BOREL SETS AND BAIRE FUNCTIONS

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CHAPTER I

INTRODUCTION

The concepts of open set, closed set, functions, continuity and sequences of functions are the backbone of modern analysis. The purpose of this thesis is to investigate a system of sets called the Borel sets and a closely related system or classification of certain real functions called the Baire functions.

The Borel sets are all the sets that can be obtained from the closed and open sets by repeated application of the set operations of union and intersection of sets.

The Baire functions are classified according to the nature or form of limit functions of a sequence of functions. The discussion begins with the somewhat familiar continuous functions and sequences of such functions.

I. ORGANIZATION

The first chapter is devoted to the definition of a few basic terms used in the discussion as well as a brief summary of several other more general topics. The two major areas of concern are those of ordinal numbers and transfinite induction.

The topic of discussion in the second chapter is the Borel sets. This section includes a definition of the Borel sets followed by statements, in the form of proven theorems, describing the most basic properties of these sets. Among the interesting results of this section are the fact that all the Borel sets, like the closed and open sets, are complements of one another. The system of Borel sets also answers a big question about what kinds of sets are obtained from closed or open sets. The major conclusion of this chapter is that the Borel sets form the smallest system of sets which contain all the open and closed sets, and the union or intersection of any denumerable number of sets in the system is also in the system. The discussion is limited to union and intersection of denumerable numbers of sets.

Chapter III is devoted to a discussion of the Baire functions and some related topics. The early part of the chapter includes a few definitions, theorems and comments concerning continuity of functions. The definition and properties of the Baire functions are the body of this section. The properties of Baire functions that are mentioned are shown to be generalizations of properties commonly known about continuous functions.

Some of the relationships that are known to exist between the Borel sets and the Baire functions are included in the fourth chapter. The chapter includes the definition of sets associated with a function and several theorems relating these sets and functions. The two major theorems of this chapter prove that a function of any prescribed class is a Baire function of that class if and only if all the sets associated with that function are Borel sets of the same finite Borel type.

The fifth and final chapter is devoted to a brief summary of Chapters II, III, and IV; conclusions, and some suggestions for further study. In addition, each chapter includes a brief summary of the important items of the chapter and some pertinent observations about the work thus far.

II. DEFINITIONS

It is assumed that the reader has had some work related to the concepts of sets, operations with sets, functions, limits, continuity, cardinal and ordinal numbers and the properties of the real number system. Any good course in analysis, topology or functions would provide the reader with a sufficient background in the above areas.

Unless otherwise stated, it is to be assumed that the domain and range of any function under discussion is the set of real numbers. All sets considered will be sets of real numbers.

It will be necessary on occasion to state additional definitions and introduce new concepts; however, a few definitions are given here as a basic foundation.

Definition 1.1 Two sets are said to be equivalent if and only if there is a one-to-one correspondence between the elements of the two sets.

Definition 1.2 A set S is said to be denumerable if and only if it is equivalent to the set of positive integers. An infinite set that is not denumerable is said to be non-denumerable.

Definition 1.3 A neighborhood of a point P is an open interval I such that $P \in I$.

Definition 1.4 A point P is an interior point of a set G if and only if there is a neighborhood I of P such that $I \subset G$.

Definition 1.5 A set G is open if and only if every point of G is an interior point.

Definition 1.6 A point P is said to be a limit point of a set S if and only if every neighborhood of P contains a point of S which is different from P.

Definition 1.7 A set H is said to be closed if and only if it contains all of its limit points. A set is also closed if its complement is open.

Definition 1.8 If S and T are sets and $T \subset S$, then the set difference, S - T, is the set of all x such that $x \in S$ and $x \notin T$.

Definition 1.9 A set $E \subset S$ is said to be open relative to S if and only if every $x \in E$ has a neighborhood I such that $(I \cap S) \subset (I \cap E)$.

Definition 1.10 A set $F \subset S$ is closed relative to S if and only if every limit point of F which is in S is in F.

III. ORDINAL NUMBERS

It is neither necessary nor desireable to give a thorough treatment of ordinal numbers; however, a few concepts and properties are needed. These concepts and properties are only mentioned here in order to give their use more meaning at a later time. For a more adequate treatment of ordinal numbers, cardinal numbers and well ordered sets the following are suggested: Goffman,[1] Wilder[2] or Hobson.[3]

The property of ordinal numbers that is of primary interest is that they form a well ordered set. This well ordering property yields several important results used in definitions and proofs throughout the course of developement. The first of these is that every subset of ordinal numbers has a first element. The subset of ordinals that is of interest here is the set of all ordinals with finite or denumerable cardinal number. The smallest non-denumerable ordinal number will be denoted by $\underline{W}$. It is a limiting ordinal number in the sense that it has no immediate predecessor. The order type or ordinal number of the set of positive integers is the first or smallest ordinal number with infinite cardinal number and is also a limiting ordinal number. Another important result of the well ordering of the ordinal numbers is the

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[1]Casper Goffman, Real Functions (New York: Rinehart & Company, Inc., 1953), Chapter 2 and Chapter 10.

[2]Raymond L. Wilder, Introduction to the Fundamentals of Mathematics (New York: John Wiley and Sons, Inc., 1960), pp. 115-124.

[3]E. W. Hobson, Theory of Functions of a Real Variable and Theory of Fourier's Series (Cambridge: Cambridge University Press, 1907), Chapters 2-6.

fact that every element has an immediate successor which gives the ordinals the induction property which follows in Theorem 1.1.

The set of all ordinal numbers of finite or denumerable cardinal number will be used frequently in the definitions, theorems, proofs, and discussions which follow. The set of all such ordinals will be used frequently as an index set in the treatment of Borel sets as well as Baire functions. The set will normally be denoted by, "the set of all $w < W$".

IV. TRANSFINITE INDUCTION

There are two forms or principles of finite induction for the positive integers. However, only the second of these is of interest here.

<u>Definition 1.11</u> <u>The Second Principle of Finite Induction</u>

If S is a set of positive integers such that:

(1) $1 \in S$

(2) If all positive integers less than n are in S implies that n is in S, then S is the set of all positive integers.

As was mentioned earlier, the set of all ordinal numbers form a well ordered set. As is shown in the following theorem, the second principle of finite induction generalizes to any well ordered set and in particular to the set of all ordinals less than a given ordinal w.

Theorem 1.1 If S is the set of all ordinal numbers less than a given ordinal w, and $T \subset S$ such that:

(1) The first ordinal, 1, is in T

(2) for all $u < w$, if all ordinals less than u are in T implies u is in T

then S = T.

Proof: Suppose $S \neq T$. Since by hypothesis $T \subset S$, we may assume that S - T is non-empty. S - T being non-empty implies there is some element a in S - T. Since S is a well ordered set and S - T is a subset of S, S - T has a first element. We have however that $1 \in T$ by hypothesis; hence $a \neq 1$. Now by part (2) of the hypothesis, if all ordinals less than a are in T, then a is in T. This contradiction to S - T being non-empty establishes the fact that S = T.

This theorem is called the principle of transfinite induction and applies to any well ordered set. This principle will be used frequently both in definition of Borel sets and Baire functions and in the proof of many of the theorems.

CHAPTER II

BOREL SETS

The Borel sets are all the sets that may be obtained from the closed and open sets by repeatedly applying the operations of union and intersection to denumerable numbers of sets.[4]

The closed sets are sets of type F_0. The union of a denumerable number of sets of type F_0 yields a set of type F_1. The sets obtained by taking the intersection of any denumerable number of sets of type F_1 are the sets of type F_2. This process is continued to define sets of type F_u, for every $u < w$. The sets of type F_w are then defined as the union or intersection of any denumerable number of sets of type F_u, for $u < w$. The operation is union if w is odd and intersection if w is even. Since the first ordinal, 1, is odd and the ordinals are alternately even and odd, this process defines the Borel sets for all $w < W$. In the case that w is the limiting ordinal it is designated as even, thus every ordinal less than W is designated as even or odd.

For $w = W$ no new sets of the type in question are obtained. This means that every set of type F_W is of type

[4]Casper Goffman, Real Functions (New York: Rinehart & Company, Inc., 1953), page 134.

F_w for some $w < W$. To see this consider any set S of type F_W. Since W is even, being the limiting ordinal, $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is of type w_n and each w_n is of type less than W. By definition of the sets themselves, each w_n is of denumerable cardinal number, hence the least upper bound of the set $\{w_n \mid n = 1,2,3,\ldots\}$ is of denumerable cardinal number and S is of type F_w for some $w < W$.

Thus by transfinite induction the sets of type F_w are defined for all ordinals $w < W$.

An analogous system of sets are obtained by beginning with the open sets which are said to be of type G_0. The intersection of any denumerable number of sets of type G_0 is a set of type G_1. The sets of type G_2 are obtained by taking the union of any denumerable number of sets of type G_1. If the sets of type G_u have been defined for all $u < w$, the sets of type G_w are formed by taking the union or intersection of a denumerable number of sets of type G_u for $u < w$. The operation is union if w is even and intersection if w is odd. By transfinite induction, the sets of type G_w are thus defined for all $w < W$.

The sets of types F_0, F_1, $F_2, \ldots F_w, \ldots$; for all $w < W$ and those of types G_0, G_1, $G_2, \ldots, G_w, \ldots$; for all $w < W$ are called the Borel sets.

It is seen from the definition that all the Borel sets are obtained ultimately from the closed and open sets. In view of this definition it seems reasonable to expect the Borel sets to retain some of the properties of the closed and open sets. This is indeed the case. The remainder of this chapter is devoted to the investigation of these somewhat generalized properties.

One form of the definition of a closed set is that a given set is closed if and only if its complement is open. This property generalizes to all the Borel sets as is shown by the following theorem.

Theorem 2.1 The complement of every set of type F_w is of type G_w, and the complement of every set of type G_w is of type F_w.

Proof: Use transfinite induction. Since by the definition the sets of type F_0 and G_0 are closed and open respectively, the statement is true for $w = 0$. Assume it holds for all $u < w$ and further that w is even. Let S be any set of type F_w. From the definition of a set of type F_w for w being even, $S = \bigcap_{n=1}^{\infty} S_n$ where each S_n is of lower type than w. (We say lower type and not preceeding type in view of the limiting ordinal). By the induction hypothesis, the complement of each S_n (denoted by $C(S_n)$) is of type G_u, for $u < w$. But $C(S) = \bigcup_{n=1}^{\infty} C(S_n)$ by DeMorgan's Law, and therefore C(S) is of type G_w, being the union of a denumerable number of sets of lower type. If w is odd, $S = \bigcup_{n=1}^{\infty} S_n$ and $C(S) = \bigcap_{n=1}^{\infty} C(S_n)$ again by DeMorgan's Law and $C(S)$ is of type G_w, hence by transfinite induction, for all $w < W$, the complement of every set of type F_w is of type G_w.

Proof: (part 2) The complement of every set of type G_w is of type F_w. Now suppose that w is even and S is any set of type G_w. By the definition of a set of type G_w for w even, $S = \bigcup_{n=1}^{\infty} S_n$ where again each S_n is of type G_u for $u < w$. By the induction hypothesis the complement of each S_n is of type F_u and $C(S) = \bigcap_{n=1}^{\infty} C(S_n)$; hence, $C(S)$ is of type F_w from the definition of a set of type F_w for w even. If w is odd, as above, replace union by intersection in the definition of S and the proof is similar. Hence the complement of every set of type G_w is of type F_w for all $w < W$ and the theorem is established.

Before proceeding, a review of some of the properties of union and intersection of denumerable sets are in order. The union of a denumerable number of sets each of which is denumerable is itself a denumerable set. This can be seen by considering any denumerable sequence of denumerable sets; $A_1, A_2, A_3, \ldots, A_n, \ldots$, where each set contains a denumerable number of elements. Then each set A_i for $i = 1,2,3,\ldots,n,\ldots$ may be written as: $A_i = a_{i1}, a_{i2}, \ldots, a_{in}, \ldots$. The familiar diagonal process may be used to illustrate the denumerability of the union of the set of A_i's. A similar result holds for the operation of intersection and follows readily from the definition of intersection of sets.

One property of open sets is that the union of any number of open sets as well as the intersection of any finite number of open sets is open. The intersection of any number as well as the union of any finite number of closed sets is closed, is a familiar property of closed sets. The extension of these properties to all of the Borel sets is accomplished in the following theorem.

<u>Theorem 2.2</u> If $w < W$ is even, the intersection of any denumerable number of sets of type F_w is of type F_w, and the union of any denumerable number of sets of type G_w is of type G_w.

<u>Proof:</u> For the sets of type F_w if w is zero, the theorem holds because of the property of closed sets cited immediately predeeding the statement of the theorem. Let w be even and let S be any set which is written, $S = \bigcap_{n=1}^{\infty} S_n$ where each S_n is of type F_w. Then by the definition of sets of type F_w for w being even, each S_n is the intersection of a denumerable number of sets of lower type which can be written as $S_n = \bigcap_{m=1}^{\infty} S_{nm}$, where each S_{nm} is of type F_u for $u < w$. Thus we have $S = \bigcap_{n=1}^{\infty} (\bigcap_{m=1}^{\infty} S_{nm})$, as the intersection of a denumerable number os sets, each denumerable and of lower

type than w, and hence S is of type F_w by definition.

For sets of type G_w, let $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is of type G_w. Then each $S_n = \bigcup_{m=1}^{\infty} S_{nm}$, where each S_{nm} is of type G_u for $u < w$. Now $S = \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} S_{nm})$, and since the union of a denumerable number of sets each denumerable is denumerable, S is the union of a denumerable number of sets of lower type and is of type G_w by definition of a set of type G_w for w even.

As a companion to this theorem we have the following one which states a similar result for odd ordinals.

<u>Theorem 2.3</u> For all odd ordinals $w < W$, the union of any denumerable number of sets of type F_w is of type F_w, and the intersection of a denumerable number of sets of type G_w is of type G_w.

<u>Proof</u>: Let $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is of type F_w. Then each S_n is the union of a denumerable number of sets of lower type; that is, $S_n = \bigcup_{m=1}^{\infty} S_{nm}$, where each S_{nm} is of type F_u for $u < w$. Hence, as in the proof of Theorem 2.2, $S = \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} S_{nm})$. Thus S, as the union of a denumerable number of sets of lower type, is of type F_w by definition of a set of that type.

The proof for sets of type G_w is again similar. All that is to be done is to replace even by odd and intersection by union and using the very same pattern that was used in the proof of Theorem 2.2.

In view of the two preceeding theorems concerning the set operations of union and intersection of Borel sets, the question arises as to the conditions under which these operations will always preserve the "type" of the set. As might be expected, the answer is very similar to that regarding the open and closed sets.

Theorem 2.4 For all $w < W$, the union and intersection of any two sets of type F_w is of type F_w.

Before proceeding with the proof, the following lemma is needed.

Lemma 2.1 For any set A and any denumerable collection of sets; $T_1, T_2, T_3, \ldots, T_n, \ldots$: (1) $A \cap (\bigcup_{n=1}^{\infty} T_n) = \bigcup_{n=1}^{\infty} (A \cap T_n)$, and (2) $A \cup (\bigcap_{n=1}^{\infty} T_n) = \bigcap_{n=1}^{\infty} (A \cup T_n)$.

Proof of lemma: Let $x \in A \cap (\bigcup_{n=1}^{\infty} T_n)$, then by the definition of the intersection of sets, $x \in A$ and $x \in \bigcup_{n=1}^{\infty} T_n$. If $x \in \bigcup_{n=1}^{\infty} T_n$, then $x \in T_i$ for some i. Now if $x \in A$ and $x \in T_i$ for some i, then $x \in A \cap T_i$ for the same i, and hence $x \in \bigcup_{n=1}^{\infty} (A \cap T_n)$. Hence $A \cap (\bigcup_{n=1}^{\infty} T_n) \subset \bigcup_{n=1}^{\infty} (A \cap T_n)$.

Now let $x \in \bigcup_{n=1}^{\infty} (A \cap T_n)$, then $x \in A \cap T_i$ for some i, again by definition of union. It follows that $x \in A$ and $x \in T_i$, and hence that $x \in A \cap (\bigcup_{n=1}^{\infty} T_i)$. Therefore we have shown that $\bigcup_{n=1}^{\infty} (A \cap T_n) \subset A \cap (\bigcup_{n=1}^{\infty} T_n)$, and by set inclusion both ways, part (1) of the lemma is established. The proof for part (2) is very similar and is not given.

Proof of Theorem 2.4: The theorem holds for the closed sets and hence for $w = 0$. Now let w be odd and assume the statement holds for all $u < w$. Let S and T be any two sets of type F_w. Then $S = \bigcup_{n=1}^{\infty} S_n$ and $T = \bigcup_{m=1}^{\infty} T_m$, where each S_n and T_m is of type $u < w$. Now $S \cap T = (\bigcup_{n=1}^{\infty} S_n) \cap (\bigcup_{m=1}^{\infty} T_m)$, and by application of Lemma 2.1, $S \cap T = \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} (S_n \cap T_m))$. Now, S_n and T_m are both of type F_u for $u < w$ and by assumption the intersection, $S_n \cap T_m$, is of type F_u. Hence $S \cap T$, as the union of a denumerable number of sets of lower type, is of type F_w by definition of such a set. Thus by transfinite induction the intersection of two sets of type F_w is a set of the same type for w being an odd ordinal. If w is even the theorem reduces to a special case of Theorem 2.2.

The case for the union of two sets of type F_w for w being even, $S \cup T$, is similar. The proof is accomplished by replacing union by intersection in each occurence and by using part (2) of the lemma. The case for the odd ordinals and the union of two sets reduces to a special case of Theorem 2.3.

Theorem 2.5 For every $w < W$, the union and intersection of two sets of type G_w is a set of type G_w.

Proof of Theorem 2.5: The theorem holds for the open sets and hence for $w = 0$. Now consider the union of two sets of type G_w. If w is even the theorem reduces to a special case of Theorem 2.2. Assume that w is odd and further that the statement holds for all $u < w$. Let S and T be any two sets of type G_w. Then $S = \bigcap_{n=1}^{\infty} S_n$ and $T = \bigcap_{m=1}^{\infty} T_m$, where S_n and T_m are of type G_u for $u < w$. Now $S \cup T = (\bigcap_{n=1}^{\infty} S_n) \cup (\bigcap_{m=1}^{\infty} T_m)$. Again by distribution of union over intersection, we get $S \cup T$ as the intersection of a denumerable number of sets each denumerable and hence is denumerable. Then $S \cup T$ as the intersection of a denumerable number of sets of lower type is of type G_w by definition of sets of type G_w for w being an odd ordinal. By transfinite induction the union of any two sets of type G_w is a set of tye same type.

The case for the intersection is again similar to the first part of the proof of Theorem 2.4, with F_w replaced by G_w, if w is even. If w is odd the theorem becomes a special case of Theorem 2.3..

The two theorems above can easily be generalized to the corresponding case for any finite number of sets by use of finite induction. To state this precisely and to summarize the above theorems, a single theorem may be stated.

Theorem 2.6 For all $w < W$ the union and intersection of any finite number of sets of type F_w (of type G_w) is a set of the same type.

The definition of open set as given in Chapter I leads immediately to the conclusion that every open set of real numbers is the union of a collection of open intervals. To see that every open set can be written as the union of a denumerable number of open intervals, it is only necessary to assign to each rational number in the set an open interval that contains it. Since the rational numbers are dense, this collection of open intervals covers the open set. Since the rational numbers are denumerable, these observations yield the result that every open set can be written as the union of a denumerable number of open intervals. In addition, every open interval may be written as the union of a denumerable number of closed intervals. To justify this it is only necessary to consider the open interval (a,b), where $a < b$. This interval can be written as the union of the closed intervals $\left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ where $n = 1,2,3,\ldots,$ and where $n > \frac{2}{b-a}$. Clearly these closed intervals are denumerable in number and their union will be (a,b).

The properties of open sets in the above paragraph, and the definition of Borel sets gives rise to the question of whether or not all of these sets are distinct or even needed. In the above arguments it was shown that every set of type G_0 could be written as a set of type F_1. Does this observation or a generalization of it hold for all of the Borel sets? It does, as is shown in the following theorem which shows that all the Borel sets are identical to the sets of types F_w only, or those of types G_w only.

Theorem 2.7 For all $w < W$, (1) every set of type G_w is of type F_{w+1}, and (2) every set of type F_w is of type G_{w+1}.

Proof, part (1): Since every open set can be written as the union of a denumerable number of closed sets the theorem holds for $w = 0$. Now let w be odd and assume that the statement holds for all $u < w$. Let S be any set of type G_w. Then $S = \bigcap_{n=1}^{\infty} S_n$ where each S_n is of type G_u for $u < w$. By hypothesis each set S_n is of type F_{u+1} for $(u+1) < (w+1)$. Since w is by assumption odd, $w + 1$ is even and $F_{w+1} = \bigcap_{n=1}^{\infty} F_{(u+1)_n}$ by definition of sets of type F_w for w being even. Thus $S = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} F_{(u+1)_n}$ is of type F_{w+1}. If w is even the proof is very similar. Observe that if w is even and S is a set of type F_w, then $w + 1$ is odd and F_{w+1} is written as the union of a denumerable number of sets of lower type, which is precisely the way S is written. Hence by transfinite induction every set of type G_w is of type F_{w+1}.

Proof, part (2): The proof is very similar to that of part (1) except for showing that every set of type F_0 is of type G_1. This follows by observing that every set of type G_0 is of type F_1 and that the complement of a set of type G_0 is a set of type F_0 and the complement of any set of type F_1 is a set of type G_1.

In working with open sets, closed sets, union and intersection of sets, the results of these operations do not always yield a set of the same "kind". For example, the intersection of the set of open intervals of the form $(-\frac{1}{n}, \frac{1}{n})$ for all positive integers n, yields a closed set, the point 0. In the preceeding paragraphs it was shown that the union of a number of closed sets was an open set. Thus the question arises concerning the structure of a system of sets, namely, under what conditions will a system of sets be "closed" with respect to the operations of union, intersection and complementation. In other words, when will the union or intersection of sets in the system of sets always yield another set of the same system. The answer lies in the system of Borel sets. This very important result, or property of the Borel sets, is summarized in the following form.

<u>Theorem 2.8</u> The Borel sets form the smallest system of sets such that:

(1) All the open and closed sets are in the system

(2) The union and intersection of any denumerable number of sets in the system is a set of the same system.

<u>Proof of Theorem 2.8</u> Let S be any system of sets which satisfies these two conditions. It will be shown that all of the Borel sets are in the system S. By part (1) of the hypothesis, all the sets of types F_0 and G_0 are in S. Assume that the sets of types F_u for all $u < w$ are in the system. Suppose w is even and let T be any set of type F_w. By definition then, $T = \bigcap_{n=1}^{\infty} T_n$ where each T_n is of lower type than F_w. By part (2) of the hypothesis, $\bigcap_{n=1}^{\infty} T_n$ and hence T, which is of type F_w, is in the system.

If w is odd and T is any set of type F_w, then by definition of F_w, $T = \bigcup_{n=1}^{\infty} T_n$ where each T_n is of lower type than F_w. Since each T_n is in the system by hypothesis or assumption and $\bigcup_{n=1}^{\infty} T_n$ is in the system by part (2) of the hypothesis, T and hence F_w is in the system. Thus by transfinite induction all the sets of type F_w are in the system for all $w < W$.

Since by Theorem 2.7 all the Borel sets are identical to the sets of types $F_0, F_1, F_2, \ldots, F_w, \ldots$, for all $w < W$, all the Borel sets are in the system S.

This theorem shows that any system of sets which is "self contained" with regard to the closed and open sets, with the operations of union and intersection on denumerable numbers of sets, must contain all the Borel sets and hence shows that the Borel sets from the smallest system of sets which satisfies those two properties.

The Borel sets are seen to be a very interesting system of sets. They are obtained by beginning with the open and closed sets, which are complements of one another, and be performing the set operations of union and intersection alternately to denumerable numbers of sets. The end result is a system of sets which behave very much like the sets from which they were obtained.

In the work that follows, the Borel sets will be shown generally to be related to the Baire functions in much the same manner as the closed and open sets are related to the continuous functions.

CHAPTER III

BAIRE FUNCTIONS

The properties of Baire functions as presented in this chapter are generally considered to be extensions or generalizations of properties of continuous functions.

No effort is made here to give a thorough treatment of continuous functions; however, a foundation of work with continuous functions is necessary. The material presented in the first section of this chapter serves only as a review for the reader and a gathering place for material needed at a later time.

1. CONTINUOUS FUNCTIONS

The work of this section depends very heavily on several concepts related to functions and in particular to the continuous functions. The most basic of these is the concept of a sequence.

<u>Definition 3.1</u> A <u>sequence</u>, $S = \{a_n\}$, of real numbers is a function which maps the set of positive integers into the set of real numbers. The n-th term of the sequence is the image of the integer n under the function.

<u>Definition 3.2</u> A sequence of real numbers, $\{a_n\}$, <u>converges</u> to a real number A if and only if given any real number $e > 0$

there is an integer N such that if $n > N$, then $|a_n - A| < e$. The number A is called the <u>limit</u> of the sequence. The standard notation for this is $\lim_{n \to \infty} a_n = A$.

If the function in the above definition has a set of functions as its range, then the sequence is called a sequence of functions. Thus for every real number a, the sequence is just a sequence of real numbers, $\{f_n(a)\}$.

<u>Definition 3.3</u> A sequence of functions, $\{f_n(x)\}$, is said to be convergent at a point a if and only if the sequence of real numbers $\{f_n(a)\}$ is convergent. The sequence converges "pointwise" on a set S if and only if it converges for every $a \in S$.

The continuous real functions or a real variable are of primary interest to those studying Calculus or analysis. Among the definitions of continuous functions that appear in textbooks are these:

<u>Definition 3.4</u> A function f(x) is <u>continuous</u> at a point a, of a set S, if and only if for every $e > 0$, there is a $d > 0$ such that whenever $|x - a| < d$, then $|f(x) - f(a)| < e$.

<u>Definition 3.5</u> A function f(x) is <u>continuous</u> at a point a, of a set S, if and only if for every sequence $\{a_n\}$ which converges to a, then the sequence $\{f(a_n)\}$ converges to f(a).

Definition 3.6 A function f(x) is continuous at a point a, of a set S, if and only if for every open set G containing f(a) there is an open set H containing a such that whenever $x \in H$ and x is in the domain of f, then $f(x) \subset G$.

Definition 3.7 In each case above the function f(x) is continuous on a set S if and only if it is continuous at every point of S.

Definition 3.8 A function f(x) is continuous on a set S if and only if for every open set G in the range of f, then $f^{-1}(G)$ is open.

It can be shown that Definitions 3.4, 3.5, 3.6 and 3.8 are equivalent for the real functions. Definitions 3.4 and 3.5 are shown to be equivalent in Goffman.[5] For the equivalence of 3.6 and 3.8 see Hall and Spencer.[6] Definitions 3.5 and 3.6 are also shown to be equivalent in Hall and Spencer.[7]

The properties of the continuous functions are many and varied. It is for this reason that the continuous functions are so important. Some of the elementary properties

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[5]Casper Goffman, Real Functions (New York: Rinehart & Co., Inc. 1953), page 82.

[6]Dick W. Hall and Guilford L. Spencer II, Elementary Topology (New York: John Wiley & Sons, Inc. 1955), page 48.

[7] Ibid., p. 47.

of continuous functions are: If $f(x)$ and $g(x)$ are both continuous and defined on the same domain, then the sum and product of the two functions is also continuous. If, in addition, $g(x)$ is never zero, then the quotient, $f(x)/g(x)$, is also continuous. Under the added condition that the range of g agrees with the domain of f, the composition function, $f(g(x))$, is also continuous. The proofs of these properties may be found in any textbook on elementary calculus.

<u>Definition 3.9</u> A sequence $\{f_n(x)\}$ of functions converges uniformly on a set S if and only if for every $e > 0$ there is an N such that for all $m,n > N$ and for every $x \in S$, it is true that $|f_m(x) - f_n(x)| < e$.

Is the limit function of a sequence of continuous functions continuous? Not necessarily, as is shown by the following example.

<u>Example 3.1</u> The function $f(x)$ defined on $(0,\infty)$ as: $f(x) = 0$ if $x = 0$, and $f(x) = 1$ if $x \neq 0$ may be used to illustrate this point. The sequence $\{f_n(x)\}$ of functions defined by: $f_n(x) = 1$ if $x \geq \frac{1}{n}$, and $f_n(x) = nx$ if $x < \frac{1}{n}$ is continuous on $(0,\infty)$. Also, $\lim_{n\to\infty} f_n(x) = f(x)$. However, the function $f(x)$ is not continuous at the point $x = 0$. Thus there is a sequence of continuous functions which converges to a function which is not continuous.

Definition 3.9 on the previous page may also be stated without using the Cauchy condition.

Definition 3.10 A sequence $\{f_n(x)\}$ of functions converges uniformly on a set S if and only if there exists a function $f(x)$ in which for every $e > 0$ there is an N such that for all $n > N$, and for all $x \in S$, $|f_n(x) - f(x)| < e$.

From the above definitions of pointwise convergence and uniform convergence, it is obvious that uniform convergence implies pointwise or ordinary convergence.

In contrast to the example on the previous page, it will be shown that if the sequence of continuous functions converges uniformly, the limit function will indeed be a continuous function.

Theorem 3.1 If $\{f_n(x)\}$ is a sequence of functions, continuous at every $a \in S$, and uniformly convergent on S to $f(x)$, then $f(x) = \lim\limits_{n\to\infty} f_n(x)$ is continuous at $a \in S$.

Proof: Let $e > 0$. Since $\{f_n(x)\}$ is uniformly convergent, there is an N for which $|f(x) - f_n(x)| < \frac{e}{3}$, for all $n > N$ and for all x. Since $f_n(x)$ is continuous at a, for every $n > N$ there is a $d > 0$ such that if $z \in S$ and $|z - a| < d$, then $|f_n(z) - f_n(a)| < \frac{e}{3}$. Now suppose $|z - a| < d$ and $z \in S$. Then

$$|f(z) - f(a)| = |f(z) - f_n(z) + f_n(z) - f_n(a) + f_n(a) - f(a)|$$

$$\leq |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)| \leq \frac{e}{3} + \frac{e}{3} + \frac{e}{3}$$

$= e$. Thus for every point $z \in S$ such that $|z - a| < d$, $|f(z) - f(a)| < e$ and $f(x)$ is continuous at the point $a \in S$.

Much of the material that follows depends on the concepts of a nowhere dense set and a set of the first category.

Definition 3.12 A set S is said to be nowhere dense if and only if its closure contains no open interval I, such that $I \subset S$.

Definition 3.13 A set S is said to be of the first category if and only if it is the union of a finite or denumerable number of nowhere dense sets. If a set is not of the first category, then it is said to be of the second category.

Theorem 3.2 The union of a finite or denumerable number of sets of the first category, is a set of the first category.
Proof: Consider $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is of the first category. Then for each n, $S_n = \bigcup_{m=1}^{\infty} S_{nm}$ where each S_{nm} is nowhere dense by definition of a set of the first category. Now, $S = \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} S_{nm})$ is the union of a denumerable number of denumerable unions which is the union of a denumerable number of sets. Since each set is nowhere dense, S is then by definition a set of the first category.

Theorem 3.3 Any set of type F_1 is either of the first category or it contains an interval as a subset.

Proof: If S is any set of type F_1, then $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is closed. (See the definition of Borel sets as given in Chapter II). If every S_n is nowhere dense, then S is of the first category by definition and the theorem is proven. If some one of the S_n is not nowhere dense, then there is a point x of S_n such that there is a neighborhood of x containing no points of S_n. (This is from the definition of a nowhere dense set). By the definition of a closed set then, x is not a point of S_n, and thus S_n contains an interval. Hence if some S_n is not nowhere dense, S_n contains an interval and so does $S = \bigcup_{n=1}^{\infty} S_n$. Thus if all the S_n are nowhere dense, S is of the first category, and if some one of them is not nowhere dense, S contains an interval and the theorem is established.

Theorem 3.4 If f(x) is any function defined on the real line, the set of points of discontinuity of f(x) is of type F_1.

Proof: Let D(f) be the set of points of discontinuity of f(x). Introduce the sets $D_n(f)$ for $n = 1,2,3,\cdots$. Let $z \in D_n(f)$ if and only if, for every neighborhood I of z, there exist x,y, elements of I, such that $|f(x) - f(y)| \geq \frac{1}{n}$. There are then two things to show: (1) $D(f) = \bigcup_{n=1}^{\infty} D_n(f)$ and (2) $D_n(f)$ is closed for every n.

<u>Proof of (1)</u>: Let $z \in \bigcup_{n=1}^{\infty} D_n(f)$, then there is an n such that $z \in D_n(f)$, by definition of union. So for every $d > 0$ there exists an x and a y such that $|x - z| < d$, $|y - z| < d$ and $|f(x) - f(y)| \geq \frac{1}{n}$. Thus there is a w such that $|w - z| < d$ and $|f(w) - f(z)| > \frac{1}{3n}$. Hence f(x) is not continuous at z and thus $z \in D(f)$.

If $x \in D(f)$ there is an $e > 0$ such that for every $d > 0$ there is an x with $|x - z| < d$ and $|f(x) - f(z)| > e$. There is an n such that $e > \frac{1}{n}$. Let I be any neighborhood of z. There is an $x \in I$ such that $|f(x) - f(z)| > e > \frac{1}{n}$. So $x \in D_n(f)$, and $x \in \bigcup_{n=1}^{\infty} D_n(f)$. As a result of the two preceeding paragraphs by set inclusion both ways, $D(f) = \bigcup_{n=1}^{\infty} D_n(f)$.

<u>Proof of (2)</u>: Let n be some positive integer and let z be a limit point of $D_n(f)$. There is by definition of limit point a sequence $\{z_m\}$, where each z_m is in $D_n(f)$ and $\lim_{m \to \infty} z_m = z$. Let I be any neighborhood of z. There is an m such that $z_m \in I$, so that I is a neighborhood of z_m. Since $z_m \in D_n(f)$, there are points $x, y \in I$ such that $|f(x) - f(y)| \geq \frac{1}{n}$. Now since I is an arbitrary neighborhood of z, it follows that $z \in D_n(f)$, and hence that $D_n(f)$ is closed since it must contain all of its limit points. Thus it has been shown that D(f) is the union of a finite or denumerable number of closed sets and is of type F_1 by definition.

<u>Definition 3.14</u> A sequence of functions $\{f_n(x)\}$ is said to

converge uniformly at a point z if and only if for every $e > 0$ there is a $d > 0$ and an N such that if $n,m > N$ and $|x - z| < d$, then $|f_n(x) - f_m(x)| < e$.

This definition says, in effect, that a sequence of functions converges uniformly at a point if and only if there is a neighborhood I of point z such that the sequence converges for all points x of I.

Theorem 3.5 If a sequence $f_n(x)$ of continuous functions converges to $f(x)$ on an open interval (a,b), it converges uniformly to $f(x)$ at some point $z \in (a,b)$.

Proof: If $z \in (a,b)$, there is an N such that for every $n,m > N$, $|f_n(z) - f_m(z)| \leq e$. Let E_N be the set of points of (a,b) for which this condition holds for N and e. Then $(a,b) = \bigcup_{N=1}^{\infty} E_N$. By a theorem which will be proven later, Theorem 4.2, for every n,m the set of points z for which $|f_n(z) - f_m(z)| \leq e$ is a closed set since $|f_n(z) - f_m(z)|$, being the sum of two continuous functions is continuous. Thus E_N is a closed set. But there is an N for which E_N is not nowhere dense; for, otherwise $(a,b) = \bigcup_{N=1}^{\infty} E_N$ would be of the first category which is not possible by Theorems 3.3 and 3.4. Since E_N is not nowhere dense it contains a closed interval. It has thus been shown that for every $e > 0$ there is an N and a closed subinterval $[a',b'] \subset (a,b)$ such that for every $z \in [a',b']$ and $n,m > N$, $|f_n(z) - f_m(z)| \leq e$.

Now there is an $[a_1,b_1] \subset (a,b)$ and an N_1 such that for $n,m > N_1$ and $x \in [a_1,b_1]$, $|f_n(x) - f_m(x)| \leq 1$. There is an $[a_2,b_2] \subset (a_1,b_1)$ and an N_2 such that for every $n,m > N_2$ and $z \in [a_2,b_2]$, $|f_n(z) - f_m(z)| \leq \frac{1}{2}$. Proceeding in this way, a sequence of closed intervals is obtained each of which is in the open interval obtained by deleting the end points of its predecessor, and a sequence of positive integers N_1, N_2, $N_3, \ldots, N_k$ such that for every k, for every $z \in (a_k,b_k)$ and for every $n,m > N_k$, $|f_n(z) - f_m(z)| \leq \frac{1}{k}$. But there is an $x \in \bigcap_{k=1}^{\infty} [a_k,b_k]$ so that $x \in \bigcap_{k=1}^{\infty} (a_k,b_k)$. The sequence $\{f_n(x)\}$ converges uniformly to f(x) at point x. To see this, let $e > 0$. There is a k such that $\frac{1}{k} < e$ and $x \in (a_k,b_k)$. For every $x \in (a_k,b_k)$ and $n,m > N_k$, $|f_n(x) - f_m(x)| \leq \frac{1}{k} < e$.[8]

<u>Theorem 3.6</u> The set of points of discontinuity of a function which is the limit of a convergent sequence of continuous functions is of the first category.

<u>Proof</u>: By Theorem 3.4 the set of points of discontinuity of such a function is of type F_1. By Theorem 3.3 any set of type F_1 is either of the first category or it contains an interval. Since the limit function of a uniformly convergent sequence of continuous functions is continuous, by

[8] Casper Goffman, <u>Real Functions</u> (New York: Rinehart & Co., Inc., 1953), pp. 108-109.

Theorem 3.5 the limit function has a point of continuity in every interval. Hence the set of points of discontinuity contains no interval, and is again by Theorem 3.3 of the first category.

II. BAIRE FUNCTIONS

The continuous functions are put into a classification of Baire class 0, which will be denoted by f_0. Functions which are limits of convergent sequences of continuous functions are of Baire class 1 or f_1. Functions which are limits of convergent sequences of functions of type f_1, are of Baire class 2 or f_2. This process is continued to define the Baire functions of Baire class w for all $w < W$. If for all $u < w$ the functions of Baire class u have been defined, the functions of type or class w are defined as limits of convergent sequences of functions of type f_u for $u < w$. By transfinite induction, this defines the Baire functions of every type or class w for all $w < W$.

The functions defined above are called the <u>Baire functions</u>. It should be noted that these Baire classes need not be disjoint in view of the way in which they were defined.

It has been proven by Lebesgue in <u>Sur les Functions Representables Analytiquement</u> that functions of each of the

Baire classes exist in the sense that it is possible to define a function of any prescribed Baire class.[9] In contrast it is also known that there do exist functions which do not belong to any Baire class.[10]

The statements and theorems that follow are a partial investigation of the properties of Baire functions. It will be seen that these properties follow quite closely those of continuous functions in a sense. The proofs of many of these properties depend rather heavily on the theory of limits which can be found in most textbooks in calculus, analysis, or topology. The proofs utilize the theory of transfinite induction and the first ordinal with nondenumerable cardinal number, W.

Theorem 3.7 For every $w < W$, the sum and product of two functions of type f_w is of type f_w.

Proof: In view of the properties on continuous functions the theorem holds for $w = 0$. Consider $w < W$ and assume the theorem holds for all $v < w$. Let $f(x)$ and $g(x)$ be any two functions of type f_w. Then by definition $f(x) = \lim_{n\to\infty} f_n(x)$

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[9] E. W. Hobson, Theory of Functions of a Real Variable and the Theory of Fourier's Series (Cambridge: Cambridge University Press, 1907), page 532.

[10] E. J. Townsend, Functions of Real Variables (New York: Henry Holt and Company, 1928), page 145.

and $g(x) = \lim_{n\to\infty} g_n(x)$, where for every n, $f_n(x)$ and $g_n(x)$ are of type f_v, for $v < w$. Now, $[f_n(x) + g_n(x)]$ and $[f_n(x) \cdot g_n(x)]$ are of type f_v, since the limit of the sum equals the sum of the limits and similarly for products. By definition then, $f(x) + g(x) = \lim_{n\to\infty} [f_n(x) + g_n(x)]$ and since $f_n(x) + g_n(x)$ is of type f_v by assumption for every n; $f(x) + g(x)$ is the limit of a convergent sequence of functions of lower type and is of type f_w by definition. The argument for the product, $f(x) \cdot g(x)$ is very similar. By transfinite induction the theorem is now proven.

Theorem 3.8 For every $w < W$, if $f(x)$ is of type w, and $f(x)$ is never 0, then $\frac{1}{f(x)}$ is of type f_w.

Proof: In view of the properties of continuous functions the theorem holds for $w = 0$, as a special case of the quotient rule for such functions. Let $f(x)$ be any function of type f_w. Then $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x)$ is of type $v < w$ for every n. Now, $\frac{1}{f(x)} = \lim_{n\to\infty} \left[\frac{1}{f_n(x)}\right]$ where each $\frac{1}{f_n(x)}$ is of type f_v for $v < w$ by assumption. Hence $\frac{1}{f(x)}$, as the limit of a convergent sequence of functions of lower type, is of type f_w by definition of a function of type f_w. By transfinite induction the proof is complete.

The properties of continuous functions may be used to suggest several other properties of functions of Baire class w for all $w < W$. Two of the most obvious properties deal with a real multiple of a Baire function and the absolute value of such functions. The next two theorems are given without proof, as they depend only on a small amount of algebraic manipulation of limits and simple application of transfinite induction.

Theorem 3.9 If $f(x)$ is any function of type f_w, then $cf(x)$ is of type f_w, where $w < W$ and c is any non-zero real number.

Theorem 3.10 For every $w < W$, if $f(x)$ is of type f_w, then $|f(x)|$ is of type f_w.

Theorem 3.11 For every $w < W$, if $f(x)$ and $g(x)$ are of type f_w, then the maximum ($f(x)$, $g(x)$) and the minimum ($f(x)$, $g(x)$) are of type f_w.

Proof: The functions $f(x) + g(x)$ and $f(x) - g(x)$ are of type f_w in view of Theorems 3.7 and 3.10. Thus the functions $\max(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}(f(x) - g(x))$ and $\min(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}(f(x) - g(x)$ are of type f_w, by using the above mentioned theorems along with Theorem 3.9.

The next concept is an extension of Theorem 3.1. This theorem states that the limit function of a uniformly convergent sequence of continuous functions is continuous. The theorem that follows here generalizes an analogous result for all Baire classes of functions.

<u>Theorem 3.12</u> For every $w < W$, the limit function of a uniformly convergent sequence of functions of type f_w, is of type f_w.

The proof of this theorem depends on the following lemma which deals with bounded Baire functions.

<u>Lemma 3.1</u> For every $w < W$, if $f(x)$ is of type f_w and $|f(x)| \leq k$ for every x, where $k > 0$; then $f(x) = \lim_{n \to \infty} f_n(x)$ where each $f_n(x)$ is of type f_u for $u < w$, and $|f_n(x)| \leq k$ for every x.

<u>Proof of lemma:</u> Since $f(x)$ is of type f_w, there is, by the definition of a function of that type, a sequence $\{g_n(x)\}$ such that $\lim_{n \to \infty} g_n(x) = f(x)$ where $g_n(x)$ is of type f_u for $u < w$. Let $h_n = \min(g_n(x), k)$ and $f_n(x) = \max(h_n(x), -k)$. By Theorem 3.11, each $f_n(x)$ is of type u for $u < w$. The process used to define $f_n(x)$ shows that $f(x) = \lim_{n \to \infty} f_n(x)$. Thus, also, $|f_n(x)| \leq k$ and the lemma is proven.

<u>Proof of Theorem 3.12:</u> In view of the similar property for continuous functions, Theorem 3.1, the theorem holds for $w = 0$. Suppose it holds for all $u < w$. Let $\{f_n(x)\}$ be a uniformly convergent sequence of functions of type f_u, and let $f(x) = \lim_{n \to \infty} f_n(x)$. Now, $f(x) = \sum_{n=1}^{\infty}(f_{n+1}(x) - f_n(x)) + f_1(x)$ where the series converges uniformly to $f(x)$. Hence there is a convergent series $\sum_{n=1}^{\infty} k_n$ of positive integers such that

for every n and x $|f_{n+1}(x) - f_n(x)| \leq k_n$.

For every n the function $(f_{n+1}(x) - f_n(x))$ is of type f_u. Hence there is a sequence $\{f_{nm}(x)\}$ of functions of lower type than f_u which converges to each function $(f_{n+1}(x) - f_n(x)$, such that for every m and every x, $|f_{nm}(x)| \leq k_n$ by Lemma 3.1. Consider the sequence $\{g_n(x)\}$ defined by: $g_1(x) = f_{11}(x)$, $g_2(x) = f_{12}(x) + f_{22}(x), \cdots,$ $g_m(x) = f_{1m}(x) + f_{2m}(x) + \cdots + f_{mm}(x), \cdots$. For every n, $g_n(x)$ is of lower type than f_u.

Next we show that $f(x) = \lim g_m(x)$, thus proving the theorem. Let $e > 0$. There is an N such that $\sum_{n=N+1}^{\infty} k_n < \frac{e}{3}$. Hence for every x, $|f(x) - f_1(x) - \sum_{n=1}^{N}(f_{n+1}(x) - f_n(x))| < \frac{e}{3}$. Now fix x. There is an N' such that for every $n > N$ and $m > N'$, $|(f_{n+1}(x) - f_n(x)) - f_{nm}(x)| < \frac{e}{3N}$. Let $m = \max(N,N')$. Then $|f(x) - g_m(x)| < |f(x) - f_1(x) - \sum_{n=1}^{N}(f_{n+1}(x) - f_n(x))$ + $\sum_{n=1}^{N}|(f_{n+1}(x) - f_n(x)) - f_{nm}(x)| + \sum_{n=N+1}^{m}|f_{nm}(x)| < \frac{e}{3} + N\frac{e}{3N} + \frac{e}{3} = e$. Thus $|f(x) - g_m(x)| < e$ and $f(x) = \lim_{m\to\infty} g_m(x)$. Since each of the $g_m(x)$ is of type f_u for $u < w$, the theorem is proven.[11]

Earlier in this chapter it was shown that every function which is the limit of a convergent sequence of continuous functions has as its set of points of discontinuity a set of the first category. Since the concept of continuity

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[11]Casper Goffman, Real Functions (New York: Rinehart & Co., Inc. 1953), pp. 138-139

is of such interest in the study of functions, it seems reasonable to ask whether the Baire functions in general are continuous in any sense or on any set. It is not possible to generalize on Theorem 3.6 for all Baire functions, as the following example illustrates.

<u>Example 3.2</u> The function $f(x) = 1$ for all x rational, and $f(x) = 0$ for x irrational is of type f_2, but is discontinuous everywhere. This can be shown in the following way: If z is any irrational number, then z is the limit of a sequence of rational numbers, $\{z_n\}$. Hence $\lim_{n \to \infty} z_n = z$, but $\lim_{n \to \infty} f(z_n) = 1$ and $f(z) = 0$. Thus $f(x)$ is not continuous at any irrational number, and similarly, it is not continuous at any rational.

It is possible; however, to retain some form of continuity for all the Baire functions. The restrictions necessary to retain this continuity are given in this next theorem.

<u>Theorem 3.13</u> For any $w < W$, if $f(x)$ is of Baire class f_w there is a set, whose complement is of the first category, such that $f(x)$ is continuous on S relative to S.

<u>Proof:</u> As a direct result of Theorem 3.6, the theorem is true for $w = 0$. Let $w < W$, and suppose it is true for all $u < w$. Then $f(x) = \lim_{n \to \infty} f_n(x)$ where each $f_n(x)$ is of type

f_u for $u < w$. By assumption, for every n there is a set S_n, whose complement is of the first category, such that $f_n(x)$ is continuous on S_n relative to S_n. Let $S = \bigcap_{n=1}^{\infty} S_n$. Then by DeMorgan's law, $C(S) = \bigcup_{n=1}^{\infty} C(S_n)$, and the complement of S is of the first category since the union of any number of sets of the first category is of the first category by Theorem 3.2. Also, $f_n(x)$ is continuous on S relative to S for every n. As a special case of Theorem 3.6, the function $f(x) = \lim_{n \to \infty} f_n(x)$ is continuous on a subset T of S relative to T, where the complement of T, relative to S, is of the first category relative to S. But the set S - T is of the first category relative to the set of all real numbers. Thus the complement of T, as the union of two sets of the first category, is of the first category. Hence, f(x) is continuous on T whose complement is of the first category, and by transfinite induction, the theorem holds for all $w < W$.

It is generally known that the number of real functions is greater than <u>c</u>, the cardinality of the real numbers. However, it can be shown that there are only <u>c</u> Baire functions. This result is given as the final theorem on the properties of Baire functions. The proof of the theorem depends on the theory of the arithmetic of transfinite cardinal numbers, and on the set 2^{Λ} which is the cardinal number of all functions on set A with values 0 or 1.

Theorem 3.14 The Baire functions are $\underline{c}$ in number.

Proof: The functions of the form $f(x) = a$ are continuous functions for all real numbers a. Thus there are at least $\underline{c}$ Baire functions. It must be shown that there are no more than $\underline{c}$ Baire functions. This can be done by transfinite induction.

Since there are $\underline{c}$ continuous functions, the functions of type f_0 are $\underline{c}$ in number. Assume there are $\underline{c}$ or fewer functions of each type f_u for every $u < w$, where $w < W$. The class of all functions whose type is less than w has cardinal number $\underline{c} \cdot \aleph_0 = \underline{c}$, since there are $\underline{c}$ functions in each of classes. But every function of type f_w is, by definition, the limit of a convergent sequence of functions of lower type. Thus the number of functions of type f_w is no more than $\underline{c}^{\aleph_0} = \underline{c}$, since there are $\underline{c}$ choices for each of $\aleph_0$ functions. By transfinite induction, it holds that every class of Baire functions contains $\underline{c}$ or fewer functions; therefore, there are no more than $\underline{c} \cdot \aleph_1 = \underline{c}$ Baire functions.

The Baire functions are seen to form a collection of functions which have many properties in common with the continuous functions. There exists a close relationship between the open sets and the continuous functions. Since the Borel sets posses many properties of the open sets and the Baire functions retain many properties of the continuous functions, it is to be expected that there is some connection between the Borel sets and Baire functions. This is indeed true, and is the topic for discussion in the next chapter.

CHAPTER IV

RELATIONSHIPS BETWEEN BOREL SETS AND BAIRE FUNCTIONS

In the study of functions in calculus and analysis, a close relationship is seen to exist between open sets and continuous functions, and between closed sets and continuous functions. In view of the definition of the Borel sets and the Baire functions, together with the properties of both, it seems reasonable to expect some relationship to exist between these two concepts. The purpose of this chapter is to show that this is indeed the case, and to give a few interesting examples of this relationship.

To avoid the necessity of undo complication in the proofs, and thus to facilitate the discussion at hand, the theorems, proofs, and illustrations will be limited to the finite ordinals.

In the preceeding chapter, some of the relationships between continuous functions and sets, both open and closed, were given in Definition 3.4, Definition 3.6 and in the statement and proof of Theorem 3.4. A generalization of the association of continuous functions with open sets can be given in the form of sets associated with a function.

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<u>Definition</u> <u>4.1</u> Associated with any real function are a

number of sets, among these sets are the following: For any real number k, $\{x \mid f(x) < k\}$, $\{x \mid f(x) > k\}$, $\{x \mid f(x) \le k\}$, $\{x \mid f(x) \ge k\}$, $\{x \mid -k < f(x) < k\}$, and many others. These sets will be denoted by: $E\,[f(x) < k]$, $E\,[f(x) > k]$, $E\,[f(x) \le k]$, and so on.

The following theorem relates the sets associated with a function specifically to the continuous functions.

<u>Theorem 4.1</u> A function f(x) defined on a set S is continuous on S relative to S if and only if, for every real number k, the sets $E\,[f(x) < k]$, and $E\,[f(x) > k]$, are open relative to S.

<u>Proof</u>: Suppose f(x) is continuous on S relative to S. Let $a \in S$ be such that $f(a) > k$. There is an $e > 0$ such that $f(a) - e > k$ by the density property of the real numbers. Since f(x) is continuous at a relative to S, there is a $d > 0$ such that if $x \in S$ and $|x - a| < d$, then $|f(x) - f(a)| < e$. Hence there is a neighborhood I, of a, such that for all $x \in I \cap S$, $f(x) > f(a) - e > k$. Thus $E\,[f(x) > k]$ is open relative to S by definition. Similarly with $f(a) + e$, we have $E\,[f(x) < k]$ is open relative to S.

Conversly, assume the sets $E\,[f(x) > k]$, and $E\,[f(x) < k]$ are open relative to S for every real number k. Let $a \in S$ and $e > 0$. Since $E\,[f(x) > f(a) - e]$ is open relative to S, there is a neighborhood I_1 of a such that for every $x \in I_1 \cap S$, $f(x) > f(a) - e$. Also since $E\left[f(x) < f(a) + e\right]$ is open relative to S, there is a neighborhood I_2 of a such that for every $x \in I_2 \cap S$, $f(x) < f(a) + e$. Now by definition the intersection, $I_1 \cap I_2$ is a neighborhood of a. Call it I. Thus for every $x \in I \cap S$, $f(x) < f(a) + e$ and $f(x) > f(a) - e$, and hence $|f(x) - f(a)| < e$. By definition, then, f(x) is continuous at a relative to S.

In view of the definition of sets associated with a function, it is quite obvious that the set $E\left[f(x) \geq k\right]$ is the complement of $E\left[f(x) < k\right]$, and $E\left[f(x) \leq k\right]$ is the complement of $E\left[f(x) > k\right]$. This observation yields the following immediate consequence of Theorem 4.1

<u>Theorem</u> <u>4.2</u> A function f(x) is continuous on S relative S if and only if the sets $E\left[f(x) \geq k\right]$ and $E\left[f(x) \leq k\right]$ are closed relative to S for every real number k.

The Borel sets as defined in Chapter II are all the sets that may be obtained from the closed and open sets by repeated application of the operations of union and intersection of sets. Theorem 4.1 shows the connection between the open sets and the continuous functions.

The continuation of the discussion will be facilitated by a slight change in terminology in the form of a more useable definition of sets associated with a function. This definition also employs the use of indexing with the ordinal numbers in order to parallel the discussion of Borel sets and that of the Baire functions.

<u>Definition</u> <u>4.2</u> For every $w < W$, a set S is said to be of type A_w if and only if there is a function $f(x)$, of type f_w, and a real number k such that $S = E\left[f(x) > k\right]$, and a set S will be said to be of type B_w if and only if there is a function $f(x)$, of type f_w, and a real number k such that $S = E\left[f(x) \geq k\right]$.

Next it will be shown that all of the sets associated with a Baire function of any prescribed finite class, are Borel sets of the same finite class; and that every Borel set of a prescribed finite type, is one of the sets associated with a Baire function of the same finite Baire class.

Before proceeding with these results, the following lemma is needed. It states specifically how the sets that are associated with a limit function are related to the sets associated with the functions of the sequence.

<u>Lemma</u> <u>4.1</u> If $f(x) = \lim_{n\to\infty} f_n(x)$, then $E\left[f(x) > k\right] = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

<u>Proof:</u> Suppose $x \in E\left[f(x) > k\right]$, then there exists an m such that $f(x) > k + \frac{1}{m}$. Since $f(x) = \lim_{n\to\infty} f_n(x)$, there is an r

such that for all $n > r$, $f_n(x) \geq k + \frac{1}{m}$, hence from the definition of union, $x \in \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$; therefore, $E\left[f(x) > k\right] \subset \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

Now suppose $x \in \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$. By definition of union, there is an m such that $x \in \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$. Also, there is an r such that $x \in \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$ and since $f(x) = \lim_{n \to \infty} f_n(x)$, for all $n > r$ $|f(x) - f_n(x)| \leq \frac{1}{m}$, and $x \in \left[f(x) > k\right]$. Thus by set inclusion both ways, we have the result that $E\left[f(x) > k\right] = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

This lemma is needed in the proof of the following theorem, the content of which was outlined above.

<u>Theorem 4.3</u> For every finite ordinal w, every set of type A_w is of type G_w and every set of type B_w is of type F_w if w is even. If w is odd, every set of type A_w is of type F_w and every set of type B_w is of type G_w.

<u>Proof</u>: By Theorems 4.1 and 4.2, the theorem holds for w = 0. Use finite induction and suppose the theorem holds for all $u < w$, and that w is even. Let S be any set of type A_w. Then there is by definition a function f(x) of type f_w and a real number k such that $S = E\left[f(x) > k\right]$. Now $f(x) = \lim_{n \to \infty} f_n(x)$ where the $f_n(x)$ are Baire functions of lower type. By Lemma 4.1 $S = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$. But, by assumption, each $E\left[f_n(x) \geq k + \frac{1}{m}\right]$ is of type G_{w-1}. Since w is even, w - 1 is odd, and the intersection of a denumerable number of sets

of type G_{w-1} is of type G_{w-1}. Hence also S is the union of a denumerable number of sets of type G_{w-1} and is of type G_w by definition.

Now suppose S is of type B_w. Then there is a function f(x) of type f_w and a real number k such that $S = E\left[f(x) \geq k\right]$. Since -f(x) is also of type f_w, C(S) is of type G_w by the argument in the above paragraph, and since the complement of every set of type G_w is of type F_w, S is of type F_w. Hence by finite induction every set of type A_w is of type G_w, and every set of type B_w is of type F_w if w is even.

In view of the results obtained concerning Borel sets in Chapter II, the proof for the case when w is odd is similar.

The next theorem states a rather surprising result, namely that every Borel Set is one of the sets associated with a Baire Function of the same type or class. Once again a Lemma is given with a proof which is very helpful in the proof of the theorem.

Lemma 4.2 For every finite ordinal w, if S is any set of type A_w or B_w there is a function f(x) of type f_{w+1} such that f(x) = 1 for every $x \in S$ and f(x) = 0 for all $x \in C(S)$.

Proof: Assume S is of type A_w. Then there is a function g(x) of type f_w such that $S = E\left[g(x) > 0\right]$. Let h(x) = max(g(x), 0). Then h(x) is also of type f_w by Theorem 3.11.

For every positive integer n, let $f_n(x) = \min\left[nh(x),1\right]$, the functions $f_n(x)$ are again of type f_w. This sequence converges everywhere and $\lim_{n\to\infty} f_n(x) = 1$, for $x \in S$ and 0 for $x \in C(S)$ and is of type F_{w+1} by definition of a function of type f_{w+1}.

Suppose S is of type B_w. Then C(S) is of type A_w. Hence, by the preceeding paragraph, there is a function f(x) of type f_{w+1} such that f(x) = 1 for all $x \in C(S)$ and f(x) = 0 for all $x \in S$. The function 1 - f(x) is of type f_{w+1} and has value 1 on S and value 0 on C(S). Thus by finite induction the theorem is established.

<u>Theorem 4.4</u> For every finite ordinal w, every set of type G_w is of type A_w and every set of type F_w is of type B_w if w is even, and every set of type F_w is of type A_w and every set of type G_w is of type B_w if w is odd.

<u>Proof</u>: Again by Theorems 4.1 and 4.2, the theorem holds for w = 0. Suppose w is odd, and that the statement holds for w - 1. Let S be any set of type F_w. Then $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is of type F_{w-1}. Since by assumption every set of type F_{w-1} is of type B_{w-1}, w - 1 is even, there is by Lemma 4.2 a function $f_n(x)$ of type f_w for every n, such that $f_n(x) = \frac{1}{2^n}$ for every $x \in S_n$, and $f_n(x) = 0$ for every $x \in C(S_n)$. Now $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is of type f_w since the series converges uniformly to f(x) and uniform convergence preserves

the class of the limit function. But $S = E\left[f(x) > 0\right]$ so S is of type A_w by definition of a set of that type. If S is any set of type G_w and w is even, the proof is very much the same except for replacing even by odd and F_w by G_w.

Now suppose w is even and the theorem holds for $w - 1$. Let S be any set of type F_w, then C(S) is of type G_w. By following the same argument as above, we obtain eventually that $C(S) = E\left[f(x) > 0\right] = E\left[-f(x) < 0\right]$, and hence $S = E\left[-f(x) \geq 0\right]$. Thus S is of type B_w by definition.

Theorems 4.3 and 4.4 can be summarized very briefly to say that for finite ordinals the sets associated with a function of any prescribed Baire class are identical with the Borel sets as defined in Chapter II. For later reference, these results are combined into a theorem.

<u>Theorem 4.5</u> For all finite ordinals w, a set S is of type A_w if and only if it is of type F_w, and it is of type B_w if and only if it is of type G_w whenever w is odd. If w is even the sets of types A_w and B_w are of types G_w and F_w, respectively.

The next two theorems now relate the Borel sets to the Baire functions. By using the definition of sets of types A_w and B_w as given in Definition 4.2, one of these relationships has already been shown; namely, that for any finite ordinal w, if f(x) is of type f_w, the sets associated with the function are Borel sets of types F_w and G_w.

Theorem 4.6 If f(x) is any function of type f_w where w is a finite ordinal number, then for every real number k, the sets $E\left[f(x) > k\right]$ and $E\left[f(x) \geq k\right]$ are of types F_w and G_w, respectively, if w is odd, and of types G_w and F_w, respectively, if w is even.

Proof: By Definition 4.2, a set S is of type A_w if and only if there exists a function f(x) of type f_w and a real number k such that $S = E\left[f(x) > k\right]$, and S is of type B_w if and only if there is a function f(x) of type f_w and a real number k such that $S = E\left[f(x) \geq k\right]$. By replacing A_w and B_w, accordingly, in Theorem 4.5, the theorem is established.

The converse of Theorem 4.6, which will be proven next, states that if the sets associated with a function are of a certain finite Borel type or class, then the function is of the same finite Baire class.

The existence of a certain type of Baire function defined on two disjoint sets is needed for the proof of the theorem, and is established with the statement and proof of the following lemma.

Lemma 4.3 If w is a finite ordinal and S and T are two disjoint sets of type B_w, there is a function g(x) of type

f_w such that $g(x) = 1$ for all x in S, $g(x) = 0$ for all x in T, and $0 < g(x) < 1$ elsewhere.

Proof: By the definition of sets of type B_w and the preceeding results of this section, there is a function $f_1(x)$ of type f_w such that $E\left[f_1(x) \geq 0\right] = S$, and an $f_2(x)$ of type f_w such that $E\left[f_2(x) \geq 0\right] = T$. Let $g_1(x) = \max(f_1(x), 0)$, and $g_2(x) = \max(f_2(x), 0)$. Then $g_1(x)$ and $g_2(x)$ are of type f_w by Theorem 3.11, and $g_1(x) = 0$ on S, $g_1(x) > 0$ on C(S); $g_2(x) = 0$ on T, and $g_2(x) > 0$ on C(T). The function $g_1(x) + g_2(x)$ is never 0 and is of type f_w by Theorem 3.7. Now let $g(x) = \dfrac{g_2(x)}{g_1(x) + g_2(x)}$, thus $g(x)$ is of type f_w by Theorem 3.8; and $g(x) = 1$ for every $x \in S$, $g(x) = 0$ for every $x \in T$, and $0 < g(x) < 1$ for every other x.

Theorem 4.7 If w is a finite odd ordinal and f(x) is such that for every real number k the sets $E\left[f(x) > k\right]$ and $E\left[f(x) \geq k\right]$ are of type F_w and G_w, respectively, then f(x) is a function of type f_w. If w is even and the sets are of type G_w and F_w, respectively, then f(x) is of type f_w.

Proof: In view of Theorem 4.6, and the comments preceeding it concerning sets of type A_w and B_w, the proof reduces to showing that if the sets $E\left[f(x) > k\right]$ and $E\left[f(x) \geq k\right]$ are of type A_w and B_w, respectively, then f(x) is of type f_w.

By Theorem 4.1, the theorem holds for $w = 0$. For every real number k, suppose the sets $E\left[f(x) > k\right]$ and

$E\left[f(x) \geq k\right]$ are of type A_w and B_w. Then the sets $E\left[f(x) \leq k\right]$, as complements of sets of type A_w, are of type B_w. This is because every set of type A_w is of type F_w or G_w, and every set of type F_w is the complement of a set of type G_w by Theorem 2.1.

Suppose, for now, that $0 < f(x) < 1$ for every x. Let n be a positive integer. For every $m = 0,1,2,\ldots,n-1$, the sets $E\left[f(x) \leq \frac{m}{n}\right]$ and $E\left[f(x) \geq \frac{m+1}{n}\right]$ are of type B_w. Hence by Lemma 4.3 there is a $g_m(x)$ of type f_w such that $g_m(x) = 0$ for all $x \in E\left[f(x) \geq \frac{m+1}{n}\right]$, $g_m(x) = 1$ for all $x \in E\left[f(x) \leq \frac{m}{n}\right]$, and $0 < g_m(x) < 1$ for all other values of x. Let $g(x) = \frac{1}{n}\left[g_0(x) + g_1(x) + \cdots + g_{n-1}(x)\right]$. Suppose that $\frac{m}{n} \leq f(x) < \frac{m+1}{n}$. Then $g_0(x) = g_1(x) = \cdots = g_{n-1}(x) = 1$, $0 \leq g_m(x) \leq 1$, and $g_r(x) = 0$ for every $r > m$. As a result of these arguments, $\left|f(x) - g(x)\right| < \frac{1}{n}$ for every x. Now $g(x)$, as the sum of a finite number of functions of type f_w, is itself of type f_w by Theorem 3.7. Thus $f(x)$, as the limit of a uniformly convergent sequence of functions of type f_w, by Theorem 3.12 is itself of type f_w.[12]

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[12]Casper Goffman, Real Functions (New York: Rinehart & Company, Inc., 1953), page 143.

There are other relationships which exist between the Borel sets and the Baire functions. It is not practical to investigate them all. However, one last such relationship will be given. Further statements relating these two concepts may be found in the final chapter.

Theorem 4.8 The characteristic function of every Borel set is a Baire function.

Proof: By Lemma 4.2, for every set S of type A_w or B_w there is a Baire function of type f_{w+1} such that f(x) has value 1 for all $x \in S$ and value 0 for all $x \notin S$. Theorems 4.4 and 4.5 can be combined to state that a given set is of type A_w or B_w if and only if it is of type F_w or G_w if w is odd and of type G_w or F_w if w is even. Thus a set is of type A_w or B_w if and only if it is a Borel set. Hence for every Borel set of type F_w or G_w there is a function f(x) of type f_{w+1} such that f(x) is the characteristic function of the given set.

In summary, here are several important results of this chapter. Associated with every function are a large number of sets which were defined as being sets of type A_w or B_w depending on the relation of the function values with real numbers. In this way all the values in the domain of the function are classified into sets associated with a function. It was shown that these sets can be classified into types or categories depending on the type or class of the Baire function with which they are associated. The next two theorems illustrated the fact that the sets associated with a function are indeed Borel sets. The major result of the section, and one of the most striking of the entire project, was the fact that a given function is a Baire function of a finite Baire type if and only if all the sets associated with the function are Borel sets of the same finite type or class.

CHAPTER V

SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FURTHUR STUDY

I. SUMMARY

The significance of the Borel Sets is best stated in the final theorem of Chapter II. The Borel sets, from their properties, answer a number of questions which normally arise in working with the open and closed sets of real numbers. The Borel sets form the smallest system of sets which contains all the open and closed sets and also contains all the sets that can be obtained from the closed and open sets by taking the union or intersection of any finite or denumerable number of sets that are in the system. This answers the question, for example, of what kind of set is obtained from taking the denumerably infinite union of a collection of closed sets, or the intersection of any number of open sets.

The Baire functions have much value in the properties that they possess that are extensions of properties of continuous functions. In almost all work in analysis the continuous functions capture the spotlight. It is very interesting and informative to find other functions which have interesting properties that are not continuous functions.

This classification of discontinuous functions discussed here was first introduced by Rene Louis Baire (1874 -1932) in the year 1899.[13]

As this thesis unfolded it became quite evident, from the work encountered in regard to open and closed sets as related to continuous functions and from the definition of Borel sets and Baire functions, that there must exist some connection between the Borel sets and the Baire functions. This actual relationship is specifically stated in the final theorem of Chapter IV. The theorem states that: A function is of a prescribed finite Baire Class if and only if all the sets associated with the function are of the same finite Borel type or class.

II. CONCLUSIONS

The closed sets and the open sets, as complements of one another, are not the only sets of real numbers that possess useful properties and intricate relationships to functions. The most commonly known properties of closed and open sets generalize to a system of sets, the Borel

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[13]Lawrence M. Graves, Theory of Functions of a Real Variable (New York, Toronto, London: McGraw Hill Book Co., 1956), page 127.

sets, which are far more inclusive. Another surprising result is that all the Borel sets may be obtained from the closed sets or the open sets only. It is also noteworthy that the Borel sets can be defined by beginning with only compact sets, or by beginning with the open and bounded sets.[14] These observations naturally give rise to the question of how much real difference exists between these types of sets.

The properties of continuous real functions serve to classify all real functions into two disjoint classes, those that are continuous and those that are not. The definition of Baire functions makes it obvious that many functions which are not continuous do possess properties which facilitate a classification of those functions by some standard. The fact that there are functions which do not belong to any Baire class leaves at least one avenue of investigation wide open.

In undergraduate work with continuous functions many properties of these functions are given which are not essential to the concept of continuity. As a result, in the mind of the undergraduate, frequent misconceptions occur concerning the properties that are unique to continuous functions.

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[14]Edward J. McShane and Truman R. Botts, Real Analysis (Princeton: E. Van Norstrand Co., Inc., 1959), page 159.

The properties referred to here are those concerning the sum, product, quotient, etc., of continuous functions.

The observation that the Baire functions in general possess many of the properties that have been traditionally attributed to continuous functions, at least to the immature mathematical mind, naturally gives a person a clearer picture of the properties essential to continuity itself.

The study of Baire functions seems to be the first step in a vast investigation of non-continuous functions. This study will obviously have as an end result a much clearer and more precise intuitive feeling for continuity.

In any academic pursuit in the area of analysis a reference is made, and a partial investigation given, to the relationships that exist between continuous functions and open sets. Once again it is refreshing to discover that this type of relationship is not one that is unique to the open sets or to the continuous functions. Under the condition of the proper classification of functions and the proper structure of a set, an analogous result generalizes to all finite types of Baire functions in relationship to the same class of Borel sets.

III. SUGGESTIONS FOR FURTHUR STUDY

The vast majority of current works or writings in the area of analysis emphasize very strongly the usefulness and necessity of measure theory as applied to sets and functions. There are a large number of types of measure that are used, but most are very similar in the original content.

The relationships that exist between the measureability of sets of real numbers and real valued functions, according to several definitions of such measures, can be found in most textbooks dealing with real analysis or measure theory.

This approach could be used very effectively as an alternate method of investigation of the Borel sets, the Baire functions and their relationships to one another. It has been proven, for example, that all Borel sets are measureable sets.[15] Also, if sets of measure zero are neglected, the Borel sets are all of the measureable sets of real numbers.[16] It is also easily established that all continuous functions are measureable functions.[17]

The results of the preceeding paragraph can be combined with the fact that: every function which is a limit of a convergent sequence of measureable functions is itself

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[15] Paul R. Halmos, Measure Theory (Toronto, New York, London: D. Van Nostrand Co., Inc., 1950), page 62.

[16] Casper Goffman, Real Functions (New York; Rinehart & Co., Inc., 1953), page 64.

[17] Richard R. Goldberg, Methods of Real Analysis (New York, Toronto, London: Blaisdell Publishing Co., 1964), page 283.

a measureable function, to produce the following general results: (1) all Baire functions are measureable, and (2) all the sets associated with a function are measureable if and only if the function is a Baire function.[18] This last result follows immediately from Theorem 4.7 of Chapter IV. It is worthy of notation that this last result relating sets to functions is not limited to the finite classes of Baire functions as is the discussion contained in Chapter IV of this paper.

[18] *Ibid.*, page 287.

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