THE PYTHAGOREAN THEOREM

AND RELATED TOPICS

A Thesis

Presented to

the Faculty of the Department of Mathematics

Kansas State Teachers College of Emporia

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts in Mathematics

by

Ronald L. Iman

August 1965

Approved by the Department of Mathematics

Approved by the Graduate Division

ACKNOWLEDGEMENTS

The author is indebted to Mr. Lester Laird for his time and patience in directing the writing of this thesis, and to Dr. Marion Emerson for his assistance.

TABLE OF CONTENTS

CHAPTER PAGE

CHAPTER I

INTRODUCTION

Although the Pythagorean Theorem is over 2000 years old, it still continues to fascinate millions of people all over the world. This is evidenced by the fact that there exist a large number of proofs of the theorem. Much has been written about the Pythagorean Theorem and its related topics, and one of the most important reasons for this study was to collect and organize some of this material into a report under a single cover. It must be kept in mind that while there are no primary sources covering Greek mathematics some sort of consistent, although largely hypothetical, account has been compiled.

PURPOSE OF STUDY

The purpose of this study was to provide an enriching resource of material on the Pythagorean Theorem. The main points that will be covered are (1) historical background information on the Pythagorean Theorem, (2) a variety of proofs of the theorem, (3) an analysis and classification of these proofs, (4) a background for work on primitive Pythagorean triples, (5) an application of the Pythagorean Theorem to some famous figure-cutting problems, and (6) an investigation into the feasibility of a Pythagorean relationship for spherical triangles.

ORGANIZATION

Chapter two provides a historical background on Pythagoras, the Pythagorean School, and the Pythagorean Theorem. Included are some of the contributions of the Pythagoreans in the fields of religion, theory of numbers, geometry, and astronomy, also, the early work on the theorem among the Greek, Egyptian, and Chinese.

Given in chapter three is a variety of the most well known and different proofs of the Pythagorean Theorem. The chapter is concluded with a proof which was developed by the author of this paper.

Chapter four gives an analysis and classification of the proofs presented in chapter three. The proofs are analysed as to the fundamental properties on which they were based and classified as to their similarities and differences.

In chapter five conditions necessary for the selection of two integers which generate a primitive Pythagorean triple are stated and proved.

The application of the Pythagorean Theorem to the last four of fifteen figure-cutting problems originally presented by the mathematics staff of the University of Chicago is given in chapter six.

Included in chapter seven is an investigation into the feasibility of a Pythagorean relationship for spherical triangles. A summary of the paper is given in chapter eight.

CHAPTER II

PYTHAGORAS, THE PYTHAGOREAN SCHOOL, AND THE DEVELOPMENT OF THE PYTHAGOREAN THEOREM

This chapter has been devoted to a historical account of Pythagoras' life and the development of the Pythagorean School. It includes some of the main contributions of the school and early work on the Pythagorean Theorem in Greece, China, and Egypt.

It must be kept in mind when reviewing the literature of the primary sources covering Greek mathematics, that the reviewer must rely chiefly on manuscripts and accounts dating from Arabian and Christian times. Scholars have reliably restored many of the original texts, such as those of Euclid, Apollonius, Archimedes, and others. From many fragments and scattered writings by later authors and philosophers, some sort of consistent, although largely hypothetical, account of the history of early Greek mathematics has been compiled.

The topics discussed in this chapter are the early life of Pythagoras, the formation of the Pythagorean society, the Pythagorean contributions in the fields of religion, theory of numbers, geometry, and astronomy, and the early work on the theorem in Greece, China, and Egypt.

I. PYTHAGORAS

The Greek philosopher, Pythagoras, was born about 572 B.C. on the Aegean Island of Samos off the coast of Asia Minor. In his early life he was a student of Thales. Thales had traveled in Egypt and learned much from the priests of Egypt, and he strongly advised his pupil, Pythagoras, to pay them a visit. Pythagoras heeded this advice and traveled and gained a wide experience. This experience benefited him when he later had disciples of his own, and he became even more famous than his teacher. It is supposed that, besides traveling to Egypt, he traveled also to Babylon and perhaps on the Greek mainland.

Returning home he found Samos under the tyranny of Polycrates and Ionia under the dominion of the Persians. He then migrated from Samos to Croton in Southern Italy in 530 B.C. There he lectured on philosophy and mathematics. His lecture room was crowded with enthusiastic hearers of all ranks, and many of the upper classes attended. Women broke a law which forbade them to attend public meetings to hear him. Among them was Theano, the beautiful young daughter of Pythagoras' host, Milo. Pythagoras later married Theano, who wrote a biography of him. This manuscript was lost.

At the time of Pythagoras' arrival, Croton had suffered a crushing defeat by the hand of the Locrians. The moral and political reform which he promoted was evidenced by the fact that Croton was able to defeat and destroy the much more populous and powerful city of Sybaris just twenty years later in 510 B.C.[1]

II. PYTHAGOREAN SOCIETY

So remarkable was the influence of Pythagoras that the more attentive of his pupils gradually formed themselves into a society of brotherhood. This newly formed order, the Pythagorean Brotherhood, had much in common with the Orphic communities, which sought by rites and abstinences to purify the believer's soul and enable it to escape from the "wheel of birth."[2] This new order was soon exercising a great influence across the Grecian world, though its influence was more religious than political.

Members of the society shared everything in common, held the same philosophical beliefs, engaged in the same pursuits, and bound themselves with an oath not to reveal the secrets and teachings of the school. "When, for example,

[1] "Pythagoras," Encyclopedia Americana (1960 ed.), XXIII, p. 47.

[2] "Pythagoras," Encyclopedia Britannica (14th ed.), XVIII, p. 803.

Hippasus perished in a shipwreck, was not his fate due to a broken promise? For he had revealed one of the secrets of the brotherhood."[3]

In the course of time this order spread to other Italian cities. The order was most outstanding in the cities of Metapontum, Rhegium, and Locri. The order probably never ruled any of these cities directly, but rather exercised its influence through members who had attained leading political positions.

In addition to the internal reforms which it promoted everywhere, the order also worked for a political and economic alliance between the cities in which it was dominant. The success of this policy is shown by the coins of the period. Many of them had the emblem of Croton on one side and the emblem of one of the other cities on the reverse side, thus indicating a monetary agreement in which Croton had the leading part.[4]

In time the influence and aristocratic tendencies of the brotherhood became so great that the democratic forces of southern Italy destroyed the buildings of the school and caused the society to disperse. The first reaction against the Pythagoreans was led by Cylon. This action stemmed from

---

[3]James R. Newman, The World of Mathematics (New York: Simon and Schuster, 1956), Vol. I, p. 83.

[4]"Pythagoras," Encyclopedia Americana (1960 ed.), XXIII, p. 47.

the victory of Croton over Sybaris in 510 B.C. The civic disturbances which followed resulted in a setback to Pythagorean power in Croton. According to one report, Pythagoras fled to Metapontum where he later died, maybe through murder, at the advanced age of seventy-five to eighty years.

An act of violence against the Pythagoreans worthy of mention was "the house of Milo" in Croton. Here fifty to sixty Pythagoreans were surprised and slain. Those who survived took refuge at Thebes and other places.[5] The brotherhood, although scattered, continued to exist for at least two centuries longer.

III. PYTHAGOREAN CONTRIBUTIONS

Some of the important contributions of Pythagoras and the Pythagorean School were religion, theory of numbers, geometry, and astronomy. These four subjects merit discussion here.

Religion. One of the most advanced of the religious doctrines of the school was the theory of the immortality and transmigration of the soul. Pythagorean teaching on this point was connected with the primitive belief in the kinship of men and beasts. The Pythagorean rule of abstinence

---

[5]"Pythagoras," Encyclopedia Britannica (14th ed.), XVIII, p. 803.

from flesh was thus, in its origin, a taboo resting on the blood-brotherhood of men and beasts. Likewise, a number of the Pythagorean rules of life which were found embodied in the different traditions appeared to be genuine taboos belonging to a similar level of primitive thought. The moral and religious application which Pythagoras gave to the doctrine of transmigration continued to be the teaching of the school.[6]

Theory of numbers. The scientific doctrines of the Pythagorean school had no apparent connection with the religious mysticism of the society or their rules of living. Their discourses and speculations all connect themselves with the mystical assumption that the whole number was the cause of the various qualities of men and matter. This oriental outlook may have been acquired by Pythagoras in his eastern travels. It led to the exaltation and study of number relations and to a perpetuation of numerological nonsense that lasted even into modern times.[7]

It has been found that much of the Pythagorean study was of an unscientific nature. In spite of this, however,

---

[6] Ibid., p. 803.

[7] Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Holt, Rinehart, and Winston, 1958), p. 12.

members of the society contributed a good deal of sound mathematics during the two hundred or so years following the founding of their organization. Aristotle said that the Pythagoreans "applied themselves to the study of mathematics and were the first to advance that science; insomuch that, having been brought up in it, they thought that its principle must be the principles of all existing things."[8]

Pythagoras is said to have attached supreme importance to arithmetic, which he advanced and took beyond the realm of commercial use. He also made geometry a part of a liberal education, examining the principle of the science and treating the theorems from an immaterial and intellectual standpoint.

Perhaps Pythagoras' greatest discovery was that of the dependence of the musical intervals on certain arithmetical ratios of lengths of string at the same tension, 2:1 giving the octave, 3:2 the fifth, and 4:3 the fourth. This discovery must have contributed powerfully to the idea that "all things are numbers." According to Aristotle, the theory in its original form did not regard numbers as relations predictable of things, but as actually constituting their essence or substance. He said numbers seemed to the

---

[8] "Pythagoras", Encyclopedia Britannica (14th ed.), XVIII, p. 803.

Pythagoreans to be the first things in the whole of nature, and they supposed the elements of numbers to be the elements of all things and the whole heaven to be a musical scale and a number. Later, in the fragmentary writings of Philolaus, things were spoken of, not as being numbers, but as having number and thereby becoming knowable.[9]

The development of these ideas into a comprehensive metaphysical system was probably the work of Philolaus. According to the Pythagoreans, the elements of numbers referred to by Aristotle were the odd and the even, which they identified with the limit and the unlimited. The unlimited and therefore the limit also, was conceived as spatial (of or pertaining to space). Numbers were thus spatially regarded, and "one" was identified with the point, which was a unit having position and magnitude; "two" was similarly identified with the line; "three" with surface; and "four" with solid.[10]

The odd and even and the limit and unlimited were the first two of a set of ten fundamental oppositions postulated by the Pythagoreans. The remaining eight were the following: one and many, right and left, male and female, rest and motion, straight and curved, light and darkness,

---

[9] Ibid., p. 803.

[10] James R. Newman, The World of Mathematics (New York: Simon and Schuster, 1956), Vol. I, p. 85.

good and evil, and square and oblong. To the Pythagoreans the universe was in a sense the realization of these opposites.

Some further speculations of the Pythagoreans on the subject of number rested mainly on fanciful analogies. "Five" suggested marriage because it was the union of the first masculine and the first feminine number (3 + 2, unity not being considered a number); "one" was identified with reason because it was unchangeable; "two" with opinion because it was unlimited and indeterminate; "four" with justice because it was the first square number, the product of equals.[11]

Pythagoras pictured numbers as having characteristic designs. There were the triangular numbers, one, three, six, ten, and so on. Ten was known as the Holy Tetractys and was highly revered by the brotherhood. The triangular numbers were represented by figures of the following kind:

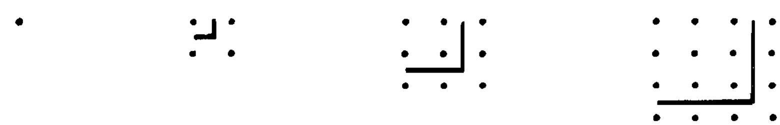
```
                                        .
                          .            . .
             .           . .          . . .
  .         . .         . . .        . . . .
```

which represent respectively one, three, six, and ten. The figures show at a glance the composition of the triangular numbers, for example 10 = 1 + 2 + 3 + 4. To add a row of five dots to "ten" gave the next triangular number with 5

---

[11] Ibid., p. 85.

as the side, and so on, showing that the sum of any number of the series of natural numbers beginning with 1 was a triangular number. The sum of any number of the series of odd numbers beginning with 1 was similarly seen to be a square, so the square numbers were represented by figures like the following.

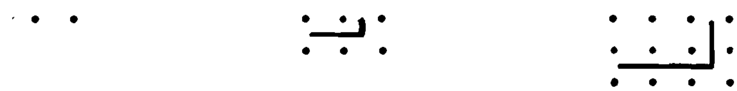


Each of these square numbers could be derived from its predecessor by adding an L-shaped border. Great importance was attached to this border; it was called a gnomon (carpenter's rule). If the gnomon added to a square was itself a square number, e.g., 9, there resulted a square number which was the sum of two squares: thus $1 + 3 + 5 + 7 = 16$ or 4^2, and the addition of 9 or 3^2 gave 25 or 5^2, thus $3^2 + 4^2 = 5^2$. Pythagoras himself was credited with a general formula[12] for finding two square numbers the sum of which was also a square. Namely, (if m is any odd number), $m^2 + \left[\frac{1}{2}(m^2 - 1)\right]^2 = \left[\frac{1}{2}(m^2 + 1)\right]^2$. Letting m be a number of the form $2k + 1$ where k is an integer, and then simplifying the formula, shows that it is an identity.

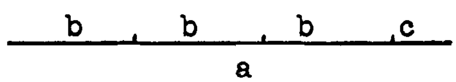
---

[12] "Pythagoras", Encyclopedia Britannica, (14th ed.) XVIII, p. 804.

Another pattern was obtained by taking the sum of any number of even numbers beginning with 2. These were called the oblong numbers, and they were represented by figures of the following kind.



Geometry. One of the greatest contributions to geometry by Pythagoras was the discovery of the irrational. In other words, he proved that it was not always possible to find a common measure for two given lengths *a* and *b*. The practice of measuring one line against another must have been very primitive. Given below is a long line *a*, into which the shorter line *b* fits three times, with a still shorter piece *c* left over.



Today it is expressed by the equation $a = 3b + c$, or more generally by $a = nb + c$. If there is no such remainder *c*, the line *b* measures *a*; and *a* is called a multiple of *b*. If, however, there is a remainder *c*, further subdivision might perhaps account for each length *a*, *b*, *c* without remainder: experiment might show, for instance, that in tenths of inches, $a = 17$, $b = 5$, $c = 2$. At one time it was thought that it was always possible to reduce lengths *a* and *b* to such multiples of a smaller length. It appeared to be simply a question of patient subdivision, and sooner or later the desired measure would be found. So the required subdivision, in the present example, was found by measuring b with c. For c fit twice into b with a remainder d; and d fit exactly twice into c without remainder. Consequently d measured c, and also measured b and also a. This was how the numbers 17, 5, and 2 came to be attached to a, b, and c; because a contained d 17 times.

This reduction of the comparison on a line a with a line b to that of the number 17 with 5, or speaking more technically, this reduction of the ratio a:b to 17:5 would have been agreeable to the Pythagoreans. It exactly fitted in with their philosophy; for it helped to reduce space and geometry to pure number. Then came the discovery by Pythagoras himself that the reduction was not always possible, that something in geometry eluded whole numbers. Exactly how this discovery of the irrational took place was not related, although two early examples can be cited. First when a was the diagonal and b was the side of a square, no common measure could be found; nor could it be found in a second example, when a line a was divided in golden section into parts b and c. This meant that the ratio of a, the whole line, to the part b was equal to the ratio of b to the other part c. Here c could be fitted once into b with remainder d; then d could be fitted once into c with remainder e, and so on. It was not hard to prove that such lengths a, b, c, d, . . . form a geometrical progression without end, and that the desired common measure could never be found.[13]

The reason why such a problem came to be studied was to be found in the star pentagram. It was the badge of the Pythagorean Brotherhood, and each line in it was divided into this golden section. The star had five lines, each cut into three parts, the lengths of which can be taken as a, b, a. As for the ratio of the diagonal to the side of a square, Aristotle suggested that the Pythagorean proof of its irrationality was substantially as the following:

> If the ratio of diagonal to side is commensurable, let it be p:q, where p and q are whole numbers prime to one another. Then p and q denote the number of equal subdivisions in the diagonal and the side of a square respectively. But since the square on the diagonal is double that on the side, it follows that $p^2 = 2q^2$. Hence p^2 is an even number, and p itself must be even. Therefore p may be taken to be 2r, p^2 to be $4r^2$, and consequently q^2 to be $2r^2$. This requires q to be even; which is impossible because two numbers p, q, prime to each other cannot both be even. So the original supposition is untenable: no common measure can exist; and the ratio is therefore irrational.[14]

Other contributions to geometry by the Pythagoreans include the following. (1) Credit was usually given to

---

[13] James R. Newman, The World of Mathematics (New York: Simon and Schuster, 1956), Vol. I, p. 87-88.

[14] Ibid., p. 89.

Pythagoras for formulating geometry. (2) The Pythagoreans proved that the sum of the three angles of any triangle was equal to two right angles. Their proof, like Euclid's, used the property of parallels; hence they knew the theory of parallels. (3) They discovered the powerful method in geometry of the application of areas, including application with excess and defect (Euclid, vi, 28-29) which amounted to the geometrical solution of any quadratic equation in algebra having real roots. (4) Pythagoras himself is said to have discovered the theory of proportion and the three means: arithmetic, geometric, and harmonic. In Babylon, Pythagoras is said to have learned the "perfect proportion":

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b$$

which involved the arithmetical and harmonical means of two numbers as its middle terms. Particular cases being 12:9, 8:6, from the terms of which the three musical intervals can be obtained. The Pythagorean theory of proportion was arithmetical (after the manner of Euclid, book vii) and did not apply to incommensurable magnitudes; it must not, therefore, be confused with the general theory due to Eudoxus, which was expounded in Euclid V.

In the field of geometric constructions, the assumption that the Pythagoreans could construct a regular pentagon was confirmed by the fact that the star pentagram was used by the Pythagoreans as a symbol of recognition between the members of the school and was called by them health. The Pythagoreans also discovered how to construct a rectilineal figure equal to one and similar to another rectilineal figure.[15]

In summing up the Pythagorean geometry, one can say that it covered the bulk of the subject matter of Euclid's books i, ii, iv, vi, (and probably iii), with the qualification that the Pythagorean theory of proportion was inadequate in that it did not apply to incommensurable magnitudes.

Astronomy. Pythagoras was one of the first to hold that the earth and the universe were spherical in shape. He realized that the sun, moon, and planets had a motion of their own independent of the daily rotation and in the opposite sense. It was unlikely that Pythagoras himself was responsible for the astronomical system known as Pythagorean, which disposed the earth from its place in the center and made it a planet like the sun, the moon, and the other planets. For Pythagoras apparently the earth was still at the center.

The Pythagorean system was attributed alternatively to Philolaus and to Hicetas, a native of Syracuse. They

---

[15] "Pythagoras", Encyclopedia Britannica (14th ed.) XVIII, p. 804.

believed that the universe was spherical in shape and finite in size. Outside it was infinite void, which enabled the universe to breathe. At the center was the central fire, called the hearth of the universe, wherein was situated the governing principle, the force which directed the movement and activity of the universe. In the universe there revolved around the central fire the following bodies: nearest to the central fire was the counterearth which always accompanied the earth; next in order (reckoning from the center outward) was the earth, then the moon, then the sun, then the five planets and then, last of all, the sphere of the fixed stars. The counterearth, revolving in a smaller orbit than the earth, was not seen by them because the hemisphere in which they lived was always turned away from the counterearth (the analogy of the moon which always turned the same side to them may have suggested this). This part of the theory involved the assumption that the earth rotated about its own axis in the same time as it took to complete its orbit around the central fire; and as the latter revolution was held to produce day and night, it was a logical inference that the earth was thought to revolve around the central fire in a day and a night, or in twenty-four hours.[16]

---

[16] *Ibid.*, p. 804.

EARLY DEVELOPMENT

Greek Development. Since Pythagoras' teaching was entirely oral and it was the custom of the brotherhood to refer all discoveries back to the revered founder, it is difficult to know just which mathematical findings and which philosophical viewpoints should be credited to Pythagoras, and which to the other members of the fraternity. However, tradition has unanimously ascribed to Pythagoras the independent discovery of the theorem which bears his name; namely, that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs. Others may have known of the theorem before Pythagoras, but he may well have given the first general proof of it.

Egyptian Development. The Egyptian geometrical knowledge seems to have been of a wholly practical nature. The Egyptians knew of special cases of the Pythagorean Theorem, but they didn't offer a general proof of it. An illustration of the way they used the theorem is given by the following example.

The Egyptians were very particular about the exact orientation of their temples. They had to obtain a north and south line and an east and west line with accuracy. They observed the points on the horizon where a star rose and set, took a plane midway between them, and obtained a north and south line. To get an east and west line, which had to be drawn at right angles to this, they used a rope ABCD, divided by knots or marks at B and C, so that the lengths AB, BC, CD were in the proportion 3:4:5. The length BC was placed along the north and south line, and pegs P and Q inserted at the knots B and C. The piece BA was then rotated around the peg P, and the piece CD was rotated around the peg Q until the ends A and D coincided. The point thus indicated was marked by a peg R. The result was to form a triangle PQR whose angle at P was a right angle, and the line PR gave an east and west line.

The Egyptians probably knew that this theorem was true for a right-angled triangle when the sides which contained the right angle were equal; for it would be obvious if a floor were paved with tiles of that shape. But these facts cannot be said to show that geometry was then studied as a science.[17]

Chinese Development. The Chinese in the time of Chou-Kong had known of the Pythagorean Theorem. Although it was not enunciated in such a concise geometrical form as was given by Euclid, there can be no denying the fact that it was soundly established by the Chinese.

---

[17]James R. Newman, The World of Mathematics (New York: Simon and Schuster, 1956) Vol. I, p. 11.

The Chinese mathematical treatise now in existence next in age to the Chou-pei is doubtless the Chiu-chang Saun-shu, or the "Arithmetic in Nine Sections." It was written by Ch'ang Ts'ang around 176 B.C.

The ninth and last section is on the "kou-ku." The kou was one side of a right triangle, and the ku another side. The term "kou-ku" meant, therefore, the right triangle. The problems in the ninth section were mostly those that could be solved by means of the Pythagorean Theorem. The theorem was enunciated in the following words:

> Square the first side and the second side, and add them together; then the square root (of the sum) is the hsien or the hypotenuse.
>
> Again, when the square of the second side is subtracted from the square of the hypotenuse, the square root of the remainder is the first side.
>
> Again, when the square of the first side is subtracted from the square of the hypotenuse, the square root of the remainder is the second side.[18]

---

[18]Yoshio Mikami, The Development of Mathematics in China and Japan (New York: G. E. Strechert and Co., 1913), p. 21.

CHAPTER III

PROOFS OF THE PYTHAGOREAN THEOREM

This chapter has been devoted to a presentation of some of the more noteworthy proofs of the Pythagorean Theorem. These proofs range from the one that was thought to have been given by Pythagoras to an original one derived by the author of this paper.

The materials containing proofs of the Pythagorean Theorem are quite abundant, and in fact there exist some very large collections of the proofs of the theorem. For example, in the second edition of his book, The Pythagorean Proposition, E. S. Loomis has collected and classified 370 demonstrations of the famous theorem. It was not, however, the intent of this chapter to exhaust the supply of available proofs, but rather to present a variety of some of the more noteworthy and different proofs of the theorem.

Proofs included in this chapter start with a dissection proof that might have been offered by Pythagoras and another dissection proof given by Bhaskara. Also included will be a second proof by Bhaskara which was rediscovered by John Wallis in the seventeenth century, and the proof given by Euclid in his famous Elements. A proof by James A. Garfield and some proofs from a collection by William W. Rupert comprise the bulk of the remainder of this chapter. In concluding the chapter the author of this paper has given his original proof of the theorem, which was developed during the writing of this paper.

<u>The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs</u>. This theorem has remained one of the most famous geometrical theorems of all time, and has fascinated millions of people all over the world. This has been evidenced by the fact that there exist so many proofs of it. There has been much conjecture as to the proof Pythagoras might have offered, but it has generally been felt that it was a dissection type of proof like the one on the following page.

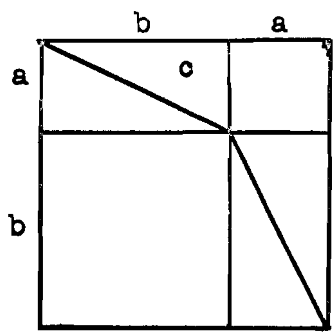


Figure 1

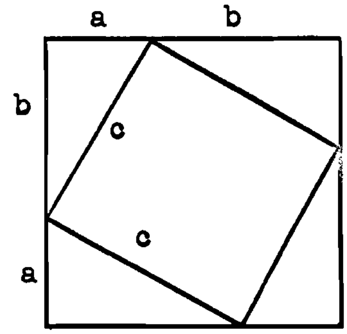


Figure 2

Proof No. 1. Let a, b, and c denote the legs and hypotenuse of the given right triangle, and consider the two squares in the accompanying figure, each having a + b as a side.

To prove that the central piece of the second dissection is actually a square of side c, employ the fact that the sum of the angles of a right triangle is equal to two right triangles. The rest of the proof is as follows:

The area of the square 1 is given by,

$$a^2 + b^2 + 4(ab/2).$$

The area of square 2 is given by,

$$c^2 + 4(ab/2).$$

Hence,

$$a^2 + b^2 + 4(ab/2) = c^2 + 4(ab/2).$$

Or,

$$a^2 + b^2 = c^2.$$

Q. E. D.[1]

[1]Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart, and Winston, 1960), p. 59.

A second dissection proof was given by Bhaskara, the famous Hindu mathematician. In this proof the square on the hypotenuse was cut, as indicated in Figure 3, into four triangles each congruent to the given triangle and a square. The pieces were easily rearranged to give the sum of the squares on the two legs. Bhaskara drew the figure and offered no further explanation than the word "Behold!" The proof is as follows.

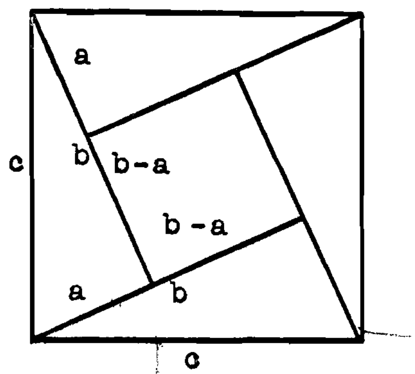


Figure 3

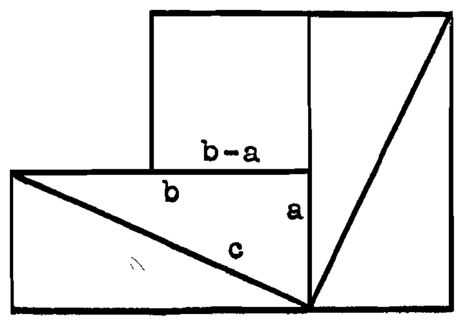


Figure 4

Proof No. 2. If c is the hypotenuse and a and b are the legs of the triangle, the area of the square in Figure 3 is c^2.

The area of the figure formed by reassembling the pieces is,

$$4(ab/2) + (b - a)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2.$$

Therefore, $c^2 = a^2 + b^2$.

Q. E. D.[2]

---

[2]*Ibid.*, p. 187.

Bhaskara also gave a second demonstration of the Pythagorean Theorem, which was rediscovered by John Wallis in the seventeenth century. This proof has been used in many of the present day high school geometry texts.[3] In the following figure, the altitude h is constructed on the hypotenuse c of the given right triangle.

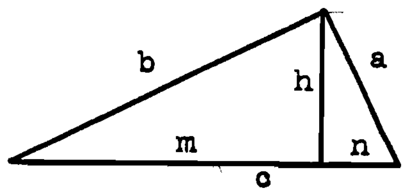


Figure 5

Proof No. 3. From similar right triangles,

$$c/b = b/m, \text{ and } c/a = a/n;$$

or,

$$cm = b^2, \text{ and } cn = a^2.$$

Then by adding,

$$a^2 + b^2 = c(m + n) = c^2.$$

Q. E. D.[4]

---

[3] A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson, Plane Geometry (Boston: Ginn and Company, 1958), p. 417.

[4] Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart, and Winston, 1960), p. 59.

Probably the most well-known proof of the Pythagorean Theorem was given by Euclid, as Proposition 47 of Book I, in his Elements. The complete proof is given below.

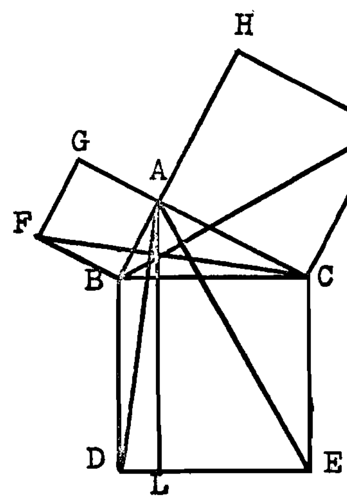


Figure 6

Proof No. 4. In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right-angled triangle, with the right angle at A; I say that the square on BC is equal to the squares on BA, AC. For let there be described on BC the square BDEC, and on BA, AC the squares GB, HC; through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG.

For the same reason, BA is also in a straight line with AH. And, since the angle DBC is equal to the angle FBA, for each is right, let the angle ABC be added to each; therefore, the whole angle DBA is equal to the whole angle FBC.

And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively; and the angle ABD is equal to the angle FBC; therefore, the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC.

Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels BD, AL.

And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC.

But the doubles of equals are equal to one another.

Therefore, the parallelogram BL is also equal to the square GB. Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC; therefore, the whole square BDEC is equal to the two squares GB, HC.

And the square BDEC is described on BC, and the squares GB, HC on BA, AC.

Therefore, the square on the side BC is equal to the squares on the sides BA, AC.

Q. E. D.[5]

The above proof has also been included in many high school textbooks.[6] However, some of the terminology used in the translation of the proof as presented here would no doubt seem strange to many students of modern day mathematics.

---

[5]Robert M. Hutchins (ed.), Great Books of the Western World (Chicago: William Benton, 1952), XI, 28-29.

[6]A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson, Plane Geometry (Boston: Ginn and Company, 1958), p. 252.

A very beautiful proof of the Pythagorean Theorem was given by General James A. Garfield. It appeared in the New England Journal of Education in 1876, five years before General Garfield became president. Garfield's proof utilizes the area of a trapezoid.

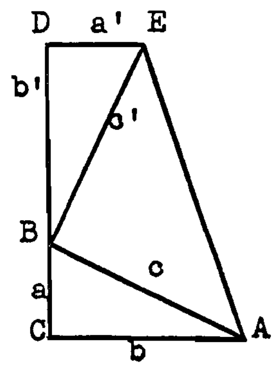


Figure 7

Proof No. 5. ABC is the given right-angled triangle. CB is extended to D, making b' = b. ED is constructed perpendicular to BD, making a' = a. BE and AE are drawn. The area S of the trapezoid CAED is given by the formula:

$$S = \tfrac{1}{2}(a + b')(b + a').$$

$$S = \tfrac{1}{2}(a + b)(b + a), \text{ since } b' = b, \text{ and } a' = a.$$

$$S = \tfrac{1}{2}(a^2 + 2ab + b^2).$$

$$S = \tfrac{1}{2}a^2 + ab + \tfrac{1}{2}b^2.$$

Considering the areas of the triangles of the trapezoid,

$$S = \tfrac{1}{2}ab + \tfrac{1}{2}c^2 + \tfrac{1}{2}ab.$$

$$S = ab + \tfrac{1}{2}c^2.$$

Therefore,

$$ab + \tfrac{1}{2}c^2 = \tfrac{1}{2}a^2 + ab + \tfrac{1}{2}b^2.$$

Or,

$$c^2 = a^2 + b^2.$$

Q. E. D.[7]

Some very fine proofs of the Pythagorean Theorem were collected by William W. Rupert and were published by D. C. Heath and Company in 1900. Several of these proofs have been chosen for presentation in this chapter. In all fairness to Mr. Rupert, it should be known that each of these proofs has been rewritten, adding to them and rewording them where the line of thought could be better transmitted. In some of the proofs, congruency of areas is involved. In these proofs, for the sake of brevity, no effort was made to show all of the steps involved in proving these congruences. However, they can be supplied readily by the reader. Also, for continuity and easiness of reading, these proofs will be presented one to a page.

---

[7] *Ibid.*, p. 253.

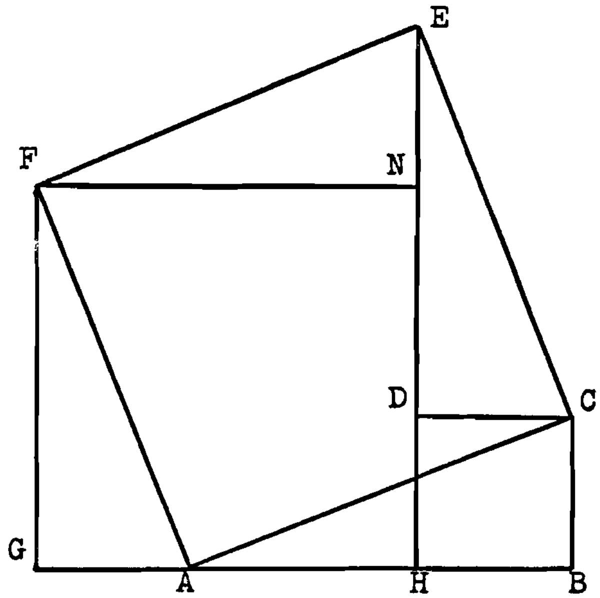


Figure 8

Proof No. 6. ABC is a right-angled triangle. The four triangles ABC, AGF, FEN, and EDC are equal to each other. HNFG is a square, and is equal to the square on AB.

(1) Area of GBCEF = $\overline{AC}^2 + \frac{1}{2}\overline{GA}\cdot\overline{FG} + \frac{1}{2}\overline{AB}\cdot\overline{BC}$.

(2) Also, area of GBCEF = $\overline{GH}^2 + \frac{1}{2}\overline{EN}\cdot\overline{FN} + \frac{1}{2}\overline{DC}\cdot\overline{ED} + \overline{BC}^2$.

Since triangles ABC, AGF, FEN, and EDC are equal:

(3) $\frac{1}{2}\overline{GA}\cdot\overline{FG} = \frac{1}{2}\overline{AB}\cdot\overline{BC} = \frac{1}{2}\overline{EN}\cdot\overline{FN} = \frac{1}{2}\overline{DC}\cdot\overline{ED}$.

From (1) and (2)

$$\overline{AC}^2 = \overline{GH}^2 + \overline{BC}^2.$$

Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, since $\overline{GH} = \overline{AB}$.

Q. E. D.[8]

---

[8]William W. Rupert, Famous Geometrical Theorems and Problems (Boston: D. C. Heath and Company, 1960), p. 21.

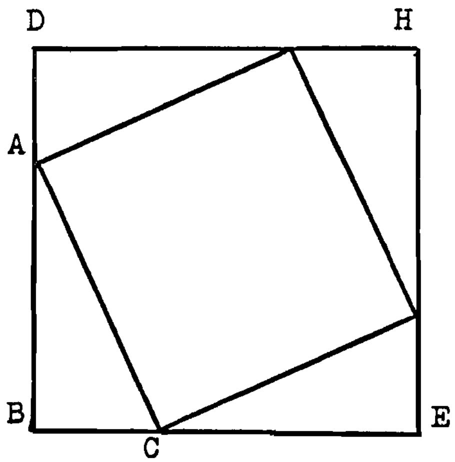


Figure 9

Proof No. 7. In right triangle ABC, BA is extended to D, making AD = BC; also BC is extended to E, making CE = AB, and the square is completed. A square is erected on AC. Then $(AB + AD)^2$ = area of square BEHD. But this area is composed of the area of the four triangles, which are equal to each other, and the square of AC. Hence,

(1) Square BEHD = $4\left[(AB \times AD)/2\right] + \overline{AC}^2$.

(2) Then $\overline{DB}^2 = (AB + AD)^2$,

$= \overline{AB}^2 + 2(AB \times AD) + \overline{AD}^2$.

From (1) and (2),

(3) $2(AB \times AD) + \overline{AC}^2 = \overline{AB}^2 + 2(AB \times AD) + \overline{AD}^2$.

Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{AD}^2$.

But AD = BC.

$\therefore\ \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

Q. E. D.[9]

---

[9] Ibid., p. 23.

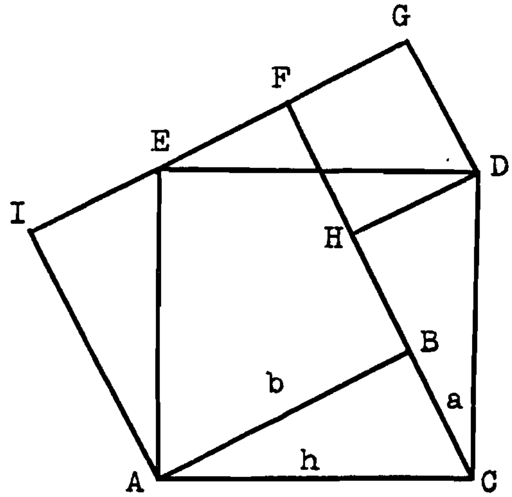


Figure 10

<u>Proof No. 8</u>. ABC is a right-angled triangle. The squares ABFI and ACDE are constructed. At D the perpendiculars are constructed to IF extended, (G), and to FC, (H). The triangles ABC, DHC, EGD, and AIE are equal, and therefore the side of square FGDH is equal to BC. Let AB = b, BC = a, and CA = h.

Then, AEFHDC = AEDC + DEG - FGDH,

$$= h^2 + \tfrac{1}{2}ab - a^2. (1)$$

Also, AEFHDC = ABC + ABFI + CDH - AEI,

$$= \tfrac{1}{2}ab + b^2 + \tfrac{1}{2}ab - \tfrac{1}{2}ab. (2)$$

Equating (1) and (2), and solving for h^2,

$$h^2 = a^2 + b^2.$$

Or, $$\overline{CA}^2 = \overline{BC}^2 + \overline{AB}^2.$$

Q. E. D.[10]

---

[10] *Ibid.*, p. 32.

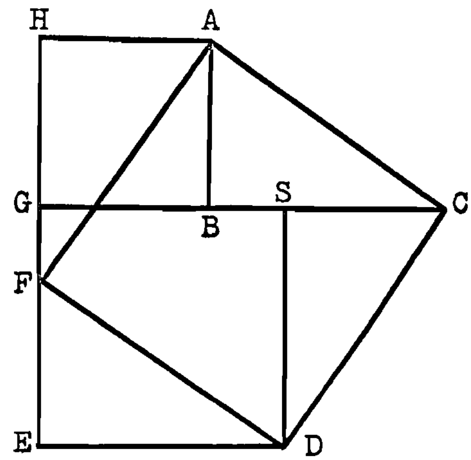


Figure 11

Proof No. 9. ABC is a right-angled triangle. The square ABGH is constructed, and also the square GEDS with side equal to BC, and square AFDC is completed. It is easily shown that F lies on HE. Therefore, square AFDC = triangle ABC + triangle DSC + figure FABSD.

But since AB = AH and AC = AF, triangle ABC $\cong$ triangle AHF.

Also since ED = DS and FD = DC, triangle DSC $\cong$ triangle FED.

Therefore, square AFDC = triangle AHF + triangle FED + figure FABSD.

Also triangle AHF + triangle FED + figure FABSD = square GEDS + square ABGH = $\overline{BC}^2 + \overline{AB}^2$.

Therefore, square AFDC = $\overline{BC}^2 + \overline{AB}^2$. Or, $\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2$.

Q. E. D.[11]

---

[11] Ibid., p. 27.

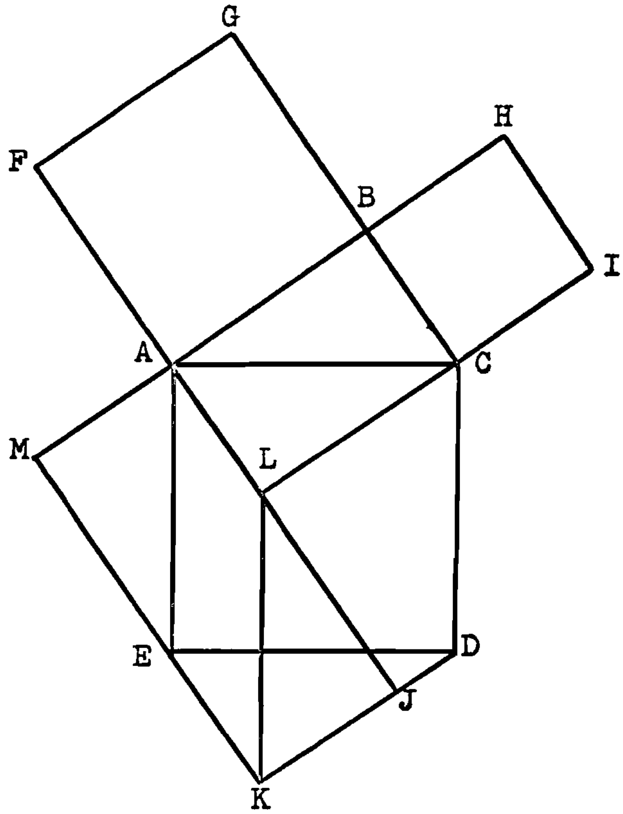


Figure 12

<u>Proof</u> <u>No</u>. <u>10</u>. ABC is a right-angled triangle, with squares constructed on its sides. DK is constructed parallel to BA extended, and MK is constructed perpendicular to MA and KD. AL is extended to J and LK is drawn. It is evident that square ACDE = parallelogram CDKL + parallelogram AEKL (as triangle ALC $\cong$ triangle EKD).

But parallelogram CDKL = square ABGF (as CL = AB = LJ), and parallelogram AEKL = square BCIH (as BC = AL = MA).

Therefore, square ACDE = square ABGF + square BCIH. Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. Q. E. D.[12]

---

[12] <u>Ibid</u>., p. 31.

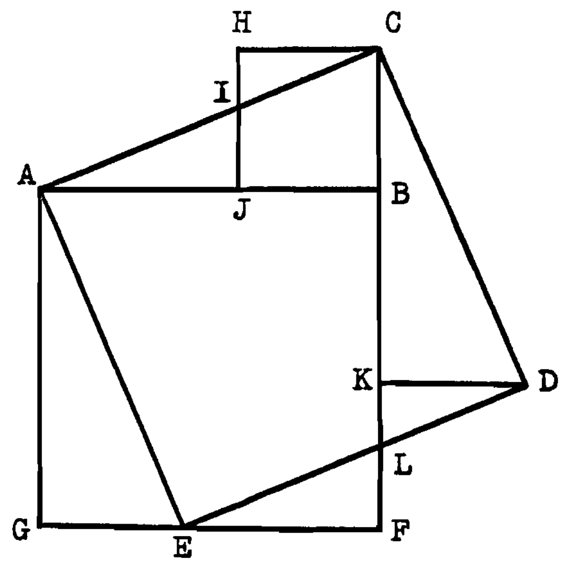


Figure 13

<u>Proof</u> <u>No</u>. <u>11</u>. ABC is a right-angled triangle, with squares constructed on the sides. DK is constructed perpendicular to CL.

Square AEDC = triangle AJI + triangle CKD + triangle DKL = figure AELCIJ. Square AGFB + square BCHJ = triangle CHI + triangle AGE + triangle EFL + figure AELCIJ.

But, triangle AGE $\cong$ triangle CKD (as AE = AC = CD and AG = AB = CK) and, triangle EFL $\cong$ triangle AJI (as angle JAI = angle FEL and AJ = EF) and, triangle CHI = triangle DKL (as angle KLD = angle HIC and KL = HI).

Therefore, square AEDC = square AGFB + square BCHJ.

Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

Q. E. D.[13]

---

[13]<u>Ibid</u>., p. 32.

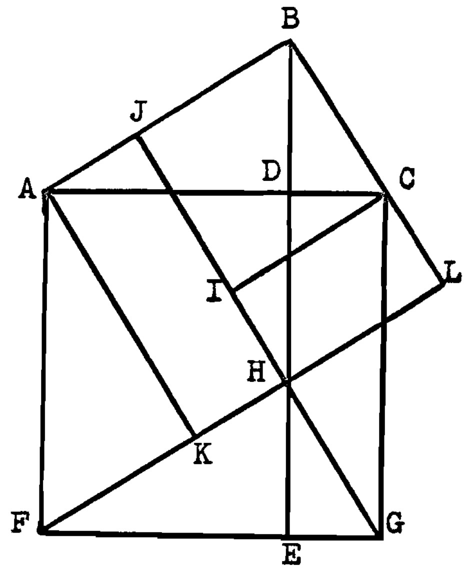


Figure 14

<u>Proof</u> <u>No</u>. <u>12</u>. ABC is a right-angled triangle, with squares constructed on the sides. KL and JI are extended to F and G, respectively, and BHE is drawn.

Square AFGC = rectangle AFED + rectangle CGED.

The square AKLB = parallelogram AFHB,

= rectangle AFED, (as they have the same bases in the same sets of parallels).

The square BJIC = parallelogram BHGC,

= rectangle CGED, (same as above).

Therefore, AFGC = AKLB + BJIC.

Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

Q. E. D.[14]

---

[14]<u>Ibid</u>., p. 28

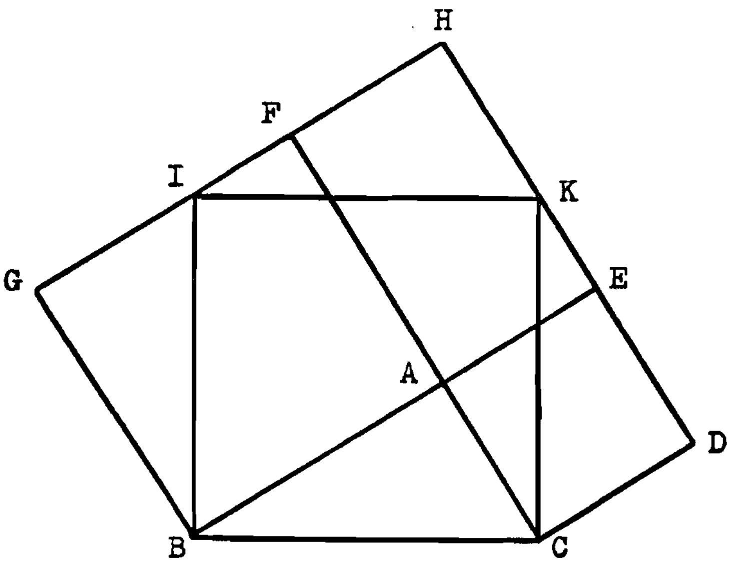


Figure 15

Proof No. 13. ABC is a right-angled triangle, with the right angle at A. The squares ABGF and ACDE are completed, and GF and DE are extended until they meet at H. Let BI, CK be perpendicular to BC and join IK. The square BCKI on the hypotenuse has now been completed. The three triangles BGI, IHK, KDC are equal to each other, and to the triangle ABC. (as BI = IK = KC = CB and the corresponding acute angles are equal.) Hence, the square on BC = figure BGHDC - 3 times the area of triangle ABC.

Now the rectangle AEHF = AE x HE

= 2 times the area of triangle ABC.

(As AE = AC and HE = KD = BC.)

And, BGHDC - (rectangle AEHF + triangle ABC) = $\overline{AB}^2 + \overline{AC}^2$.

That is, BGHDC - 3 times area of triangle ABC = $\overline{AB}^2 + \overline{AC}^2$.

Therefore, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$. Q. E. D.[15]

---

[15]*Ibid.*, p. 24

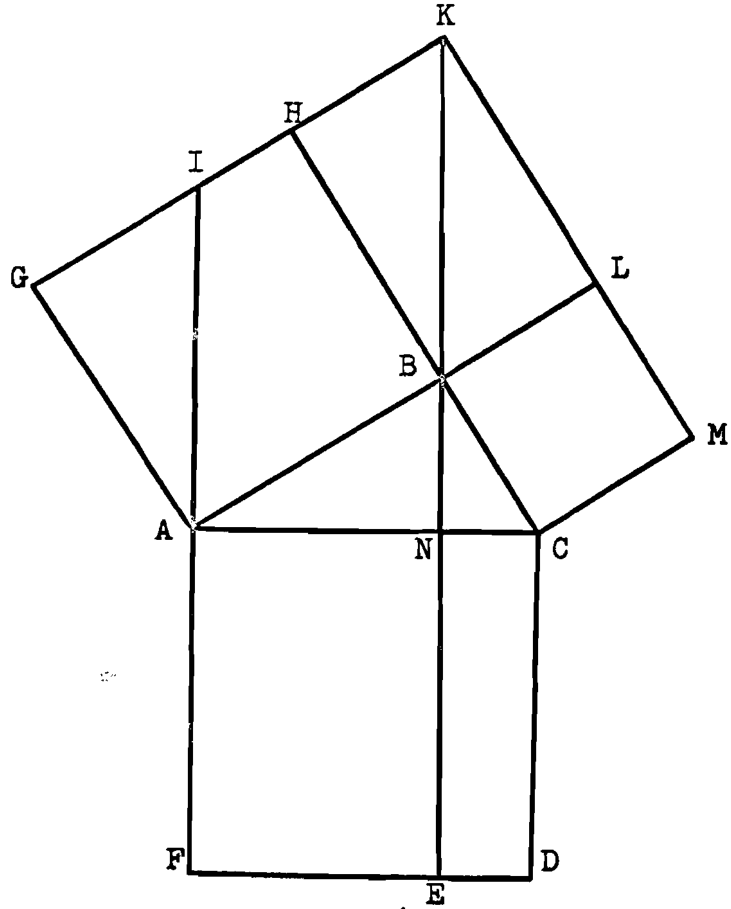


Figure 16

Proof No. 14. ABC is a right-angled triangle with the right angle at B. The squares on the sides and rectangle BLKH are completed. FA is extended to I, and KE is drawn parallel to CD and FAI. The square AGHB equals parallelogram AIKB, as they have the same base, AB, in the same set of parallels; which in turn is equal to rectangle ANEF, as they have equal bases, AI = AC = AF, in the same set of parallels. In a like manner CBLM = NCDE.

Therefore, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. Q. E. D.[16]

---

[16] *Ibid.*, p. 22.

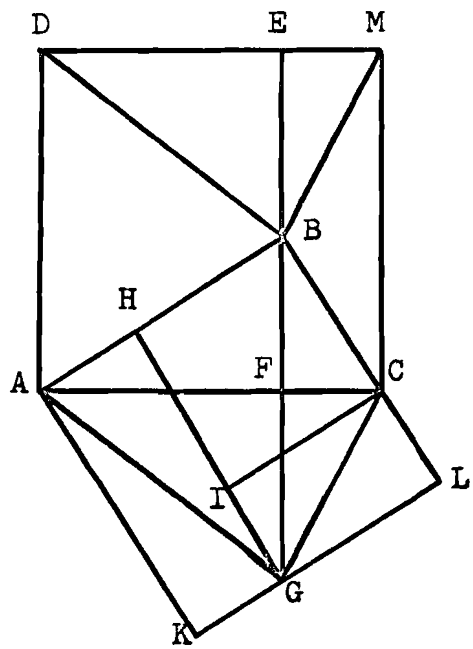


Figure 17

Proof No. 15. ABC is a right-angled triangle with squares constructed on its sides. DB and AG are parallel as are BM and CG; also EG is parallel to DA and MC. The rectangle ADEF equals parallelogram ADBG, as they have the same base DA, and are in the same set of parallels.

The area of triangle ABG = ½ of square AKLB. (As triangle AKG ≅ triangle AHG and triangle BHG ≅ triangle BLG.)

But triangle ABG = ½ parallelogram ADBG.

∴ square AKLB = parallelogram AGBD = rectangle ADEF.

In like manner square BCIH = parallelogram BGCM = rectangle FEMC.

But ADEF + FEMC = square on AC.

$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

Q. E. D.[17]

---

[17] Ibid., p. 26.

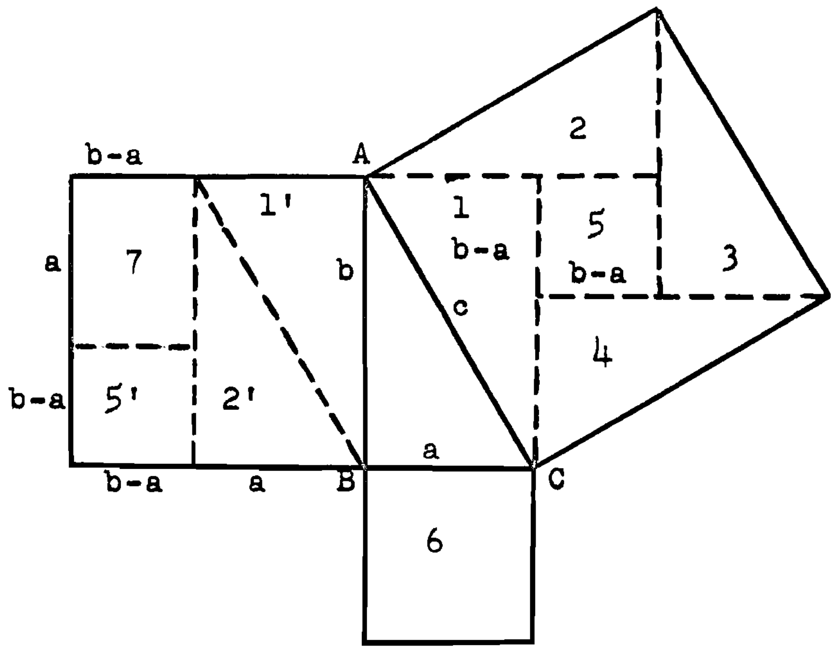


Figure 18

<u>Proof</u> <u>No</u>. <u>16</u>. ABC is a right-angled triangle, with squares constructed on its sides. In square AC triangles 1, 2, 3, and 4 all constructed equal to triangle ABC. 5 is a square with a side of b-a. In square AB triangles 1' and 2' are constructed equal to triangle ABC. Square 5' is completed with a side of b-a. Rectangle 7 will have the dimensions of a and b-a.

$\therefore$ triangle 1 = triangle 1', triangle 2 = triangle 2', and square 5 = triangle 5'.

But, triangle 3 + triangle 4 = $\frac{1}{2}ab + \frac{1}{2}ab = ab$.

Also, square 6 + rectangle 7 = $a^2 + a(b-a) = ab$.

$\therefore$ triangle 3 + triangle 4 = square 6 + rectangle 7.

$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. Q. E. D.[18]

---

[18] <u>Ibid</u>., p. 33.

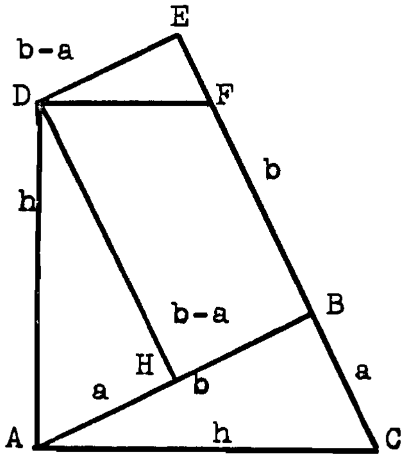


Figure 19

Proof No. 17. ABC is a right-angled triangle. AD is erected perpendicular to and equal to AC, and BE perpendicular to and equal to AB. DE is drawn. DF is drawn perpendicular to AD, and DH perpendicular to AB. Let AB = b, BC = a, AC = h.

Triangle ADH $\cong$ triangle ABC, and AH = BC = a, and BH = DE = b-a.

Since the triangles ABC and DEF are similar, DF = h(1-a/b), and EF = a(1-a/b).

Obviously, ADEC = ADEB + ABC,

$$= \tfrac{1}{2}b\left[b + (b-a)\right] + \tfrac{1}{2}ab = b^2. \qquad (1)$$

Also, ADEC = ADFC + DEF,

$$= \tfrac{1}{2}h\left[h(1-a/b) + h\right] + \tfrac{1}{2}ab(1-a/b)^2. \qquad (2)$$

Equating (1) and (2), and solving for h^2,

$$(2b-a)h^2 = (2b-a)(a^2 + b^2).$$

Or, $h^2 = a^2 + b^2$.

$\therefore \quad \overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2$. Q. E. D.[19]

---

[19] Ibid., p. 35.

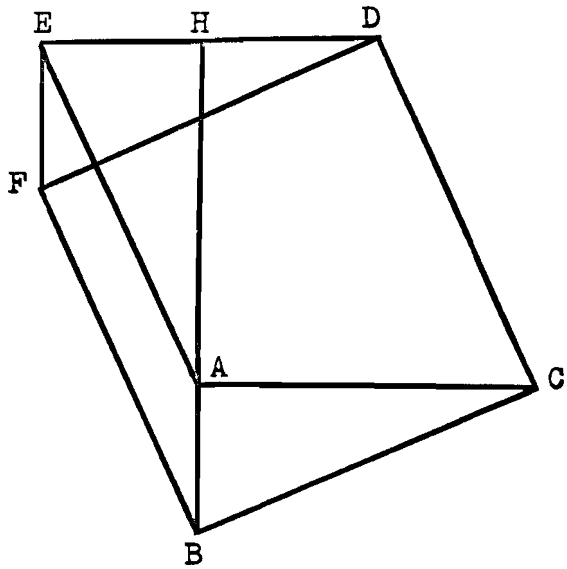


Figure 20

<u>Proof</u> <u>No</u>. <u>18</u>. ABC is a right-angled triangle. BCDF is constructed on the hypotenuse. AE is drawn parallel and equal to BF (and CD). ED and EF are drawn and BA is extended to H.

EABF and EACD are parallelograms.

Triangle FED $\cong$ triangle BAC,

$\therefore$ BFEDCA = square BFDC.

Triangle AHE $\cong$ triangle DEF.

Since the altitude of BAEF = HE = AB,

$$BAEF = \overline{AB}^2.$$

Since the altitude of ACDE = AH = AC.

$$ACDE = \overline{AC}^2.$$

$\therefore$ BFEDCA = $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

Q. E. D.[20]

---

[20] <u>Ibid</u>., p. 38.

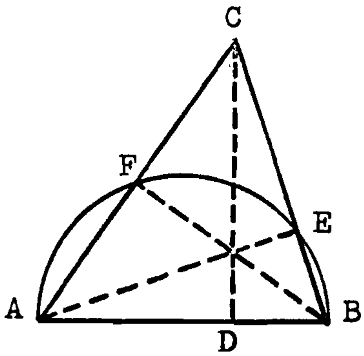


Figure 21-a

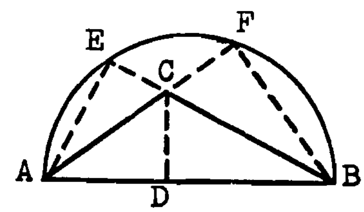


Figure 21-b

<u>Proof</u> <u>No</u>. <u>19</u>. ABC is <u>any</u> triangle. Let AE, BF, and CD be the three perpendiculars from the angles upon the opposite sides, or upon the sides produced. Since an angle inscribed in a semi-circle is a right angle, a circumference described on any side as a diameter passes through the feet of two of the perpendiculars. From theorems relating to secants and to intersecting chords,

$AB \times AD = AC \times AF = \overline{AC}^2 \pm AC \times FC$, and

$AB \times DB = BC \times EB = \overline{BC}^2 \pm BC \times CE$.

Adding, $AB^2 = AC^2 + BC^2 \pm 2AC \times FC$ (or $2BC \times CE$).

The + sign being taken when C is obtuse, and the - sign when C is acute. If, however, C is a right angle, CE and CF become 0.

$\therefore \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Q. E. D. [21]

[21] <u>Ibid</u>., p. 35-36.

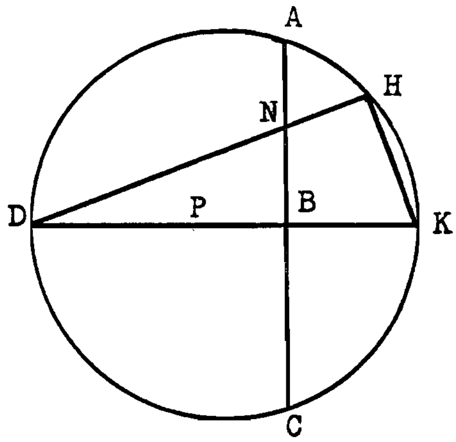


Figure 22

<u>Proof</u> <u>No. 20</u>. Let AC be any chord in the circle DAC with center at P. The diameter DK is drawn perpendicular to AC, and any point in the circumference, as H, is joined with D and K. From similar right triangles, DN/DB = DK/DH.

$\therefore$ DN x DH = DB x DK.

But, DK = DB + BK.

Hence, DN x DH = DB(DB + BK).

$= \overline{DB}^2 + DB \times BK.$

But from interesting chords in a circle,

$DB \times BK = AB \times BC = \overline{AB}^2.$

$\therefore DN \times DH = \overline{DB}^2 + \overline{AB}^2.$

Now conceive point H to revolve about P as a center until the point coincides with the point A. Then,

DN = DH = DA,

and $\overline{DA}^2 = \overline{DB}^2 + \overline{AB}^2.$

Q. E. D.[22]

---

[22]<u>Ibid</u>., p. 28.

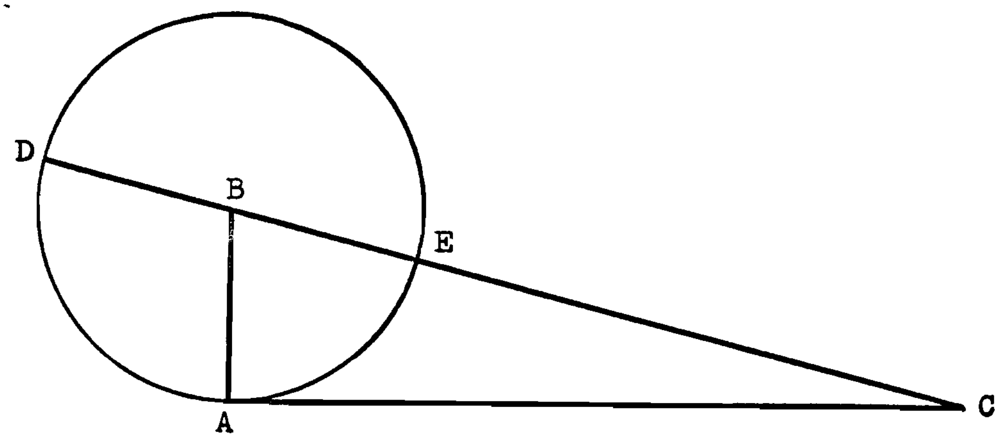


Figure 23

Proof No. 21. From an external point C, the tangent CA and the secant CD are drawn to the given circle having B as its center.

$\therefore$ EC/AC = AC/DC.

But DC = BC + BD,

= BC + AB.

Also EC = BC - BE.

= BC - AB.

$\therefore$ (BC - AB)/AC = AC/(BC + AB).

Or, $\overline{BC}^2 - \overline{AB}^2 = \overline{AC}^2$.

Or, $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2$.

Q. E. D.[23]

---

[23]Ibid., p. 22.

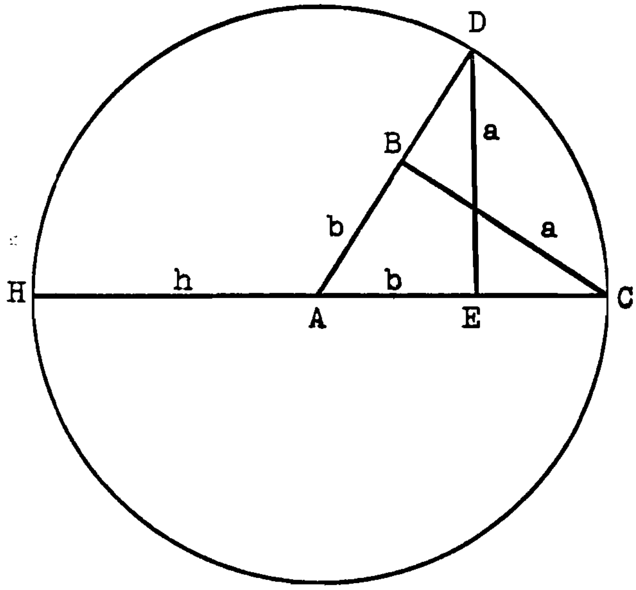


Figure 24

Proof No. 22. ABC is a right-angled triangle. Using the hypotenuse AC as a radius and with the center at A, a circle is constructed. AB is extended until it meets the circumference in D. DE is constructed perpendicular to AC. Triangle AED = triangle ABC. Let a represent DE and CB, h represent HA and CA, b represent EA and BA, HE = h + b, and CE = h - b.

Since DE is a mean proportional between HE and CE,

$\overline{DE}^2 = HE \times CE.$

$\therefore a^2 = (h + b)(h - b).$

$= h^2 - b^2.$

Or, $h^2 = a^2 + b^2.$

$\therefore \overline{CA}^2 = \overline{CB}^2 + \overline{BA}^2.$ Q. E. D.[24]

---

[24] Ibid., p. 30.

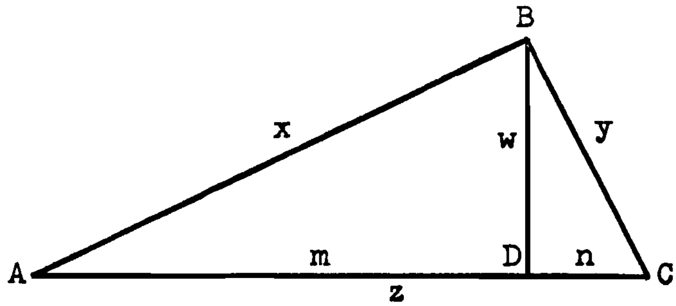


Figure 25

<u>Proof No. 23</u>. ABC is a right-angled triangle with altitude BD. Let AB = x, BC = y, CA = z, AD = m, CD = n, and BD = w.

(1) Then $x^2/y^2 = \frac{1}{2}mw/\frac{1}{2}nw$ (since $x/w = y/n$ and $x/m = y/w$).

(2) Also $z^2/y^2 = \frac{1}{2}xy/\frac{1}{2}nw$ (since $z/y = y/n$ and $z/x = y/w$).

Adding one to both sides of (1),

(3) $(x^2 + y^2)/y^2 = \frac{1}{2}(m + n)w/\frac{1}{2}nw$.

Since $\frac{1}{2}xy = \frac{1}{2}(m + n)w$, proportion (2) becomes,

(4) $z^2/y^2 = \frac{1}{2}(m + n)w/\frac{1}{2}nw$.

From (3) and (4),

$$(x^2 + y^2)/y^2 = z^2/y^2,$$

or, $y^2z^2 = y^2(x^2 + y^2)$.

Therefore, $z^2 = x^2 + y^2$,

or, $\overline{CA}^2 = \overline{AB}^2 + \overline{BC}^2$.

Q. E. D.[25]

---

[25] <u>Ibid</u>., p. 29.

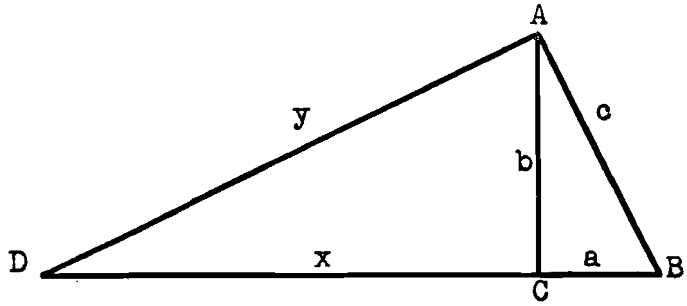


Figure 26

<u>Proof</u> <u>No</u>. <u>24</u>. ABC is a right-angled triangle. At A, a perpendicular is constructed to AB, and BC is extended until it intersects this perpendicular at some point D. Now triangle DAB will also be a right triangle, in which AC is an altitude. Let AD = y, DC = x, AC = b, BC = a, and AB = c.

Therefore, by similar triangles,

(1) $x/b = b/a$ or $b^2 = ax$.

(2) Also $(a + x)/c = c/a$.

(3) From (2) $c^2 = a^2 + ax$.

(4) From (1) and (3) $c^2 = a^2 + b^2$.

Or, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2$.

Q. E. D.

CHAPTER IV

ANALYSIS AND CLASSIFICATION OF THE PROOFS OF CHAPTER III

This chapter is devoted to an analysis and classification of the proofs presented in chapter three. First, each proof has been analysed as to the fundamental property on which it was based, and then the proofs were classified as to their similarities and differences.

Two definitions need to be presented at this time. The terms outward and inward refer to the manner in which the squares were constructed on the sides of a right triangle. If a square is constructed outward on the side of a right triangle, then it lies entirely outside of the triangle. If a square is then rotated 180° about the side on which it is constructed so that part of the interior of the square coincides with part or all of the interior of the triangle, then it is constructed inward.

I. ANALYSIS OF EACH PROOF

In this section each proof of chapter three has been analysed as to the fundamental property on which each was based.

Proof No. 1. Two congruent squares were dissected differently, and the Pythagorean relationship was obtained by setting the areas of the two squares equal to each other.

Proof No. 2. The square on the hypotenuse was dissected into four congruent triangles and a square. The pieces were reassembled to give the sum of the squares on the legs.

Proof No. 3. An altitude was constructed to the hypotenuse of the given right triangle, and from similar right triangles proportions were derived which when simplified give the Pythagorean relationship.

Proof No. 4. The squares on the sides of a right triangle were constructed outward. The squares on the legs were then divided into parts, the sum of whose areas is equal to the square on the hypotenuse.

Proof No. 5. A trapezoid was constructed with one of the legs of the given right triangle as a base. The Pythagorean relationship was then derived by use of formulas for the area of a trapezoid and a triangle.

Proof No. 6. The square on the hypotenuse of the given right triangle was constructed inward, and the square of one of the legs was constructed outward. The square of the remaining side was constructed so as to have vertices in common with the other two squares. The squares on the legs were divided into parts, the sum of whose areas is equal to the square on the hypotenuse.

Proof No. 7. Four congruent right triangles were constructed along the interior sides of a square with a square remaining in the middle. The Pythagorean relationship was obtained by setting the area of the original square equal to the sum of the areas of the four triangles and the square in the middle.

Proof No. 8. The square on the hypotenuse of a given right triangle was constructed inward, and the square of one of the legs was constructed outward. The square of the remaining side was constructed so as to have vertices in common with the other two squares. The squares on the legs were divided into parts, the sum of whose areas is equal to the square on the hypotenuse.

Proof No. 9. As in proof 8, the square on the hypotenuse of a given right triangle was constructed inward, and the square of one of the legs was constructed outward. The square of the remaining side was constructed so as to have vertices in common with the other two squares. The squares on the legs were divided into parts, the sum of whose areas is equal to the square on the hypotenuse.

Proof No. 10. The squares were constructed outward on all sides of the right triangle, with two triangles congruent to the given right triangle constructed on two of the sides of the square on the hypotenuse. With the aid of some auxiliary lines the area of the square on the hypotenuse was shown to be equal to the sum of the area of the other two squares.

Proof No. 11. The squares on the hypotenuse and one of the sides were constructed inward, and the square on the other side was constructed outward. The area of the square on the hypotenuse was shown to be equal to the sum of the areas of the other two squares.

Proof No. 12. The squares on the two sides of the right triangle were constructed inward, and the square on the hypotenuse was constructed outward. With some auxiliary lines, the area of the square on the hypotenuse was shown to be equal to the sum of the areas of the other two squares.

Proof No. 13. The squares on the legs of the given right triangle were constructed outward. The sides of these squares were extended to form a rectangle which was adjacent to both of the squares. The square on the hypotenuse was constructed inward, and the sum of the parts of the areas of the squares on the legs was shown to be equal to the area of the square on the hypotenuse.

Proof No. 14. The squares on the legs of the given right triangle are constructed outward. The sides of these squares were extended to form a rectangle which was adjacent to both of the squares. The square on the hypotenuse was constructed outward and with the aid of some auxiliary lines, the sum of the parts of the areas of the squares on the legs was shown to be equal to the area of the square on the hypotenuse.

Proof No. 15. The squares on all the sides were constructed inward. With the aid of auxiliary lines, the sum of the parts of the areas of the squares on the legs was shown to be equal to the area of the square on the hypotenuse.

Proof No. 16. All of the squares on the sides were constructed outward. Each of the squares were dissected into congruent triangles, squares, and a rectangle. Some of the pieces were reassembled to form parts of the square on the hypotenuse. The remaining area of the square on the hypotenuse was shown to be equal to the area of the remaining parts of the other two squares.

Proof No. 17. An irregular quadrilateral was formed by constructing a rectangle and a right triangle on one leg of the given right triangle. The area of the figure was found two different ways by summing different parts of the figure. By setting these two sums equal to each other, the Pythagorean relationship was derived.

<u>Proof</u> <u>No</u>. <u>18</u>. The square on the hypotenuse was constructed inward, and on the opposite side of the square another triangle congruent to and in the same relative position as the given triangle was constructed. With the aid of auxiliary lines, parallelograms and triangles were constructed to show that the square on the hypotenuse was equal to the sum of the squares on the legs.

<u>Proof</u> <u>No</u>. <u>19</u>. A semi-circle was constructed on the hypotenuse of a right triangle. The Pythagorean relationship was derived by obtaining proportions relating to secants and intersecting chords.

<u>Proof</u> <u>No</u>. <u>20</u>. A right triangle was inscribed in a circle with its hypotenuse acting as the diameter of the circle. From similar right triangles and intersecting chords in a circle, proportions were derived giving the Pythagorean relationship.

<u>Proof</u> <u>No</u>. <u>21</u>. A right triangle was formed by constructing a tangent and a secant to a circle from an external point, together with a radius of the circle. Proportions which involved secants and tangents to a circle provided the Pythagorean relationship.

<u>Proof</u> <u>No</u>. <u>22</u>. Two congruent right triangles were constructed in a circle with the hypotenuse of one of the

triangles coinciding with a radius of the circle. From mean proportional relationships, the Pythagorean relationship was derived.

Proof No. 23. An altitude was constructed upon the hypotenuse of a right triangle, and from similar right triangles the Pythagorean relationship was obtained.

Proof No. 24. A right triangle was constructed upon one of the legs of a right triangle so that the two right angles of the triangles were adjacent. From similar right triangles, proportions were obtained which provided the Pythagorean relationship.

II. CLASSIFICATION AS TO SIMILARITIES AND DIFFERENCES

In the classification of the proofs of chapter three, the proofs were classified in one of three categories, and then the proofs in each group were compared. The categories are dissection, sum of the parts, and proportions.

Dissection. In dissection proofs a figure of known area is dissected and the pieces reassembled in a different manner to give the desired areas. Proofs 1 and 2 are of this type and differ only in the manner in which the dissection was made.

Sum of the Parts. The majority of the proofs presented in chapter three depended upon the premise that the area of the whole was equal to the sum of the areas of the parts. These proofs were numbered 4-18 inclusive. Of this group, proof number 4, the one given by Euclid, was probably the classic in regard to the length of the proof. The Pythagorean relationship was obtained in proof number 5 by use of the parts of a trapezoid, while proof number 7 used the parts of a square. Proofs 6, 8-15, and 18 are very much alike. They all depended upon constructing the squares on the sides or upon constructing some of the squares on the sides together with some auxiliary lines to obtain the "parts." The differences of these proofs lie mainly in the manner in which the constructions were accomplished, i.e. whether the squares were constructed outward or inward or some combination thereof, and the manner in which the parts were associated together to get the sum of the parts. Proof 16 was a combination of dissection and the sum of the parts, as some of the parts were dissected to be reassembled while the remaining parts were shown to have equal sums.

Proof number 17 of this group had two triangles and a trapezoid erected on one leg of the given triangle to form an irregular polygon, rather than constructing the squares upon the sides.

Proportions. The second largest classification of the proofs of chapter three was the one based upon proportion. Proofs numbered 3, 23, and 24 all employed the same general diagram and the necessary proportions were derived by use of similar triangles. The three proofs differ only in the use of the similar triangles which were used to derive the necessary proportions.

Proofs numbered 19-22 of this group all relied upon the circle to obtain the necessary proportions. Proof 19 had the given right triangle inscribed in a semi-circle with the hypotenuse coinciding with the diameter of the circle, while proof 20 used the same diagram with the exception that it had another diameter constructed perpendicular to the first one. Proof 21 depended upon a tangent to a circle with a radius drawn to the point of tangency to form the right angle, while a secant was drawn from the point of origin of the tangent through the center of the circle to complete the necessary right triangle. Proof 22 had the right triangle constructed with its hypotenuse coinciding with the radius, rather than the diameter of the circle.

CHAPTER V

PRIMITIVE PYTHAGOREAN TRIPLES

The subject of Pythagorean triples has fascinated many students of mathematics. Pythagorean triples are three positive integers a, b, and c which satisfy the Pythagorean relationship $a^2 + b^2 = c^2$. Examples of the triples which are given quite often are (3,4:5), (5,12;13), and (7,24;25).

An account of Pythagorean triples is presented in this chapter. Included is a proof of the restrictions that are necessary for the selection of two positive integers u and v, which will generate a primitive Pythagorean triple.

There are an infinite number of these triples, and various methods of determining how to find these numbers have been advanced. One such method was given by Euclid as lemmas 1 and 2 of Proposition 28 of Book X. The algebraic conclusion of these lemmas is that the values of a^2, b^2, and c^2 must always be of the form

$$a^2 = mp^2mq^2, \quad b^2 = \left(\frac{mp^2 - mq^2}{2}\right)^2, \quad c^2 = \left(\frac{mp^2 + mq^2}{2}\right)^2,$$

with the further condition that mp^2 and mq^2 must both be either even or odd simultaneously.[1]

A triple is primitive if the greatest common divisor (g.c.d.) of a, b, and c is unity. For example consider the

---

[1]Ben Moshan, "Primitive Pythagorean Triples", The Mathematics Teacher, November, 1959, p. 541.

triple (6,8;10). Since the g.c.d. is 2 which is greater than 1, this triple is not primitive, whereas the triple (3,4;5) is primitive.

Another and more frequently used method of determining Pythagorean triples is to chose two positive intergers u and v such that u is greater than v. Then a, b, and c will be determined by the following:

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

Verification of these formulas for a, b, and c is given in the following:

(1) $b = 2uv.$

(2) $b^2 = (2uv)^2 = 4u^2v^2.$

(3) $a = u^2 - v^2.$

(4) $a^2 = (u^2 - v^2)^2 = u^4 - 2u^2v^2 + v^4.$

(5) $c = u^2 + v^2.$

(6) $c^2 = (u^2 + v^2)^2 = u^4 + 2u^2v^2 + v^4.$

(7) $a^2 + b^2 = 4u^2v^2 + u^4 - 2u^2v^2 + v^4.$ (From 2 and 4).

(8) $a^2 + b^2 = u^4 + 2u^2v^2 + v^4.$

(9) $a^2 + b^2 = c^2.$ (From 6 and 8).

As an illustration of this method let $u = 6$ and $v = 4$, then: $a = 6^2 - 4^2 = 36 - 16 = 20.$

$b = 2(6)(4) = 48.$

$c = 6^2 + 4^2 = 36 + 16 = 52.$

Since $a^2 + b^2$ is to equal c^2,

$(20)^2 + (48)^2 = (52)^2.$

Or, $400 + 2304 = 2704.$

Although in this example the values of u and v produced a Pythagorean triple, it was not a primitive triple as the g.c.d. of a, b, and c is 4. Therefore more restrictions must be placed on the choice of u and v to insure that the triple will be primitive. The restrictions were given as a theorem by Ben Moshan.[2]

Theorem. If u and v are two positive integers which determine values of a, b, and c as follows: $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ and (1) u and v are "relatively prime" integers, i.e. they have no common divisor greater than 1, (2) u and v are of "opposite parity", i.e., one of them is even and the other is odd, (3) u is greater than v to insure that a is positive, then a, b, and c will form a primitive Pythagorean triple.

[2] *Ibid.*, p. 541

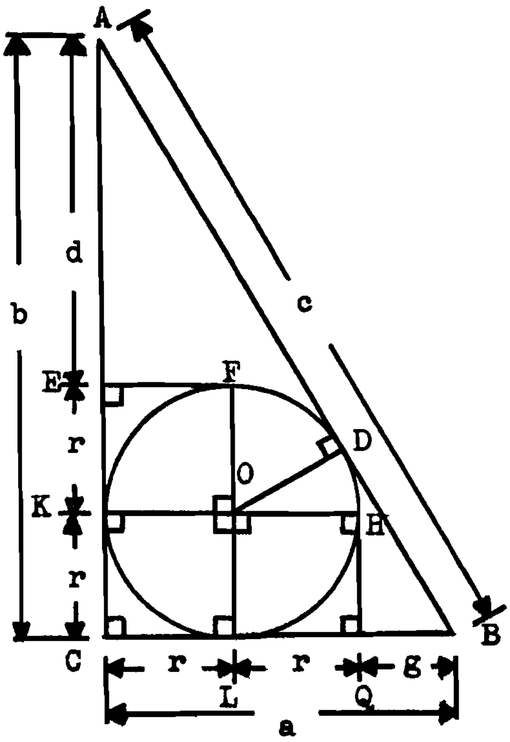


Figure 1

Proof: Let triangle ABC be a right triangle, labeled as in figure 1; r is the radius of the inscribed circle, known as the inradius. FE and QH are tangents to the circle, such that $FE \perp AC$, and $QH \perp BC$.

From the diagram the following are evident:

(1) $c^2 = a^2 + b^2$.

(2) $a = 2r + g$, or $g = a - 2r$.

(3) $b = 2r + d$, or $d = b - 2r$.

Now, $c = AD + BD = AK + BL$, since the tangents from an external point to a circle are equal. But $AK = d + r$ and $BL = g + r$. Therefore,

(4) $c = 2r + g + d$.

By substituting (2) and (3) in (4),

(5) $c = a + b - 2r.$

From (2) and (5) it follows that,

(6) $g = c - b.$

From (3) and (5) it follows that,

(7) $d = c - a.$

Now substituting (2), (3), and (4) in (1):

$$(2r + g + d)^2 = (2r + g)^2 + (2r + d)^2,$$

which simplifies to,

(8) $gd = 2r^2.$

If a, b, and c are integers which satisfy the equation (1), it is obvious from (5), (6), and (7) that 2r, g, and d are integers. And since g and d are integers, it follows from (8), $2r^2$ is also an integer.

It will now be proved that since both 2r and $2r^2$ are integers, r is also an integer. If r is not an integer, and 2r is, then r must be of the form $r = k + \frac{1}{2}$, where k is an integer. It then follows that $2r^2 = 2(k + \frac{1}{2})^2 = 2k^2 + 2k + \frac{1}{2}$, which is impossible since $2r^2$ is an integer.

Now it will be proved that if the triple is primitive, then g and d are relatively prime and of opposite parity. If g and d have a common divisor k which is greater than 1, then every divisor of k would be a common divisor of g and d, or even if k is prime, g and d can be expressed as $g = kg_1$, $d = kd_1$.

Equation (8) would then be,

$$k^2 g_1 d_1 = 2r^2 \text{ or } g_1 d_1 = 2(r/k)^2,$$

thus k is a factor of r, say $r = kr_1$.

Then from (2), (3), and (4),

$$a = 2kr_1 + kg_1, \quad \text{or} \quad (a/k) = 2r_1 + g_1;$$

$$b = 2kr_1 + kd_1, \quad \text{or} \quad (b/k) = 2r_1 + d_1;$$

$$c = 2kr_1 + kg_1 + kd_1, \text{ or } (c/k) = 2r_1 + g_1 + d_1.$$

Thus, k divides a, b, and c, which is contrary to the fact that the triple is primitive.

Every composite integer r can be expressed as a product of primes in one and only one way if no distinction is made between arrangements of the same prime factors, say,

$$r = 2^t p_1{}^m \cdot p_2{}^n \cdot p_3{}^w \cdot \cdot \cdot p_n{}^z,$$

where $p_1, p_2, p_3, \ldots, p_n$ are distinct odd primes and $t, m, n, w, \ldots, z$ are positive integers.

Thus, $2r^2$ can be expressed as a product of distinct primes,

$$2r^2 = 2(2^t \cdot p_1{}^m \cdot p_2{}^n \cdot \cdot \cdot p_n{}^z)^2 = gd,$$

where $p_1, p_2, \ldots, p_n$ are distinct odd primes only and $t, m, n, \ldots, z$ are positive integers.

All of the factors of $2r^2$, since $2r^2 = gd$, must be in g and d taken together, but each distinct prime can only be in g or d since g and d are relatively prime. Therefore, $2(2^t)^2$ can only appear in either g or d, and since all of the other distinct primes of $2r^2$ are odd, it follows that either g or d is even and the other is odd. No loss of generality results if g is odd and d even. Let $g = (G)^2$, where G is the product only of distinct odd primes in $2r^2$, thus, G is always odd. Then, let $d = 2(n)^2$, where n is the product of the remaining distinct primes in $2r^2$.

Then from (8) it follows,

(9) $gd = 2r^2 = 2n^2G^2$, and

(10) $r = nG$.

From (2), (3), and (4),

(11) $a = 2r + g = 2nG + G^2$.

(12) $b = 2r + d = 2nG + 2n^2$.

(13) $c = 2r + g + d = 2nG + G^2 + 2n^2$.

Using the transformation, $u = G + n$ and $v = n$, statements (11), (12), and (13) can be written as $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$ respectively. Inspection of the three conditions on u and v in the theorem, will show that the values of G and n in the transformation satisfy the conditions also.

The necessary conditions for a primitive triple have now been proven. It is now necessary to prove that the values of a, b, and c as expressed in equations (11), (12), and (13), are always primitive when G is odd and n and G are relatively prime.

By substituting these values in equation (1), one has

$$(2nG + G^2)^2 + (2nG + 2n^2)^2 = (2nG + G^2 + 2n^2)^2,$$

which simplifies to an identity and thus equation (1) is satisfied.

The next condition for a primitive triple is that G be odd. For if G is even, it is apparent from (11), (12), and (13) that a, b, and c are also all even and the resulting triple is not primitive.

It now remains to be proved that the triple is always primitive when n and G are relatively prime. Since G must be odd, from (11), (12), and (13), a and c must be odd and b must be even, and as such they do not have the common factor 2. They also do not have an odd common factor k which is greater than 2. For if they did then every prime divisor of k would be a common divisor of a, b, and c; or even if k is prime let $a = ka_1$, $b = kb_1$, and $c = kc_1$, then from (4), (2), and (3),

$$kc_1 = 2nG + G^2 + 2n^2, \quad kc_1 = 2nG + G^2 + 2n^2,$$
$$ka_1 = 2nG + G^2, \text{ and } \quad kb_1 = 2nG + 2n^2.$$

Subtracting,

$$k(c_1 - a_1) = 2n^2, \quad k(c_1 - b_1) = G^2,$$

or,

$$c_1 - a_1 = (2n^2/k), \quad c_1 - b_1 = (G^2/k).$$

Since k is greater than 2, it would follow that k divides n and G, which is contrary to the fact that n is prime to G.

Q. E. D.[3]

---

[3] *Ibid.*, p. 542-543.

CHAPTER VI

APPLICATION OF THE PYTHAGOREAN THEOREM TO FIGURE-CUTTING PROBLEMS

This chapter is devoted to showing how the Pythagorean Theorem was applied to four of fifteen well-publicized problems on figure-cutting. The fifteen problems were written by the mathematics staff of the University of Chicago. The problems appeared as a series of six articles in The Mathematics Teacher during the years 1956 to 1958.

The term "transform" as used in this chapter means to cut a figure into parts using straight lines only and then to rearrange these parts to form a new figure.

In this chapter each of the original fifteen problems is stated, and then the Pythagorean Theorem applied to obtain a solution of the last four.

I. STATEMENT OF THE PROBLEMS

(1) Given three congruent squares, to transform them into a single square.

(2) Given a square, to transform it into three congruent squares.

(3) Given two squares (congruent or not), to transform them into a single square.

(4) Given a square, to transform it into an equilateral triangle.

(5) Given an equilateral triangle, to transform it into a square.

(6) Given a (non-rectangular) parallelogram, to transform it into a square.

(7) Given a square, to transform it into n congruent equilateral triangles (where n is some natural number greater than 1 and fixed in advance.)

(8) Given one or more rectangles, to transform them into a square.

(9) Given a regular hexagon, to transform it into a square.

(10) Given a regular pentagon, to transform it into a square.

(11) A pin consists of three congruent silver squares soldered at the vertices in such a way that the sides of one square are extensions of the sides of another. It is required to cut this pin along two pairs of parallel lines and from the resulting parts to assemble a brooch having the shape of a rhombus.

(12) Given a right triangle ACB whose longer side BC is less than twice its shorter side AC, to cut it into no more than four parts that reassemble into a square.

(13) Given a square, by cutting it into at most four parts to transform it into a right triangle whose longer side is less than twice the shorter side.

(14) Given a square, to transform it into two squares of which one has an area twice that of the other.

(15) Given a square, to transform it into three squares whose areas are in the ratio 2:3:4.[1]

---

[1]Mathematics Staff of the University of Chicago, The Mathematics Teacher, XLIX, May, 1956, p. 332; XLIX, October, 1956, p. 442; XLIX, December, 1956, p. 585; L, February, 1957, p. 125; L, May, 1957, p. 330; LI, February, 1958, p. 96.

II. APPLICATION OF PYTHAGOREAN THEOREM TO PROBLEMS 12-15

In the January, 1962, issue of The Mathematics Teacher, Frank Piwnicki has shown how the Pythagorean Theorem can be applied to the solution of problems 12-15. As the solutions of problems 14 and 15 are somewhat simpler, they will be presented first. The solutions of these problems are as follows.

Problem 14: Given a square, to transform it into two squares of which one has an area twice that of the other.

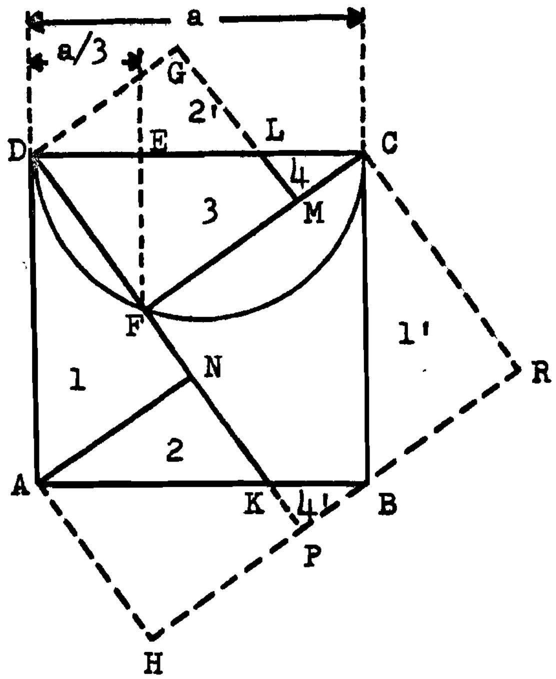


Figure 1

ABCD is the given square (Figure 1). On the side DC a semi-circle is constructed with a radius equal to DC/2 and DE is equal to DC/3. From point E a perpendicular is constructed to intersect the arc DC at F. Now triangle DFC is a right triangle with the hypotenuse AC and legs DF and FC. With respect to this right triangle, the square on DC is inward. The square on DF is then constructed inward, and the square on FC outward.

Using the parallel lines formed in Figure 1, the following right triangles can be shown to be congruent. Triangle AND is congruent to triangle BRC, triangle ANK is congruent to triangle DGL, triangle DFC is congruent to triangle AHB, and triangle LMC is congruent to triangle KPB.

Therefore, the square ABCD on the hypotenuse of triangle DFC, is cut into five parts by the segments DK, CF, AN, and LM, all lying within the area of the square ABCD. From these five parts the two required squares can be constructed as shown in Figure 1. The squares completed on the sides of triangle DFC indicate how the Pythagorean Theorem is used in the solution of this problem.

To verify that square FPRC is equal to twice square AHPN, it is necessary to show only that $\overline{FC}^2 = 2\overline{DF}^2$. This can be shown by applying the Pythagorean relationship to the right triangles DFC, DFE, and FEC. The proof is as follows:

(1) $\overline{FC}^2 = \overline{DC}^2 - \overline{DF}^2$,

(2) or, $\overline{FC}^2 = 9\overline{DE}^2 - \overline{DF}^2$, since $\overline{DC} = 3\overline{DE}$.

(3) Therefore $3\overline{DE}^2 = (1/3)\overline{FC}^2 + (1/3)\ \overline{DF}^2$.

(4) Also, $\overline{DF}^2 = \overline{EF}^2 + \overline{DE}^2$ or $\overline{EF}^2 = \overline{DF}^2 - \overline{DE}^2$,

(5) and $\overline{FC}^2 = \overline{EF}^2 + \overline{EC}^2$.

(6) Therefore, $\overline{FC}^2 = \overline{DF}^2 - \overline{DE}^2 + \overline{EC}^2$. From 4 and 6.

(7) $\overline{FC}^2 = \overline{DF}^2 - \overline{DE}^2 + 4\overline{DE}^2$, since $\overline{EC} = 2\overline{DE}$;

(8) or $3\overline{DE}^2 = \overline{FC}^2 - \overline{DF}^2$.

(9) Therefore, $\overline{FC}^2 - \overline{DF}^2 = (1/3)\ \overline{FC}^2 + (1/3)\ \overline{DF}^2$.

From 3 and 8.

(10) Or, $\overline{FC}^2 = 2\overline{DF}^2$.

Q. E. D.

Before starting on exercise 15 it will be necessary to construct several figures, which will all be grouped together for convenience when referring to them.

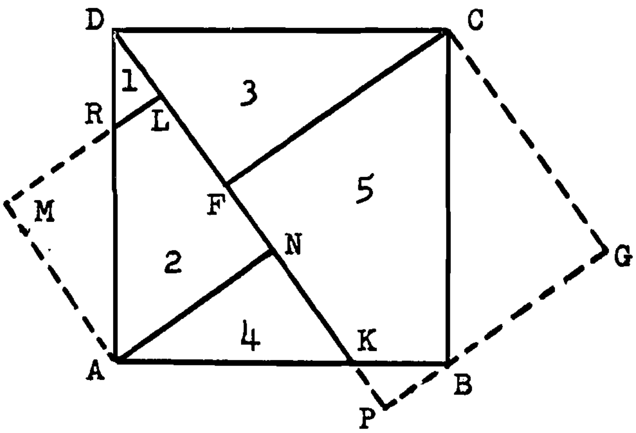


Figure 2

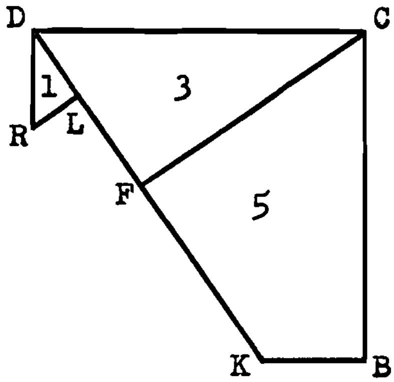


Figure 3

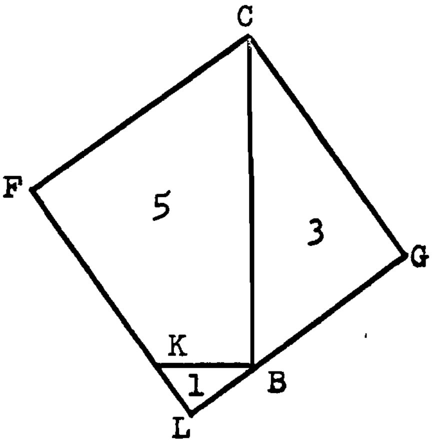


Figure 4

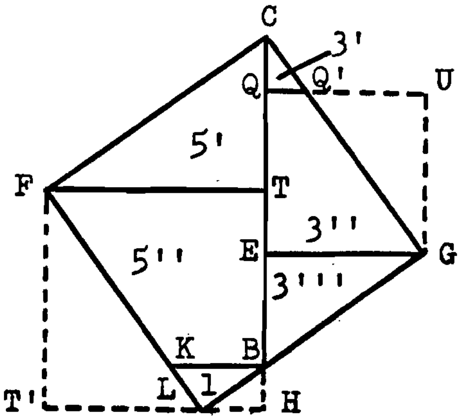


Figure 5

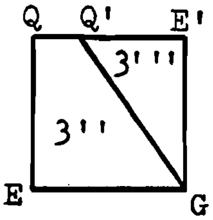


Figure 6c

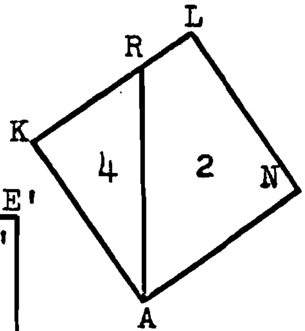


Figure 6b

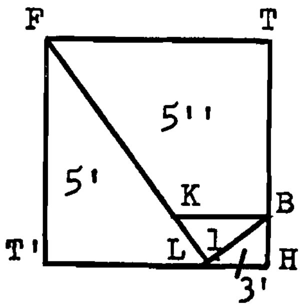


Figure 6a

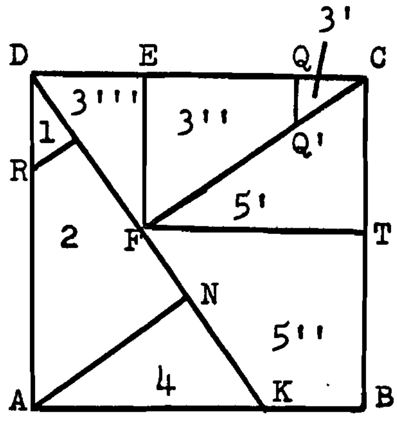


Figure 7

Problem 15. Given a square, to transform it into three squares whose areas are in the ratios 2:3:4.

This problem has a two part solution, the first of which is a division of the square into two squares in the ratio 1:2. This part is a repetition of the solution just given to Problem 14. Figure 2 differs from Figure 1 in that the square on DF, the smaller arm, is constructed, not on that arm, but on AN which is equal to DF. The use of Figure 2 in place of Figure 1 is justified by the fact that triangle DFC and LMC of Figure 1 are congruent respectively to triangle AND and RLD of Figure 2. The trapezoid 2 and triangle 4 make up the square shown in Figure 6b, as in Figure 2 triangle ANK is congruent to triangle AMR. What remains of the square ABCD now is Figure 3, which, with the pieces rearranged, becomes Figure 4. The fact that Figure 4 is a square is apparent since $\overline{DC} = \overline{CB}$ and $\overline{RD} = \overline{KB}$.

Now if the 1:2 ratio of the squares in Figures 6b and 4 is regarded as 3:(2 + 4), then the square in Figure 4 has to be cut into squares in the ratio 2:4, which is 1:2. This gives the same ratio and the same operation as applied in Figure 2. In Figure 5 FT is constructed perpendicular to CB. Triangle FTC here is similar to triangle CFD in Figure 2, and therefore, the ratio of the squares on its arms TC and FT is 1:2. The whole of Figure 5 is similar to Figure 2 except for the difference in size and the presence of triangle KLB. Hence from the pieces of Figure 5, the squares QG (Figure 6c) and FH (Figure 6a) can be assembled.

If the parts in Figure 2 of Figure 3, numbered 3 and 5, are replaced by parts from Figure 5, numbered 3' + 3'' + 3''' and 5' + 5'' respectively, then Figure 2 is reassembled showing all the necessary cuts to assemble the three required squares, Figures 6c, 6b, and 6a which areas are in the ratio 2:3:4. Figure 7 illustrates the necessary cuts. Again the use of the Pythagorean Theorem in the solution of this problem is indicated by the squares on the sides of the right triangle.

Before the solution of problems 12 and 13 are given, it is necessary to illustrate how to do two constructions used in the solution of problems 12 and 13. First consider the transformation of a right triangle into a square and the reverse. The Pythagorean Theorem establishes the following relation between certain elements of the right triangle: the area of the square on an arm of the right triangle is equal to the area of a rectangle constructed of the hypotenuse and the projection of the arm on the hypotenuse.

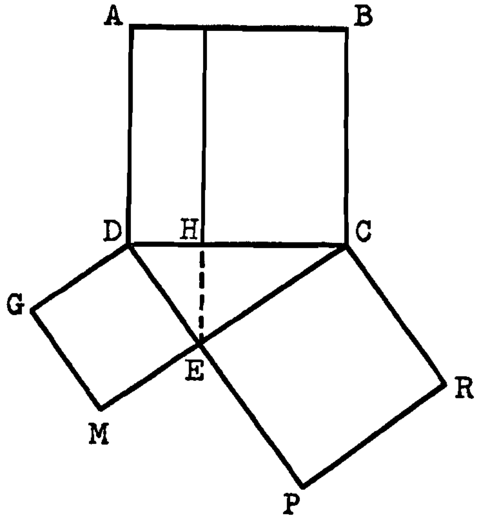


Figure 8

By applying proof number 4 of chapter 3 to Figure 8, the following is evident: $(DE)^2 = (HD)(AD) = (HD)(DC)$, and $(EC)^2 = (HC)(CB) = (HC)(DC)$.

Examination of Figure 8 suggests methods (1) of transforming the rectangle into a square of equal area, and (2) of transforming a square into a rectangle of equal area.

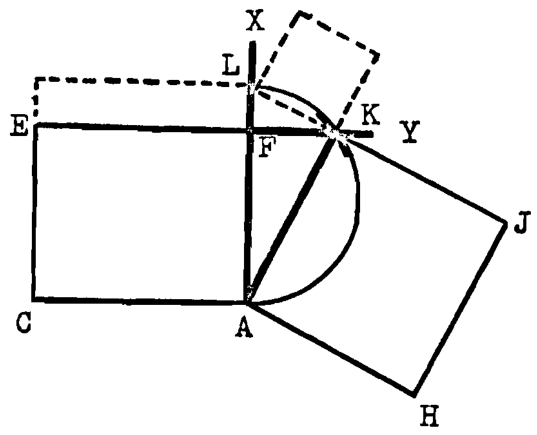


Figure 9

Problem: Given a rectangle to transform it into a square of the same area.

Let the given rectangle be AFEC of Figure 9, then the solution is obtained by the following:

(1) Extend AF to X.

(2) Construct AL equal to AC on AX.

(3) Construct a semi-circle with AL as diameter.

(4) Extend EF to intersect the semi-circle at K.

(5) Draw lines KL and KA, resulting in the right triangle AKL.

Then the line AK is the side of the desired square, and $(AK)^2 = (AF)(AL) = (AF)(AC)$. The heavy lines represent the necessary construction elements to find the side AK of the square. Broken lines relate the diagram to the Pythagorean Theorem.

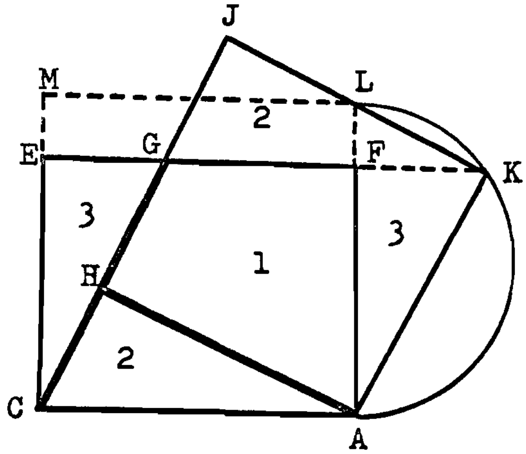


Figure 10

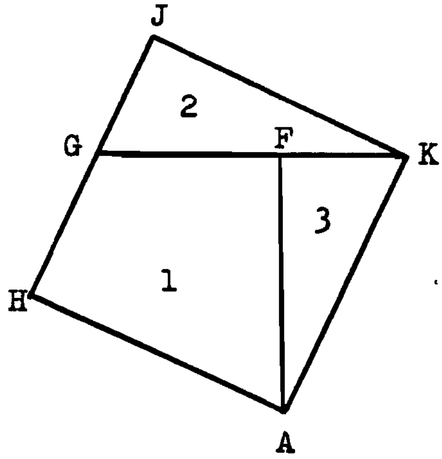


Figure 10a

Figure 10 shows basically the same operation as Figure 9 with one significant change. In Figure 9 the required square was constructed outward of the right triangle AKL, and in Figure 10 the required square AKJH was constructed inward on the triangle AKL. If JH is extended it will pass through the point C, which with HA, divides the rectangle AFEC into three parts from which the square AKJH of Figure 10a can be assembled. Numerals indicate the congruence of triangles, and quadrilateral 1 is common.

Problem: Given a square to transform it into a rectangle of the same area.

Examination of Figure 10a suggests a simple method of cutting a square into three parts from which a rectangle can be assembled. From K let line KG intersect JH at G, and erect AF perpendicular to KG. KG and AF now divide the square into three parts from which a rectangle can be assembled (compare Figure 10).

Figure 10 shows that a given rectangle can be transformed into one and only one square, but that a given square can be transformed into an unlimited number of rectangles. Therefore, when transforming a square into a rectangle it is necessary to have a side of the desired rectangle given. In Figure 12 a point G may be chosen anywhere on JH. If G coincided with J, then KG would be equal to KJ, and this would merely reconstruct the square. If G coincided with H, then KG would equal KH, and square AKJH would be transformed into a rectangle whose dimensions would be $a\sqrt{2}$ and $a\sqrt{2}/2$ where a is the side of the square. Thus, the given method is correct for transforming a square into a rectangle only if the ratio of the sides of the rectangle is between 1 and 2.

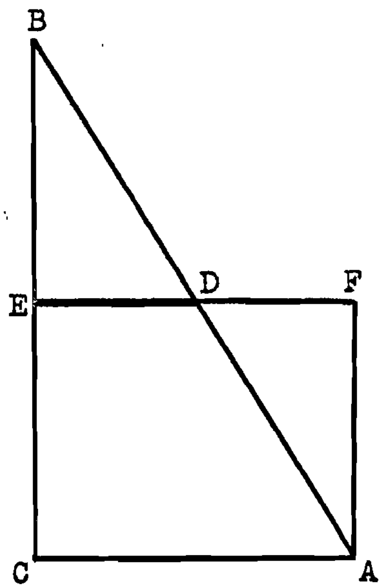


Figure 11

Before the problem of transformation of a triangle into a rectangle may be solved, one more step is necessary. In right triangle ACB (Figure 11), if CB and AB are bisected at E and D respectively, then ED will cut the triangle into two parts from which the rectangle AFEC can be assembled.

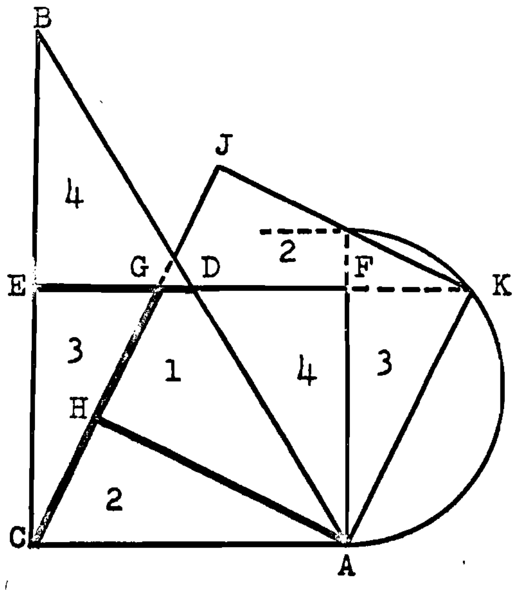


Figure 12

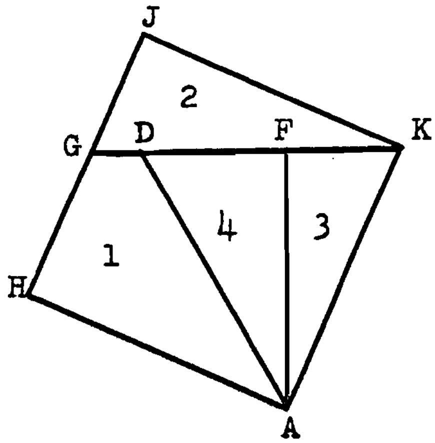


Figure 12a

<u>Problem</u> <u>12</u>: Given a right triangle ACB whose longer side BC is less than twice its shorter side AC, to cut it into no more than four parts that reassemble into a square.

In Figure 11 one cut, ED, transformed triangle ACB into rectangle AFEC. In Figure 10 is shown how the same rectangle AFEC, by two cuts, CG and AH, is transformed into square AKJH. Figure 12 is a composite of Figures 10 and 11, and shows how right triangle ACB was cut with segments ED, CG, and AH, to reassemble the square AKJH.

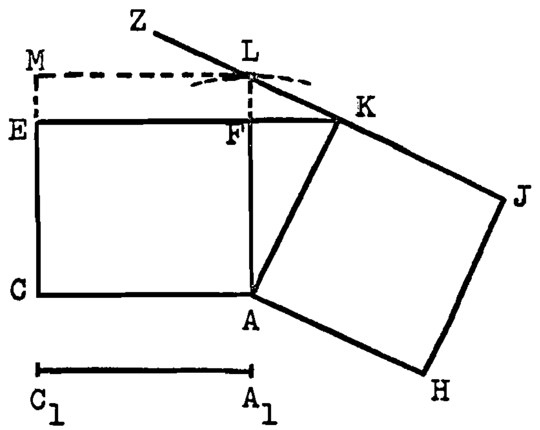


Figure 13

<u>Problem 13</u>. Given a square, by cutting it into at most four parts to transform it into a right triangle whose longer side is less than twice the shorter side.

The reverse procedure of problem 12 is applied to solve problem 13, i.e., (1) transform the square into a rectangle, and (2) transform the rectangle into a triangle. Step 2 has already been shown in Figure 11. A_1C_1 is the shorter side of the desired right triangle.

Step 1 is shown in Figure 13, where a square AHJK is given. The given square is to be transformed into a rectangle whose area is equal to that of the square. One side A_1C_1 is given.

The complete solution of problem 13 is as follows:

On side AK of square AKJH at K, the right angle AKZ is constructed by extending JK. With an arc of radius $AL = A_1C_1$ and center at A, KZ is intersected at L. KF is constructed perpendicular to AL; then AF is the shorter side of the required rectangle. To construct that rectangle, the square ALMC is constructed, and KF is extended to E. The area of the rectangle AFEC is equal to the area of the given square AHJK. In Figure 13 the square is constructed outward on the right triangle AKL. If it is placed inward on the triangle, then it is the diagram of Figure 10. In Figure 10a segments AF and CG cut the square into three parts from which the rectangle AFEC can be assembled. Figure 12 shows how to transform this rectangle into a right triangle. In Figure 12a AD will be the third cut. Figure 12a shows all the cuts of the square to transform it into a right triangle.[2]

[2]Frank Piwnicki, "Application of the Pythagorean Theorem in the Figure-Cutting Problem," *The Mathematics Teacher*, LV, January, 1962, p. 44-51.

CHAPTER VII

EXTENSIONS OF THE PYTHAGOREAN RELATIONSHIP

This chapter is devoted to a brief discussion of an extension of the Pythagorean relationship to non-right plane triangles and to right spherical triangles.

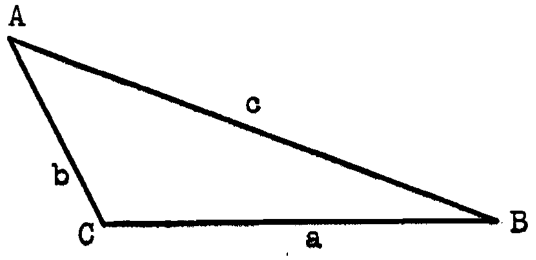


Figure 1

I. LAW OF COSINES

The Law of Cosines of plane trigonometry for any triangle, such as the one in Figure 1, states that:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

When C is a right angle, $\cos C = 0$, and the Law of Cosines reduces to:

$$c^2 = a^2 + b^2.$$

Therefore, the Law of Cosines is an extension of the Pythagorean Theorem for any triangle of plane geometry.

II. RIGHT SPHERICAL TRIANGLES

This section considers the possibility of an extension of the Pythagorean relationship to right spherical triangles.

Several definitions and propositions from spherical geometry need to be given here for reference.

1. A great circle of a sphere is the intersection of the sphere and a plane through the center of the sphere.

2. A spherical polygon is a closed line on a sphere consisting of three or more arcs of great circles.

3. A spherical polygon of three sides in which each side lies between 0° and 180° is a spherical triangle.

4. Each side of a spherical triangle is less than the sum of the other two sides.

5. The sum of the sides of a spherical triangle is less than 360°.

6. A right spherical triangle is one which has only one angle equal to 90°.

7. The sum of the angles of a spherical triangle is greater than 180° and is less than 540°.

8. If a spherical triangle has one right angle, the other angles may both be acute, both obtuse, or one acute and the other obtuse.

Consideration will be given only to those spherical triangles that contain only one right angle.

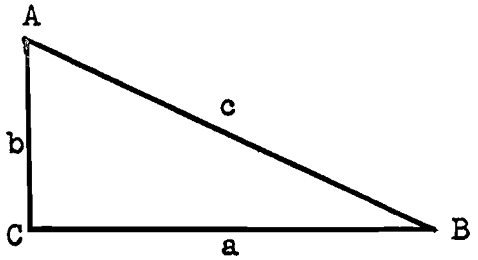


Figure 2

From plane trigonometry, if A and B are the two acute angles of a right triangle (Figure 2), then $\sin^2 A + \sin^2 B = 1$, which when expressed in terms of the ratios of the sides gives: $a^2/c^2 + b^2/c^2 = 1$, or $a^2 + b^2 = c^2$. This then is the Pythagorean relationship.

Therefore one approach to the problem is, what is the value of $\sin^2 A + \sin^2 B$, where A and B represent the non-right angles of a right spherical triangle. There are three possible values for this relationship which are: (1) $\sin^2 a + \sin^2 b = 1$, (2) $\sin^2 a + \sin^2 b > 1$, and (3) $\sin^2 a + \sin^2 b < 1$. If (1) is found to be true then the Pythagorean relationship holds for right spherical triangles. If either or both of (2) and (3) hold true then there is no Pythagorean relationship.

By property (8) there are three cases to consider for A and B. These cases are when they both are acute, when one is acute and the other obtuse, and when they are both obtuse. First considering the case where $A < 90°$ and $B < 90°$, and $90° < A + B < 180°$. Selecting some specific values for A and B gives the following table.

TABLE I

A	B	A + B	$\sin^2 A + \sin^2 B$
89°	2°	91°	1.001
50°	43°	93°	1.052
60°	60°	120°	1.500
70°	80°	150°	1.843
89°	89°	178°	1.999

Now considering the case where $A < 90^\circ$, $90^\circ < B < 180^\circ$, and $90^\circ < A + B < 270^\circ$.

TABLE II

A	B	A + B	$\sin^2 A + \sin^2 B$
8°	95°	103°	1.011
60°	95°	155°	1.742
8°	150°	158°	0.269
89°	95°	184°	1.991
89°	150°	239°	1.249
89°	175°	264°	1.075

Next considering the case where $90^\circ < A < 180^\circ$, $90^\circ < B < 180^\circ$, and $180^\circ < A + B < 360^\circ$.

TABLE III

A	B	A + B	$\sin^2 A + \sin^2 B$
95°	95°	190°	1.984
95°	150°	245°	1.242
95°	178°	273°	1.011
150°	150°	300°	0.500
175°	175°	350°	0.152

Examination of the three tables shows that condition (2) or (3) holds for possible choices of A and B.

The results of the three tables suggest that: If angles A and B are both less than 90°, then their sum will be greater than 90°, and the value of $\sin^2 A + \sin^2 B$ will be greater than 1 and less than 2. If either or both of the angles are greater than 90° but less than 180°, then the value of their supplement(s) (or related angle(s)) gives the value of $\sin^2 a + \sin^2 b$. In this case, if the sum of the two angles, or related angles, exceeds 90°, then $1 < \sin^2 A + \sin^2 B < 2$, and if the sum is less than 90°, then $0 < \sin^2 A + \sin^2 B < 1$. Therefore, this approach does not yield an extension of the Pythagorean relationship for the sides of right spherical triangles.

CHAPTER VIII

I. SUMMARY

In chapter one the purpose of this paper was stated to be: to provide a resource of enrichment material on the Pythagorean Theorem. This resource of material was to cover five areas: (1) historical background, (2) proofs, (3) primitive Pythagorean triples, (4) figure-cutting, and (5) spherical triangles. It was the authors intent to cover each of these areas and keep the report brief.

Chapter two provided a historical background on Pythagoras and the Pythagorean School which included some of their main contributions in the fields of religion, theory of numbers, geometry, and astronomy. Also early work on the Pythagorean Theorem by the Greek, Egyptian, and Chinese was considered.

Chapter three contained a wide variety of proofs of the Pythagorean Theorem. The proofs varied from one thought to have been given by Pythagoras to an original one developed by the author of this paper.

In chapter four the proofs were analyzed as to the fundamental property on which each was based. These properties were classified into three areas: (1) dissection, (2) sum of parts, and (3) proportion. The proofs were then classified as to their similarities and differences.

Necessary conditions for the selection of two integers that will generate a primitive Pythagorean triple, were stated and proved in chapter five.

Chapter six demonstrated how the Pythagorean Theorem could be applied to the solution of figure-cutting problems. The problems considered in this chapter were the last four of a set of fifteen problems written by the Mathematics Staff of the University of Chicago.

In chapter seven the Law of Cosines was considered as an extension of the Pythagorean relationship. Also, the relationship of $\sin^2 A + \sin^2 B$, where A and B are the non-right angles of a right spherical triangle, was used to show that the Pythagorean Theorem does not hold for the sides of a right spherical triangle.

BIBLIOGRAPHY

BIBLIOGRAPHY

Butler, Charles H., and F. Lynwood Wren. Trigonometry For Secondary Schools. Boston: D. C. Heath and Company, 1957.

Eves, Howard. An Introduction to the History of Mathematics. New York: Holt, Rinehart, and Winston, 1960.

Eves, Howard and Carroll V. Newsom. An Introduction to the Foundations and Fundamental Concepts of Mathematics. New York: Holt, Rinehart, and Winston, 1958.

Hart, Walter W., Veryl Schult, and Henry Swain. Plane Geometry and Supplements. Boston: D. C. Heath and Company, 1959.

Hutchins, Robert M. (ed.). Great Books of the Western World. 54 Vols. Chicago: William Benton, 1952.

Mathematics Staff of the University of Chicago. "A Problem On the Cutting of Squares," The Mathematics Teacher, XLIX (May, 1956), 332-343.

Mathematics Staff of the University of Chicago. "More On the Cutting of Squares," The Mathematics Teacher, XLIX (October, 1956), 442-454.

Mathematics Staff of the University of Chicago. "Still More On the Cutting of Squares," The Mathematics Teacher, XLIX (December, 1956), 585-596.

Mathematics Staff of the University of Chicago. "New Exercises in Plane Geometry," The Mathematics Teacher, L (February, 1957), 125-135.

Mathematics Staff of the University of Chicago. "More New Exercises in Plane Geometry," The Mathematics Teacher, L (May, 1957), 330-339.

Mathematics Staff of the University of Chicago. "Four More Exercises In Cutting Figures," The Mathematics Teacher, LI (February, 1958), 96-104.

Mikami, Yoshio. The Development of Mathematics In China and Japan. New York: G. E. Strechert and Company, 1913.

Moshan, Ben. "Primitive Pythagorean Triples," The Mathematics Teacher, LII (November, 1959), 541-545.

Nelson, Alfred L., and Karl W. Folley. Plane and Spherical Trigonometry. New York: Harper and Brothers, 1936.

Newman, James R. (ed.). The World of Mathematics. 4 Vols. New York: Simon and Schuster, 1956.

Piwnicki, Frank. "Application of the Pythagorean Theorem In the Figure-Cutting Problem," The Mathematics Teacher, LV (January, 1962), 44-51.

"Pythagoras," Encyclopedia Americana (1960 ed.), XXIII, 47.

"Pythagoras," Encyclopedia Britannica (14th ed.), XVIII, 803-804.

Rothrock, David A. Elements of Plane and Spherical Trigonometry. New York: The Macmillan Company, 1925.

Rupert, William W. Famous Geometrical Theorems and Problems. Boston: D. C. Heath and Company, 1900.

Welchons, A. M., and E. R. Krickenberger. New Solid Geometry. Boston: Ginn and Company, 1955.

Welchons, A. M., W. R. Krickenberger, and Helen R. Pearson. Modern Trigonometry. Boston: Ginn and Company, 1962.

Welchons, A. M., W. R. Krickenberger, and Helen R. Pearson. Plane Geometry. Boston: Ginn and Company, 1958.