CUT POINTS OF PLANE CONTINUA

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GRADUATE COUNCIL APPROVAL

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Robert L. Poe

TO

My parents and my wife, Kay.

ACKNOWLEDGEMENT

The writer takes this opportunity to express his sincere appreciation to Dr. R. Poe for suggestion of this topic and for his guidance in the completion of this paper.

G. G. B.

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INTRODUCTION

Often in topology, it is desirable to limit basic ideas to a plane continuum. This thesis is a study of the properties of a plane continuum in terms of cut points, and especially the development of simple continuous arcs, simple closed curves and continuous curves.

I. THE PROBLEM

Statement of the problem. The purpose of this study is (1) to describe a plane continuum in terms of cut points and non-cut points, and (2) to show examples of the theorems developed.

Limitations. Since there has been extensive development of this topic, this paper will deal only with a few aspects of the problem.

The thesis will be restricted to three axioms and they will be introduced as needed.

II. AXIOMS AND BACKGROUND DEFINITIONS AND THEOREMS

The relationship between undefined object "point" and "region" in the study of topology is given by the following axioms:

Axiom 0. Every region is a point set.

Axiom 1. There exists a sequence $\{G_n\}$ such that (1) for each n, G_n is a collection of regions covering S, (i.e. $S \subset \bigcup_{\alpha \varepsilon \Lambda} G_\alpha$), (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is any region whatsoever, $\{x\}$ is a point of R and $\{y\}$ is a point of R, identical with $\{x\}$ or not, then there exists a positive integer m such that if g is any region belonging to the collection G_m, such that $\{x\} \varepsilon g$, then $\bar{g}$ is a subset of $R - \{y\} \cup \{x\}$, and (4) if $\{M_i\}$ is a sequence of closed point sets such that for each i, M_{i+1} is contained in M_i is a subset of $\bar{g}_i$, then there is at least one point common to all point sets of the sequence $\{M_i\}$. ($\bar{g}$ denotes the closure of set g.)

Axiom II will be added when needed for development of this paper.

For the most part, it is assumed the reader is familiar with the elementary properties of point set theory. However, definitions, as well as some explanation, will be given when necessary for clarity. The definitions and unproved theorems which follow are presented to help refresh the memory of the reader as well as to establish a background for the paper. These definitions and theorems are to be considered for the Euclidean plane, E^2.

Definition 1. Two point sets are separated if neither contains a point or limit point of the other.

Definition 2. A point set is said to be compact if every infinite subset of M has a limit point belonging to M.

If M is compact and K is an infinite subset of M, then K is compact; in fact, every subset of M is compact.

Definition 3. The sequence of regions $\{R_n\}$ is said to close down on the point p if it satisfies with respect to p all the conditions of the following: If p is a point, there exists an infinite sequence of regions R_n such that (1) p is the only point they have in common, that is, $p = \bigcap_{n=1}^{\infty} R_n$ (2) for each n, $\bar{R}_{n+1}$ is a subset of R_n, and (3) if R is a region containing p, then there is an integer n such that $\bar{R}_n$ is a subset of R.

Definition 4. The point p is said to be a sequential limit point of the sequence $\{p_n\}$if for each region R containing p there is a positive integer N such that if n is an integer greater than N then p_n is a point of R. If p is a sequential limit point of the sequence $\{p_n\}$, $\{p_n\}$ is said to converge to p and is said to be a convergent sequence.

Theorem 1. If p is a limit point of the point set M, then every region that contains p contains infinitely many points of M and there is an infinite sequence of points of M, distinct from each other and from p such that p is a sequential limit point of this sequence.

A sequential limit point of a sequence of distinct points $\{p_n\}$ is also considered to be a limit point of the point set $\bigcup_{n=1}^{\infty} p_n$.

Theorem 2. If for each n, p_n is a point and the point set $\bigcup_{n=1}^{\infty} p_n$ is compact, then there is an infinite sequence of positive integers $\{n_i\}$ such that the subsequence $\{p_{n_i}\}$ is convergent to a point of $\{p_n\}$.

Definition 5. A set is said to be closed if it contains all its limit points. A point set is said to be conditionally compact if every infinite subset of it has a limit point.

Theorem 3. If the point p is not a limit point of any one of a finite number of point sets, it is not a limit point of their union.

Definition 6. The point set D is said to be open if for each point p of D there is a region containing p which is a subset of D.

Theorem 4. No sequence has more than one sequential limit point, and if p is a sequential limit point of a sequence, no other point is a limit point of the union of the points of the sequence.

Theorem 5. If the point set whose elements are the points of an infinite sequence is conditionally compact, and there do not exist two distinct points such that each of them is a sequential limit point of a subsequence of the given sequence, then the given sequence converges.

Definition 7. The subset K of M is said to be an open subset of M if for each point p of K there is a region R containing p such that $R \cap M$ is a subset of K.

Definition 8. A point set M is said to be separable if it contains a countable point set K such that every point of M belongs to K or is a limit point of K.

Definition 9. A point set M is said to be locally compact if, for each point p of M, there is a conditionally compact open subset of M containing p, (or if for each open subset R of M containing p, there is a region U which contains p such that $[U \cap M] \subset R$ and $\overline{U \cap M}$ is compact).

Theorem 6. No locally compact closed point set M is the union of a countable number of closed point sets such that if g is any one of them then every point of g is a limit point of M-g.

Definition 10. The point set is said to be completely separable if and only if there exists a countable set H such that each element of H is an open set of M and if D is any open set of M and p is a point of D then there exists an open set of M belonging to H, containing p, and lying in D.

Definition 11. A point set is said to be connected if it is not the union of two separated non-empty sets.

Definition 12. A point set which is both closed and connected is said to be a continuum.

Theorem 7. If G is a collection of sets such that if H is a monotonic subcollection of G there is a set K of the collection G which is a subset of every set of the collection H, then there exists a set of the collection G which contains K but no other set of the collection G.

Definition 13. A maximal connected subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

Definition 14. A subset K of a set M is said to be a component of M if and only if it is a maximal connected subset of M.

Theorem 8. If M is a connected point set and L is a point set consisting of M together with some or all of its limit points, then L is connected.

Definition 15. If H and K are two disjoint point sets, the continuum M is said to be an irreducible continuum from H to K if M contains both a point of H and a point of K but no proper subcontinuum of M contains both a point of H and a point of K.

Theorem 9. If the open subset D of the continuum M is a proper subset of M and $\bar{D}$ is compact, then the boundary, with respect to M, of D contains at least one limit point of every maximal connected subset of D.

Definition 16. If $\{M_i\}$ is a sequence of point sets, then the limiting set of this sequence is the set of all points p such that if R is a region containing p there exists infinitely many positive integers n such that M_n contains a point of R.

Theorem 10. If $\{M_i\}$ is a sequence of connected point sets such that $\overline{\cup M_i}$ is compact and there exists a convergent sequence of points, $\{a_i\}$, such that for each i, a_i belongs to M_i, then the limiting set of the sequence $\{M_i\}$ is a continuum.

Definition 17. The point set M is said to be the sequential limiting set of the sequence $\{M_i\}$ if M is the limiting set of every infinite subsequence of $\{M_i\}$ and $\{M_i\}$ is said to converge to M.

Theorem 11. If the limiting set of a sequence $\{M_i\}$ is compact, then some subsequence of $\{M_i\}$ has a sequential limiting set.

Definition 18. A point set is said to be degenerate if it consists of less than two points; otherwise, it is said to be nondegenerate.

Definition 19. A point set is said to be totally disconnected if it contains no nondegenerate connected point sets.

Theorem 12. If H and K are two separated point sets, every connected subset of $H \cup K$ is a subset either of H or of K.

Definition 20. If H, K and T are proper subsets of the connected point set M, then T is said to separate H from K in M if M-T is the union of two separated sets containing H and K respectively.

Theorem 13. If A, B, X, and Y are closed point sets of the connected point set M, and B separates A from X in M, and X separates B from Y in M, then B separates A from Y in M.

Definition 21. If a and b are two points of a connected set M, and there exists at least one point of M that separates a from b in M, then the set of all such points is called the segment ab of M and segment ab together with the points a and b is called the interval ab of M. The points a and b are called endpoints of the segment ab and of the interval ab of M.

An example of segment, interval and endpoint is the connected set (a,b) of the reals, E^1. Let p be any point which separates a from b. Then the set of all such points which separates a from b is $\{p|a<p<b\}$ and this set is called a segment. However, if a and b are included with the segment, the result is an interval. The points a and b are called endpoints.

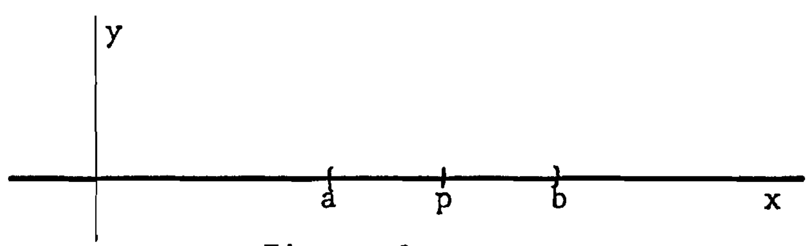

Figure 1

Definition 22. The point set M is said to be perfectly compact if and only if it is true that if G is a monotonic collection of subsets of M, then either the elements of G have a common point or they have a common limit point.

Theorem 14. If T is a component of the open set D relative to the continuum M, D is a proper subset of M, and $\bar{D}$ is perfectly compact, then the boundary of D with respect to M, contains at least one limit point of T.

Theorem 15. Every compact point set is perfectly compact.

CHAPTER II

PROPERTIES OF CUT POINTS OF A PLANE CONTINUA

The establishment of an order relation, <, and cut points, of a connected set of E^2, followed by the properties of a plane continuum in terms of cut points, is the procedure followed in this chapter. Simple continuous arcs, simple closed curves, and continuous curves will be defined, and basic theorems and examples will be developed from these definitions. The definitions, theorems and examples of this chapter are confined to the Euclidean plane, E^2.

Definition 23. If ab is an interval of a connected point set M in E^2, the point x is said to precede the point y, in M, on ab, (or in the order from a to b in M on ab), if x and y are distinct points of M and either (1) y is b, or (2) y separates $a \cup x$ from b in M. The point x is said to follow the point y in M on ab if y precedes x, in M, on ab. A point is said to be between the points x and y in M on ab if it precedes one of these points and follows the other, in M, on ab.

The above definition established the order of points on any interval of E^2. This thesis assumes that the starting point of any interval is the first point, or ab, as illustrated by Figure 1. The points a and b are considered to be end points of the connected set of E^2.

The idea of cut and non-cut points will be introduced and used in the following theorems. The following definition will be needed for the description of the plane continuum in terms of cut points.

Definition 24. The point set T of E^2 is said to separate the point set H from the point set K in the connected point set M if and only if T is a subset of M and M-T is the union of two separated point sets, one containing H and the other containing K. The statement that T separates H from K means that it separates H from K in M. The point p will be called a cut point of the connected point set M if M-p is not connected.

A simple continuous arc is usually defined as a set of points homeomorphic with [0,1], but for purposes of this thesis the arc will be described in terms of cut points. The simple arc is the first plane continua to be described in terms of cut points in this thesis.

Definition 25. A simple continuous arc of E^2 is a compact nondegenerate continuum that does not have more than two non-cut points.

The continuum pictured in Figure 2 is an arc ab with endpoints a and b. (The points a and b are also non-cut points of the arc ab.) The removal of any point x of ab, except point a or b, disconnects ab; such a point is called a cut point of ab.

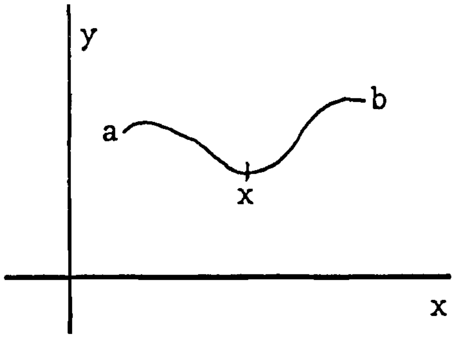

Figure 2

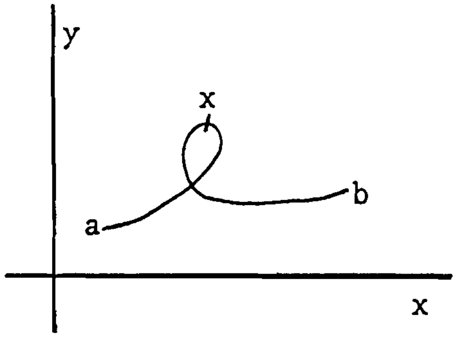

Figure 3

In Figure 3, the points a and b are non-cut points. Further, the point x does not disconnect the continuum of Figure 3; therefore, the point x is called a non-cut point of this continuum. Figure 3 does not represent an arc.

The following definition will serve to introduce a principle which will be used in the proofs of several of the following theorems.

Definition 26. If for the point o of the connected set M of E^2, the set M-o is the union of two separated point sets H and K, then H and K are called sects of M from o, and M will be said to be separated into two sets, H and K, by the omission of o. If p is a point of M distinct from o, then in case there is only one sect of M from o that contains p, that sect will be called a sect of op.

The following theorems describe properties of cut-points.

Theorem 16. Every nondegenerate compact continuum of E^2 has at least two non-cut points.

Proof. Suppose M is a nondegenerate compact continuum and e is a point of M such that every other point of M is a cutpoint of M. Let G denote a set such that G' belongs to the set if and only if G' is the closure of a sect of M that neither contains e nor has e as a starting point. Suppose H is a monotonic subcollection of G. By Theorem 15, the continua of the collection H have a point p in common. The point set M-p is the union of two separated sets M_{pe} and K such that M_{pe} contains e. Suppose H' is an element of the set H. There exists a point x such that M-x is the union of two separated point sets M_{xe} and H'-x, where M_{xe} contains e. Since H' belongs to H, it contains p; so p does not belong to the connected point set $M_{xe} \cup x$. Hence, it is a subset of M_{pe}. Therefore, H' contains $K \cup p$. But $K \cup p$ belongs to G. So the element $K \cup p$ of G is a subset of every element of H. Hence, by Theorem 7, there exists an element G_o of G which contains $K \cup p$ but no other element of G. There exists a point e_o such that $M-e_o$ is the union of two separated point sets such that one of them is G_o-e_o, and the other one contains e. Let y denote some point of G_o. The point set M-y is the union of two separated point sets M_{ye} and T such that T does not contain e. The point set $T \cup y$ is a subset of G_o-e_o. This involves a contradiction since $T \cup y$ is an element of G. So, for every point e of M, there is a non-cut point of M distinct from e. It follows that M has at least two non-cut points.

The simple continuous arc is an example of a nondegenerate compact continuum of E^2 with only two non-cut points. An example of more than two non-cut points would be a circle in E^2. A circle also illustrates the above theorem. It is a nondegenerate compact continuum where every point is a non-cut point.

Lemma 1 is needed in the proof of the following two theorems.

<u>Lemma 1</u>. If T is a connected subset of the connected point set M of E^2, and M-T is the union of two separated point sets H and K, then $H \cup T$ and $K \cup T$ are connected.

<u>Proof</u>. Suppose, on the contrary, that $H \cup T$ is the union of two separated point sets L and N where L intersects T. The set T is connected, and by Theorem 12, it is a subset of L. Therefore, N is a subset of H, and H and K are separated. Therefore, N and K are separated, and so are N and L. Thus, $K \cup L$ and N are separated and M is not connected, contrary to hypothesis, that $H \cup T$ is the union of two separated point sets L and N where L intersects T. The proof of $K \cup T$ is similar.

<u>Theorem 17</u>. If ab is a simple continuous arc from a to b of E^2, and p is any point of ab except a and b, and ab-p is the union of two separated point sets H and K, then one of the sets, say H, contains a, and K contains b.

Proof. Suppose K contains neither a nor b. By Theorem 16 and Lemma 1, the compact continuum $K \cup p$ contains a point x distinct from p such that $(K \cup p)-x$ is connected. The point set $H \cup p$ is also connected, and the union of these two connected point sets is ab-x. Hence, ab-x is connected. This contradicts that ab is a simple continuous arc.

Theorem 18. If M is a simple continuous arc from a to b in E^2, and p is a point of M distinct from a and b, and M-p is the union of two separated point sets H and K, and H contains a, then $H \cup p$ is an arc from a to p and H is connected.

Proof. Suppose that x is a point of $H \cup p$ distinct from a and from p. Then, by Theorem 17, M-x is the union of two separated sets T and L, where T contains a and L contains b. The connected set $K \cup p$ is a subset of L. Let $U = [(H \cup p) - x] \cap T$ and $V = [(H \cup p) - x] \cap L$. The sets T and L are separated and U and V are subsets of T and L respectively; then U and V are separated, but $(H \cup p) - x = U \cup V$. Thus, the continuum $H \cup p$ by Lemma 1 is disconnected by the omission of any one of its points except a and p. Therefore, it is an arc from a to p. It follows that $(H \cup p) - p$, that is to say H, is connected.

Theorems 17 and 18 have described some of the properties of a simple continuous arc. Consider the simple continuous arc ab in Figure 4. The point p is any point of ab such that $p \neq a$ and $p \neq b$, and this point is a cut point separating ab into two point sets H and K. The set H contains a and the union of H with the point p forms an arc ap.

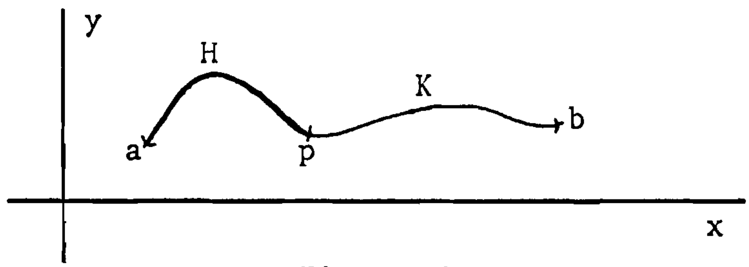

Figure 4

Theorem 19. If M is a simple continuous arc from a to b of E^2, and α is a sequence of points $\{p_n\}$, belonging to M, such that for every n, p_n, precedes p_{n+1}, in the order from a to b on M, then this sequence has a sequential limit point p and every term of it precedes p in that order on M.

Proof. The set M is compact, then; by Theorem 2 there exists an infinite increasing sequence $\{n_i\}$ such that the sequence $\{p_{n_i}\}$ has a sequential limit point p, and by Theorem 4 p is a sequential limit point of α. Unless every p_n precedes p, then there exists a number j such that p precedes p_{n_j}. The point set $\bigcup p_{n_j}$ lies wholly on the interval $p_{n_j}b$ of ab. Then the point p is not a limit point of this point set and neither is it a limit point of the finite point set $\bigcup p_{n_k}$ for $k < j$. Then by Theorem 3, p is not a limit point of $\bigcup p_{n_i}$. Thus, the supposition that not every p_n precedes p leads to the contradiction that p is a sequential limit point of $\bigcup p_{n_i}$.

Theorem 20. If M is a subset of the simple continuous arc ab of E^2, then the point p of ab is a limit point of M if and only if every segment of ab that contains p also contains a point of M distinct from p, (i.e., that every interval of ab that contains p also contains a point of M distinct from p, according as p is, or is not, distinct from a and from b).

Proof. Suppose p is neither a nor b and that every segment of ab that contains p contains a point of M distinct from p. Then either every such segment contains a point of M preceding p or every one contains a point of M following p. Suppose that the former of the alternatives is true. The point p is a limit point of the segment ap. It follows, with the help of Theorems 1, 2, 4, and 19, that p is the sequential limit point of some sequence α of points $p_1, p_2, p_3, \ldots$ all lying on ap such that for each n, p_n precedes p_{n+1}. Let H denote the set of all points of M that precede p. The segment p_1p contains a point x_1 of H. Let n_1 denote the smallest n such that p_n is between x_1 and p. The segment $p_{n_1}p$ contains a point x_2 of H. This process may be continued indefinitely. Thus, there exists a sequence β of points $x_1, p_{n_1}, x_2, p_{n_2}, x_3, p_{n_3}, \ldots$ such that if one point precedes another one in this sequence, it also precedes it in the order from a to b on ab. It follows, by Theorem 19 that the sequence β has a sequential limit point z. But the sequence α and the sequence β have the infinite subsequence $\{p_{n_i}\}$ in common. Hence, z is p. Then the point p is the sequential limit point of the sequence of distinct points $\{x_i\}$; but these points all belong to M. Therefore, p is a limit point of M.

Suppose, secondly, that p is distinct from a and b and that p is a limit point of M, and also that x is a point of ab preceding p, and y is one following p in the order from a to b on ab. If the segment xy of ab contains no point of M distinct from p, then M-p is a subset of the union of the two intervals ax and yb of ab, but these intervals are closed point sets and neither of them contains p. Then p is not a limit point of their union and is not a limit point of M-p. Hence, it is not a limit point of M. Thus, the supposition that there is a segment of ab containing p but no point of M leads to a contradiction that p is a limit point of M.

Since a simple continuous arc is a compact connected subset of a continuum, the proof for either end point of the arc ab, a limit point of M, has been accomplished, i.e., every segment which contains an end point of the arc is a segment of the continuum containing arc ab.

An example of Theorem 19 and 20 is readily shown by using the subset [0,1] of the real line. Let $p_n = \{\frac{1}{n}\}$, where p and n are real numbers, such that n = 1, 2, 3, · · · . The sequence $\{p_n\}$ converges to 0, which is an endpoint of the subset [0,1]. An example of a sequence which does not converge to an endpoint is $p_n = \{\frac{n}{2n+2}\}$, where p and n are real numbers, such that n = 1, 2, 3, · · · . This sequence converges to $\frac{1}{2}$ and also satisfies the conditions of Theorems 19 and 20.

The Dedekind-cut proposition is an important theorem and can be found in many point set theory books.[1] The proposition is introduced to facilitate an understanding of the properties of order of a simple continuous arc.

Theorem 21. (DEDEKIND-CUT PROPOSITION.) If H and K are two subsets of the arc ab of E^2 such that each point of ab belongs either to H or to K, and each point of H precedes each point of K in the order from a to b on ab, then there is either a last point of H or a first point of K in that order.

Proof. Since ab is not a connected point set, one of the sets H and K contains a limit point of the other one. Suppose H contains a point p which is a limit point of K, and x is a point of ab following p. Then by Theorem 20, the segment ax of ab contains a point y which belongs to K. Since y precedes x, then x belongs to K. Thus, every point that follows p belongs to K. Therefore, p

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[1] Dick Wick Hall and Guilford L. Spencer II, Elementary Topology, (New York: John Wiley and Sons, 1955), p. 50.

is the last point of H. Similarly, if K contains a point p which is a limit point of H, then p is a first point of K.

The Dedekind-cut proposition will be used in proving the following theorem.

Theorem 22. If K is a closed point set lying on the arc ab of E^2, there is a first point of K in the order from a to b on ab.

Proof. If a belongs to K it is the first point of K. Suppose a does not belong to K. Let H denote the set of all points x of ab such that x precedes every point of K, and let T denote the point set ab-H. Every point of H precedes every point of T. Suppose x is a point of H. The set K is closed and does not contain x; then, by Theorem 20, there is a point y between x and b such that the interval xy of ab contains no point of K. The point y precedes every point of K, and y belongs to H. Thus, for every point x of H there is a point y of H following x. In other words, there is no last point of H. Then, by Theorem 21, there is a point p which is the first point of T. If p did not belong to K it would precede every point of K and would belong to H, which is impossible since it belongs to T. The point p cannot belong to T-K since H denotes all points that precede every point of K, and if p preceded K, it would belong to H. Hence, p belongs to K and is the first point of K.

An example of Theorem 22 is shown by a closed, bounded subset of the real line. No matter where this closed and bounded subset is chosen, there will be a first point. Let [1,2] be this subset; then there is a first point of this set in order from one to two on [1,2].

<u>Theorem 23</u>. If H is a connected proper subset of the compact continuum K, then K-H contains a non-cut point of K.

<u>Proof</u>. Suppose, on the contrary, that every point of K-H is a cut point of K. Let p denote some definite point of K-H. Then K-p is the union of two separated point sets, K_1 and K_2. Since H is a connected subset of K-p, it must lie wholly in one of these sets. Let K_1 contain H and let K_2 denote the other one. By Theorem 16, K_2 contains a point o whose omission does not disconnect $K_2 \cup p$. Since the connected point sets $(K_2 \cup p)$-o and $K_1 \cup p$ have the point p in common, their union is connected, but their union is K-o. Thus, K is not disconnected by the omission of o, and since o belongs to K_2 it belongs to K-H.

Figure 5 is an example of the above theorem where interval ab is a compact continuum of E^2. Let interval cd be a connected proper subset of interval ab; then ab-cd contains a non-cut point of interval ab, namely a or b.

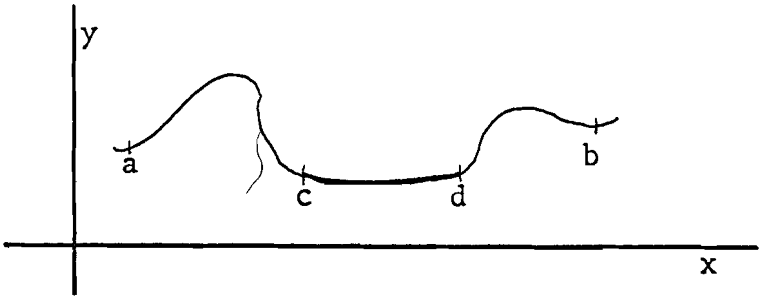

Figure 5

Lemmas 2 and 3 are needed for the proof of the theorems which follow.

Lemma 2. If H and K are two separated point sets of E^2, every connected subset of $H \cup K$ is a subset either of H or of K.

Proof. Suppose, on the contrary, that T is a connected subset of $H \cup K$ containing both a point of H and a point of K. Then T is the union of the two separated point sets $H \cap T$ and $K \cap T$, and thus, T is not connected, contrary to hypothesis.

Lemma 3. If M is a connected point set of E^2, there do not exist an open point set D and an uncountable collection H of disjoint closed subsets of M such that $D \cap M$ is a separable point set containing $\bigcup_{\alpha \in \Lambda} H_\alpha$ and if H_{α_1} is any element of H, then $M - H_{\alpha_1}$ is the union of two separated point sets of which one contains $(\bigcup_{\alpha \in \Lambda} H_\alpha) - H_{\alpha_1}$.

Proof. Suppose, on the contrary, that there exists such an open point set D and the uncountable collection H of disjoint closed subsets of M, and for each point set H_{α_1} of the uncountable collection H, $M - H_{\alpha_1}$ is the union of two separated point sets T_{α_1} and K_{α_1}, where K_{α_1} contains $(\bigcup_{\alpha\varepsilon\Lambda} H_\alpha)-H_{\alpha_1}$ and T_{α_1} contains no point of $\bigcup_{\alpha\varepsilon\Lambda} H_\alpha$. Suppose that H_{α_i} and H_{α_j} are two elements of H where $i \neq j$ and p_{α_i} and p_{α_j} are points of M, such that the point H_{α_i} separates the point p_{α_i} from H_{α_j} in M and H_{α_j} separates H_{α_i} from p_{α_j} in M. Then, by Theorem 13, H_{α_i} separates p_{α_i} from p_{α_j} in M. Hence, p_{α_i} and p_{α_j} are distinct points of M. Thus, a sequence of distinct points $\{p_\alpha\}$ has been defined such that a $p_\alpha \varepsilon T_\alpha$ for every $\alpha\varepsilon\Lambda$. Now let G denote the collection of all point sets T_{α_1} for all sets H_{α_1} of H, and let N denote a countable subset of D such that D is a subset of $\overline{N}$. Since each point set T_{α_1} of the collection G contains a point p of the sequence of distinct points $\{p_\alpha\}$ of M then, for each element T_α of G, $G_\alpha \cap D$ contains a point of N. But there are uncountably many sets, T_α, since H is uncountable. This involves a contradiction that N is countable and therefore that D is separable.

Theorem 24. No separable and connected point set M of E^2 contains a connected subset K which contains an uncountable set of points each of which disconnects M but not K.

Proof. Suppose, on the contrary, that there does exist such a set K and an uncountable set of points H, a subset of K, each of which disconnects M but not K. Then, by Lemma 2, if h is an element of H, M-h is the union of two separated point sets of which one contains K-h. But this is contrary to Lemma 3.

Consider Figure 6 as an illustration of the above theorem. Let interval ab be a set of points with rational coordinates of E^2 such that ab contains a countable and connected subset K where $ab \subset \bar{K}$. Therefore, interval ab is a separable and connected point set of E^2. Theorem 24 has proven that K does not contain an uncountable set of points each of which disconnect ab, but not K.

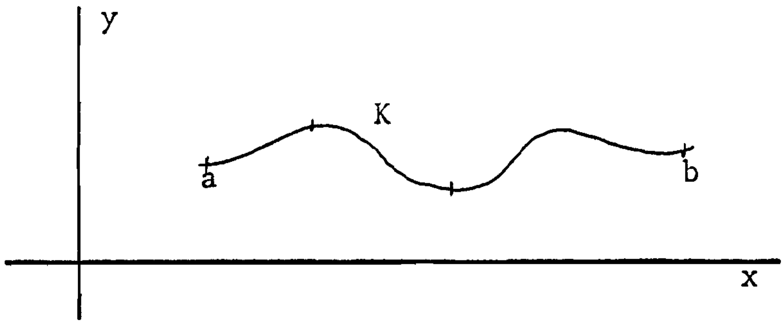

Figure 6

Further properties of a simple continuous arc are proven in the following theorems.

<u>Theorem 25</u>. No separable and connected point set M of E^2 contains a set of disjoint connected point sets M_1, M_2, M_3, $\cdots$ and a point set K such that (1) the sequence $\{M_i\}$ has a sequential limiting set that contains K and (2) there exists an uncountable collection H of disjoint closed subsets of K each of which separates M but intersects no set of this sequence.

Proof. Suppose, on the contrary, that there exists a point set M which is separable, connected and in E^2. Suppose h is an element of H. The point set M-h is the union of two separated point sets T_h and K_h where K_h contains a point p of K. Since K_h contains p, and p belongs to the sequential limiting set of the sequence $\{M_i\}$, then there are infinitely many point sets of this sequence each intersecting K_h. Each M_n is connected, and if a connected subset of M-h contains a point of K_h, then it is a subset of K_h. There is an infinite subsequence α of the sequence $\{M_i\}$ such that every point set of the sequence α is a subset of K_h. The set K is a subset of $\bar{E} \cap M$, where E is the union of all the point sets of the sequence α, and $\bar{E} \cap M$ is a subset of $K_h \cup h$. Hence, K-h is a subset of K_h. Thus, if h is any element of the uncountable collection H, then M-h is the union of two separated point sets where K_h contains K-h. This contradicts Lemma 3.

Theorem 26. If M is a simple continuous arc of E^2, and H and K are disjoint closed point sets, there do not exist infinitely many disjoint segments of M, each having one endpoint in H and the other in K.

Proof. Suppose there exists an infinite sequence of such disjoint segments of M such that each has one endpoint in H and the other in K. By Theorem 11 there exists a convergent subsequence β of α. By Theorem 10 the sequential limiting set of β is a continuum L.

The continuum L contains a point of H and a point of K, and is therefore nondegenerate. By Theorem 6, L is uncountable. Hence, the set L contains more than two non-cut points of M, contrary to the definition of an arc.

All the previous work of this chapter has been concerned with a simple continuous arc of E^2. The following pages will be concerned with describing simple closed curves and continuous curves in terms of cut points.

The simple closed curve, or Jorden curve, is usually defined as a set of points which is the homeomorph of a circle; but for purposes of this thesis the simple closed curve will be described in terms of cut points.

Definition 27. A simple closed curve of E^2 is a nondegenerate compact continuum which is disconnected by the omission of any two of its points.

A simple closed curve can be illustrated by a circle as in Figure 7. Any point can be omitted and the continuum remains connected, but the omission of any two points disconnects it. Now consider Figure 8. This continuum is not a simple closed curve. Select x and y as cut points and the continuum remains connected, therefore not satisfying the definition of a simple closed curve.

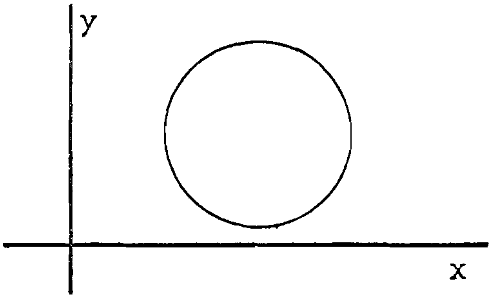

Figure 7

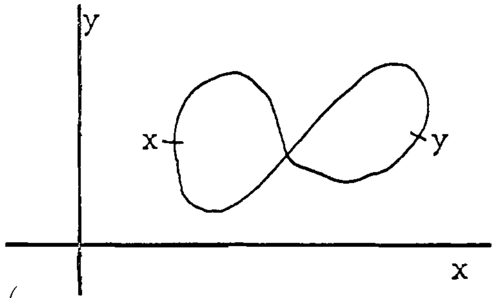

Figure 8

Theorem 27. If a and b are two points of the simple closed curve M, then M is the union of two simple continuous arcs which have in common only their endpoints a and b.

Proof. By definition, $M - (a \cup b)$ is the union of two separated point sets H and K. Suppose the closed point set $H \cup a \cup b$ is not connected. Then it is the union of two disjoint closed point sets H_a and H_b where H_a contains a. If H_b did not contain b, then M would be the union of two separated point sets $K \cup H_a$ and H_b. Hence, H_b contains b and the point set H_a is connected. Otherwise, it would be the union of two separated point sets L and N such that L contains a, and M would be the union of two separated sets $K \cup H_b \cup L$ and N. The point set $K \cup a \cup b$ is connected since, if it were not, it would be the union of two separated point sets K_a and K_b, containing a and b respectively, and thus M would be the union of two separated point sets $H_a \cup K_a$ and $H_b \cup K_b$.

Figure 9 is an illustration of Theorem 26. Choose points a and b where $a \neq b$, then the simple closed curve is two simple continuous arcs with the same endpoints a and b. Thus the union of the two arcs in Figure 9 form a simple closed curve.

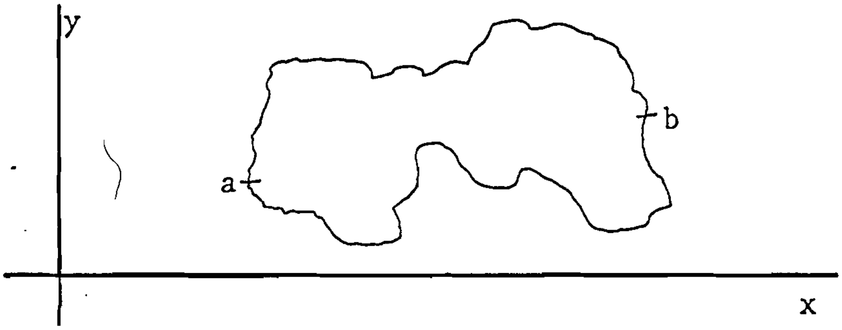

Figure 9

<u>Theorem 28</u>. If p is a point of a simple closed curve M of E^2, then M-p is connected.

<u>Proof</u>. There exists a point o belonging to M but distinct from p. By Theorem 27, M is the union of two simple arcs pxo and pyo having only their endpoints in common. The point sets pxo-p and pyo-p are connected. Since they have o in common, and therefore their union is connected, their union is M-p.

A simple continuous arc is not the homeomorph of a simple closed curve because the connection of any simple arc is destroyed by removing any point except the endpoints; but, by Theorem 28, the connection of a simple closed curve cannot be destroyed by the removal of one point.

The continuous curve is the last plane continuum to be described in terms of cut points in this thesis. Axioms 0 and 1 were used to describe the simple continuous arc and the simple closed curve, but to describe the properties of the continuous curve also requires Axiom II.

Axiom II. If p is a point of a region R, there exists a nondegenerate connected open set containing p and lying wholly in R.

Definition 28. The point set M is said to be connected im kleinen at the point p if p belongs to M, and for every open set D of M that contains p, there is an open subset of M that contains p and which is a subset of a component of D. If M is connected im kleinen at every one of its points, then M is said to be connected im kleinen.

Definition 29. The point set is said to be locally connected at the point p if p belongs to M and every open subset of M that contains p contains a connected open subset of M containing p. If the point set M is locally connected at every one of its points, M is said to be locally connected.

Definition 30. A connected im kleinen continuum is called a continuous curve.

The study of the continuous curve dates back to the late 19th century when C. Jordan is credited with doing the original work of the continuous curve. Jordan defined a plane continuous curve as a set of points (x,y) which may be obtained by functions x = f(t),

$y = g(t)$ which are continuous in the real variable t as t varies from 0 to 1.[2]

Figure 10 is an example of a continuum in E^2 which is not a continuous curve. Let H denote the interval of the y axis whose extremities are the points $(0,1)$ and $(0,-1)$ and let K be the graph of $y = \sin \frac{1}{x}$ $(0<x\leq 1)$, H and K are both connected and connected im kleinen, but $H \cup K$, though a continuum, is not a continuous curve. $H \cup K$ is not connected im kleinen at any point of H.

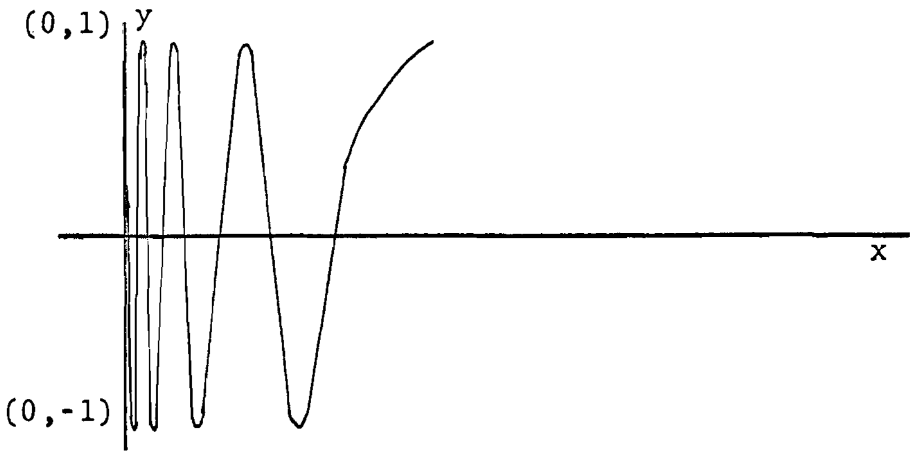

Figure 10

The following theorems summarize some basic properties of a continuous curve.

Theorem 29. If D is an open set of the point set M and M is connected im kleinen at every point of D in E^2, then every component of D is an open set of M.

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[2]Raymond Louis Wilder, *Topology of Manifolds*, (New York: American Mathematical Society, 1949), p. 69.

Proof. Let T denote a component of the open set D of the point set M. Suppose p is a point of T. By hypothesis, there exists an open set of M which contains p and belongs to some component of D and therefore to T. Hence, T is an open set of M.

Theorem 30. The point set M of E^2 is connected im kleinen at the point p if it is locally connected at that point; and if M is connected im kleinen at every point of some open set of M that contains p, then M is locally connected at p.

Proof. If a point set is locally connected at a point, then it is connected im kleinen there, since locally connected and connected im kleinen at a point are defined the same. Suppose that the point set M is connected im kleinen at every point of N, and some open set of M contains p. Suppose D is an open set of M containing p. Let T denote the component of $D \cap N$ that contains p. By Theorem 29, T is an open set of M. But T is connected. Hence, M is locally connected at p.

Theorem 31. If p is a point of the locally compact continuum M, and M is not connected im kleinen at the point p, then if R is a region containing p, there exists a connected open set D containing p and lying in R, an infinite sequence of points p_i, converging to p, and an infinite sequence of disjoint continua $\{M_i\}$ such that (1) $M \cap \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cap \overline{D}-p$ containing p_n and a point of the boundary of D, and (3) the sequence $\{M_i\}$ converges to a subcontinuum of M which contains p.

<u>Proof</u>. There exists an open set D and a locally compact continuum containing p such that (1) $M \cap \overline{D}$ is a compact proper subset of M, and (2) if Q is a subset of the open set D containing p, then there exists a point of Q which cannot be joined to p by a connected subset or component of $M \cap \overline{D}$. Let $\{R_i\}$ denote a sequence of regions closing down on p such that R_1 is a subset of D. There exists in R_1 a point p_1 which belongs to M but not to the component of $M \cap \overline{D}$ that contains p. Let N_1 denote the component of $M \cap \overline{D}$ that contains p_1. By Theorems 14 and 8, N_1 contains a point of the boundary of D. There exists a number n_1 such that R_{n_1} contains no point of the continuum N_1. There exists, in R_{n_1}, a point p_2 belonging to M but not to the component of $M \cap \overline{D}$ that contains p. Let N_2 denote the component of $M \cap \overline{D}$ that contains p_2. The point set N_2 contains a point of the boundary of D. There exists a number n_2 such that R_{n_2} contains no point of N_2. The point set $M \cap R_{n_2}$ contains a point p_3 not lying in the component of $M \cap \overline{D}$ that contains p. Let N_3 denote the component of $M \cap \overline{D}$ that contains p_3. This process may be continued. Thus, there exists a sequence of points $\{p_i\}$ converging to 0, and a sequence of disjoint continua $\{N_i\}$ such that, for each n, N_n is a component of $M \cap \overline{D}$ containing both p_n and

a point of the boundary of D. By Theorems 10 and 11, there is a subsequence of $\{N_i\}$ which converges to some continuum. This sequence fulfills all the requirements of Theorem 31.

CHAPTER III

SUMMARY

In this thesis, plane continua have been described by listing a set of axioms and defining the necessary terms needed in proving theorems about cut points and non-cut points of continua in E^2. The development of the thesis has been to list theorems which describe properties of cut points and non-cut points of simple continuous arcs, simple continuous curves and continuous curves, and then to prove these theorems by using only definitions and axioms along with the theorems previously listed in the paper.

There are many possibilities for further study of the topic, such as describing properties of continuous curves; which includes open curves, rays and acyclic curves of E^2, in terms of cut points and non-cut points. The development of these ideas could then be used to meet the conditions of Peano spaces.

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