qquad (26)$$

$$\tan \tfrac{1}{2}B = \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin s \sin(s-b)}} \quad . \qquad (27)$$

Interchanging B and C and b and c in (25) through
7), the resulting half angle formulas for angle C are:

$$\sin \tfrac{1}{2}C = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} \quad ; \qquad (28)$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\sin s \ \sin(s-c)}{\sin a \ \sin b}} \ ; \qquad (29)$$

$$\tan \tfrac{1}{2}C = \sqrt{\frac{\sin(s-a) \ \sin(s-b)}{\sin s \ \sin(s-c)}} \ [14;177]. \qquad (30)$$

A particular advantage in using the half angle formu-
s is that rules of species need not be referred to. An
ute value and an obtuse value will be obtained for half
the angle being sought. The obtuse value is rejected
nce this would make the whole angle greater than 180°.
e acute value of the half angle is doubled to obtain the
lution to the angle. Another advantage of the half angle
rmulas is their ease of computation with logarithms. A
sadvantage is their specialized application, that is, all
ree sides must be known. This is not really a disadvan-
ge since other formulas will be developed for solving
her cases.

II. FORMULAS FOR HALF SIDES

The formulas for half sides give the solution to
lf of any side of a spherical triangle when all three
gles of the triangle are known. Half of any side can be
und by the sine, cosine, or tangent functions. These formu-
s are particularly useful in a case II solution.

The derivation of the half side formula for side a
arts with formula (15) solved for cos a, that is

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} .$$

Then,

$$1 - \cos a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C} .$$

Using the addition identity $\cos(x+y) = \cos x \cos y -$ in $x \sin y$ on the numerator of the right hand side of the ast equation, the result is

$$1 - \cos a = \frac{-\cos(B+C) - \cos A}{\sin B \sin C} .$$

Using the product formula $\cos x + \cos y =$ cos $\frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$ on the numerator of the last equaon, the result is

$$1 - \cos a = \frac{-2 \cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C} .$$

But, $1 - \cos x = 2 \sin^2 \frac{1}{2}x$, so the last equation beomes

$$\sin^2 \tfrac{1}{2}a = \frac{-\cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C} .$$

This result will be condensed after expressions for os^2 $\frac{1}{2}a$ and $\tan^2 \frac{1}{2}a$ have been developed.

Proceeding as above it is seen that

$$1 + \cos a = \frac{\cos B \cos C + \sin B \sin C + \cos A}{\sin B \sin C} .$$

Using $\cos(x-y) = \cos x \cos y + \sin x \sin y$, the reult is

$$1 + \cos a = \frac{\cos(B-C) + \cos A}{\sin B \sin C} \quad ; \text{ or}$$

$$2 \cos^2 \tfrac{1}{2}a = \frac{2 \cos \frac{1}{2}(A+B-C) \cos \frac{1}{2}(B-C-A)}{\sin B \sin C} \quad ; \text{ which is}$$

$$\cos^2 \tfrac{1}{2}a = \frac{\cos \frac{1}{2}(A+B-C) \cos \frac{1}{2}(A+C-B)}{\sin B \sin C} .$$

Thus,

$$\tan^2 \tfrac{1}{2}a = \frac{-\cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\cos \frac{1}{2}(A+B-C) \cos \frac{1}{2}(A+C-B)} .$$

Let A+B+C = 2S,

$$\left.\begin{aligned} \text{then} \quad \tfrac{1}{2}(A+B+C) &= S, \\ \tfrac{1}{2}(B+C-A) &= S - A, \\ \tfrac{1}{2}(A+C-B) &= S - B, \\ \text{and} \quad \tfrac{1}{2}(A+B-C) &= S - C. \end{aligned}\right\} \tag{31}$$

If the appropriate formulas in group (31) are sub-ituted in the results obtained for $\sin^2 \frac{1}{2}a$, $\cos^2 \frac{1}{2}a$, and n^2 $\frac{1}{2}a$; and the principal square root of each side is taken, e resulting half side formulas for side a are:

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \; ; \tag{32}$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}} \; ; \tag{33}$$

$$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B) \cos(S-C)}} \; \left[15;210\right]. \tag{34}$$

By symmetry, that is interchanging A and B and a and in formulas (32) through (34), the half side formulas for de b are:

$$\sin \tfrac{1}{2}b = \sqrt{\frac{-\cos S \cos(S-B)}{\sin A \sin C}}\ ; \qquad (35)$$

$$\cos \tfrac{1}{2}b = \sqrt{\frac{\cos(S-A) \cos(S-C)}{\sin A \sin C}}\ ; \qquad (36)$$

$$\tan \tfrac{1}{2}b = \sqrt{\frac{-\cos S \cos(S-B)}{\cos(S-A) \cos(S-C)}}\ . \qquad (37)$$

Interchanging B and C and b and c in (35) through (37), the resulting half side formulas for side c are:

$$\sin \tfrac{1}{2}c = \sqrt{\frac{-\cos S \cos(S-C)}{\sin A \sin B}}\ ; \qquad (38)$$

$$\cos \tfrac{1}{2}c = \sqrt{\frac{\cos(S-A) \cos(S-B)}{\sin A \sin B}}\ ; \qquad (39)$$

$$\tan \tfrac{1}{2}c = \sqrt{\frac{-\cos S \cos(S-C)}{\cos(S-A) \cos(S-B)}}$$ [15;210-211]. (40)

The half side formulas have the same advantages and disadvantages stated for the half angle formulas, they include: all three angles must be known; the computation is readily carried out with logarithms; and the quadrant of the side found is known without reference to rules of species.

III. GAUSS'S EQUATIONS OR DELAMBRE'S ANALOGIES

The four equations,

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C, \qquad (41)$$

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C, \qquad (42)$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C, \qquad (43)$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C, \qquad (44)$$

are called Gauss's Equations, or Delambre's Analogies [3;153].

The derivation of (41) is as follows: By an addition identity,

$$\cos \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B,$$

and substituting in formulas (23), (26), (22), and (25), the result is:

$$\cos \tfrac{1}{2}(A+B) = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c} \cdot \frac{\sin s \sin(s-b)}{\sin a \sin c}} - \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c} \cdot \frac{\sin(s-a) \sin(s-c)}{\sin a \sin c}} ;$$

$$\text{or } \cos \tfrac{1}{2}(A+B) = \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} - \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} ;$$

$$\text{or } \cos \tfrac{1}{2}(A+B) = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} \left(\frac{\sin s - \sin(s-c)}{\sin c}\right) .$$

Upon using (28) and the identity $\sin x - \sin y = 2 \cos \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(s-y)$, the last equation simplifies to

$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \left(\frac{2 \cos \tfrac{1}{2}(s+s-c) \sin \tfrac{1}{2}(s-s+c)}{\sin c}\right) .$$

But, $\cos \tfrac{1}{2}(s+s-c) = \cos(s-\tfrac{1}{2}c)$, $\sin \tfrac{1}{2}(s-s+c) = \sin \tfrac{1}{2}c$, and $\sin c = 2 \sin \tfrac{1}{2}c \cos \tfrac{1}{2}c$. Thus,

$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \left(\frac{2 \cos(s-\tfrac{1}{2}c) \sin \tfrac{1}{2}c}{2 \sin \tfrac{1}{2}c \cos \tfrac{1}{2}c}\right) ; \text{ or}$$

$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \left(\frac{\cos(s-\tfrac{1}{2}c)}{\cos \tfrac{1}{2}c}\right) .$$

Also, $s - \tfrac{1}{2}c = \tfrac{1}{2}(a+b)$, and the last equation becomes

$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \left(\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \right) \; ; \text{ or}$$

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C,$$

ch is equation (41) [15;212].

Proceeding as above with $\sin \frac{1}{2}(A+B)$, the derivation (42) is as follows:

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \; ;$$

$$\text{in } \tfrac{1}{2}(A+B) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \cdot \frac{\sin s \sin(s-b)}{\sin a \sin c}} + \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c} \cdot \frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}} \; ;$$

$$\text{in } \tfrac{1}{2}(A+B) = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} \left(\frac{\sin(s-b) + \sin(s-a)}{\sin c} \right) ;$$

$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}C \left(\frac{2 \sin \frac{1}{2}(2s-a-b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \right);$$

$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}C \left(\frac{\sin \frac{1}{2}c \cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \right) \; ;$$

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C \; ,$$

ch is equation (42). Likewise,

$$\cos \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B + \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \; ;$$

$$\cos \tfrac{1}{2}(A-B) = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c} \cdot \frac{\sin s \sin(s-b)}{\sin a \sin c}} + \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \cdot \frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}} \; ;$$

$$\cos \tfrac{1}{2}(A-B) = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}} \left(\frac{\sin s + \sin(s-c)}{\sin c} \right) ;$$

$$\cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \left(\frac{2 \sin \frac{1}{2}(2s-c) \cos \frac{1}{2}c}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \right) \; ;$$

$$\cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \left(\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} \right) ;$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}C ,$$

ich is equation (43). As before,

$$\sin \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B - \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B ;$$

$$\sin \tfrac{1}{2}(A-B) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \cdot \frac{\sin s \sin(s-b)}{\sin a \sin c}} -$$

$$\sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c} \cdot \frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}} ;$$

$$\text{in } \tfrac{1}{2}(A-B) = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} \left(\frac{\sin(s-b) - \sin(s-a)}{\sin c} \right) ;$$

$$\sin \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2}C \left(\frac{2 \cos \frac{1}{2}(2s-a-b) \sin \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \right) ;$$

$$\sin \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2}C \left(\frac{\cos \frac{1}{2}c \sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \right) ;$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C ,$$

ich is equation (44). Notice that equations (41) through 4) each involve all six parts of spherical triangle ABC. is involvement of all six parts of a spherical triangle kes these formulas most useful in checking solutions of spheral triangles [6;194].

Mitra [11;60] states that equations (41) through (44) re discovered by Delambre in 1807, although not published til 1809, and are sometimes improperly called Gauss's Equaons. Hart [6;193], Taylor [14;182], and Crockett [3;153] give th [illegible] credit for the formulas. Wentworth [15;212] gives e credit entirely to Gauss.

IV. NAPIER'S ANALOGIES

Napier's Analogies provide a useful means for obtaining partial solution to cases III or IV. Involving five parts spherical triangle ABC, they are:

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}C \;; \qquad (45)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}C \;; \qquad (46)$$

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2}c \;; \qquad (47)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2}c \,. \qquad (48)$$

The derivation of formulas (45) through (48) is not all difficult having developed Gauss's Equations(Delamre's Analogies). For (45) is the result of dividing (42) (41), (46) is the result of dividing (44) by (43), (47) the result of dividing (43) by (41), and (48) is the esult of dividing (44) by (42). There are other forms for ne analogies of Napier, depending upon the parts of the riangle that are known [15;213]. For example, if A, b, and are the known parts, (45) can be written

$$\tan \tfrac{1}{2}(B+C) = \frac{\cos \tfrac{1}{2}(b-c)}{\cos \tfrac{1}{2}(b+c)} \cot \tfrac{1}{2}A \,,$$

nere the parts A, B, C, a, and b were replaced by the parts , C, A, b, and c respectively.

V. PROOF OF RULES OF SPECIES FOR OBLIQUE SPHERICAL TRIANGLES

On page 58 of Chapter III, two rules of species were ated with the admission that the proofs of these rules uld be given after the necessary formulas had been de- loped. These rules will be repeated and proofs given at is time.

Rule 1. If a side(an angle)differs more than another side(angle)from 90°, it is of the same species as its opposite angle(side).

Proof Of Rule 1. It is necessary to show that cos a d cos A have the same sign when the difference between a d 90° is numerically greater than the difference between and 90°. Solving (12),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

r cos A, the result is

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad .$$

The denominator of the right side of the last equa- on is always positive so that the sign of cos A and of s a - cos b cos c must be the same. But, $|90 - a| >$ $0 - b| \rightarrow |\cos a| > |\cos b| > |\cos b \cos c|$. Thus, e sign of the numerator of the fraction is determined by e sign of cos a. Then the sign of the fraction agrees with

ne signs of cos a and cos A so that a and A must be of
ne same species.

By a similar process, solving (15),

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

or cos a, the result is

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad .$$

The denominator of the fraction is always positive so hat the sign of cos a agrees with the sign of the numeraor. But, if $|90 - A| > |90 - B|$, then $|\cos A| > |\cos B| > |\cos B \cos C|$, and as before the sign of cos A determines he sign of the numerator and of the fraction. Therefore, and A are of the same species when A differs from 90° y more than B differs from 90° [3;153-154].

Rule 2. Half the sum of any two sides is of the same species as half the sum of the opposite angles.

<u>Proof Of Rule 2</u>. To show that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ re of the same species. From the first of Napier's Analgies,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C , \qquad (45)$$

t is seen that $\frac{1}{2}C$ and $\frac{1}{2}(a-b)$ must each be less than 90°, ince C and a are each less than 180°, so that $\cos \frac{1}{2}(a-b)$ nd $\cot \frac{1}{2}C$ are always positive. Thus, $\tan \frac{1}{2}(A+B)$ and os $\frac{1}{2}(a+b)$ must agree in sign. But, A+B and a+b must each less than 360°, so that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are each less an 180°. In order for $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ to each be less an 180° and tan $\frac{1}{2}(A+B)$ and cos $\frac{1}{2}(a+b)$ to agree in sign, A+B) and $\frac{1}{2}(a+b)$ must be of the same species [3;154].

VI. SOLUTIONS AND CHECKS BY SPECIAL METHODS

Two examples will be considered, one of case II and e of case VI, so that the half side, half angle, and pier's Analogies can be illustrated. Gauss's Equations ll be used to check the results.

Case II. Given all three angles. Let ABC be a spher-al triangle in which the following parts are known:

$$A = 100^\circ 50'; \ B = 80^\circ 50'; \ C = 74^\circ.$$

Half side formula (32),

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \quad ,$$

ll be used to solve for a. From the following formulas group (31):

$$S = \tfrac{1}{2}(A+B+C) \ , \quad S - A = \tfrac{1}{2}(A+B-C) \ ,$$

is seen that $S = 127^\circ 50'$ and $S - A = 27^\circ$. The computa-on is:

$$\begin{aligned}
\text{(n) } \log\cos S &= \log\cos 52^\circ 10' &&= 9.7877 - 10\\
\log\cos(S-A) &= \log\cos 27^\circ &&= 9.9499 - 10\\
\text{colog}\sin B &= \text{colog}\sin 80^\circ 50' &&= 0.0056\\
\text{colog}\sin C &= \text{colog}\sin 74^\circ &&= 0.0172\\
&\quad 2\log\sin\tfrac{1}{2}a &&= 19.7604 - 20\\
&\quad \log\sin\tfrac{1}{2}a &&= 9.8802 - 10\\
&\quad \tfrac{1}{2}a &&= 49^\circ 22' \text{ or } \cancel{130^\circ 38'}.
\end{aligned}$$

Napier's Analogies, equations (47) and (48) in terms

b and c,

$$\tan\tfrac{1}{2}(b+c) = \frac{\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}\tan\tfrac{1}{2}a, \text{ and}$$

$$\tan\tfrac{1}{2}(b-c) = \frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}\tan\tfrac{1}{2}a,$$

ll be used to solve for b and c. The computation is:

$$\begin{aligned}
\log\cos\tfrac{1}{2}(B-C) &= \log\cos 3^\circ 25' &&= 9.9992 - 10\\
\text{colog}\cos\tfrac{1}{2}(B+C) &= \text{colog}\cos 77^\circ 25' &&= 0.6618\\
\log\tan\tfrac{1}{2}a &= \log\tan 49^\circ 22' &&= 0.0664\\
&\quad \log\tan\tfrac{1}{2}(b+c) &&= 0.7274\\
&\quad \tfrac{1}{2}(b+c) &&= 79^\circ 23' \text{ or } \cancel{100^\circ 37'}.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B-C) &= \log\sin 3^\circ 25' &&= 8.7751 - 10\\
\text{colog}\sin\tfrac{1}{2}(B+C) &= \text{colog}\sin 77^\circ 25' &&= 0.0106\\
\log\tan\tfrac{1}{2}a &= \log\tan 49^\circ 22' &&= 0.0664\\
&\quad \log\tan\tfrac{1}{2}(b-c) &&= 8.8521 - 10\\
&\quad \tfrac{1}{2}(b-c) &&= 4^\circ 4' \text{ or } \cancel{175^\circ 56'}.
\end{aligned}$$

Thus, $b = \frac{1}{2}(b+c) + \frac{1}{2}(b-c) = 83^\circ 27'$, and

$$c = \tfrac{1}{2}(b+c) - \tfrac{1}{2}(b-c) = 75^\circ 19'.$$

Therefore, the solution to triangle ABC is:

$a = 98^\circ 44'$; $b = 83^\circ 27'$; and $c = 75^\circ 19'$.

Using the fourth equation of Gauss, (44),

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C,$$

he check is as follows:

$$\begin{aligned} \log \sin \tfrac{1}{2}(A-B) &= \log \sin 10^\circ &&= 9.2397 - 10 \\ \log \sin \tfrac{1}{2}c &= \log \sin 37^\circ 40' &&= \underline{9.7861 - 10} \\ & && \quad 9.0258 - 10 \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b) &= \log \sin 7^\circ 38' &&= 9.1233 - 10 \\ \log \cos \tfrac{1}{2}C &= \log \cos 37^\circ &&= \underline{9.9023 - 10} \\ & && \quad 9.0256 - 10. \end{aligned}$$

This completes the check and the problem.

<u>Case VI. Given two angles and the side opposite one</u> <u>f them.</u> Let ABC be a spherical triangle in which the ollowing parts are known:

$A = 104^\circ 40'$; $B = 80^\circ 15'$; $a = 126^\circ 50'$.

From the law of sines,

$$\sin b = \sin a \sin B / \sin A,$$

he computation for b is as follows:

$$\begin{aligned} \log \sin a &= \log \sin 53^\circ 10' &&= 9.9033 - 10 \\ \log \sin B &= \log \sin 80^\circ 15' &&= 9.9937 - 10 \\ \text{colog} \sin A &= \text{colog} \sin 75^\circ 20' &&= \underline{0.0144} \\ & \log \sin b &&= 9.9114 - 10 \\ & b &&= 54^\circ 38' \text{ or } \\ & && \quad 125^\circ 22'. \end{aligned}$$

Now to check for the number of solutions. $B < A$ mplies that b must be less than a and it is seen that both lues of b, $54^\circ 38'$ and $125^\circ 22'$, are less than a, $126^\circ 50'$, two solutions will be possible. The first rule of ecies is of no help since B is closer to 90° than is A. e second rule supports the conclusion that two solutions e possible since $\frac{1}{2}(A+B)$ is of the same species as $\frac{1}{2}(a+b)$ r either value of b.

Let $b_1 = 54^\circ 38'$ and $b_2 = 125^\circ 22'$, so the two solu-ons can be referred to without confusion. Two of Napier's alogies, (46) and (48), solved for cot $\frac{1}{2}C$ and tan $\frac{1}{2}c$ will used to compute C and c in each solution.

$$\cot \tfrac{1}{2}C = \tan \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}(a+b)/\sin \tfrac{1}{2}(a-b) \qquad (46)$$

$$\tan \tfrac{1}{2}c = \tan \tfrac{1}{2}(a-b) \sin \tfrac{1}{2}(A+B)/\sin \tfrac{1}{2}(A-B) \qquad (48)$$

In preparation:

$A = 104^\circ 40'$;	$a = 126^\circ 50'$;	$a = 126^\circ 50'$;
$B = 80^\circ 15'$;	$b_1 = 54^\circ 38'$;	$b_2 = 125^\circ 22'$;
$A+B = 184^\circ 55'$;	$a+b_1 = 181^\circ 28'$;	$a+b_2 = 252^\circ 12'$;
$A-B = 24^\circ 25'$;	$a-b_1 = 72^\circ 12'$;	$a-b_2 = 1^\circ 28'$;
$\frac{1}{2}(A+B) = 92^\circ 28'$;	$\frac{1}{2}(a+b_1) = 90^\circ 44'$;	$\frac{1}{2}(a+b_2) = 126^\circ 6'$;
$\frac{1}{2}(A-B) = 12^\circ 12'$;	$\frac{1}{2}(a-b_1) = 36^\circ 6'$;	$\frac{1}{2}(a-b_2) = 0^\circ 44'$.

In the first solution, using $b_1 = 54^\circ 38'$, the com-utation for C_1 and c_1 is:

$$\begin{aligned}
\log \tan \tfrac{1}{2}(A-B) &= \log \tan 12^\circ 12' = 9.3348 - 10 \\
\log \sin \tfrac{1}{2}(a+b_1) &= \log \sin 89^\circ 16' = 10.0000 - 10 \\
\text{colog} \sin \tfrac{1}{2}(a-b_1) &= \text{colog} \sin 36^\circ 6' = 0.2297 \\
& \log \cot \tfrac{1}{2}C_1 = 9.5645 - 10 \\
& \tfrac{1}{2}C_1 = 69^\circ 51' \\
& C_1 = 139^\circ 42'.
\end{aligned}$$

$$\begin{aligned}
\log \tan \tfrac{1}{2}(a-b_1) &= \log \tan 36^\circ 6' &&= 9.8629 - 10 \\
\log \sin \tfrac{1}{2}(A+B) &= \log \sin 87^\circ 32' &&= 9.9996 - 10 \\
\text{colog} \sin \tfrac{1}{2}(A-B) &= \text{colog} \sin 12^\circ 12' &&= \underline{0.6750} \\
&\quad \log \tan \tfrac{1}{2}c_1 &&= 0.5375
\end{aligned}$$

$$\tfrac{1}{2}c_1 = 73^\circ 49'$$

$$c_1 = 147^\circ 38'.$$

In the second solution, using $b_2 = 125^\circ 22'$, the computation for C_2 and c_2 is:

$$\begin{aligned}
\log \tan \tfrac{1}{2}(A-B) &= \log \tan 12^\circ 12' &&= 9.3348 - 10 \\
\log \sin \tfrac{1}{2}(a+b_2) &= \log \sin 53^\circ 54' &&= 9.9074 - 10 \\
\text{colog} \sin \tfrac{1}{2}(a-b_2) &= \text{colog} \sin 0^\circ 44' &&= \underline{1.8928} \\
&\quad \log \cot \tfrac{1}{2}C_2 &&= 1.1350
\end{aligned}$$

$$\tfrac{1}{2}C_2 = 4^\circ 11'$$

$$C_2 = 8^\circ 22'.$$

$$\begin{aligned}
\log \tan \tfrac{1}{2}(a-b_2) &= \log \tan 0^\circ 44' &&= 8.1072 - 10 \\
\log \sin \tfrac{1}{2}(A+B) &= \log \sin 87^\circ 32' &&= 9.9996 - 10 \\
\text{colog} \sin \tfrac{1}{2}(A-B) &= \text{colog} \sin 12^\circ 12' &&= \underline{0.6750} \\
&\quad \log \tan \tfrac{1}{2}c_2 &&= 8.7818 - 10
\end{aligned}$$

$$\tfrac{1}{2}c_2 = 3^\circ 28'$$

$$c_2 = 6^\circ 56'.$$

Therefore, the two solutions to triangle ABC are:

$b_1 = 54^\circ 38'$; $C_1 = 139^\circ 42'$; $c_1 = 147^\circ 38'$; and

$b_2 = 125^\circ 22'$; $C_2 = 8^\circ 22'$; $c_2 = 6^\circ 56'$.

The checks will each be made with equation (44) of Gauss,

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}C.$$

The computation of the check for solution one is:

$$\begin{aligned}
\log \sin \tfrac{1}{2}(A-B) &= \log \sin 12^\circ 12' &&= 9.3250 - 10 \\
\log \sin \tfrac{1}{2}c_1 &= \log \sin 73^\circ 49' &&= \underline{9.9825 - 10} \\
& && 9.3075 - 10
\end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b_1) &= \log \sin 36^\circ 6' &= 9.7703 - 10 \\ \log \cos \tfrac{1}{2}C_1 &= \log \cos 69^\circ 51' &= \underline{9.5372 - 10} \\ & & 9.3075 - 10. \end{aligned}$$

This completes the check of solution one. The com-tation of the check for solution two is:

$$\begin{aligned} \log \sin \tfrac{1}{2}(A-B) &= \log \sin 12^\circ 12' = 9.3250 - 10 \\ \log \sin \tfrac{1}{2}c_2 &= \log \sin \ \ 3^\circ 28' = \underline{8.7815 - 10} \\ & \qquad\qquad\qquad\qquad 8.1065 - 10 \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b_2) &= \log \sin 0^\circ 44' = 8.1072 - 10 \\ \log \cos \tfrac{1}{2}C_2 &= \log \cos 4^\circ 11' = \underline{9.9988 - 10} \\ & \qquad\qquad\qquad\qquad 8.1060 - 10 \end{aligned}$$

The check, although not in complete agreement, is ry close considering the size of the angles and the great riance of the sine function for angles near zero degrees.

VIII. SPHERICAL EXCESS

The spherical excess, E, of a spherical triangle ABC number of degrees by which the angles of the triangle ceed 180°. Thus,

$$E = A+B+C-180^\circ \quad . \qquad (49)$$

Let H be the area of spherical triangle ABC. In herical geometry, it is proved that H is equal to the area a lune with angle $\frac{1}{2}E$. But the area of a lune is propor-on to the surface area of the sphere as its angle, $\frac{1}{2}E$, is 360°. The surface area of a sphere of radius r being πr^2, the following results are obtained:

$$\frac{H}{4\pi r^2} = \frac{\frac{1}{2}E}{360} \quad , \text{ or}$$

$$H = \pi r^2 E/180 \ [6;199]. \qquad (50)$$

As an example, suppose $A = 120^\circ$, $B = 100^\circ$, and $C = 85^\circ$ a sphere of radius 15 in. Then,

$$E = 120^\circ + 100^\circ + 85^\circ - 180^\circ = 125^\circ, \text{ and}$$

$$H = \pi(15)^2\ (125)/180 = 156.25\pi \text{ sq. in.}$$

Therefore, the area of this spherical triangle is 6.25π square inches.

In (50), E is expressed in degrees. If each angle triangle ABC were expressed in radians, formula (49) uld be

$$E = A+B+C-\pi,$$

ere E would be in radians; and formula (50) would become

$$\frac{H}{4\pi r^2} = \frac{\frac{1}{2}E}{2\pi}, \text{ or}$$

$$H = r^2E. \qquad (51)$$

This final equation, which is equivalent to (50), referred to as Girard's Theorem by Mitra [11;115-116].

It is possible, knowing the radius of the sphere, compute the area of a spherical triangle in any of cases through VI by first solving the triangle for all of its gles. This is not, in most cases, a very efficient method. st of the rest of this chapter will be devoted to devel- ing formulas that can be used to compute the spherical ex- ss of a triangle directly from the given parts, at least the cases of the given being the three sides or the given ing the two sides and the included angle. From the spher-al excess the area is easily computed by the method above.

uillier's Theorem, Spherical Excess From The Three Sides

Lhuillier's Theorem,

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \ \tan \tfrac{1}{2}(s-a) \ \tan \tfrac{1}{2}(s-b) \ \tan \tfrac{1}{2}(s-c)} \quad , (52)$$

termines the spherical excess of a spherical triangle from e three sides of the triangle. The derivation of formula 2) is as follows:

$$\tan \tfrac{1}{4}E = \frac{\sin \tfrac{1}{4}E}{\cos \tfrac{1}{4}E} = \frac{\sin \tfrac{1}{4}(A+B+C-180^\circ)}{\cos \tfrac{1}{4}(A+B+C-180^\circ)} \quad ;$$

$$\tan \tfrac{1}{4}E = \frac{\sin \tfrac{1}{4}(A+B+C-180)}{\cos \tfrac{1}{4}(A+B+C-180)} \cdot \frac{2 \cos \tfrac{1}{4}(A+B+180^\circ-C)}{2 \cos \tfrac{1}{4}(A+B+180^\circ-C)} \quad ;$$

ich, upon using the product formulas

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y) \quad \text{and}$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y) \quad , \text{ becomes}$$

$$\tan \tfrac{1}{4}E = \frac{\sin \tfrac{1}{2}(A+B) + \sin \tfrac{1}{2}(C-180^\circ)}{\cos \tfrac{1}{2}(A+B) + \cos \tfrac{1}{2}(C-180^\circ)} \quad ; \text{ or}$$

$$\tan \tfrac{1}{4}E = \frac{\sin \tfrac{1}{2}(A+B) - \sin \tfrac{1}{2}(180^\circ-C)}{\cos \tfrac{1}{2}(A+B) + \cos \tfrac{1}{2}(180-C)} \quad .$$

But, $\sin(90-x) = \cos x$ and $\cos(90-x) = \sin x$, so that

$$\tan \tfrac{1}{4}E = \frac{\sin \tfrac{1}{2}(A+B) - \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}(A+B) + \sin \tfrac{1}{2}C} \quad .$$

The problem is to get $\tan \tfrac{1}{4}E$ expressed in terms of b, and c. Using formulas (41) and (42) of Gauss to sub-itute in for $\sin \tfrac{1}{2}(A+B)$ and $\cos \tfrac{1}{2}(A+B)$, the result is:

$$\tan \tfrac{1}{4}E = \frac{\dfrac{\cos \frac{1}{2}(a-b) \cos \frac{1}{2}C}{\cos \frac{1}{2}c} - \cos \frac{1}{2}C}{\dfrac{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}C}{\cos \frac{1}{2}c} + \sin \frac{1}{2}C} \quad ; \quad \text{or}$$

$$\tan \tfrac{1}{4}E = \frac{\cos \frac{1}{2}(a-b) - \cos \frac{1}{2}c}{\cos \frac{1}{2}(a+b) + \cos \frac{1}{2}c} \cdot \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C} \quad ;$$

ich, upon using the product formulas, becomes

$$\tan \tfrac{1}{4}E = \frac{-2 \sin \frac{1}{4}(a-b+c) \sin \frac{1}{4}(a-b-c)}{2 \cos \frac{1}{4}(a+b+c) \cos \frac{1}{4}(a+b-c)} \cdot \cot \tfrac{1}{2}C \quad .$$

Dividing numerator and denominator by two, recalling

n(-x) = -sin x, $s - b = \frac{1}{2}(a+c-b)$, $s - a = \frac{1}{2}(b+c-a)$,

$- c = \frac{1}{2}(a+b-c)$, and $s = \frac{1}{2}(a+b+c)$, the result is:

$$\tan \tfrac{1}{4}E = \frac{\sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-a)}{\cos \frac{1}{2}s \cos \frac{1}{2}(s-c)} \cdot \cot \tfrac{1}{2}C \ .$$

Using the reciprocal of formula (30) of the half

gle formulas for cot $\frac{1}{2}C$, squaring both sides, and using

e identity sin 2x = 2 sin x cos x, the result is:

$$\tan^2 \tfrac{1}{4}E = \frac{\sin^2 \frac{1}{2}(s-b) \sin^2 \frac{1}{2}(s-a)}{\cos^2 \frac{1}{2}s \cos^2 \frac{1}{2}(s-c)} \cdot$$

$$\frac{2 \sin \frac{1}{2}s \cos \frac{1}{2}s \cdot 2 \sin \frac{1}{2}(s-c) \cos \frac{1}{2}(s-c)}{2 \sin \frac{1}{2}(s-a) \cos \frac{1}{2}(s-a) \cdot 2 \sin \frac{1}{2}(s-b) \cos \frac{1}{2}(s-b)} \quad ; \text{ or}$$

$$\text{n}^2 \tfrac{1}{4}E = \frac{\sin \frac{1}{2}s \cdot \sin \frac{1}{2}(s-a) \cdot \sin \frac{1}{2}(s-b) \cdot \sin \frac{1}{2}(s-c)}{\cos \frac{1}{2}s \cdot \cos \frac{1}{2}(s-a) \cdot \cos \frac{1}{2}(s-b) \cdot \cos \frac{1}{2}(s-c)} \quad ; \text{ or}$$

$$\tan \tfrac{1}{4}E = \pm\sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}$$

3;181-182].

Examples using Lhuillier's Theorem. Let ABC be a herical triangle in which

$$a = 49^\circ 20', \; b = 38^\circ 40', \text{ and } c = 63^\circ 30'.$$

Then, $s = \frac{1}{2}(a+b+c) = 75^\circ 45'$; $\frac{1}{2}s = 37^\circ 52'$;

$- a = 26^\circ 25'$; $s - b = 37^\circ 5'$; $s - c = 12^\circ 15'$;

$s-a) = 13^\circ 12'$; $\frac{1}{2}(s-b) = 18^\circ 32'$; $\frac{1}{2}(s-c) = 6^\circ 8'$.

Using Lhuillier's Theorem,

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)} \;,$$

e computation is:

$$\begin{aligned}
\log \tan \tfrac{1}{2}s &= \log \tan 37^\circ 52' = 9.8907 - 10 \\
\log \tan \tfrac{1}{2}(s-a) &= \log \tan 13^\circ 12' = 9.3702 - 10 \\
\log \tan \tfrac{1}{2}(s-b) &= \log \tan 18^\circ 32' = 9.5253 - 10 \\
\log \tan \tfrac{1}{2}(s-c) &= \log \tan 6^\circ 8' = \underline{9.0312 - 10} \\
2 \log \tan \tfrac{1}{4}E &= 17.8174 - 20 \\
\log \tan \tfrac{1}{4}E &= 8.9087 - 10 \\
\tfrac{1}{4}E &= 4^\circ 38'. \\
E &= 18^\circ 32'.
\end{aligned}$$

us, the spherical excess of triangle ABC is $18^\circ 32'$.

Suppose triangle ABC is on a sphere of radius six eet. Then, the area, H, of triangle ABC is given by

$$H = \pi r^2 E/180 \; ; \text{ or}$$

$$H = (3.1416)(6^2)(18.53)/180 = 11.64 \text{ sq. ft.}$$

gnoli's Theorem, Spherical Excess From The Three Sides

Cagnoli's Theorem,

$$\sin \tfrac{1}{2}E = \sqrt{\frac{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}{2 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}} \;, \qquad (53)$$

so determines the spherical excess of a spherical triangle om the three sides of the triangle. The derivation of e formula is as follows:

From

$$\tfrac{1}{2}(A+B+C) = S \text{ and } A+B+C-\pi = E,$$

$$\sin \tfrac{1}{2}E = \sin(S-\tfrac{1}{2}\pi) = -\sin(\tfrac{1}{2}\pi-S) = -\cos S \text{ ; or}$$

$$\sin \tfrac{1}{2}E = -\cos \tfrac{1}{2}(A+B+C) =$$

$$\sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}C.$$

e last expression was obtained by using an addition identi-. Replacing $\sin \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(A+B)$ by their equiva-nts from Gauss's Equations, the result is:

$$\sin \tfrac{1}{2}E = \frac{\cos \frac{1}{2}(a-b) \cos \frac{1}{2}C \sin \frac{1}{2}C}{\cos \frac{1}{2}c} -$$

$$\frac{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}c} \; ;$$

$$\sin \tfrac{1}{2}E = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}c} \left(\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2}(a+b) \right) \; ;$$

ich, upon using a product formula on the expression in-de the brackets, becomes

$$\sin \tfrac{1}{2}E = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}c} \left(-2 \sin \tfrac{1}{2}a \sin(-\tfrac{1}{2}b) \right) \; ;$$

, substituting the half angle formulas for $\sin \frac{1}{2}C$ and s $\frac{1}{2}C$ and simplifying, is

$$\sin \tfrac{1}{2}E = \sqrt{\frac{\sin(s-a) \sin(s-b) \sin s \sin(s-c)}{\sin a \sin b \sin a \sin b}} \cdot$$

$$\left(\frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} \right) \; ;$$

$$\sin \tfrac{1}{2}E = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b}{\sin a \sin b \cos \frac{1}{2}c} \; .$$

$$\sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)} \; .$$

But, $\sin 2x = 2 \sin x \cos x$, so that

$$\text{in } \tfrac{1}{2}E = \frac{2\sin \frac{1}{2}a \sin \frac{1}{2}b \sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{2 \sin \frac{1}{2}a \cos \frac{1}{2}a \cdot 2 \sin \frac{1}{2}b \cos \frac{1}{2}b \cos \frac{1}{2}c} \; ,$$

hich reduces to formula (53) [11;118].

<u>Example using Cagnoli's Theorem.</u> Let ABC be a spheri-al triangle in which

$$a = 110°4', \; b = 74°32', \text{ and } c = 56°30'$$

Then, $s = 120°33'$; $s - a = 10°29'$; $s - b = 46°1'$; $- c = 64°3'$; $\frac{1}{2}a = 55°2'$; $\frac{1}{2}b = 37°16'$; $\frac{1}{2}c = 28°15'$.

Using Cagnoli's Theorem,

$$\sin \tfrac{1}{2}E = \frac{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \; ,$$

he computation is:

$$\begin{aligned}
\log \sin s &= \log \sin 59°27' &&= 9.9351 - 10 \\
\log \sin (s-a) &= \log \sin 10°29' &&= 9.2599 - 10 \\
\log \sin (s-b) &= \log \sin 46°1' &&= 9.8570 - 10 \\
\log \sin (s-c) &= \log \sin 64°3' &&= \underline{9.9539 - 10} \\
& 2 \log(\text{numerator}) &&= 19.0059 - 20 \\
\\
& \log(\text{numerator}) &&= 9.5030 - 10 \\
& \text{colog } 2 &&= 9.6990 - 10 \\
\text{colog} \cos \tfrac{1}{2}a &= \text{colog} \cos 55°2' &&= 0.2418 \\
\text{colog} \cos \tfrac{1}{2}b &= \text{colog} \cos 37°16' &&= 0.0992 \\
\text{colog} \cos \tfrac{1}{2}c &= \text{colog} \cos 28°15' &&= \underline{0.0550} \\
& \log \sin \tfrac{1}{2}E &&= 9.5980 - 10 \\
\\
& \tfrac{1}{2}E &&= 23°21' \\
\\
& E &&= 46°42'.
\end{aligned}$$

Thus, the spherical excess of triangle ABC is °42'. If the radius of the sphere was known, the area the triangle could be computed using the formula

$$H = \pi r^2 E/180 \ .$$

herical Excess In Terms Of Two Sides And Their Included gle.

Let ABC be a spherical triangle in which the measures a, b, and C are known. To show that

$$\tan \tfrac{1}{2}E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C} \ . \qquad (54)$$

From the definition of E,

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}(A+B+C-180^\circ)}{\cos \frac{1}{2}(A+B+C-180^\circ)} \ ;$$

ich upon using $\sin(x-90^\circ) = -\sin(90^\circ-x) = -\cos x$,

$$\cos(x-90^\circ) = \cos(90-x) = \sin x \ ,$$

d the addition formulas, becomes

$$\tan \tfrac{1}{2}E = \frac{-\cos \frac{1}{2}(A+B+C)}{\sin \frac{1}{2}(A+B+C)} =$$

$$\frac{\sin \frac{1}{2}(A+B) \sin \frac{1}{2}C - \cos \frac{1}{2}(A+B) \cos \frac{1}{2}C}{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}C + \cos \frac{1}{2}(A+B) \sin \frac{1}{2}C} \ .$$

Using Gauss's Equations to substitute in for n $\frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(A+B)$, the result is:

$$n \ \tfrac{1}{2}E = \frac{\dfrac{\cos \frac{1}{2}(a-b) \cos \frac{1}{2}C \sin \frac{1}{2}C}{\cos \frac{1}{2}c} - \dfrac{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}c}}{\dfrac{\cos \frac{1}{2}(a-b) \cos \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}c} + \dfrac{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}C \sin \frac{1}{2}C}{\cos \frac{1}{2}c}}$$

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C \left(\cos \frac{1}{2}(a-b) - \cos \frac{1}{2}(a+b)\right)}{\cos \frac{1}{2}(a-b) \cos^2 \frac{1}{2}C + \cos \frac{1}{2}(a+b) \sin^2 \frac{1}{2}C} .$$

The identities,

$$\cos^2 \tfrac{1}{2}C = \tfrac{1}{2}(1+\cos C) \text{ and } \sin^2 \tfrac{1}{2}C = \tfrac{1}{2}(1-\cos C) \quad ,$$

d the product formulas make the last result

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C \left(\cos \frac{1}{2}(a-b) - \cos \frac{1}{2}(a+b)\right)}{\cos \frac{1}{2}(a-b)\left(\frac{1}{2}(1+\cos C)\right) + \cos \frac{1}{2}(a+b)\left(\frac{1}{2}(1-\cos C)\right)} ;$$

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C \left(-2 \sin \frac{1}{2}a \sin \frac{1}{2}(-b)\right)}{\frac{1}{2}\left(\cos\frac{1}{2}(a-b)+\cos\frac{1}{2}(a+b)\right) + \frac{1}{2}\left(\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)\right)\cos C} ;$$

$$\tan \tfrac{1}{2}E = \frac{2 \sin \frac{1}{2}C \cos \frac{1}{2}C \left(\sin \frac{1}{2}a \sin \frac{1}{2}b\right)}{\cos \frac{1}{2}a \cos \frac{1}{2}(-b) + \left(-\sin \frac{1}{2}a \sin \frac{1}{2}(-b)\right) \cos C} ;$$

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin C}{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C} ;$$

ere $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, and $\sin 2x =$ $\sin x \cos x$ were used to simplify the result. Dividing ch term of the fraction by $\cos \frac{1}{2}a \cos \frac{1}{2}b$, the desired re- lt is formula (54) [3;182].

<u>Example--Given two sides and the included angle.</u>

t ABC be a spherical triangle in which

$$a = 83^\circ 20'; \; b = 125^\circ 40'; \text{ and } C = 110^\circ 38' .$$

Using formula (54),

$$\tan \tfrac{1}{2}E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C} ,$$

computation is:

$$
\begin{aligned}
\log \tan \tfrac{1}{2}a &= \quad \log \tan 41^\circ 40' = 9.9494 - 10 \\
\log \tan \tfrac{1}{2}b &= \quad \log \tan 62^\circ 50' = 0.2897 \\
\log \cos C &= (n) \log \cos 69^\circ 22' = 9.5470 - 10 \\
\log(\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \sin C) &= 9.7861 - 10
\end{aligned}
$$

$$
\begin{aligned}
\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \sin C &= -0.6111 \\
1 + \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \sin C &= 0.3889
\end{aligned}
$$

$$
\begin{aligned}
\log 0.3889 &= 9.5898 - 10 \\
\text{colog } 0.3889 &= 0.4102
\end{aligned}
$$

$$
\begin{aligned}
\log \tan \tfrac{1}{2}a = \log \tan 41^\circ 40' &= 9.9494 - 10 \\
\log \tan \tfrac{1}{2}b = \log \tan 62^\circ 50' &= 0.2897 \\
\log \sin C = \log \sin 69^\circ 22' &= 9.9712 - 10 \\
\text{colog } 0.3889 &= 0.4102 \\
\log \tan \tfrac{1}{2}E &= 0.6205 \\
\tfrac{1}{2}E &= 76^\circ 32' \\
E &= 153^\circ 4'.
\end{aligned}
$$

us, the spherical excess of triangle ABC is $153^\circ 4'$.

Suppose triangle ABC is on a sphere of diameter 18 ches. Again using the formula $H = \pi r^2 E/180$, the area of iangle ABC is:

$$H = 3.1416(9^2)(153.07)/180 = 216.40 \text{ sq. in.}$$

$$
\begin{aligned}
\log 3.1416 &= 0.49715 \\
\log 81 &= 1.90849 \\
\log 153.07 &= 2.18489 \\
\text{colog } 180 &= 7.74473 - 10 \\
\log H &= 2.33526 \\
H &= 216.40
\end{aligned}
$$

In the next chapter some of the applications of herical trigonometry will be examined. The applications spherical trigonometry are so many in number and some so tensive that only a few will be examined and some to a ry limited degree.

CHAPTER V.

APPLICATIONS OF SPHERICAL TRIGONOMETRY

The applications of spherical trigonometry are large
n number and include many fields. Perhaps the most commonly
iown applications are those of distances on the earth, navi-
ation, and astronomy. These applications as well as some
the not so commonly known applications will be examined.
ome examples will be given to illustrate these applications.

I. LATITUDE AND LONGITUDE

In Plane Analytical Geometry, it is customary to give
ne location of any point in a plane by a pair of coordinates
nich indicate distances and directions from the coordinate
xes. A similar procedure is followed for locating any
oint on the earth's surface.

Let A be any point on the
arth's surface, which can be
ssumed to be a sphere of radi-
s 3959 miles. (Figure 27) Let N
epresent the north pole, S the
outh pole, O the center of the
arth, and great circle WBCE
epresent the equator. The

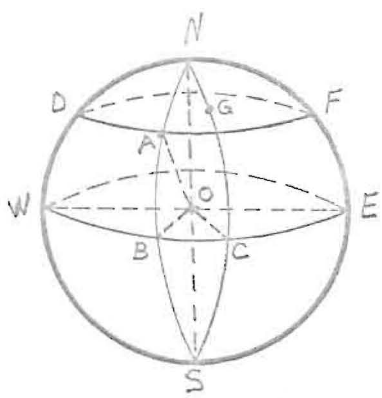


Figure 27

oles of the equator are N and S. For an observer standing

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I. LATITUDE AND LONGITUDE

In Plane Analytical Geometry, it is customary to give the location of any point in a plane by a pair of coordinates which indicate distances and directions from the coordinate axes. A similar procedure is followed for locating any point on the earth's surface.

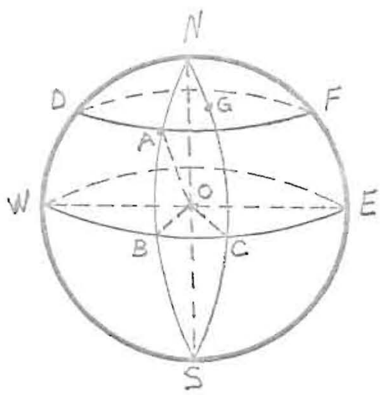

Figure 27

Let A be any point on the earth's surface, which can be assumed to be a sphere of radius 3959 miles. (Figure 27) Let N represent the north pole, S the south pole, O the center of the earth, and great circle WBCE represent the equator. The poles of the equator are N and S. For an observer standing

A and facing N, the direction of N is north and that of is south, while the perpendicular direction to the right called east and the opposite direction is called west. e meridian circles are those great circles containing and S. The <u>meridian</u> of a point A is the semicircle NABS ving N and S as endpoints. The <u>prime meridian</u> is the ridian NGCS containing G, Greenwich, England. The locaon of any point on the earth's surface is given in referce to Greenwich for its east-west location and in referce to the equator for its north-south location. The ordinates of a point on the earth's surface are called e longitude and latitude of the point.

The <u>longitude</u> of a point A is the angle of intersecon of the meridian of A and of the prime meridian of G, eenwich. Longitude is measured from 0° to 180° incluve, east or west of Greenwich. The longitude is the smallt angle of intersection of the planes of the meridians of and G, which is equal to the measure of angle BOC or of e minor arc BC on the equator.

<u>Latitude</u> is the angular measure of the arc of the ridian of A from the equator to A, AB. Angle AOB also ves the latitude of A. Latitude is measured from 0° to $^\circ$, north or south of the equator. The <u>co-latitude</u> of a int A is the complement of the latitude of A, or the angu-

r measure of arc AN or that of angle AON. All points on ne earth's surface having the same latitude lie on a small ircle called a <u>parallel</u> <u>of</u> <u>latitude</u>. Circle ADF is a paral- el of latitude through A in Figure 27 [6;201-202].

<u>Nautical</u> <u>mile</u> <u>and</u> <u>knot</u>. A nautical mile is defined s the length of an arc of one minute measured along a great ircle of the earth. A nautical mile is approximately 6080 eet or 1.15 ordinary land miles.

A knot is a rate of speed and indicates a movement f one nautical mile per hour [8;122]. For example, an air- lane flying at 200 knots is flying approximately 230 miles er hour.

<u>Bearing</u> <u>and</u> <u>azimuth</u>. Let NAS be the meridian through point A. (Figure 28) The azimuth r course of a point, as B, from is the spherical angle NAB easured in a clockwise direc- ion using AN as the initial ide. Azimuth is an angle easured from 0° to 360°, in- luding 0° but not 360°. Thus, he approximate azimuth of B from is 60°; of C from A is 130°; nd of E from A is 280°. All arcs are arcs of great circles.

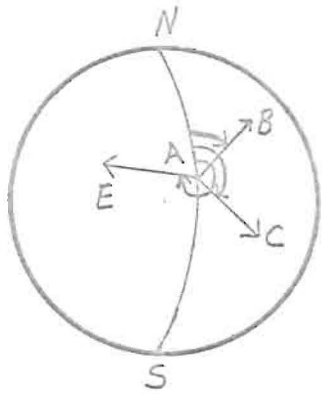

Figure 28

The bearing of a point B from A is defined to be the
ortest great circle path from A to B[6;202-203]. Bearing
given as the acute spherical angle from either N or S to
Using the approximations of azimuths, the bearing of B
om A is N 60° E; of C from A is S 50° E; and of E from A
N 80° W. The azimuth of a point is more commonly given
an its bearing[8;176].

II. GREAT CIRCLE DISTANCES

The shortest distance between any two points A and B
the earth's surface is the length of the minor arc of the
eat circle connecting the two points. In angular measure
is distance cannot exceed 180°, and the distance in nauti-
l miles is equal to the angular measure expressed in minutes.
eat circle distances are of particular value in geography
d navigation, whether by sea or by air. Also of value is
e course or azimuth of any point on the great circle route,
rticularly at A and at B. The initial course is given by
e azimuth at A and the final course is given by the azimuth
the direction at B[6;203]. The following example will
lustrate how spherical trigonometry can be used to compute
eat circle distance as well as the initial and final course
two points.

Example. To compute the great circle distance in naucal miles, the initial course, and the final course in aveling from Sydney to San Francisco if the latitude and ngitude of the two cities are:

Sydney($33°52'$ S, $151°12'$ E);

San Francisco($37°47'$ N, $122°26'$ W).

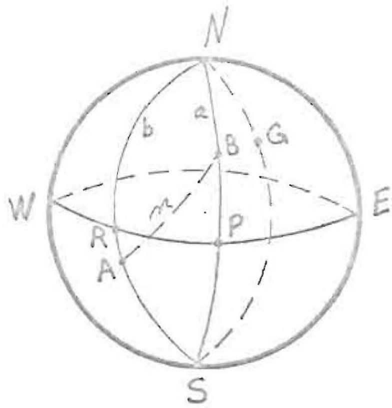

Figure 29

The numerical information s obtained from Hart [6;204].

Let A represent Sydney d B represent San Francisco. igure 29) Then, in spherical iangle ABN,

$$AN = 33°52' + 90° = 123°52' ;$$

$$ANB = 360° - (151°12' + 122°26') = 86°22' ;$$

$$BN = 90° - 37°47' = 52°13' ;$$

AB = great circle distance between A and B ;

NAB = initial course ;

ABN = 180° - (final course from A to B) .

Napier's first two analogies,

$$\tan \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}N / \cos \tfrac{1}{2}(a+b) \text{ and}$$

$$\tan \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}N / \sin \tfrac{1}{2}(a+b) ,$$

ll be used to solve for A and B. The computation is:

$$a+b = 176°5' ; \quad \tfrac{1}{2}(a+b) = 88°2' ;$$

$$a-b = -71°39'; \quad \tfrac{1}{2}(a-b) = -35°50';$$

$$N = 86°22' ; \quad \tfrac{1}{2}N = 43°11' .$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(a-b) &= \log\cos 35^\circ 50' = 9.9089 - 10 \\ \log\cot\tfrac{1}{2}N &= \log\cot 43^\circ 11' = 0.0276 \\ \text{colog}\cos\tfrac{1}{2}(a+b) &= \text{colog}\cos 88^\circ 2' = 1.4645 \\ \log\tan\tfrac{1}{2}(A+B) &= 1.4010 \end{aligned}$$

$$\tfrac{1}{2}(A+B) = 87^\circ 44'.$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(a-b) &= (n)\log\sin 35^\circ 50' = 9.7675 - 10 \\ \log\cot\tfrac{1}{2}N &= \log\cot 43^\circ 11' = 0.0276 \\ \text{colog}\sin\tfrac{1}{2}(a+b) &= \text{colog}\sin 88^\circ 2' = 0.0003 \\ (n)\log\tan\tfrac{1}{2}(A-B) &= 9.7954 - 10 \end{aligned}$$

$$\tfrac{1}{2}(A-B) = -31^\circ 59'.$$

Thus, $A = 55^\circ 45'$ and $B = 119^\circ 43'$. The law of sines n be used to solve for n. Thus,

$$\sin n = \sin b \sin N / \sin B ,$$

r which the computation is:

$$\begin{aligned} \log\sin b &= \log\sin 123^\circ 52' = \log\sin 56^\circ 8' = 9.9192 - 10 \\ \log\sin N &= \log\sin 86^\circ 22' = \log\sin 86^\circ 22' = 9.9991 - 10 \\ \text{log}\sin B &= \text{colog}\sin 119^\circ 43' = \text{colog}\sin 60^\circ 17' = 0.0612 \\ \log\sin n &= 9.9795 - 10 \end{aligned}$$

$$n = \sout{72^\circ 32'} \text{ or } 107^\circ 28'.$$

The obtuse value of n can be reasoned to be the cor-ct value in the following manner. A below the equator and above the equator implies that n must be greater than the gular measure between the meridians of A and B measured ong the equator, which is RP in Figure 29. But RPN is an osceles quadrantal triangle making RP and N have the same asure. N is greater than $72^\circ 32'$, thus $n = 107^\circ 28'$ is the rrect value.

The distance AB in nautical miles is found by changing

107°28', into minutes. Thus,

$$AB = 107(60) + 28 = 6448 \text{ nautical miles.}$$

The initial course from A to B is given by

$$NAB = A = 55°45' .$$

The final course from A to B is given by

$$180° - ABN = 180° - B = 60°17' .$$

Had only the distance between A and B been desired, rmula (20) of the haversine formulas, with c and C re-aced by n and N, would have determined the angular value n directly from the given parts.

Sailing great circle routes in actual practice. The eat circle distance between any two points on the earth's rface always being the shortest distance between those two ints makes traveling a great circle route seem to be the gical choice to make. In the case of sailing, islands y make a great circle route impossible. But disregarding at possibility, the navigation of a great circle route uld requ[illegible] that the course be in a constant state of change nce any great circle, except a meridian or the equator, osses successive meridians at a constantly changing angle. r most great circle routes, the navigation is so difficult at an approximation to a great circle route is used when direct route is desirable.

This approximation to a great circle route is accomished by navigating a series of short curves or arcs own as rhumb lines. A rhumb line or loxodrome is a curve the earth's surface which cuts all meridians at the same gle. The closer the approximation to a great circle route required the shorter the rhumb lines are made and thus e more rhumb lines are required. In practice, the term ailing the great circle route" refers to sailing a series rhumb lines which approximate to a great circle route.

In laying out the approximation to a great circle ute just referred to the navigator uses two types of maps charts; the great circle chart and the Mercator's chart.

A great circle chart distorts a portion of the earth's rface so that all great circle routes appear to be straight nes. Figure 30 shows the arc AB of a great circle route a line segment. Likewise, all meridians are straight.

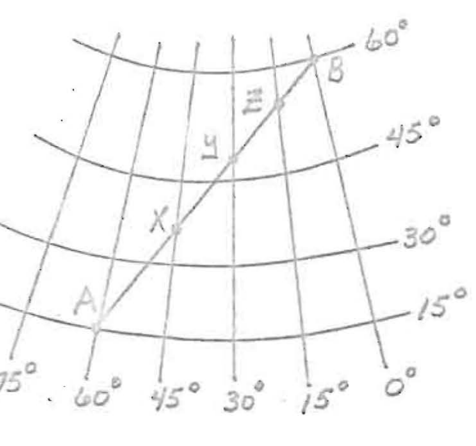

Figure 30

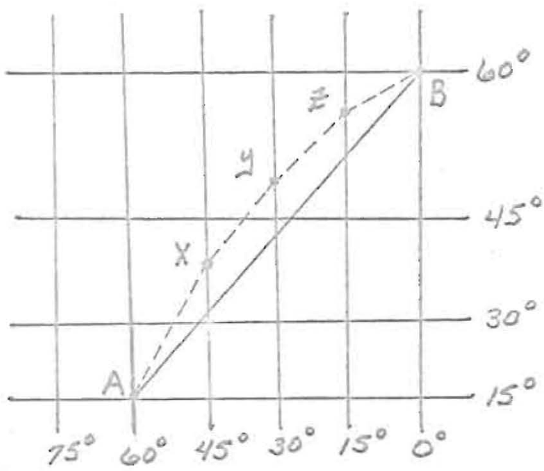

Figure 31

A Mercator's chart(Figure 31)distorts the earth's sur-
ce so that all rhumb lines appear as straight lines. Fig-
e 31 represents the same portion of the earth's surface
Figure 30. Straight line-segment AB represents the rhumb
ne connecting A and B. Equally spaced meridians appear as
ually spaced parallel lines. Equally spaced parallels of
titude appear as parallel lines perpendicular to the meri-
ans but not equally spaced. An important property of a
rcator's chart is that all angles of the earth's surface
e reproduced accurately.

The laying off of an approximation to a great circle
ute is accomplished in the following manner. On the great
rcle chart the great circle route between A and B is in-
cated by line segment AB. If the route does not hit any
lands or any other nonnavigable objects, a great circle
oute may be used. The coordinates of several convenient
oints are noted(X, Y, and Z). These points are then plotted
the Mercator's chart using the same coordinates. The
roken(dotted)line AXYZB on the Mercator's chart indicates
e successive rhumb lines that give an approximation to
e great circle route AB. If a closer approximation is
esired, the plotting of more points on the great circle
hart and then on the Mercator's chart will accomplish this
nd.

Since the Mercator's chart preserves angle size, and

owing the coordinates of the points plotted, the length
d direction of each of the rhumb lines AX, XY, YZ, and
, can be computed which completely determines the route
 be taken. On a large scale Mercator's chart the size of
e angles and the lengths of the chords can be read fairly
curately, however, mathematical computation can be used if
greater degree of accuracy is desired [8;183-185].

III. THE CELESTIAL SPHERE

The celestial sphere is a sphere, whose center is the nter of the earth, with a radius so large that by compari- on the radius of the earth is of negligible length. The rth is sometimes referred to as the center of this sphere. utical Astronomy is not concerned with the actual size of is sphere but rather with the location of and the movement heavenly bodies upon its surface. All the heavenly bodies: n, moon, stars, and planets visible from the earth are agined to lie on the celestial sphere's surface. Their lative positions are indicated in angular measure and from ese measures spherical triangles are formed and their solu- ons can be found. In the following discussion references de will be to Figure 32.

The <u>celestial north pole</u> P and <u>south pole</u> P' are the oints where the earth's axis, NS, meets the celestial sphere en this axis is extended. To any observer, at A, the celesial sphere appears to rotate from east to west about the xis PP', because the earth rotates from west to east on its xis. If A is some given terrestrial position on the earth's urface, the point vertically above A on the celestial sphere s the zenith, Z, of A. The point Z' diametrically opposite he zenith, Z, on the celestial sphere is called the nadir. f M represents any one of the heavenly bodies the point G irectly under M on the earth's surface is called the geo-raphic position(GP)of M. Geometrically, points M, G, and are collinear as are points Z, A, O, and Z', where O is the arth's center.

The positions of the celestial bodies may be expressed y a celestial coordinate system very similar to the latitude-ongitude system on the earth's surface. In one such coor-inate system, the basic lines of reference are the celestial quator and the celestial prime meridian, each of which is etermined by extending the plane of the earth's equator and he plane of the terrestrial prime meridian. Other systems nclude: the altitude-azimuth coordinate system; the declina-ion-hour-angle system of coordinates; and the right ascen-ion-declination coordinate system[6;207]. In Figure 32, 'JC'T' is the celestial equator, and PT'LP' is the celestial rime meridian. A celestial meridian is a great circle con-aining P and P'. The celestial meridian PMFD'P' which asses through M is called the hour circle. The spherical

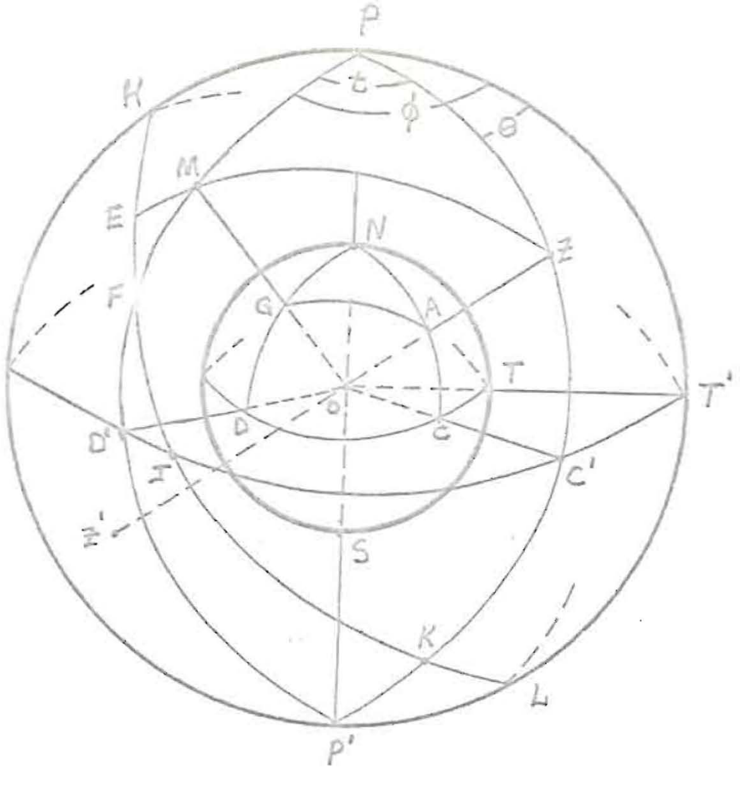

Figure 32

: celestial north pole
': celestial south pole
P': celestial axis
: celestial body
: GP of M
: a given terrestrial point
: zenith of A
': nadir of A
: LHA of M
: GHA of M
: GHA of Z
'M: Dcl. of M (Dcl. of M = lat. of G)
G: latitude of G
M: latitude of M(EM never exceeds 90°)
Z: zenith distance of M(EM + MZ = 90°)
M: polar distance of M
ngle PZM: azimuth of M
CT: terrestrial equator
'JC'T': celestial equator
TS: terrestrial prime meridian
T'LP': celestial prime meridian
MFD'P': hour circle of M
ZC'KP': zenith meridian
EFJKL: celestial horizon of A(Z is pole)
ME: vertical circle of M with respect to Z
ph. triangle PMZ: astronomical triangle
ph. triangle NGA: terrestrial triangle

Abbreviations:
GP: geographic position
LHA: local hour angle
GHA: Greenwich hour angle
Dcl.: declination
sph.: spherical

gle t formed by the hour circle and the celestial meridian rough Z is called the local hour angle(LHA)of M. Spherical gle ϕ formed by the hour circle and the celestial prime ridian is the Greenwich hour angle(GHA)of M. Spherical gle θ is the GHA of Z. The GHA for any celestial body or int is measured westward from the celestial prime meridian a positive angle up to 360°. The declination(Dcl.)of a lestial point is the angular distance of the point from e equator, measured along the hour circle of that point. us, the declination of M is arc D'M. Note that the declina- on of a celestial point is merely the latitude of its GP. e two coordinates of the coordinate system just described e Dcl. and GHA.

Solar noon, 12:00 noon, is the instant when the sun on the zenith meridian of a given point A; that is, when e hour circle of the sun and the zenith meridian of A coin- ide the apparent time at A is 12:00 noon. The rotation of e earth on its own axis makes it appear as if the sun cir- les the earth every twenty-four hours. Using this, an in- rease of 15° in the sun's GHA would indicate an elapse in ime of one hour. If the sun, M, is west of the zenith of , Z, the time is after noon; if the sun is east of the zenith f A, the time is before noon.

The great circle HEFJKL which has the zenith, Z, and he nadir, Z', of a terrestrial point A as poles is the elestial horizon of A. The visible horizon is the apparent ine of separation of the earth from the sky. The plane of he celestial horizon intersects the center of the earth hile the plane of the visible horizon is parallel to that f the celestial horizon but intersects the earth in a small ircle.

The vertical circle of a point M with respect to the enith, Z, of A is the great circle containing M and Z. The ltitude of M is then the angular distance M is above the elestial horizon measured along the vertical circle. Thus, M is the altitude of M. The zenith distance of M is the istance between M and Z on the vertical circle. Thus, Z = 90° - EM is the zenith distance of M. The polar distance of M is arc PM.

The azimuth of M is the angle PZM between the meridian hrough Z and the vertical circle of M. The azimuth angle f a point is sometimes called the bearing of the point [7;79].

Spherical triangle PMZ is known as the astronomical riangle. Spherical triangle NGA on the earth is the same riangle, with respect to angular measure. Solution of spherical triangle PMZ gives information as to latitude, ongitude, time of day, and so forth [8;187-189], [6;205-208].

pplications Of The Astronomical Triangle

Only a few of the possible examples will be considered s there are so many ways in which the data can be given that t would be impossible to illustrate each type of application.

To find the time of day(solar time)of an observer hose latitude is known, when the sun's declination and al- itude are given. Example: To find the solar time in Chi- ago(latitude: 41°50' N)at a moment in the morning when the un's declination is 14°16' if the sun's altitude is observed o be 38°14'[6;208, 211-prob. 1].

In Figure 32, C'Z = 41°50'; D'M = 14°16'; and EM = 8°14'. Thus, in astronomical triangle PMZ,

PZ = 48°10'; MP = 75°44'; and MZ = 51°46' .

The local hour angle, t or MPZ, when converted into its quivalent in hours will give the solar time desired. Any f the half angle formulas with the proper substitutions ill get the desired results. Let A = M, B = Z, and C = P. hen, formula (28) becomes

$$\sin \tfrac{1}{2}P = \sqrt{\frac{\sin(s-PZ)\ \sin(s-MP)}{\sin PZ\ \sin MP}}\quad, \text{ where}$$

$s = \frac{1}{2}(PZ + MP + MZ) = 87°50'$; $s - PZ = 39°40'$; and

$s - MP = 12°6'$.

The computation is:

$$\begin{aligned}
\log \sin(s-PZ) &= \log \sin 39^\circ 40' = 9.8050 - 10\\
\log \sin(s-MP) &= \log \sin 12^\circ 6' = 9.3214 - 10\\
\text{colog} \sin PZ &= \text{colog} \sin 48^\circ 10' = 0.1278\\
\text{colog} \sin MP &= \text{colog} \sin 75^\circ 44' = 0.0136\\
2 \log \sin \tfrac{1}{2}P &= 19.2678 - 20\\
\log \sin \tfrac{1}{2}P &= 9.6339 - 10\\
\tfrac{1}{2}P &= 25^\circ 30'\\
P &= 51^\circ .
\end{aligned}$$

viding $P = 51^\circ$ by 15° per hour gives $P = 3.4$ hours. But 4 hours = 3 hours, 24 minutes to the nearest minute. The servation was taken in the morning so that the solar time 3 hours, 24 minutes before noon or 8:36 A.M.

<u>To find the solar time of sunrise or sunset in a ven latitude when the declination of the sun is known.</u>

is case is a special case of the last example where the titude of the sun is 0°. Example: Find the Chicago time sunrise on the morning in the last example, and the bear- g of the sun at sunrise [6;209, 211-prob. 3].

In Figure 32, $C'Z = 41^\circ 50'$; $D'M = 14^\circ 16'$; and $EM = 0^\circ$. hus, in astronomical triangle PMZ,

$$PZ = 48^\circ 10';\ MP = 75^\circ 44';\ \text{and } MZ = 90^\circ.$$

The parts of the triangle desired are angles MPZ and ZM. Triangle PMZ is quadrantal so that the solution can est be carried out in polar triangle P'M'Z'(P' and Z' not be confused with those of Figure 32). In spherical tri- ngle P'M'Z',

$P' = 90^\circ$; $M' = 131^\circ 50'$; and $Z' = 104^\circ 16'$.

Let m', p', and z' represent the supplements of M, and Z, respectively. Using Napier's rules to solve for and z', the required formulas are:

$\cos p' = \cot M' \cot Z'$; and $\cos z' = \cos Z'/\sin M'$.

The computation is:

$$\log \cot M' = (n) \log \cot 48^\circ 10' = 9.9519 - 10$$
$$\log \cot Z' = (n) \log \cot 75^\circ 44' = 9.4053 - 10$$
$$\log \cos p' = 9.3572 - 10$$

$$p' = 76^\circ 49'.$$

$$\log \cos Z' = (n) \log \cos 75^\circ 44' = 9.3917 - 10$$
$$\text{colog} \sin M' = \text{colog} \sin 48^\circ 10' = 0.1278$$
$$\log \cos z' = 9.5195 - 10$$

$$z' = 70^\circ 41' \text{ or } 109^\circ 19'.$$

Thus, $P = 103^\circ 11'$ and $Z = 70^\circ 41'$. Divide $P = 103^\circ 11'$ 103.18° by 15° to get P = 6.88 hours or 6 hours, 53 min-es before noon. Therefore, the sun will rise at 5:07 A.M. a bearing(azimuth)of $70^\circ 41'$.

<u>To find the lengths of the longest and shortest days a known terrestrial point and the bearing of the sun at nrise on each of these days</u>. Essential to the solution an example of this type is having the information that e sun's declination is $23^\circ 27'$ on the longest day of the ar and $-23^\circ 27'$ on the shortest day of the year, for the rthern hemisphere of the earth. Fairbanks, Alaska, has a

atitude of 64°51' N[6;211].

In Figure 32, computing for the longest day, D'M = 3°27'; C'Z = 64°51'; and EM = 0°. Thus, in astronomical riangle PMZ,

MP = 66°33'; PZ = 25°9'; and MZ = 90°.

Thus, in polar triangle P'M'Z',

P' = 90°; Z' = 113°27'; and M' = 154°51' .

The parts of triangle PMZ desired are angles P and Z. sing Napier's rules to solve for their supplements, p' and ', in polar triangle P'M'Z' requires the relationships:

cos p' = cot M' cot Z'; and cos z' = cos Z'/sin M' .

The computation is:

log cot M' = (n) log cot 25°9'	=	0.3283
log cot Z' = (n) log cot 66°33'	=	9.6372 - 10
log cos p'	=	9.9655 - 10
p'	=	22°32'.

log cos Z' = (n) log cos 66°33'	=	9.5998 - 10
colog sin M' = colog sin 25°9'	=	0.3717
log cos z'	=	9.9715 - 10
z'	=	~~20°32'~~ or 159°28'.

Thus, P = 157°28' and Z = 20°32'. Converting P to ours and minutes gives P = 10 hours, 30 minutes. This epresents the number of hours before solar noon that the sun rises and also the number of hours from solar noon until sunset. Therefore, the length of the longest day is 21 hours,) minutes and the bearing of the sun at sunrise is 20°32'.

In Figure 32, computing for the shortest day,

'M = -23°27'; C'Z = 64°51'; and EM = 0°. Thus, in astronom-
cal triangle PMZ,

MP = 113°27'; PZ = 25°9'; and MZ = 90°.

Again using the relationships:

$$\cos p' = \cot M' \cot Z \text{ and } \cos z' = \cos Z'/\sin M',$$

o solve for p' and z' in polar triangle P'M'Z', in which

P' = 90°; Z' = 66°33'; and M' = 154°51',

he computation is:

$$\begin{aligned} \log \cot M' &= (n)\ \log \cot 25°9' &&= 0.3283 \\ \log \cot Z' &= \log \cot 66°33' &&= 9.6372 - 10 \\ && \log \cos p' &= 9.9655 - 10 \\ && p' &= 22°32' \text{ or } 157°28'. \end{aligned}$$

$$\begin{aligned} \log \cos Z' &= \log \cos 66°33' &&= 9.5998 - 10 \\ \text{colog} \sin M' &= \text{colog} \sin 25°9' &&= 0.3717 \\ && \log \cos z' &= 9.9715 - 10 \\ && z' &= 20°32'. \end{aligned}$$

Thus, P = 22°32' and Z = 159°28'. P is equivalent o 1.5 hours, which is just half of the daylight hours. herefore, the length of the shortest day is 3 hours, 0 min- tes and the bearing of the sun at sunrise is 159°28'.

<u>To find the latitude of the observer if the altitude, our angle, and declination of a celestial object are known.</u>

xample: To find the latitude of an observer if at 9:28 A.M. he sun's altitude is 37°26' and the sun's declination is

20'[6;210, 211-prob. 7].

In Figure 32, the given parts are:

= 2 hours, 32 minutes before noon; EM = $37°26'$; and

M = $6°20'$.

t, 2 hours, 32 minutes is equivalent to $38°$ so that, in

iangle PMZ,

$$MPZ = 38°;\ MZ = 52°34';\ \text{and}\ MP = 83°40'\ .$$

The altitude of Z, C'Z, is the required part. Solving

r its complement, PZ, in triangle MPZ can be accomplished

solving for Z by the law of sines and then using Napier's

alogies to solve for PZ. From the law of sines,

$$\sin Z = \sin P \sin MP / \sin MZ\ .$$

e computation is:

$$\begin{aligned}
\log \sin P &= \log \sin 38° &= 9.7893 - 10 \\
\log \sin MP &= \log \sin 83°40' &= 9.9973 - 10 \\
\operatorname{colog} \sin MZ &= \operatorname{colog} \sin 52°34' &= 0.1001 \\
& \quad \log \sin Z &= 9.8867 - 10 \\
& \quad Z &= 50°23' \text{ or } \\
& & 129°37'.
\end{aligned}$$

Rules of species do not eliminate either one of these

alues of Z. Thus, two solutions appear to be possible. In

e first solution $Z_1 = 50°23'$ and in the second $Z_2 = 129°37'$.

ormula (48) of Napier's Analogies, solved for tan $\frac{1}{2}c$ and

ith the proper substitutions, can be used to solve for PZ

n triangle PMZ. Thus,

$$\tan \tfrac{1}{2}PZ = \tan \tfrac{1}{2}(MP-MZ) \sin \tfrac{1}{2}(P+Z)/\sin \tfrac{1}{2}(Z-P)\ .$$

The computations are:

$$\begin{aligned}
\log\tan\tfrac{1}{2}(MP-MZ) &= \log\tan 15^\circ 33' = 9.4445 - 10\\
\log\sin\tfrac{1}{2}(Z_1+P) &= \log\sin 44^\circ 12' = 9.8434 - 10\\
\operatorname{colog}\sin\tfrac{1}{2}(Z_1-P) &= \operatorname{colog}\sin 6^\circ 12' = 0.9666\\
\log\tan\tfrac{1}{2}PZ &= 0.2545\\
\tfrac{1}{2}PZ &= 60^\circ 54'\\
PZ &= \sout{121^\circ 48'}.
\end{aligned}$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}(MP-MZ) &= \log\tan 15^\circ 33' = 9.4445 - 10\\
\log\sin\tfrac{1}{2}(Z_2+P) &= \log\sin 83^\circ 49' = 9.9975 - 10\\
\operatorname{colog}\sin\tfrac{1}{2}(Z_2-P) &= \operatorname{colog}\sin 45^\circ 49' = 0.1444\\
\log\tan\tfrac{1}{2}PZ &= 9.5864 - 10\\
\tfrac{1}{2}PZ &= 21^\circ 06'\\
PZ &= 42^\circ 12'.
\end{aligned}$$

An ambiguity about a second solution can now be elimi-ated since PZ must be less than or equal to 90° [6;211]. Thus, Z = 42°12' and the latitude of the observer must be 47°48'.

<u>In a given latitude, to find the altitude and azi-uth of a celestial object whose declination and hour angle re known</u>. Example: To find the altitude and azimuth of a tar M as observed from A whose latitude is 22°25' N at an nstant of time when the declination of M is 76°25' and the ocal hour angle of M is 4 hours, 40 minutes, west [6;209, 11-prob. 13].

In astronomical triangle PMZ,

PZ = 67°35'; PM = 13°35'; and P = 70° west.

he unknown parts are MZ(to find EM)and Z. Expressing one f the haversine formulas in terms of the known parts results in

$$\text{hav } MZ = \text{hav}(PZ-PM) + \sin PZ \sin PM \text{ hav } P.$$

The computation is:

$$\text{hav}(PZ-PM) = \text{hav } 54^\circ = 0.20611$$

$$\begin{aligned} \log \sin PZ &= \log \sin 67^\circ 35' = 9.96588 - 10 \\ \log \sin PM &= \log \sin 13^\circ 35' = 9.37081 - 10 \\ \log \text{hav } P &= \log \text{hav } 70^\circ = 9.51718 - 10 \\ &\log \text{product} = 8.85387 - 10 \end{aligned}$$

$$\text{product} = 0.07143$$

$$\text{hav } MZ = 0.27754$$

$$MZ = 63^\circ 35'.$$

Thus, the altitude of $M = EM = 90 - 63^\circ 35' = 26^\circ 25'$.

Using the law of sines to solve for Z results in the relationship

$$\sin Z = \sin P \sin MP / \sin MZ .$$

The computation is:

$$\begin{aligned} \log \sin P &= \log \sin 70^\circ = 9.9730 - 10 \\ \log \sin MP &= \log \sin 13^\circ 35' = 9.3708 - 10 \\ \text{colog} \sin MZ &= \text{colog} \sin 63^\circ 35' = 0.0479 \\ &\log \sin Z = 9.3917 - 10 \end{aligned}$$

$$Z = 14^\circ 16'.$$

But M is west of Z, so the azimuth of M is $360^\circ - Z = 345^\circ 44'$.

<u>Time when the sun will reach the prime vertical(position where bearing is due west or east), the latitude and declination being known.</u> Example: Determine the time when the sun will be on the prime vertical, both morning and evening, if the declination of the sun is 18° N and the latitude

f an observer is 46° N [7;78, 82-prob. 6a].

The sun, M, will be due east or due west of Z when = 90°. The morning and afternoon times will be equally paced before and after solar noon [7;81]. In astronomical riangle PMZ,

$$Z = 90^\circ;\ PM = 72^\circ;\ \text{and}\ PZ = 44^\circ.$$

Using Napier's rules on right triangle PMZ to find results in the use of the relationship:

$$\cos P = \cot PM \tan PZ\ .$$

he computation is:

$$\begin{aligned} \log \cot PM = \log \cot 72^\circ &= 9.5118 - 10 \\ \log \tan PZ = \log \tan 44^\circ &= \underline{9.9848 - 10} \\ \log \cos P &= 9.4966 - 10 \end{aligned}$$

$$P = 71^\circ 43'.$$

onverting $P = 71^\circ 43'$ to hours and minutes, the result is = 4 hours, 47 minutes. Thus, the sun's bearing will be ue west at 4:47 P. M. and due east at 7:13 A. M.

Many other astronomical applications of the spherical riangle are given in textbooks. Some of these use the ight ascension coordinate system(see Crockett [3;189-192]) hile others go into much more detail than is possible in his work. Several applications of spherical geometry will e considered in the remainder of this chapter that could e classified as geometric applications.

IV. INSCRIBED AND CIRCUMSCRIBED CIRCLES

Circles can be described touching the sides of a iven spherical triangle, or passing through the vertices f the triangle. The circle may be inside the spherical riangle or the triangle may be inside the circle. The purose of this section is to derive formulas for the angular adii of the incircle, the excircle, and the circumcircle.

he Incircle

The circle which can be inscribed within a given pherical triangle so as to touch each of the sides of the riangle internally is called the inscribed circle or incircle. (Figure 33) The pole (nearest pole) of the incircle is he point of concurrency of the angle bisectors of the given riangle. The radius of the incircle will be denoted by r [11;135].

A formula for r. In Figure 33, P is the pole of the ircle inscribed in spherical triangle ABC. AP, BP, and CP re the angle bisectors of angles , B, and C, respectively. PD, E, and PF are the perpendicular rcs to the sides a, b, and c, respectively, of triangle ABC. ach of these arcs is equal to r.

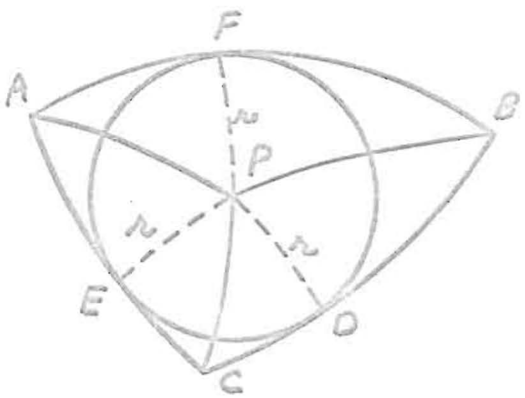

Figure 33

e pairs of triangles, APE and APF; BPD and BPF; and CPD and E, are symmetric in pairs and right-angled at D, E, and

In right spherical triangle BPD, using Napier's rules,

$$\tan PD = \tan r = \tan \tfrac{1}{2}B \sin BD.$$

e pairs of triangles being symmetric implies that AE = AF,) = BF, and CD = CF. Thus, s = BD + AE + EC and

$$BD = s - (AE+EC) = s - b \ .$$

bstituting s - b for BD in the above relationship results

$$\tan r = \tan \tfrac{1}{2}B \sin(s-b) \ .$$

bstituting in the expression for tan ½B from the half angle rmula for tan ½B gives

$$\tan r = \sqrt{\frac{\sin(s-a)\ \sin(s-c)}{\sin s\ \sin(s-b)}} \cdot \sin(s-b) \text{ , or}$$

$$\tan r = \sqrt{\sin(s-a)\ \sin(s-b)\ \sin(s-c)/\sin s} \ . \qquad (55)$$

Formula (55) expresses the tangent of the radius of e inscribed circle in terms of the sides of spherical trigle ABC. The same result would have been obtained had y one of the other right-angled triangles in Figure 33 een used [11;136, 137].

ne Excircle

A circle which touches one side of a spherical triangle and the other two sides extended is called the esribed circle or excircle.(Figure 34)The pole(nearest pole) f an excircle is the point of concurrency of the bisectors f the exterior angles of the riangle. Every spherical tri-ngle has three excircles asso-iated with it, which are actu-lly incircles of the colunar riangles of the given tri-ngle. Let r_1, r_2, and r_3 epresent the angular radii f the incircles of colunar riangles A'BC, B'AC, and C'AB, espectively [11;135, 136].

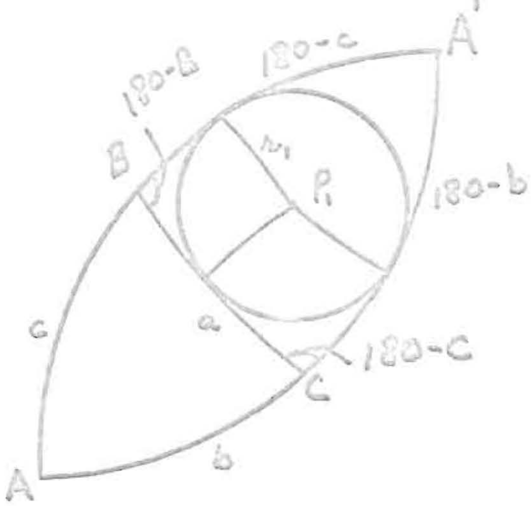

Figure 34

<u>Formulas for r_1, r_2, and r_3.</u> In colunar triangle A'BC,

$$s_1 = \tfrac{1}{2}(a+180-b+180-c) ;$$

$$s_1 = \tfrac{1}{2}(360+a-b-c) ;$$

$$s_1 = 180 - (s-a) ;$$

nd the other parts are as shown in Figure 34.

Using these results, formula (55) becomes

an $r_1 = \sqrt{\sin(s_1-a)\ \sin(s_1-180+b)\ \sin(s_1-180+c)/\sin s_1}$;

an $r_1 = \sqrt{\sin(180-s)\ \sin(a+b-s)\ \sin(a+c-s)/\sin(s-a)}$;

an $r_1 = \sqrt{\sin s\ \sin(s-c)\ \sin(s-b)/\sin(s-a)}$. (56)

In a similar manner, formulas for r_2 and r_3 can be derived. Thus,

$$\tan r_2 = \sqrt{\sin s \sin(s-a) \sin(s-c)/\sin(s-b)} \quad ; \text{ and } (57)$$

$$\tan r_3 = \sqrt{\sin s \sin(s-a) \sin(s-b)/\sin(s-c)} \quad . \quad (58)$$

Formulas (56), (57), and (58) express the radii of the three excircles in terms of the sides of the given triangle [11;139, 140].

The Circumcircle

The circle which passes through each of the three vertices of a given spherical triangle is called the circumscribing circle or circumcircle. The pole of the circumcircle of a spherical triangle is the point of intersection of the arcs of great circles perpendicular to the sides of the triangle at their midpoints. Let R denote the radius of the circumcircle [11;136].

A formula for R. In Figure 35, P is the pole of the circle circumscribing spherical triangle ABC. DP, EP, and FP are the perpendicular bisecting arcs of the sides of triangle ABC. Thus, AE = CE; AF = BF; and BD = CD. The pairs of triangles, AEP and CEP; AFP and BFP; BDP and CDP, are symmetric

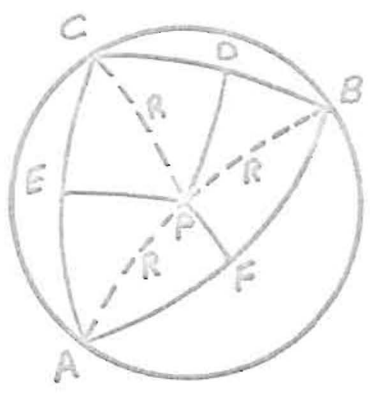

Figure 35

n pairs and right-angled at D, E, and F. Therefore, P = BP = CP = R, and the following pairs of angles are qual:

$$EAP = ECP;\ FAP = FBP;\ \text{and}\ DBP = DCP\ .$$

In right spherical triangle APF, using Napier's rules,

$$\tan AP = \tan AF/\cos PAF\ .$$

ut, AP = R, AF = $\frac{1}{2}$c, and PAF = S - C since FAP + ECP + CP = S, or FAP = S - C. Therefore, the last expression for an AP becomes

$$\tan R = \tan \tfrac{1}{2}c/\cos(S-C)\ ,\ \text{which}$$

pon substituting from the half side formulas, becomes

$$\tan R = \sqrt{\frac{-\cos S \cos(S-C)}{\cos(S-A)\cos(S-B)}} \cdot \frac{1}{\cos(S-C)}\ ,\ \text{or}$$

$$\tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}\ . \qquad (59)$$

Formula (59) expresses the tangent of the radius of he circumscribed circle in terms of the angles of spherical riangle ABC. The same result would have been obtained had ny one of the right-angled triangles in Figure 35 been used 11;140, 141].

V. OTHER GEOMETRIC APPLICATIONS

<u>Distances from a point within a tri-rectangular spher-cal triangle to its vertices.</u> Let ABC be a tri-rectangular pherical triangle(all sides and angles equal to 90°); let

, q, and r be the great circle distances from any point
within the triangle to the vertices A, B, and C, re-
pectively.(Figure 36)Then,

$$\cos^2 p + \cos^2 q + \cos^2 r = 1 \ . \quad (60) \ [4;55].$$

Proof: Extend each of
he great circle arcs AP, BP,
nd CP to meet the sides of
he triangle at D, E, and F,
espectively. A, B, and C be-
ng poles of the opposite sides
f the triangle makes the arcs
t D, E, and F perpendicular.
hus, any triangle having D,
, or F as a vertex will be
ight-angled at that vertex.
ach of arcs AD, BE, and CF is
qual to 90°.

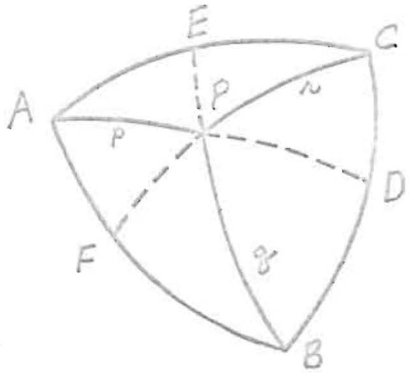

Figure 36

In right spherical triangle AFP, using Napier's
ules,

$$\sin PF = \sin FAP \sin p \text{ , which,}$$

ince PF = 90 - r, is

$$\cos r = \sin FAP \sin p \ . \qquad (1)$$

In right spherical triangle AEP, again using Napier's
ules,

$$\sin EP = \sin EAP \sin p \text{ , or}$$

$$\cos q = \cos FAP \sin p \,, \qquad \text{(ii)}$$

ince $EP = 90 - q$ and $EAP = 90 - FAP$.

Squaring (i) and (ii) and adding, gives

$$\cos^2 r + \cos^2 q = \sin^2 FAP \sin^2 p + \cos^2 FAP \sin^2 p \,.$$

actoring on the right side of the last equation, substi-uting $1 - \cos^2 p$ for $\sin^2 p$ and 1 for $\sin^2 FAP + \cos^2 FAP$, nd transposing gives

$$\cos^2 p + \cos^2 q + \cos^2 r = 1 \,,$$

hich is formula (60).

Let O be the center of the sphere upon which tri-rec-angular triangle ABC is placed. The arcs p, q, and r measure he angles which OP makes with lines OA, OB, and OC. In nalytic Geometry the angles p, q, and r are called the di-ection angles of line OP and the relation expressed in ormula (60) is fundamental [14;197].

<u>Distances from a point within a tri-rectangular spher-cal triangle to its sides</u>. Let P be any point within a tri-ectangular spherical triangle and x, y, and z represent he distances from P to A, B, and C, respectively. (Figure 37) hen,

$$\sin^2 x + \sin^2 y + \sin^2 z = 1 \,. \qquad (61)$$

Proof: Figure 37 is equal in all respects to Figure 6 with the exception of x, y, and z to indicate arcs DP, P, and FP, respectively. In right spherical triangles

P and BFP,

$\cos p = \cos AF \cos z$ and $\cos q = \sin AF \cos z$,

nce AF + BF = 90°. Squaring

ch side of both equations,

ding the equations and simpli-

ing gives

$s^2 p + \cos^2 q = \cos^2 z$. (i)

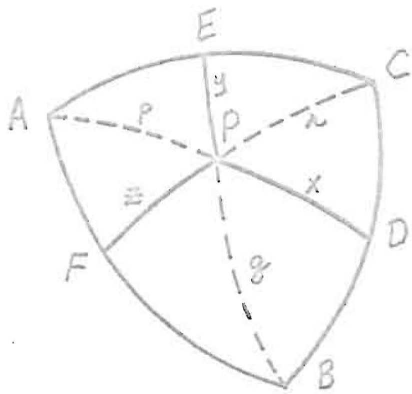

Figure 37

Likewise, considering

iangles AEP and CEP,

$s\ p = \cos AE \cos y$ and

$s\ r = \sin AE \cos y$, or

$s^2 p + \cos^2 r = \cos^2 y$. (ii)

Also, considering triangles BDP and CDP,

$\cos q = \cos BD \cos x$ and $\cos r = \sin BD \cos x$, or

$$\cos^2 q + \cos^2 r = \cos^2 x . \qquad \text{(iii)}$$

Addition of (i) through (iii) gives

$$2(\cos^2 p + \cos^2 q + \cos^2 r) = \cos^2 x + \cos^2 y + \cos^2 z.$$

bstituting from (60) on the left side and using the iden-
ty $1 - \sin^2 x = \cos^2 x$ for x, y, and z on the right side
ves

$$2(1) = 3 - (\sin^2 x + \sin^2 y + \sin^2 z)$$

ich is equivalent to formula (61) [4;56].

Angle between two directions. Let ABC be a tri-rec-angular spherical triangle on a sphere with center O. Let and P' be any two points within triangle ABC and p, q, r, ', q', and r' be the arcs drawn to the vertices as in Figure 3. Then, if $x = PP'$,

$$\cos x = \cos r \cos r' + \cos p \cos p' + \cos q \cos q'. \quad (62)$$

Proof: Extend arcs CP nd CP' to meet side AB at Q nd Q', respectively. Let Q = y and AQ' = y'. Then ngle PCP' = y' - y. Thus, n triangle PCP', by the law f cosines,

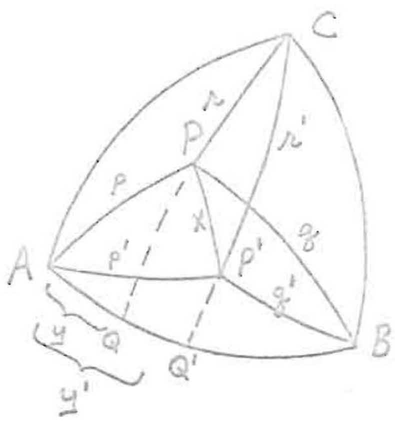

Figure 38

$\cos x = \cos r \cos r' +$
$\sin r \sin r' \cos(y'-y).$

sing an addition identity,

$$\cos x = \cos r \cos r' + \sin r \sin r' \cos y' \cos y + \sin r \sin r' \sin y' \sin y. \quad \text{(i)}$$

In right spherical triangles APQ, AP'Q', BPQ, and P'Q',

$$\left.\begin{aligned} \cos p &= \cos PQ \cos y = \sin r \cos y \ ; \\ \cos p' &= \cos P'Q' \cos y' = \sin r' \cos y' \ ; \\ \cos q &= \cos PQ \cos BQ = \sin r \sin y \ ; \\ \cos q' &= \cos P'Q' \cos BQ' = \sin r' \sin y' \ . \end{aligned}\right\} \text{(ii)}$$

ubstituting the results of (ii) into (i) gives

$\cos x = \cos r \cos r' + \cos p \cos p' + \cos q \cos q'$

hich is the desired result [4;57, 58].

Since x measures angle POP', formula (62) is the ormula for the cosine of the angle between two lines in pace, OP and OP', in terms of the direction cosines of hese lines [14;198].

<u>Distance from point between two great circles to</u> <u>heir intersection</u>. Let P be a point between two great cirles AC and BC. (Figure 39) Let p nd q represent the distances rom P to AC and BC, respectively. Also, PC = x, angle CB = C, angle ACP = y, and ngle BCP = C - y. The problem s to find an expression for x n terms of p, q, and C.

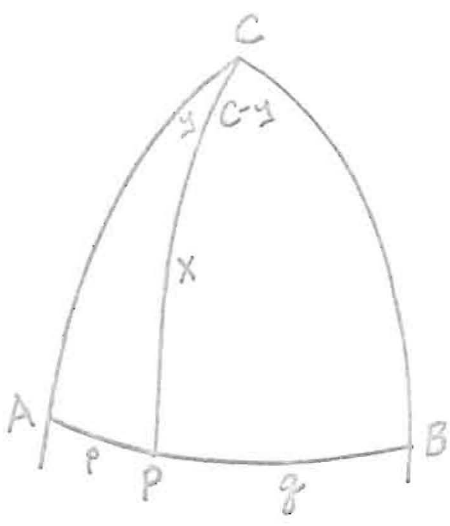

Figure 39

In right spherical triangle CAP, using Napier's rules,

$\sin p = \sin x \sin y$, or

$\sin y = \sin p / \sin x$. (i)

Likewise, in right spherical triangle CBP,

$$\sin q = \sin x \sin(C-y) \text{ , or}$$

$$\sin q = \sin x \sin C \cos y - \sin x \cos C \sin y \text{ .} \qquad \text{(ii)}$$

From (i), $\cos y = \dfrac{\sqrt{\sin^2 x - \sin^2 p}}{\sin x}$,

ere upon substituting this in (ii) and also sin p/sin x
r sin y the result is

$$\sin q = \sin C \sqrt{\sin^2 x - \sin^2 p} - \cos C \sin p.$$

Thus,

$$\sin q + \cos C \sin p = \sin C \sqrt{\sin^2 x - \sin^2 p} \;;$$

$$(\sin q + \cos C \sin p)^2 = \sin^2 C(\sin^2 x - \sin^2 p) \;;$$

$$\frac{(\sin q + \cos C \sin p)^2}{\sin^2 C} + \sin^2 p = \sin^2 x \;;$$

$$n\ x = \frac{1}{\sin C}\sqrt{\sin^2 q + 2\sin q\cos C\sin p + \cos^2 C\sin^2 p + \sin^2 C\sin^2 p} \;; \text{ or}$$

$$n\ x = \frac{1}{\sin C}\sqrt{\sin^2 p + \sin^2 q + 2 \sin p \sin q \cos C}\ . \quad (63)$$

Formula (63) is the desired result[4;60, 61].

Horizontal angle between two points. Points A and B
e observed from O in a horizontal plane A'OB'. The angles
elevation of A and of B can
measured with a sextant, as
ll as the inclined angle AOB.
find the horizontal angle
OB' between the points A and
as seen from O.(Figure 40)

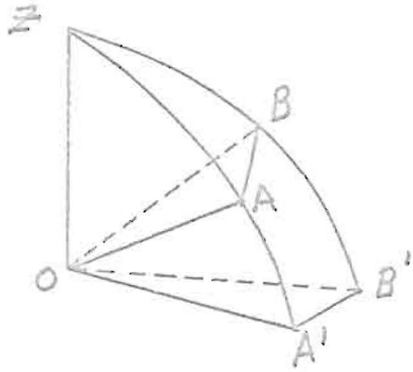

Figure 40

Let OZ be the vertical
ne of intersection of the
rtical planes AOA' and BOB'.
gles AOA', BOB', and AOB are

own from the readings with a sextant from which AZ, BZ, d AB, respectively, are easily obtained.

Angle A'OB', being the desired quantity, is equal to e measure of arc A'B' which in turn is equal in measure to gle A'ZB'. But the three sides of spherical triangle(con- idering O as the center of the sphere)ABZ being known make e determination of angle AZB the required solution[14;196]. gle AZB could be computed using a half angle formula.

<u>To find the angle between two chords of a spherical riangle</u>. Let ABC be a spher- cal triangle on the sphere ith O as center.(Figure 41) o find an expression for the lane angle BAC when A, b, and are the known parts of spher- cal triangle ABC. Let A' = lane angle BAC and A = spher- cal angle BAC.

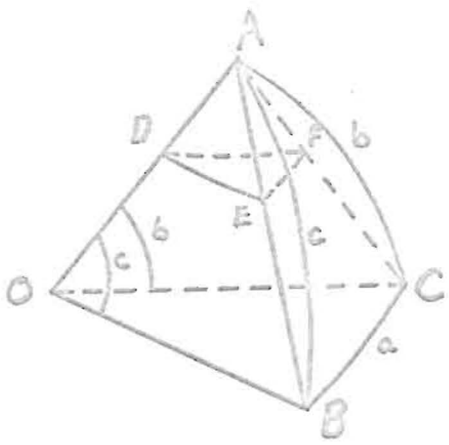

Figure 41

Describe a sphere about , A the center, intersecting lanes AOB, AOC, and ABC in arcs E, DF, and EF, respectively. The sphere with O as center as OA, OB, and OC as radii. Thus, plane angles OAB and BA are equal in measure as are OAC and OCA. Therefore,

$$DE = OAB = (180-c)/2 = 90 - \tfrac{1}{2}c \quad ;$$

$$DF = OAC = (180-b)/2 = 90 - \tfrac{1}{2}b \quad ;$$

$$EDF = A \text{ ; and}$$

$$EF = EAF = A' .$$

pherical angles EDF and A are equal since both have the easure of dihedral angle B - OA - C.

In spherical triangle DEF, using the law of cosines,

$$\cos EF = \cos DF \cos DE + \sin DF \sin DE \cos EDF .$$

ubstituting in the relationships established above gives he desired result of

$$\cos A' = \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c + \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos A \text{ [14;198]}. \quad (64)$$

<u>Surface area and volume of a parallelopiped</u>. Given he lengths of three edges of a parallelopiped that meet in point, and the angles formed y them, to find the surface rea and volume of the parallelopiped in Figure 42. Let D be the perpendicular from to the base BOCE of parallelopiped OG. Then AOD is a lane perpendicular to base OCE. Let the known angles nd edges be represented as ollows:

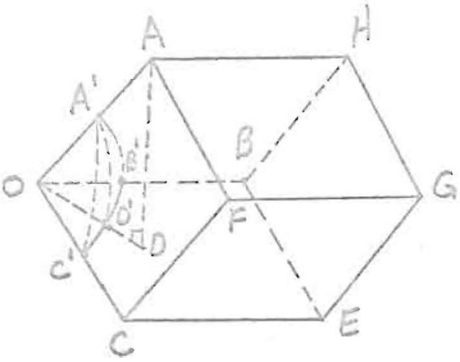

Figure 42

$$BOC = a \; ; \quad AOC = b \; ;$$

$$AOB = c \; ; \quad OA = l \; ;$$

$$OB = m \; ; \quad OC = n \; .$$

ith O as center, describe a unit sphere intersecting the lanes of the faces as shown with primed letters.

Thus, the <u>surface</u> <u>area</u> is given by

$$S = 2 \; OBEC + 2 \; OAFC + 2 \; OAHB \; ;$$

$$S = 2(mn \sin a + ln \sin b + lm \sin c) \; . \qquad (65)$$

In spherical triangle A'B'C', A'B', A'C', and B'C' are qual to c, b, and a, respectively. In right-angled tri-ngle A'B'D', right-angled at D', using Napier's rules,

$$\sin A'D' = \sin A'B' \sin A'B'D' \; ; \text{ or}$$

$$\sin A'D' = \sin c \sin A'B'D' \; . \qquad \text{(i)}$$

rc A'D' = angle AOD, thus

$$AD = OA \sin AOD \; ;$$

$$AD = l \sin A'D' \; ;$$

$$AD = l \sin c \sin A'B'D' \; . \qquad \text{(ii)}$$

From plane trigonometry, $\sin 2x = 2 \sin x \cos x$, hus

$$\sin A'B'D' = 2 \sin \tfrac{1}{2}A'B'D' \cos \tfrac{1}{2}A'B'D' \; .$$

sing half angle formulas,

$$\sin \tfrac{1}{2}A'B'D' = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}} \; , \text{ and}$$

$$\cos \tfrac{1}{2}A'B'D' = \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}} \; . \text{ Hence,}$$

$$\sin A'B'D' = \frac{2}{\sin a \sin c}\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)} \ . \quad (iii)$$

The volume of the parallelopiped is given by

$$V = (\text{area OBEC})\ AD \ ;$$

$$V = mn \sin a(l \sin c \sin A'B'D') \ ;$$

$$V = 2lmn\sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)} \quad [3;175, 176]. \quad (66)$$

Volume of a regular polyhedron. In Figure 43, AB is the edge in which two adjacent faces of the regular polyhedron intersect, D the midpoint of AB, C and E the centers of the polygonal faces where the inscribed sphere with center O is tangent to those faces.

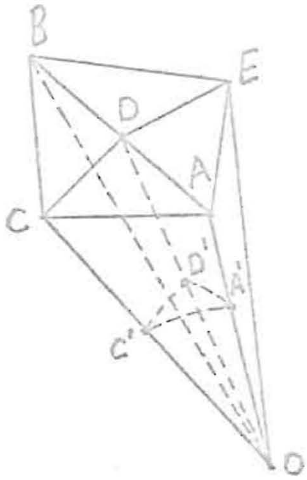

Figure 43

Then

DC = DE ; DA = DB ;

ADC = BDC = BDE = ADE = 90° ;

DCO = DEO = 90°.

Let a = length of an edge AB; s = number of sides of each polygonal face; n = number of faces meeting at a vertex of the polyhedron; N = number of faces of the polyhedron; E = edge angle CDE of the polyhedron.

The volume of a regular polyhedron is equal to the sum
the volumes of the N congruent right regular pyramids
ich can be formed from a regular polyhedron using the
ces of the polyhedron as bases of the pyramids and the cen-
r of the inscribed sphere as the common vertex. Thus, the
lume of a regular polyhedron is given by

$$V = \frac{1}{3}\,(CO)(A)(N)\ , \qquad \text{(i)}$$

ere CO in Figure 43 represents the altitude of one of the
ramids, A the area of one of the faces of the polyhedron
base of one of the pyramids, and N the number of faces
the polyhedron.

Then,

$$CD = AD \cot ACD = \tfrac{1}{2}a \cot \frac{180^\circ}{s} \ ; \qquad \text{(ii)}$$

$$CO = CD \tan CDO = \tfrac{1}{2}a \cot \frac{180}{s} \tan \tfrac{1}{2}E\ . \qquad \text{(iii)}$$

The area, A, of each face is given by

$$A = (CD)(AD) \cdot s \ ;$$

$$A = \tfrac{1}{4}\, a^2 s \cot \frac{180^\circ}{s} \ . \qquad \text{(iv)}$$

Thus, from (i) through (iv), the volume of a regu-
ar polyhedron is given by

$$V = \frac{1}{24}\, a^3 sN \cot^2 \frac{180^\circ}{s} \tan \tfrac{1}{2}E\ . \qquad (67)$$

n order for formula (67) to be useful, it is necessary to
e able to compute $\frac{1}{2}E$ from s, n, and N. Spherical trigo-
ometry is used to establish the desired relationship. If
sphere, with unit radius, is described about O, its inter-

ection with planes AOC, COD, and AOD will form spherical
riangle A'C'D'. Then,

$$A'C'D' = ACD = \frac{180^\circ}{s} \ ;$$

$$A'D'C' = ADC = 90^\circ \ ;$$

$$C'A'D' = CAD = \tfrac{1}{2}(360^\circ/n) = \frac{180^\circ}{n} \ .$$

By Napier's rules,

$$\cos C'A'D' = \cos C'D' \sin A'C'D' \ ; \text{ or}$$

$$\cos \frac{180^\circ}{n} = \cos C'D' \sin \frac{180^\circ}{s} \ .$$

at,

$$\cos C'D' = \cos COD = \cos(90^\circ - CDO) = \sin CDO = \sin \tfrac{1}{2}E.$$

herefore,

$$\cos \frac{180^\circ}{n} = \sin \tfrac{1}{2}E \sin \frac{180^\circ}{s} \ , \text{ or}$$

$$\sin \tfrac{1}{2}E = \cos \frac{180^\circ}{n} / \sin \frac{180^\circ}{s} \ . \qquad (68)$$

ormula (68) is used to find $\frac{1}{2}E$ and this result used in form-
la (67) to compute the volume of the regular polyhedron
3;176, 177].

Example: To find the volume of a regular octahedron
hose edge is eight inches. A regular octahedron is made up
f eight equilateral triangles, four meeting at a vertex.
hus,

$$a = 8 \text{ inches} \ ; \quad s = 3 \ ;$$

$$n = 4 \ ; \quad N = 8 \ .$$

From formula (68),

$$\sin \tfrac{1}{2}E = \cos 45^\circ / \sin 60^\circ = \sqrt{2/3} \quad ; \text{ or}$$

$$\tan \tfrac{1}{2}E = \sqrt{2} \quad .$$

Thus, the volume, as given by formula (67), is

$$V = \frac{1}{24}\,(8^3)(3)(8)\,\cot^2 60^\circ \tan \tfrac{1}{2}E \;;$$

$$V = 8^3(1/3)\sqrt{2} = \frac{512}{3}\sqrt{2} \quad ; \text{ or}$$

$$V = 241.32 \text{ cubic inches.}$$

This concludes the development of spherical trigonometry ich was made from the postulates and theorems of Euclidean ometry. The next chapter will develop some of the formulas spherical trigonometry from the not so familiar Non-clidean basis. The results will prove interesting.

CHAPTER VI

ELLIPTIC TRIGONOMETRY OF A RIGHT TRIANGLE

The first five chapters were devoted to developing the rigonometry of the sphere using Euclidean Geometry. This hapter will develop spherical trigonometry using the postu-ates and theorems common to Riemannian(Elliptic)Geometry. he sphere is a model for Riemannian Geometry [12;119] so that e could expect the trigonometry of Riemannian Geometry to e equivalent to that developed for the sphere. Only the form-las for right triangles will be developed in this chapter. or convenient reference, the postulates and some of the theo-ems, lemmas, and corollaries commonly included in a develop-ent of Riemannian Geometry will be stated, without proof, at his time.

I. RIEMANNIAN GEOMETRY

<u>Postulates</u>. The postulates that will be used as a asis for Riemannian Geometry are those of the familiar Hilbert ystem for Euclidean Geometry(see Wolfe [16;12-16])with the sual major changes as required for Riemannian Geometry. The ostulates dealing with the uniqueness of a line as deter-ined by two points, the infinitude of a line, and the parallel ostulate are changed to:

1. Two distinct points determine at least one straight line.

2. Every straight line is boundless.

3. Two straight lines always intersect one another.

e other changes required in the Hilbert System are consid-
ed to be familiar to the reader.

Theorems. The theorems listed below are actually
mmas, theorems, and corollaries commonly included in devel-
ments of Riemannian Geometry. They are all listed below as
eorems as an aid to future reference.

Theorem 1. In any triangle which has one of its
gles a right angle, each of the other two angles is less than,
ual to, or greater than a right angle, according as the
de opposite it is less than, equal to, or greater than q,
d conversely [16;180]. Note: The distance from any point
a line to the pole of that line is q.

Theorem 2. The line joining the midpoints of the base
d summit of a Saccheri quadrilateral is perpendicular to
th of them, and the summit angles are equal and obtuse [16;181].

Theorem 3. In a trirectangular quadrilateral(Lambert
adrilateral)the fourth angle is obtuse and each side adjacent
this angle is smaller than the side opposite [16;181].

Theorem 4. If any triangle has one of its angles a
ght angle, then the angle-sum of the triangle is greater
an two right angles [16;182].

Theorem 5. The sum of the angles of any triangle is
eater than two right angles [16;182].

Theorem 6. The sum of the angles of every quadriateral is greater than four right angles [16;183].

It will also be necessary to use those propositions f Euclid whose proofs are valid without the parallel postuate and do not demand the infinitude of the line [16;17, 18, 77]. It is for this reason that most investigations will be arried out restricting the lengths of segments less than r equal to q.

uclidean, Hyperbolic, and Elliptic Geometries.

Gauss used the name non-Euclidean to describe a sysem of geometry which differs from Euclid's in its properties f parallelism [2;vii]. During the time of Saccheri(1667-1773) he favorite starting-point in demonstrating a postulate on arallels was to conceive of parallels as equidistant straight ines. Saccheri, recognizing the error in that approach, tarted with two equal perpendiculars AC and BD at the ends f a segment AB. When the ends, C and D, are joined it is asily proven that the angles at C and D are congruent but t is not easily proven that they are right angles. With an pen mind, Saccheri proposes three hypotheses:

(1) The hypothesis of the right angle.

(2) The hypothesis of the obtuse angle.

(3) The hypothesis of the acute angle.

Saccheri was trying to prove the dependence of the ifth Postulate of Euclid. His method of proof was to be inrect, that is, he wanted to show that in assuming either of ne last two hypotheses he would eventually find a contra- iction and thus prove the first hypothesis which is equiva- ent to the Fifth Postulate of Euclid. His work left him hort of his goal but he did establish a number of noteworthy heorems, some of which are:

> If one of the three hypotheses is true in any one case, the same hypothesis is true in every case.
>
> On the hypothesis of the right angle, the obtuse angle, or the acute angle, the sum of the angles of a triangle is equal to, greater than, or less than two right angles.
>
> On the hypothesis of the right angle two straight lines intersect, except in the one case in which a transversal cuts them at equal angles.
>
> On the hypothesis of the obtuse angle two straight lines always intersect.
>
> On the hypothesis of the acute angle there is a whole pencil of lines through a given point which do not intersect a given straight line, but have a common perpendicular with it, and these are separated from the pencil of lines which cut the given line by two lines which approach the given line more and more closely, and meet it at infinity.

Saccheri managed to demolish the hypothesis of the ob- use angle by showing that the sum of two angles of a triangle eed not be less than two right angles. After nearly twenty iditional theorems he managed to demolish the hypothesis of ne acute angle by showing that two lines which meet in a oint at infinity can be perpendicular at that point to the ame straight line. He did not seem to be satisfied with his roof, and he offered another proof in which he lost himself n the infinitesimal.

Saccheri's lack of sufficient imagination and his beng bound by tradition, plus a firm belief that Euclid's hypothesis was the only true one, kept him from the discovery century later of the two non-euclidean geometries which ollow from his hypotheses of the obtuse and acute angle.

Lambert(1728-1777)also fell just short of the disovery. Using a starting point very similar to Saccheri's, e also named the three hypotheses, but he added to Saccheri's iscoveries. With the hypothesis of the obtuse angle he howed that the area of a triangle is proportional to the excess of the sum of its angles over two right angles, which is he case for the geometry on the sphere. He also concluded hat the hypothesis of the acute angle would be verified on sphere of imaginary radius.

He dismissed the hypothesis of the obtuse angle since t required that two straight lines enclose a space. His rgument against the hypothesis of the acute angle included he non-existence of similar figures [13;11-14].

That the three hypotheses, right angle, obtuse angle, nd acute angle, eventually led to the discovery of three ystems of geometry called the Parabolic, the Elliptic, and he Hyperbolic Geometries is a matter of record [10;27]. lein in 1871 attached those names to the systems that were veloped respectively by Euclid, Riemann-Cayley, and Bolyai-batschewsky [2;vii]. Riemann developed the differential ometry of spherical space. Cayley(1821-1895)considered ace "in the large" and defined distance in terms of homoneous coordinates [2;13].

Two Elliptic Geometries. If a is a line in Elliptic ometry it is frequently taken that each line has two disnct poles, call them A and A'. In this geometry each pair lines intersect in exactly two antipodal points, and every ne is of length 4q. This geometry is the spherical type d has the character of a two-sided surface. The plane of is geometry is referred to as the Double Elliptic Plane. emann most likely had this plane in mind [16;179, 180].

If the points A and A' are taken to be the same point ch line then has exactly one pole. Klein is credited with ving suggested this perfectly consistent geometry. In this ometry, two points always determine a line, two lines interect in only one point, and straight lines are finite and osed, but of length 2q. The distinguishing feature of this pe of Elliptic Geometry is that a line does not separate e plane into two regions, that is, it is always possible to ss from one side of a line to the other without crossing the ne. The one-sided nature of the plane of this geometry can illustrated with a Mobius strip. This plane is called e Single Elliptic Plane [16;64, 179, 180].

II. THE TRIGONOMETRIC FUNCTIONS OF AN ANGLE

The remainder of this chapter will be devoted to eveloping some of the basic formulas of spherical trigo- ometry using the basis of Riemannian Geometry as stated in rt one of this chapter. Sommerville(see [13;114-122])de- elops the trigonometric formulas of the elliptic plane by ssuming that euclidean geometry holds in the infinitesimal omain of the elliptic plane. Sommerville uses the differ- tial and the integral calculus to a great extent in devel- ping the necessary relationships. The plan that will be ollowed in the remaining sections of this chapter is pat- erned after the approach used by Wolfe(see [16;185-200]).

<u>Sine and cosine of an angle T</u>. The sine of T and e cosine of T will be defined only for acute values of T. et T(Figure 44), with vertex t O, be any acute angle. From y point P on the one side of T raw PQ perpendicular to the ther side labeling the point n the other side Q. Let r, , and y represent the lengths f OP, OQ, and PQ, respectively. s an added restriction to P,

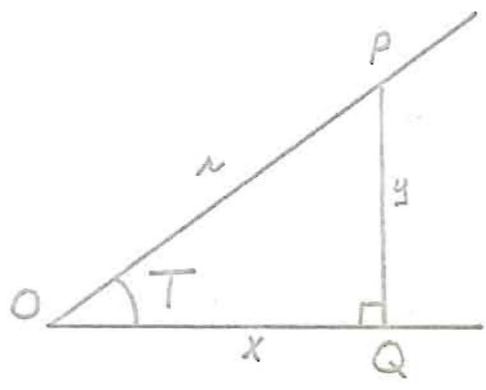

Figure 44

ck P so that r is less than q. The ratios y/r and x/r are t constant, when the value of r(position of P) is changed, t they do approach finite limits as r approaches zero. It these limits which are defined to be the sine and cosine T. Thus,

$$\sin T = \lim_{r \to 0} \frac{y}{r}$$

d

$$\cos T = \lim_{r \to 0} \frac{x}{r} ,$$

ere the other trigonometric functions follow in the con- ntional manner [16;185, 186].

<u>Theorems concerning the parts of triangle OPQ.</u> The llowing theorems will be needed in the work to follow.

Theorem 7. As r decreases, the angle OPQ decreases.

Proof: Let P_1 and P_2 igure 45)be any two positions P(Figure 44)such that OP_1 less than OP_2 and OP_2 is ess than q. P_1Q_1 and P_2Q_2 are erpendiculars from P_1 and Q_1 espectively. It is sufficient o show angle OP_1Q_1 less than ngle OP_2Q_2.

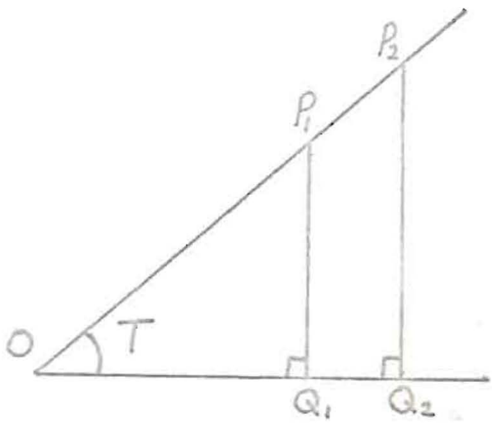

Figure 45

Angles OP_1Q_1 and $Q_1P_1P_2$ are supplementary making their um equal to two right angles. $P_1Q_1Q_2P_2$ is a quadrilateral that the sum of the angles of the quadrilateral at P_1, Q_1, Q_2, and P_2 must exceed four right angles(Theorem 6). But the angles at Q_1 and Q_2 are right angles since P_1Q_1 and P_2Q_2 are perpendiculars from P_1 and P_2. Thus, the sum of the angles $Q_1P_1P_2$ and $P_1P_2Q_2$ must exceed two right angles. Therefore, angle OP_2Q_2 is greater than angle OP_1Q_1 and the proof is complete.

A result of this theorem is that the angle at P (Figure 44)approaches a right angle as OP approaches q [16;186]. If OP is made equal to q(Figure 46)and a perpendicular to OP at P is drawn, it will intersect OQ at a point, call it Q'. Now to show Q' = Q. OP = q and OP perpendicular to PQ' implies that O is a pole of line PQ'. Thus, PQ' is perpendicular to OQ'. But, PQ is perpendicular to OQ, so that if Q ≠ Q' then P is a pole of line QQ' and PQ = PQ' = q. This is impossible. Angle T is given acute, thus, Theorem 1 implies that PQ as also PQ' must be less than q. Hence, P cannot be a pole of line QQ' and therefore Q = Q'. Q = Q' makes O a pole of line PQ and P a right angle.

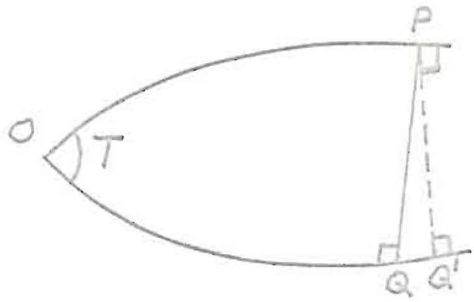

Figure 46

Theorem 8. As r decreases continuously, so also does T.(Figure 44)

Proof: As before, let OP_1(Figure 47)be less than

2. Exactly one of the following must be true:

$P_1Q_1 = P_2Q_2$; $P_1Q_1 > P_2Q_2$; or $P_1Q_1 < P_2Q_2$.

indirect proof will be used.

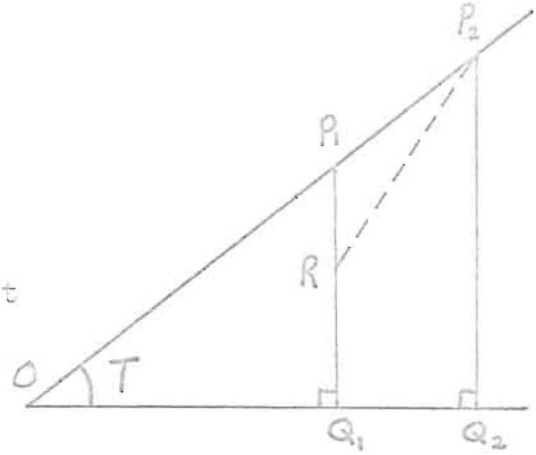

Figure 47

Suppose $P_1Q_1 = P_2Q_2$.

en $P_1Q_1Q_2P_2$ is a Saccheri adrilateral and angle $P_1P_2Q_2$ obtuse. Angle OP_2Q_2 is a right gle when $OP_2 = q$. But, OP_2 less than q, thus, by eorem 7 angle $P_1P_2Q_2$ is acute. is contradicts the supposition $Q_1 = P_2Q_2$. Therefore, $Q_1 \neq P_2Q_2$.

Suppose $P_1Q_1 > P_2Q_2$. If $P_1Q_1 > P_2Q_2$ there exists a oint R on segment P_1Q_1 such that $RQ_1 = P_2Q_2$. Then $RQ_1Q_2P_2$ s a Saccheri Quadrilateral and angle RP_2Q_2 is obtuse so that ngle OP_2Q_2 must also be obtuse. But, as in the last case, P_2 less than q and Theorem 7 imply that angle OP_2Q_2 is acute. ence, $P_1Q_1 > P_2Q_2$ is also contradicted. Therefore, the only emaining possibility, $P_1Q_1 < P_2Q_2$, must be true. That is, decreases as r decreases [16; 36].

Theorem 9. As r decreases continuously, so also does he ratio x/r. (Figure 44)

Proof: Divide OQ into n equal parts, Q_1, Q_2, Q_3, ..., $_{n-1}$ being the points of division. (Figure 48) The perpen-

culars at these points intersect OP at P_1, P_2, P_3, ...,
$_{-1}$. It will be shown that these points of intersection

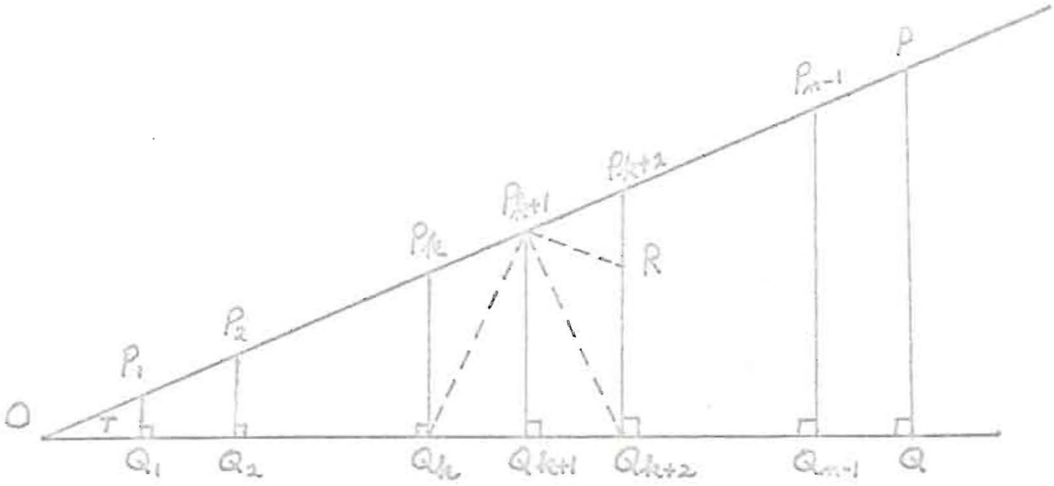

Figure 48

n OP do not divide OP into n equal parts. Consider any hree consecutive perpendiculars, say P_kQ_k, $P_{k+1}Q_{k+1}$, and $_{k+2}Q_{k+2}$. Theorem 8 gives assurance that P_kQ_k is less than $_{k+2}Q_{k+2}$. From point R on $Q_{k+2}P_{k+2}$, where $Q_{k+2}R$ equals P_kQ_k, raw segment $P_{k+1}R$. Likewise, draw segments Q_kP_{k+1} and $_{k+1}Q_{k+2}$. Triangles $Q_kQ_{k+1}P_{k+1}$ and $Q_{k+2}Q_{k+1}P_{k+1}$ have right ngles at Q_{k+1}, the side $P_{k+1}Q_{k+1}$ in common, and sides $_kQ_{k+1}$ and $Q_{k+1}Q_{k+2}$ congruent so they are congruent by side-ngle-side. Thus, by corresponding parts, $Q_kP_{k+1} = Q_{k+2}P_{k+1}$ nd angle $P_{k+1}Q_kQ_{k+1}$ is congruent to angle $P_{k+1}Q_{k+2}Q_{k+1}$. onsequently, triangles $P_kQ_kP_{k+1}$ and $RQ_{k+2}P_{k+1}$ have the angles t Q_k and Q_{k+2} congruent, the sides P_kQ_k and RQ_{k+2} congruent s are the sides Q_kP_{k+1} and $Q_{k+2}P_{k+1}$. Hence, triangles

$_kP_{k+1}$ and $RQ_{k+2}P_{k+1}$ are also congruent by side-angle-side.
is, the pairs of corresponding parts, P_kP_{k+1} and RP_{k+1}, and
les $Q_kP_kP_{k+1}$ and $Q_{k+2}RP_{k+1}$, are congruent. Theorem 6
lies that the sum of angles $Q_kP_kP_{k+1}$ and $Q_{k+2}P_{k+2}P_{k+1}$ is
eater than two right angles. Angles $Q_{k+2}RP_{k+1}$ and $P_{k+1}RP_{k+2}$
supplementary and have a sum of two right angles. These
results and the previous result of angles $Q_kP_kP_{k+1}$ and
$_{2}RP_{k+1}$ being congruent imply that angle $P_{k+1}P_{k+2}R$ must be
eater than angle $P_{k+1}RP_{k+2}$. This important result implies,
ice each side of triangle $RP_{k+1}P_{k+2}$ is less than q, that side
$_{1}R$ is longer than side $P_{k+1}P_{k+2}$ [16;177]. Therefore,
$_{k+1}$ is greater than $P_{k+1}P_{k+2}$, that is, as x receives equal
rements r receives decreasing increments.

Taking the results above to the ratio x/r, observe
t, as r increases, the ratio x/r decreases. When r equals
, OP_1 is less than q and angle OP_1Q_1 is a right angle only
en $OP_1 = q$. Then, by Theorem 7, angle OP_1Q_1 must be acute.
en, in triangle OP_1Q_1 angle P_1 acute and angle Q_1 a right
le implies that OP_1 is greater than OQ_1. Thus the ratio
must be less than one when r is less than q. Then for any
positions of P, for which the x's are commensurable, the
responding ratios are not equal, the larger ratio corres-
nding to the larger value of r. If the x's are not commen-
able the same conclusion can be reached by a limiting pro-

ess. Thus, in summary, the ratio x/r increases continuously s r increases and therefore decreases as r decreases [16;187, 188].

Theorem 10. In Figure 48, if OP = OQ = q, then the atio $\frac{PP_{n-1}}{QQ_{n-1}}$ increases as QQ_{n-1} does and decreases as QQ_{n-1} ecreases [16;188].

Proof: If OP = OQ = q, O is a pole of line PQ and P and OQ are perpendicular to PQ. Angle $P_{n-1}Q_{n-1}Q$ is a right ngle so quadrilateral $PQQ_{n-1}P_{n-1}$ is a Lambert Quadrilateral. y Theorem 3 PP_{n-1} is less than QQ_{n-1}. Consider Lambert Quad- ilateral $P_{n-2}Q_{n-2}QP$. Theorem 3 again implies that PP_{n-2} is ess than QQ_{n-2}. But QQ_{n-1} equals $Q_{n-1}Q_{n-2}$ and, by Theorem , PP_{n-1} is less than $P_{n-1}P_{n-2}$, so, reasoning as before, the atio $\frac{PP_{n-1}}{QQ_{n-1}}$ must increase as QQ_{n-1} increases since, as QQ_{n-1} eceives equal increments, PP_{n-1} receives increasing incre- ents. Likewise, the ratio must decrease as QQ_{n-1} decreases.

Theorem 11. As r decreases continuously, the ratio /r increases.

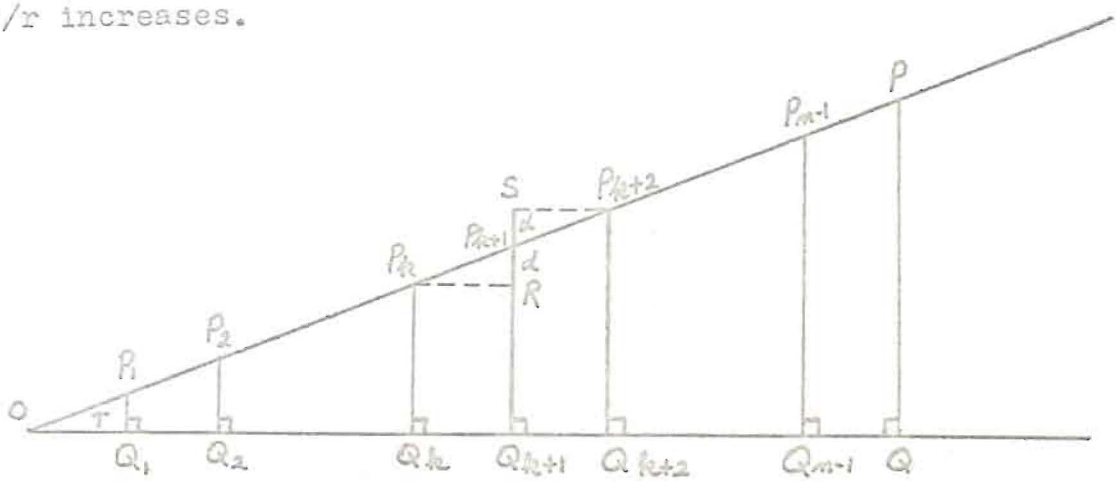

Figure 49

Proof: Divide OP in n equal parts, with P_1, P_2, P_3, .., P_{n-1} being the points of division. (Figure 49) Draw the erpendiculars P_1Q_1, P_2Q_2, P_3Q_3, ..., $P_{n-1}Q_{n-1}$ to OQ. Conider any three consecutive perpendiculars, say P_kQ_k, $_{k+1}Q_{k+1}$, and $P_{k+2}Q_{k+2}$. P_kQ_k being the smallest as a result f Theorem 8, draw the perpendicular through P_k to $Q_{k+1}P_{k+1}$ eeting $Q_{k+1}P_{k+1}$ at R, R between Q_{k+1} and P_{k+1}. Call the ength of RP_{k+1}, d. Since $Q_{k+1}P_{k+1}$ is likewise shorter than $_{k+2}P_{k+2}$, extend $Q_{k+1}P_{k+1}$ through P_{k+1} to S so that $P_{k+1}S = d$. ow, $P_kP_{k+1} = P_{k+1}P_{k+2}$, $RP_{k+1} = P_{k+1}S$, and vertical angles $_kP_{k+1}R$ and $SP_{k+1}P_{k+2}$ are congruent, hence triangles $P_kP_{k+1}R$ nd $P_{k+2}P_{k+1}S$ are congruent by side-angle-side. Thus, angle $_kRP_{k+1}$ a right angle implies angle $P_{k+2}SP_{k+1}$ is also a right ngle. $P_kQ_kQ_{k+1}R$ is a Lambert quadrilateral, P_k the obtuse ngle, so that

$$P_kQ_k < RQ_{k+1} = P_{k+1}Q_{k+1} - d$$

r

$$P_{k+1}Q_{k+1} - P_kQ_k > d. \qquad \text{(i)}$$

Likewise, $SQ_{k+1}Q_{k+2}P_{k+2}$ is a Lambert quadrilateral, $_{k+2}$ the obtuse angle, so that

$$P_{k+2}Q_{k+2} < SQ_{k+1} = P_{k+1}Q_{k+1} + d$$

r

$$P_{k+2}Q_{k+2} - P_{k+1}Q_{k+1} < d. \qquad \text{(ii)}$$

Inequalities (i) and (ii) indicate that as r receives qual increments, y receives decreasing increments. Reasoning s before, as r increases continuously the ratio y/r decreases, nd as r decreases the ratio y/r increases [16;188, 189].

It is necessary now to show that the ratios x/r and /r actually approach limits as r approaches zero. The proof f the next theorem will accomplish this end.

Theorem 12. The limit of the ratio x/r as r approaches ero and the limit of the ratio y/r as r approaches zero both xist.

Proof for x/r: Theorem 9, the proof of Theorem 9, nd the paragraph following that proof indicate that the ratio /r is less than one, always positive, and decreases as r decreases. Thus, the ratio x/r approaches a limit which may e a positive number or zero. The proof that it is not zero ill be considered in the proof of the ratio y/r.

Proof for y/r: Theorem 1 fails to show that y/r must lso approach a limit since it s not yet known that y/r for very r is bounded above. But, onsidering Figure 50, where ngle POQ is the acute angle onsidered in the previous theo- ems, and OK is constructed

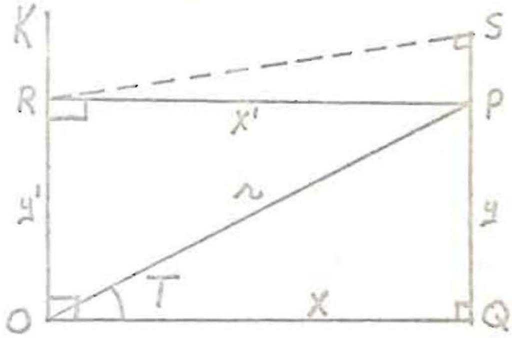

Figure 50

erpendicular to OQ, PR is drawn perpendicular to OK at R
nd PQ is perpendicular to OQ. Call the lengths of OQ, PQ,
R, and RO x, y, x', and y', respectively. Recall that r is
ess than q and angle POQ is acute so that angle ROP must also
e acute.

Lambert Quadrilateral ROQP is obtuse-angled at RPQ.
hus, by Theorem 3, x' is less than x and y is less than y'.
pplying the theorems just developed to the acute angles POQ
nd ROP, as r decreases the ratio x/r decreases while the ratio
'/r increases. But x greater than x' implies that the limit
f the ratio x/r cannot be zero since it must always be greater
han the non-negative ratio x'/r. Hence the limit of a ratio
/r is a positive number.

Likewise, as r decreases the ratio y/r increases
hile the ratio y'/r decreases. But, y' is greater than y
aking the ratio y'/r always greater than the ratio y/r. The
atio y'/r is always positive and less than one so that y/r
ust approach a finite limit less than one from below [16;189].

III. PROPERTIES OF A VARIABLE LAMBERT QUADRILATERAL

Careful investigation of Lambert Quadrilateral PQOR
of Figure 50 will serve to introduce a most important func-
tion. If side y' is fixed and side x is allowed to vary,
then x', y, r, T, and obtuse angle RPQ will be variable also.

n particular, the behavior of the ratio x'/x as x approaches
ero with y' fixed will be examined. The proof of a theorem
oncerning this function is of necessity preceded by two sup-
orting theorems.

Theorem 13. If two Lambert quadrilaterals ABCD and
'B'C'D' have their obtuse angles at A and A', and if AB and
'B' are equal, while BC is less than B'C' and each is less
han q, then angle BAD is less than angle B'A'D' [16;190].

Proof: An indirect proof will be used. Since figures
an be moved in the elliptic plane, place Lambert quadrilateral

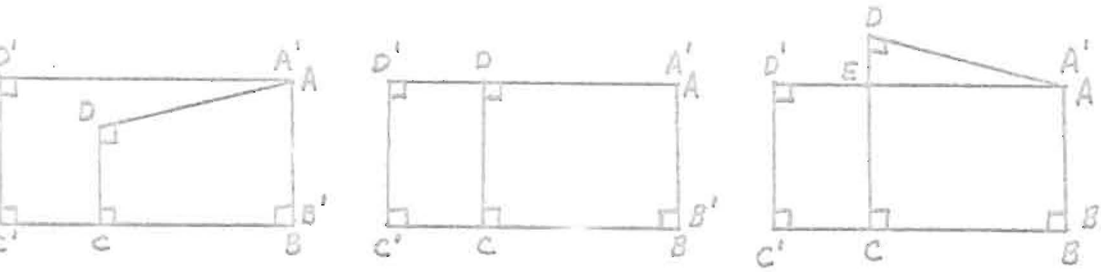

Figure 51 Figure 52 Figure 53

BCD on Lambert quadrilateral A'B'C'D' so that A is on A', B
s on B', and C is on segment B'C'. This is possible since
C is less than B'C' and AB = A'B'. Point D can take on the
hree relative positions as illustrated by Figures 51, 52, and
3 depending upon whether angle BAD is less than angle B'A'D',
ngle BAD equals angle B'A'D', or angle BAD is greater than
ngle B'A'D', respectively. It will be shown that the latter
wo possibilities lead to contradictions.

Suppose angle BAD = angle B'A'D'.(Figure 52)Then D

ill lie on A'D' and it is seen that CDD'C' is a quadrilat-
ral having all four angles right angles. This contradicts
heorem 6, hence, angle BAD is not equal to angle B'A'D'.

Suppose angle BAD is greater than angle B'A'D'.(Fig-
re 53)Since the quadrilaterals are restricted in size by the
ypothesis, CD will intersect segment A'D' in a point E. Thus,
C'D'E is a Lambert quadrilateral and so angle CED' must be
btuse. Then angle AED is also obtuse. But, angle ADC is
ight, so the sum of angles ADE and AED in triangle ADE is
reater than two right angles under this supposition. This
ontradicts Euclid I, 17 since CB less than q implies DA less
han q and Euclid I, 17 holds for restricted triangles [16;177].

Thus, the last two possibilities have been eliminated
o that it must be true that angle BAD is less than angle
'A'D'.

Theorem 14. If ABCD is a Lambert quadrilateral, ob-
use-angled at A, then, if AB is kept fixed and BC allowed to

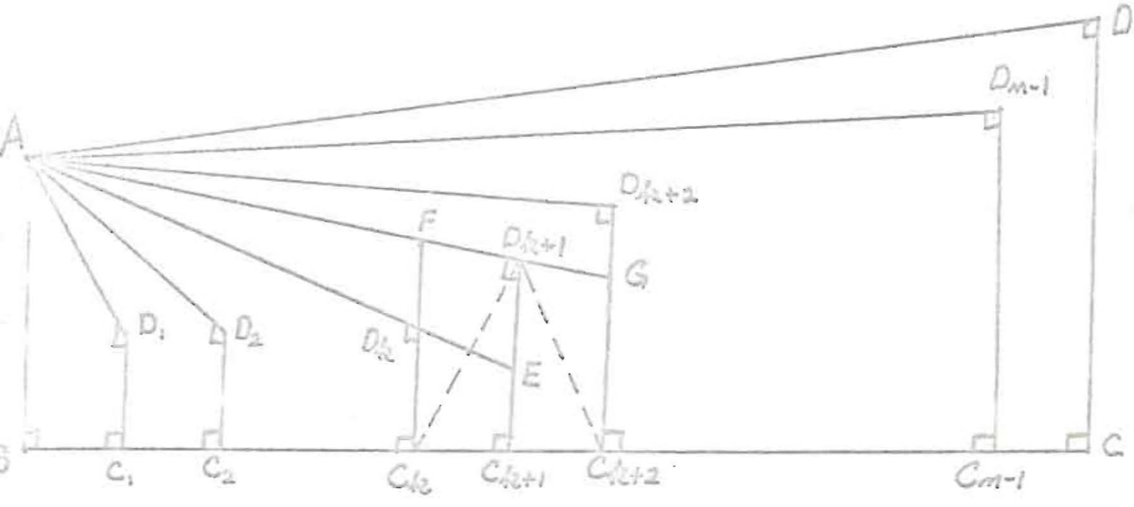

Figure 54

crease continuously and approach zero, the ratio AD/BC increases.

Proof: Divide BC into n congruent parts, C_1, C_2, C_3, ..., C_{n-1} being the points of division. (Figure 54) Construct e perpendiculars to BC at these points of division, and rough A draw the perpendiculars AD_1, AD_2, ..., AD_{n-1} to the erpendiculars from BC forming n Lambert quadrilaterals all otuse-angled at A. From Theorem 12 it is clear angle A decreases as BC decreases. Let any three consecutive Lambert uadrilaterals be represented by ABC_kD_k, $ABC_{k+1}D_{k+1}$, and $BC_{k+2}D_{k+2}$. Extend AD_k through D_k to meet $C_{k+1}D_{k+1}$ in E, D_k through D_k to meet AD_{k+1} in F, and AD_{k+1} through D_{k+1} to eet $C_{k+2}D_{k+2}$ in G. If C_kD_{k+1} and $C_{k+2}D_{k+1}$ are drawn, it s seen that triangles $C_kC_{k+1}D_{k+1}$ and $C_{k+2}C_{k+1}D_{k+1}$ are conruent by side-angle-side. Then, triangles C_kFD_{k+1} and $_{k+2}GD_{k+1}$ are congruent by angle-side-angle. Hence, FD_{k+1} is ual to $D_{k+1}G$. Angle AGC_{k+2} is obtuse making angle AGD_{k+2} ute. Angle $AD_{k+2}G$ is a right angle, thus

$$AG > AD_{k+2} .$$

Likewise, angle C_kFD_{k+1} obtuse makes angle AFD_k ute. Angle AD_kC_k is a right angle, thus

$$AF > AD_k .$$

en

$$AG = AD_{k+1} + D_{k+1}G = AD_{k+1} + FD_{k+1} > AD_{k+2} ,$$

here upon adding $-AD_{k+1}$ to both sides of the last inequality,
he result is

$$AD_{k+1} - AF > AD_{k+2} - AD_{k+1} .$$

ince $AF > AD_k$, the last inequality may be written

$$AD_{k+1} - AD_k > AD_{k+2} - AD_{k+1} .$$

easoning as in the proofs of Theorems 9 through 12, as BC akes on equal increments AD takes on decreasing increments, ence, the ratio AD/BC decreases as BC increases and increases s BC decreases [16;190, 191].

Theorem 15. If the sides of a Lambert quadrilateral re x, y, x', and y', where x and x', y and y' are pairs of pposite sides, x' and y including the obtuse angle, and if ' is kept fixed while x is allowed to approach zero, then he ratio x'/x decreases and approaches a limit indicated by (y').

Proof: By Theorem 10 the ratio x'/x decreases as x pproaches zero. Since x' is less than x, the ratio x'/x pproaches a limit less than one. It must now be shown the imit is not zero.

In Figure 50, draw RS perpendicular to QP extended hrough P. By Theorem 14, the ratio RS/x increases as x de- reases. Angle S a right angle implies that RS is less than P, hence, the ratio x'/x is always greater than the ratio S/x. Thus, the ratio x'/x has a limit which is greater than ero. This limit depends upon the length of y' and will be signated by t(y'). Note that, in the preceding discussion, is restricted to lie between zero and q. As y' approaches ro RS becomes very nearly equal to OQ so t(y') approaches e; as y' approaches q, R and S become very close together t(y') approaches zero.

It has been shown that in quadrilateral OQSR

$$t(y') > \frac{RS}{x} \quad ,$$

d in quadrilateral OQPR

$$t(y) > \frac{x'}{x} \quad .$$

nce t(y') is defined to be the limit of $\frac{x'}{x}$ as x approaches ro and $\frac{x'}{x}$ is always decreasing as x decreases, the above inuality may be written

$$t(y) > \frac{x'}{x} > t(y') \quad .$$

is last inequality will be of importance in the considerations to follow. Since y' is greater than y, it follows that x) decreases as x increases and increases as x decreases 6;190-192].

IV. THE CONTINUITY OF THE FUNCTION t(x)

Let ABCD be a Lambert quadrilateral with obtuse angle C. (Figure 55) Segments AB, BC, and DA have lengths designated x, u, and v, respectively. Theorem 15 and its proof supplies e important relationship that, if x is fixed and v is al-

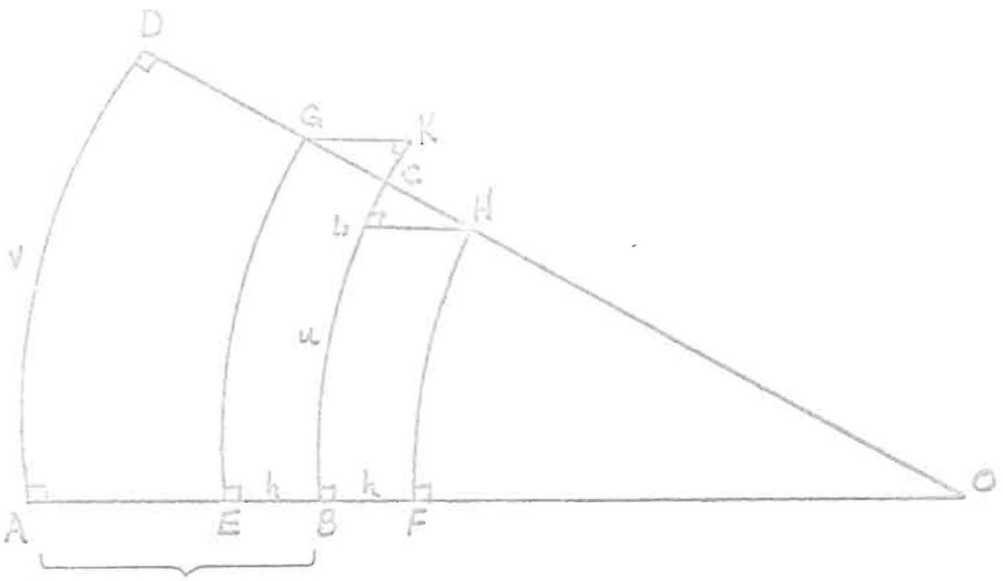

Figure 55

owed to approach zero, the ratio $\frac{u}{v}$ approaches from above a imit less than one, which is determined by the length of x nd is designated t(x). Notice that in applying Theorem 15 o Figure 55 and the preceding statement the pairs of variables x and y', u and x', and v and x must be interchanged. he pairs of variables do assume the same relative positions ith respect to the Lambert Quadrilateral and its obtuse ngle. It is the purpose of this section to prove the function t(x) is continuous. The proof follows.

Produce DC through C until it cuts AB produced through in O. The right angles at A and D make O the pole of AD nd each of AO and DO of length q. Measure off on AO, in ach direction from B, segments BE and BF of length h, where is less than the smaller of x and q-x. Draw the perpendiculars to AO at E and F intersecting DO in G and H, respecively. From G and H draw the perpendiculars GK and HL to C, BC extended in the case of GK.

Consider triangles CKG and CBO. Each have the angle t C congruent acute vertical angles. Segment CO greater than egment GC implies, as a consequence of Theorem 9, that

$$\frac{CK}{CG} < \frac{CB}{CO} \quad . \qquad \text{(i)}$$

ow CK = BK - BC = BK - u, so that making this substitution n (i), multiplying both sides of (i) by CG and dividing oth sides by v, the result is

$$\frac{BK-u}{v} < \frac{u}{v} \cdot \frac{CG}{CO} \quad . \qquad \text{(ii)}$$

hanging the left side of (ii) to the difference of two fractions, multiplying the first of those by GE/GE, and commuting n the right side gives

$$\frac{GE}{v} \cdot \frac{BK}{GE} - \frac{u}{v} < \frac{CG}{CO} \cdot \frac{u}{v} \qquad \text{(iii)}$$

ext, the limit of (iii) will be taken as v is allowed to pproach zero. But first, a closer look at Figure 55 and each f the terms of (iii) is necessary. As v approaches zero, he limit of GE/v is t(x-h). As mentioned before, the limit f u/v will be t(x). As v approaches zero, so does u. Lambert quadrilateral BKGE is obtuse-angled at G. Thus, as u pproaches zero, the limit of GE/BK is t(h). Now to analyze he ratio CG/CO. Triangle BCO has a right angle at B. As approaches zero point C approaches point B. Right triangle

CO implies CO $\geq$ CB. Theorem 11 implies that CG $\leq$ h. Thus,

$$\frac{CG}{CO} \leq \frac{h}{CB} \leq \frac{h}{q-x} \quad ,$$

nd the relationships just established makes the limit of iii) as v approaches zero

$$\frac{t(x-h)}{t(h)} - t(x) \leq \frac{h}{q-x} \cdot t(x) \ , \qquad (iv)$$

r, multiplying both sides of (iv) by t(h) gives

$$t(x-h) - t(x)\ t(h) \leq \frac{h}{q-x} \cdot t(x)\ t(h) \quad . \quad (v)$$

Inequality (v) establishes half of the desired reult. Proceeding as above, consider triangles CLH and CBO hich are right-angled at L and B, respectively. Angle BCD f Lambert quadrilateral ABCD obtuse implies angle BCO must e acute. Theorem 9, noting that segment CH is shorter than egment CO, implies

$$\frac{CL}{CH} < \frac{CB}{CO} \quad . \qquad (i')$$

ow CB = u and CL = u - BL, so that making this substitution n (i'), multiplying both sides by CH and dividing both sides y v, (i') becomes

$$\frac{u-BL}{v} < \frac{u}{v} \cdot \frac{CH}{CO} \quad . \qquad (ii')$$

gain changing the form of the left side of (ii'), multilying BL/v by FH/FH, the result is

$$\frac{u}{v} - \frac{BL}{FH} \cdot \frac{FH}{v} < \frac{u}{v} \cdot \frac{CH}{CO} \quad . \qquad (iii')$$

Before taking the limit of (iii') as v approaches ero, an examination of each term is necessary. The limit of /v as v approaches zero is t(x) and that of FH/v is t(x+h). ambert quadrilateral BLHF is obtuse-angled at H. Thus, the imit of FH/BL as v approaches zero is t(h). Now CH $\leq$ h by heorem 11 and triangle BCO having a right angle at B implies O = q - x $\leq$ CO. Thus,

$$\frac{CH}{CO} \leq \frac{h}{q-x} \quad ,$$

nd the relationships just established makes the limit of iii') as v approaches zero

$$t(x) - \frac{t(x+h)}{t(h)} \leq t(x) \cdot \frac{h}{q-x} \quad , \qquad (iv')$$

r, multiplying both sides of (iv') by t(h) gives

$$t(x)\, t(h) - t(x+h) \leq t(x)\, t(h) \cdot \frac{h}{q-x} \quad . \qquad (v')$$

Adding (v) and (v'), the result is

$$t(x-h) - t(x+h) \leq \frac{2h}{q-x} \cdot t(x)\, t(h) \; ,$$

hich, as a result of Theorem 15 and its proof, that is, since (x) and t(h) must each lie between zero and one, can be ritten

$$t(x-h) - t(x+h) < \frac{2h}{q-x} \quad .$$

This last inequality, in addition to knowing that t(x) s monotone, assures the continuity of t(x). For no matter ow small the difference between t(x-h) and t(x+h) is desired, his inequality implies there exists a value of h that will ake this difference less than any pre-assigned positive num-er. Thus, the continuity of t(x) is established [16;192, 193].

V. DEVELOPING A FUNCTIONAL EQUATION

The last section established the continuity of the unction t(x). Ultimately, it will be shown that the function (x) is equivalent to the cosine function. Before this can e done it will be necessary to establish a functional equa-ion involving t(x). It is the purpose of this section to evelop that equation.

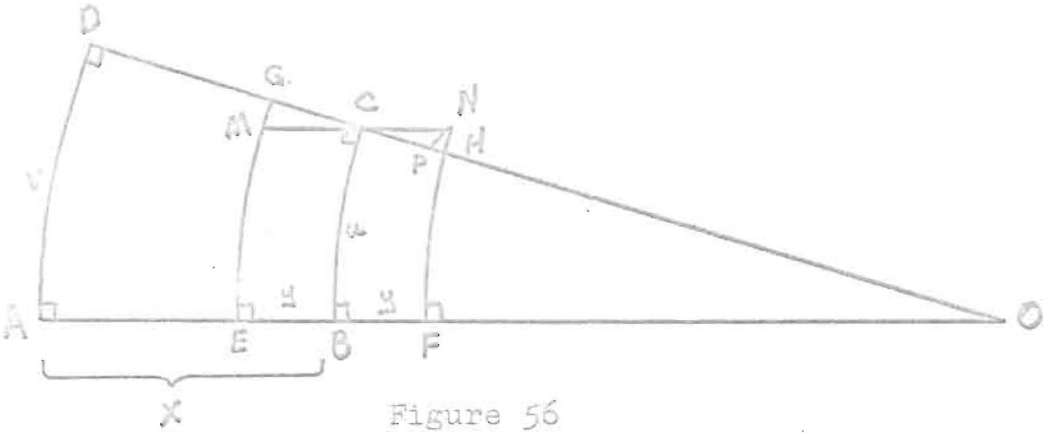

Figure 56

Proceeding in a manner similar to that of the pre-ious section, let ABCD be a Lambert Quadrilateral with ob-use angle at C. (Figure 56) Represent the lengths of AB, BC, nd AD by x, u, and v, respectively. Extend DC through C and B through B until they meet in O, the pole of AD. On seg-ent AO and at a distance y each side of B, where y is less han the smaller of x and q-x, locate points E and F as in igure 56. Draw the perpendiculars to AO at E and F, EG and H, where G and H lie on segment DO. Draw the perpendicular o BC at C intersecting EG at M and FH extended through H t N.

Consider Lambert quadrilaterals CBEM and CBFN. If egments CE and CF were drawn, triangles CBE and CBF are een to be congruent for the angles at B are right angles, E = BF, and BC = BC. Then angles CEG and CFN are congruent s are segments CE and CF and also angles ECM and FCN. Tri- ngles CEM and CFN are thus congruent by angle-side-angle aking CM = CN. As a consequence of Theorem 9, CH is longer han CG. Thus, there exists a point on segment CH, call it , where CG = CP. Draw PN. Now CP = CG, CM = CN, and the ertical angles GCM and PCN are congruent so that triangles CM and PCN are congruent by side-angle-side. Hence, GM s congruent to PN.

As v approaches zero, the angles GMC and HNC approach ight angles, angle PNH becomes infinitesimal as do segments N and HP. Wolfe([16;194])asserts, without proof, that PH s an infinitesimal of higher order than v. That assumption ill be made at this time.

Using the triangle inequality on right triangle PNH, ith right angle at P, gives

$$NH - NP < PH . \qquad (i)$$

ut, GM = MP, so (i) can be written

$$NH - GM < PH \,. \qquad \text{(ii)}$$

lso, NH = NF - HF and GM = GE - ME, so (ii) becomes

$$(NF-HF) - (GE-ME) < PH \qquad \text{(iii)}$$

ividing each term of (iii) by v and multiplying NF and ME ach by u/u gives

$$\frac{NF}{u} \cdot \frac{u}{v} - \frac{HF}{v} - \frac{GE}{v} + \frac{ME}{u} \cdot \frac{u}{v} < \frac{PH}{v} \,. \qquad \text{(iv)}$$

hus,

$$\lim_{v \to 0} \left(\frac{NF}{u} \cdot \frac{u}{v} - \frac{HF}{v} - \frac{GE}{v} + \frac{ME}{u} \cdot \frac{u}{v} \right) = 0 \,,$$

r

$$t(y)\, t(x) - t(x+y) - t(x-y) + t(y)\, t(x) = 0 \,,$$

ence,

$$t(x+y) + t(x-y) = 2\, t(x)\, t(y) \,. \qquad \text{(v)}$$

Equation (v) is the functional equation that will ake the most important conclusion of the next section possible [16;193, 194].

VI. THE FUNCTION t(x)

Several noteworthy properties of the function t(x) ave been observed. They include: t(x) is less than or qual to one; t(x) decreases as x increases; t(x) is equal o one when x is zero and is equal to zero when x is equal to ; and t(x) satisfies the condition imposed by equation (v) f the last section. These properties bring to mind the unction cos x. It is the purpose of this section to show hat

$$t(x) = \cos \frac{x}{k} \quad ,$$

here k is a constant depending upon the unit of length.

Mathematical induction will be the indicated method f proof, although the actual induction will not be carried ut here. Starting with a particular value of x, say x', ince t(x') is between zero and one, there exists a value of for which

$$t(x') = \cos \frac{x'}{k} \quad . \qquad (i)$$

Let x = px - x and y = x, then equation (v) of the ast section becomes

$$t(px-x+x) + t(px-x-x) = 2t(px-x)\, t(x) \quad , \text{ or}$$

$$t(px) = 2t\left[(p-1)x\right] t(x) - t\left[(p-2)x\right] . \qquad (ii)$$

Recall the trigonometric identity

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \quad . \qquad (iii)$$

et x = px and y = (p-2)x, then (iii) becomes

$$\cos px + \cos\left[(p-2)x\right] = 2 \cos\left[(p-1)x\right] \cos x \text{ , or}$$

$$\cos px = 2 \cos\left[(p-1)x\right] \cos x - \cos\left[(p-2)x\right] . \qquad (iv)$$

Equations (ii) and (iv) verify equation (i) when x' = px. It will be asserted, without proof, that since (ii) and (iv) hold for any value of p, that the following equation could be proven by mathematical induction. Thus,

$$t(nx') = \cos \frac{nx'}{k} \quad ,$$

here n is any positive integer. Likewise, it will be

sserted that

$$t\left(\frac{nx'}{2^m}\right) = \cos\frac{nx'}{2^m k} \quad ,$$

here m is any positive integer. Therefore, the relation

$$t(x) = \cos\frac{x}{k}$$

olds for every value of x, where $0 \leq x \leq q$, of the form $\frac{x'}{m}$. That it holds for every value of x between zero and q ollows from the continuity of the functions t(x) and cos x, nd from the realization that, by choosing m and n correctly, $\frac{x'}{m}$ can be made to differ from any value of x by as small an mount as is desired [16;194, 195].

VII. RELATING THE PARTS OF A RIGHT TRIANGLE

The relationships that will be developed in this sec-ion will be the basic relations of Elliptic Trigonometry. heir establishment will verify once again the sphere as a odel for Elliptic (Riemannian) Geometry. A detailed exami-ation of the parts of a right triangle will give the desired esults.

Let ABC be any right triangle with right angle at C nd restricted in size so that each side is less than q. In st, each segment of the figure is restricted to a length ess than q. Label the sides opposite the angles A, B, and a, b, and c, respectively. (Figure 57) Extend BA through

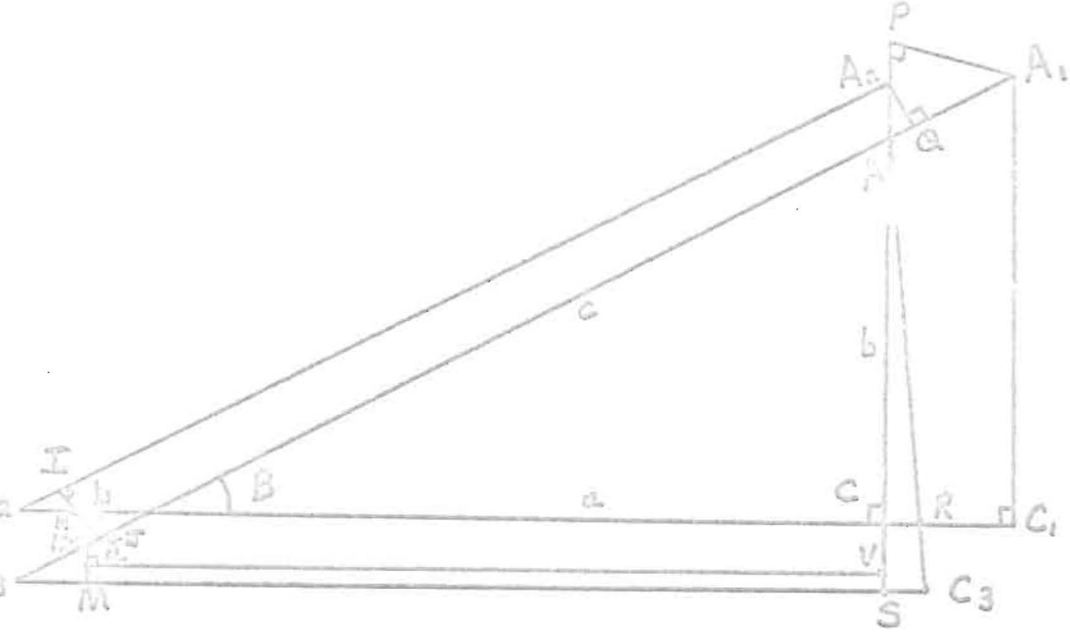

Figure 57

to a point A_1. Draw the perpendicular from A_1, A_1C_1, to C extended through C. Produce BC through B to B_2, where $C = BC_1$. Likewise, produce CA through A to A_2 so that $C = A_1C_1$. Draw A_2B_2. Now $A_1C_1 = A_2C$, angle ACB_2 = angle C_1B_1 and $BC_1 = B_2C$ implies that triangles A_2B_2C and A_1BC_1 re congruent by side-angle-side. Now $B_2C = BC_1$ so subracting BC from each of these gives $B_2B = CC_1$.

Extend AB through B to B_3 making $AB_3 = A_1B$. Construct at B_3 an angle congruent to angle A_1BC_1. Make B_3C_3, e side of angle B_3, equal to BC_1 and draw AC_3. Now

$B_3 = A_1B$, angle A_1BC_1 = angle AB_3C_3, and $BC_1 = B_3C_3$ implies that triangles A_1BC_1 and AB_3C_3 are congruent by side-angle-side. Also, $AB_3 = A_1B$ implies that $BB_3 = AA_1$.

From H, the midpoint of BB_2, draw the perpendicular HI to A_2B_2. Make J a point on segment BB_3 such that $B_2I = BJ$. Draw segment HJ. It will be shown that I, H, and J are collinear. The congruence of triangles A_2B_2C and A_1BC_1 implies that angle B_2 is congruent to angle A_1BC_1. Angle B_2BB_3, being vertical to angle A_1BC_1, is congruent to angle B_2. Also, H the midpoint of BB_2 implies $B_2H = BH$. Thus, triangles B_2IH and BJH are congruent by side-angle-side. Then angle B_2HI is congruent to angle BHJ, and since BHB_2 is a segment, these angles must be vertical angles making I, H, and J collinear. Note also that IJ is perpendicular to both A_2B_2 and AB_3.

From K, the midpoint of BB_3, draw KL perpendicular to segment BB_2. Make M a point on B_3C_3 such that $BL = B_3M$. Draw segment KM. Angle B_3 was constructed congruent to angle ABC, thus angle LBK is congruent to angle KB_3M. Also, KB_3 = KB, since K is the midpoint of BB_3, and it is seen that triangles BLK and B_3MK are congruent by side-angle-side. Then angle B_3KM is congruent to angle BKL and again, BKB_3 is a segment, it is seen that L, K, and M are collinear. Also, segment LM is perpendicular to both B_2C and B_3C_3.

Consider triangles ABC and A_1BC_1. Both have the
mon angle at B, and noting that A_1B is greater than AB,
eorem 7 then implies that angle BA_1C_1 is greater than angle
C. But, triangles AB_3C_3 and A_1BC_1 are congruent, so angle
C_3 is congruent to angle BA_1C_1 and thus angle BAC_3 is also
eater than angle BAC. This result implies that segment
$_3$ intersects segment BC_1 at a point R between C and C_1.
tend segment AC through C to intersect segment B_3C_3 in S.
nsider triangle ASC_3 which has a right angle at C_3. The
ght angle at C_3 implies that AS is greater than AC_3, but
$_3 = A_2C$, since triangles A_2B_2C and AB_3C_3 are congruent,
king CS greater than both of RC_3 and AA_2. The congruence
triangles A_2B_2C and AB_3C_3 is assured since both are con-
uent to triangle A_1BC_1. As a result, segment CV, equal to
$_2$, can be marked off on segment CS, the point V lying be-
een C and S. Draw segment VU, the perpendicular from V
segment LM, where U is between L and M. This is possible
nce V is between C and S and both of segments LC and MS
e perpendicular to segment LM, and each less than q, seg-
nt UV cannot intersect either of LC or MS between U and V,
r then a triangle with two right angles would be formed
ving all sides less than q. Finally, draw segment A_1P
rpendicular to segment AC extended through A and segment
Q perpendicular to segment AB extended through A [16;196, 197].

Equivalence of important limits. It is important to note that the lengths of segments AA_1, AA_2, BB_2, BB_3, and CC_1 are dependent upon each other, since all are determined by the choice of the length of segment AA_1.

First, consider triangles AA_1P and AA_2Q. Both have angle PAQ in common and they have right angles at P and Q, respectively. Now, as AA_1 approaches zero so does AA_2. Thus,

$$\lim \frac{A_1P}{AA_1} = \lim \frac{A_2Q}{AA_2} \quad , \qquad \text{(i)}$$

for each limit is the definition of sin A. (Section II, Chapter VI)

Next, consider triangles BLK and BJH. Both have angle HBK in common and have right angles at L and J, respectively. Now, as AA_1 approaches zero so will BB_2 and BB_3. Then also must BK and BH approach zero as AA_1 approaches zero. Thus,

$$\lim \frac{LK}{BK} = \lim \frac{JH}{BH} \quad , \qquad \text{(ii)}$$

for each is the definition of sin B. But, from previous results it was shown or given that

$$LK = KM,\ IH = HJ,\ B_2H = HB, \text{ and } B_3K = KB \ ,$$

which in turn gives

$$2LK = LM,\ 2JH = IJ,\ 2BH = BB_2, \text{ and } 2BK = BB_3 \ .$$

Thus, equation (ii) above can be written equivalently as

$$\lim \frac{LM}{BB_3} = \lim \frac{IJ}{BB_2} \quad . \qquad \text{(iii)}$$

viding equation (i) above by equation (iii) gives

$$\lim(A_1P/AA_1)/(LM/BB_3) = \lim(A_2Q/AA_2)/(IJ/BB_2) .$$

construction $AA_1 = BB_3$ and $CC_1 = BB_2$ so this last result n be written

$$\lim \frac{A_1P}{LM} = \lim \frac{A_2Q}{IJ} \lim \frac{CC_1}{AA_2} ,$$

ich can also be expressed as

$$\lim \frac{A_2Q}{IJ} = \lim \frac{A_1P}{CC_1} \lim \frac{AA_2}{LM} . \qquad \text{(iv)}$$

ch of these limits will be examined in detail. It will be ecessary to make use of the important inequality,

$$t(y) > \frac{x'}{x} > t(y') , \qquad \text{(v)}$$

stablished in the proof of Theorem 15 in Section III of is chapter. Recall that x' and y include the obtuse angle. or convenience, it will be referred to as inequality (v) what follows.

First, consider Lambert quadrilateral A_2IJQ which has e obtuse angle at A_2, since the others are known to be ght angles. Then, letting

$$y' = JQ,\ x' = A_2Q,\ x = IJ, \text{ and } y = IA_2 ,$$

nequality (v) gives

$$t(JQ) < \frac{A_2Q}{IJ} < t(IA_2) .$$

s AA_1 is allowed to approach zero both JQ and IA_2 approach B. Since the function t(x) is known to be continuous, both (JQ) and $t(IA_2)$ approach t(AB). Hence,

$$\lim \frac{A_2Q}{IJ} = t(AB) . \qquad \text{(vi)}$$

Next, consider Lambert quadrilateral A_1PCC_1 which has btuse angle at A_1, since the others are known to be right ngles. Then, letting

$$y' = PC,\ x' = A_1P,\ x = CC_1, \text{ and } y = A_1C_1 ,$$

inequality (v) gives

$$t(PC) < \frac{A_1P}{CC_1} < t(A_1C_1) .$$

As AA_1 approaches zero PC and A_1C_1 each approach AC. Again, s a result of the continuity of the function $t(x)$, both $t(PC)$ and $t(A_1C_1)$ approach $t(AC)$. Hence,

$$\lim \frac{A_1P}{CC_1} = t(AC) . \qquad \text{(vii)}$$

Finally, the third limit will be examined. The congruence of triangles A_2B_2C and AB_3C_3 established previously mplies that $A_2C = AC_3$. In triangle ACR it is seen that AC s less than AR, for angle ACR is a right angle. Thus, RC_3 s less than AA_2, and as a result

$$\frac{AA_2}{LM} > \frac{RC_3}{LM} .$$

onsider Lambert quadrilateral LMC_3R, which has obtuse angle t R, since the others are known right angles. Then, letting

$$y' = MC_3,\ x' = RC_3, \text{ and } x = LM ,$$

nd observing that as AA_1 approaches zero so will BB_3 and consequently LM, the second part of inequality (v) implies that

$$\frac{AA_2}{LM} > \frac{BC_3}{LM} > t(MC_3) . \qquad \text{(viii)}$$

Now, LU is less than LM and CV = AA_2 which results in the inequality

$$\frac{AA_2}{LM} < \frac{AA_2}{LU} = \frac{CV}{LU} .$$

Consider Lambert quadrilateral LUVC, which has obtuse angle at V, since the others are known right angles. Then, letting

$$y = UV, \ x' = CV, \text{ and } x = LU ,$$

and observing that as AA_1 approaches zero so will BB_3, LM, and LU, the first part of inequality (v) implies that

$$\frac{AA_2}{LM} < \frac{AA_2}{LU} = \frac{CV}{LU} < t(UV) . \qquad \text{(ix)}$$

As AA_1 approaches zero both UV and MC_3 approach BC. Again the continuity of t(x) implies that both $t(MC_3)$ and t(UV) approach t(BC). Thus, relations (viii) and (ix) imply the result

$$\lim \frac{AA_2}{LM} = t(BC) . \qquad \text{(x)}$$

Thus, substituting in the results of equations (vi), (vii), and (x) into equation (iv), the most important result

$$t(AB) = t(AC)\, t(BC) \qquad \text{(xi)}$$

is obtained [16;197, 198].

<u>Trigonometric relationships for the parts</u>. In

Section VI of this chapter it was shown that

$$u(x) = \cos \frac{x}{k}$$

for all x. Applying this relationship to equation (xi) above and recalling that

$$AB = c,\ AC = b,\ \text{and}\ BC = a\ ,$$

equation (xi) becomes

$$\cos \frac{c}{k} = \cos \frac{b}{k} \cos \frac{a}{k}\ . \qquad (1')$$

Equation (1') relates the three sides of a right triangle. This result leads to the important conclusion that the Pythagorean Formula holds for infinitesimal right triangles in Elliptic Geometry. A further result is that the trigonometric functions of angles, as defined in Section II, are connected by the familiar relationships of Euclidean Trigonometry, as

$$\sin^2 A + \cos^2 A = 1\ ,$$

$$\tan^2 A + 1 = \sec^2 A,\ \text{etc.}$$ [;198].

To obtain additional formulas for the parts of a right triangle ABC, extend side CA through A a convenient distance u to a point P. (Figure 58) Draw segment PQ, the perpendicular from P to side BA extended through A. Let v = PQ and w = AQ. Also, draw segment BP.

Applying equation (1') above to the sides of right triangle BCP results in

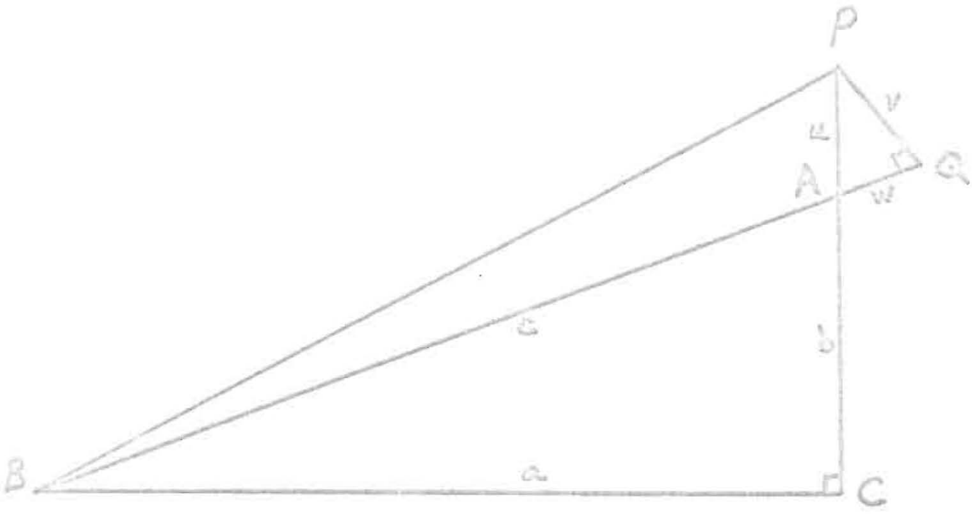

Figure 58

$$\cos(BP/k) = \cos(a/k)\cos((b+u)/k) .$$

olving (1') for cos(a/k) and substituting in that result,

lus using the trigonometric identity

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

n cos((b+u)/k), the result is

$$os(BP/k) = \frac{\cos(c/k)}{\cos(b/k)}\left[\cos(b/k)\cos(u/k) - \sin(b/k)\sin(u/k)\right] ,$$

hich, upon distributing the multiplication on the right,

ecomes

$$os(BP/k) = \cos(c/k)\cos(u/k) - \cos(c/k)\tan(b/k)\sin(u/k). \quad (i)$$

Applying equation (1') to the sides of right tri-

ngle BPQ the result is

$$\cos(BP/k) = \cos(v/k)\cos((c+w)/k) . \quad (ii)$$

pplying equation (1') to the sides of right triangle APQ

he result is

$$\cos(u/k) = \cos(w/k)\cos(v/k)\text{ , or}$$

$$\cos(v/k) = \cos(u/k)/\cos(w/k)\ . \qquad (iii)$$

bstituting the result of equation (iii) into equation (ii) r $\cos(v/k)$ and using the addition identity on $\cos((c+w)/k)$, e result is

$$s(BP/k) = \frac{\cos(u/k)}{\cos(w/k)}\left[\cos(c/k)\cos(w/k) - \sin(c/k)\sin(w/k)\right]\ ,$$

ich, upon distributing the multiplication on the right, comes

$$s(BP/k) = \cos(c/k)\cos(u/k) - \cos(u/k)\tan(w/k)\sin(c/k).(iv)$$

uating the right sides of equations (i) and (iv), the some- at simplified result is

$$s(c/k)\tan(b/k)\sin(u/k) = \cos(u/k)\tan(w/k)\sin(c/k)\ .$$

viding each side of the last result by $\sin(c/k)\sin(u/k)$ ves

$$\frac{\tan(b/k)}{\tan(c/k)} = \frac{\tan(w/k)}{\tan(u/k)}\ . \qquad (v)$$

r very small x, tan x is approximately equal to x. Thus, u is allowed to approach zero w will also approach zero d $\tan(w/k)/\tan(u/k)$ can be approximated by $(w/k)/(u/k)$, or /u), as u approaches zero. But, by definition, in triangle Q,

$$\cos A = \lim_{u \to 0} \frac{w}{u}\ .$$

nce, equation (v) becomes

$$\cos A = \frac{\tan(b/k)}{\tan(c/k)}\ . \qquad (2')$$

Equation (2') relates an acute angle of a right tri-gle and the two sides which are not opposite it. Using mmetry on equation (2'), the relationship between B, a, d c can be expressed

$$\cos B = \frac{\tan(a/k)}{\tan(c/k)} . \qquad (3')$$

Returning to equation (2'), if both sides are squared d $1 - \sin^2 A$ substituted for $\cos^2 A$, the result can be written

$$\sin^2 A = 1 - \frac{\tan^2(b/k)}{\tan^2(c/k)} = \frac{\tan^2(c/k) - \tan^2(b/k)}{\tan^2(c/k)} ;$$

, since $\tan^2 x = \sec^2 x - 1$,

$$\sin^2 A = \frac{\sec^2(c/k) - 1 - (\sec^2(b/k) - 1)}{\tan^2(c/k)} ,$$

$$\sin^2 A = \frac{\sec^2(c/k) - \sec^2(b/k)}{\tan^2(c/k)} ,$$

ich, upon multiplying by $\cos^2(c/k)/\cos^2(c/k)$ on the right de, becomes

$$\sin^2 A = \frac{1 - \cos^2(c/k)/\cos^2(b/k)}{\sin^2(c/k)} , \text{ or}$$

$$\sin^2 A = \frac{1 - \cos^2(a/k)}{\sin^2(c/k)} = \frac{\sin^2(a/k)}{\sin^2(c/k)}$$

nce $\cos(a/k) = \cos(c/k)/\cos(b/k)$ from equation (1') and $\sin^2 x + \cos^2 x = 1$. Taking the principal square root of both ides of the last result gives

$$\sin A = \frac{\sin(a/k)}{\sin(c/k)} . \qquad (4')$$

uation (4') relates an acute angle, the side opposite it and the side opposite the right angle. Using symmetry on uation (4') the relationship between B, b, and c can be expressed

$$\sin B = \frac{\sin(b/k)}{\sin(c/k)} . \qquad (5')$$

Thus,

$$\cot A = \frac{\cos A}{\sin A} = \frac{\tan(b/k)}{\tan(c/k)} \cdot \frac{\sin(c/k)}{\sin(a/k)} ,$$

ing equations (2') and (4'). Simplifying the right side of is last equation can be carried out using basic trigonometric entities and equation (1') as follows:

$$\cot A = \frac{\sin(b/k)\ \cos(c/k)\ \sin(c/k)}{\cos(b/k)\ \sin(c/k)\ \sin(a/k)} ,$$

$$\cot A = \frac{\sin(b/k)\ \cos(a/k)\ \cos(b/k)}{\cos(b/k)\ \sin(a/k)} ,$$

$$\cot A = \sin(b/k)\ \cot(a/k) . \qquad (6')$$

uation (6') relates an acute angle and the two sides not posite the right angle. By symmetry,

$$\cot B = \sin(a/k)\ \cot(b/k) . \qquad (7')$$

Consider the identity cos A = cot A sin A. From uation (6') and (4'),

$$\cos A = \frac{\sin(b/k)\ \sin(a/k)}{\tan(a/k)\ \sin(c/k)} = \cos(a/k)\ \frac{\sin(b/k)}{\sin(c/k)} ,$$

ich upon using equation (5') gives

$$\cos A = \cos(a/k)\ \sin B . \qquad (8')$$

[Eq]uation (8') rel[at]es an acute angle, its opposite side, and [t]he other acute angle. By symmetry,

$$\cos B = \cos(b/k) \sin A. \qquad (9')$$

Consider equation (1').

$$\cos(c/k) = \cos(a/k) \cos(b/k) .$$

[S]olving equations (8') and (9') for cos(a/k) and cos(b/k), [r]espectively, gives

$$\cos(a/k) = \frac{\cos A}{\sin B} \text{ and } \cos(b/k) = \frac{\cos B}{\sin A} .$$

[S]ubstituting these results into equation (1') gives

$$\cos(c/k) = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A} \text{, or}$$

$$\cos(c/k) = \cot A \cot B . \qquad (10')$$

A comparison of equations (1') through (10') with [f]ormulas (1) through (10) for right spherical triangles de[v]eloped in Chapter II shows that they are exactly the same, [a]lthough in a different order. The correspondence is as [f]ollows:

(1) corresponds to (1') ;
(2) corresponds to (4') ;
(3) corresponds to (2') ;
(4) corresponds to (6') ;
(5) corresponds to (5') ;
(6) corresponds to (3') ;
(7) corresponds to (7') ;
(8) corresponds to (9') ;
(9) corresponds to (8') ;
(10) corresponds to (10') .

[I]t must also be recognized that in (1) through (10), a, b, and [c] represent the <u>angular</u> measure of the sides of spherical tri-ngle ABC; while in (1') through (10') a, b, and c represent he actual length, in some "linear" unit, of the sides of riangle ABC. But, recall that the length of an arc of a reat circle on a sphere is given by

$$s = r\theta ,$$

here s is in linear units, r is the radius of the sphere in he same linear units, and θ is the angular measure of the entral angle in radians that subtends the arc. Solving the bove relation for θ gives

$$\theta = \frac{s}{r} ,$$

here upon replacing s by a, b or c and r by k, it is seen hat equations (1') through (10') are identical to those for right spherical triangle on a sphere of radius k [16;198-200].

VIII. CONCLUSION

The conclusion reached in the last part of the previous section is the desired goal of this work. Spherical rigonometry and Elliptic Trigonometry have been shown to be he same in the case of right triangles. Wolfe [16;196], Somerville [13;120], and Coxeter [2;232] indicate that the investigations carried out in Chapter VI would still hold without he restriction of segments to less than q and triangles to ight triangles. To the writer it seems amazing that such ontrasting starting points, Euclidean versus Elliptic, could ossibly end up with equivalent results.

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