257

AN INTRODUCTION TO THE AXIOM OF CHOICE

274

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This paper has as its primary objective the statement of several formulations of the set-theoretic principle called the "Axiom of Choice" and the proof of their equivalence.

Since the formal introduction of the axiom in the early years of this century, a considerable amount of mathematical research has been devoted to proving that a given statement implies, is implied by, or is equivalent to the Axiom of Choice. The result of this research has been the production of a formidable array of propositions, each of which is equivalent to the axiom.

The motive for showing that a statement is equivalent to the Axiom of Choice is not to lengthen an already extensive list, but is somewhat as follows. Within a certain area of mathematics it frequently happens that a particular 'law' is extremely desirable, such as the 'trichotomy law' in the arithmetic of transfinite cardinal numbers. By showing this law to be equivalent to the Axiom of Choice, one establishes that it is impossible to use the law without also endorsing the Axiom of Choice, however distasteful this may be on other grounds.

This paper deals with set theory from the naive point of view; that is, set theory is largely taken to be an intuitive body of facts of which the axioms furnish a convenient summary. The reader will find that seldom is explicit mention made of the axioms of the set-theoretic system in use (Zermelo-Fraenkel set theory). The justification of this "sin of omission" is that of conciseness of exposition. In spite of this neglect of the set-theoretic axioms, to the best of the author's knowledge, every theorem and proof appearing in this paper can be formalized within any of the standard systems of set theory in current use.

In Chapter II the author has attempted to indicate some of the different opinions that competent mathematicians have held regarding the Axiom of Choice and to describe with some care the causes of their disagreement, a point of common confusion. This is followed in Chapter III by a statement of the definitions and notational conventions used in this paper, together with an elementary development of the fundamental "facts of life" concerning ordinal numbers. Chapter IV, the main chapter of this paper, is concerned with proving the equivalence of twelve commonly encountered formulations of the Axiom of Choice. In Chapter V the author has attempted to give some informative examples of the application of the axiom to several important areas of modern mathematics. Chapter VI concerns itself with the questions of consistency and independence of the axiom and briefly considers the problems of what mathematics (in the field of classical analysis) can be developed without the axiom.

The prerequisites to a complete understanding of the contents of this paper are practically nil except, possibly, in certain parts of Chapter V. The most important requirement is that the reader possess to a certain extent that quality called 'mathematical maturity.' Set Theory is extremely general and is therefore quite abstract. However, the mathematical platitude that 'the more generally a theorem applies, the less profound it is' holds in set theory as in no other area of mathematics.

CHAPTER II

THE PHILOSOPHICAL SIGNIFICANCE OF THE AXIOM OF CHOICE

The first explicit statement of the set-theoretic principle subsequently called the "Axiom of Choice" appears to have been by Zermelo in 1904, although he was to some extent anticipated by Peano in 1890 and Levi in 1902.[1] A modern formulation of the axiom can be given in simple terms: if A is a set of non-empty sets, then there exists a mapping F such that $F(x) \in x$ for each $x \in A$. That is, there exists a mapping which 'chooses' precisely one element from each member of the collection A.

Since its formal introduction, the Axiom of Choice has generated a continuing spirited controversy among those interested in the foundations of mathematical thought; in some cases mathematicians have become so agitated by each other's peculiar attitudes toward mathematics that they have resorted to robust language in stating their views.[2] There are two fundamental reasons for the strong

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[1] E. Hobson, The Theory of Functions of a Real Variable, Vol. I (New York: Dover Publications, Inc., 1957), p. 262.

[2] For a lively account of one such battle between the mathematical giants Hilbert and Brouwer, see E. Bell, The Development of Mathematics (New York: McGraw-Hill Book Co., 1945), p. 569-570.

disagreements: first, a mathematical fear that use of the axiom might result in the appearance of contradictions within modern set theory and, secondly, an objection to the axiom on philosophical grounds. Of these, the mathematical question has largely been resolved and will be considered in the last chapter of this paper. By its nature, it is improbable that the philosophical objection will ever be disposed of to the satisfaction of mathematicians in general.

Before examining the issue on which mathematicians disagree, it is necessary to consider the manner in which sets are conventionally defined. The usual procedure in mathematics is to say that a hypothetical set 'exists' when its membership is determinate; that is, given any object, it can be determined whether or not that object is in the collection. This is done by means of propositional functions. Given a propositional function P(x), a set A is defined--namely, just those objects c for which P(c) is a true statement.[1] For example, suppose that $A_1, A_2, \ldots, A_n$ are non-empty sets. Then each set A_k has at least one element, which may be denoted by 'a_k.' Letting P(x) represent the propositional function '$(x = (A_1, a_1)$ or $x = (A_2, a_2)$ or . . . or $x = (A_n, a_n)$),' it is seen that P(x) defines

[1] Restrictions must be placed on the types of propositional functions allowed or the well-known set-theoretic paradoxes will appear. For example, the notorious "Russell Paradox" is generated by using the propositional function '$x \notin x$.' The primary goal of an axiomatic treatment of set theory is to find conditions on the propositional functions which are stringent enough to ban the known paradoxes, but flexible enough to allow set theory to serve a useful purpose.

a set consisting of n ordered pairs of the form (A_k, a_k). This example shows that the Axiom of Choice is not required when it is necessary to choose one element from each of a finite number of sets. For the propositional function P(x) given above defines a set of ordered pairs, and it is easy to see this set is actually a mapping of the type whose existence is claimed by the Axiom of Choice.

In logic, propositional functions are certain finite strings of logical operators and variables. Consequently, the process given in the proceeding paragraph for the construction of mappings cannot be extended to infinite collections of sets, since such an extension would involve propositional functions of infinite length. Nevertheless, in certain special cases the desired type of mapping can be constructed for infinite collections of sets. For example, consider an arbitrary collection A of non-empty sets of natural numbers. Taking P(B, x) to be the propositional function 'B is an element of A and x is the smallest member of B,' it is seen that the set of all ordered pairs (B, x) making the propositional function P(B, x) true is a mapping satisfying the claim of the Axiom of Choice. It should be noted that this construction owes its success to the fact that every non-empty set of natural numbers has the unusual property of having a smallest element, which allows a unique element in each set to be identified. Needless to say, not all sets contain such a 'distinguished' element.

The aspect of the Axiom of Choice which has been primarily responsible for the furor in the ranks of those interested in the foundations of mathematical thought can now be considered. According to the axiom, if A is any collection of non-empty sets, there "exists" a mapping F such that $F(x) \in x$ for each $x \in A$. Since a mapping is a certain type of set, the axiom asserts the existence of a special type of set. The fact is, no one knows of a general method to define such a set; that is, a propositional function P(x) which will define a set of the required type has yet to be found. Furthermore, it is extremely unlikely that such a propositional function will be found in the future. One might, of course, attempt to infer the existence of such a set by reductio ad absurdum, but no one has succeeded in doing so. Thus the Axiom of Choice appears to postulate the "existence" of undefinable sets. But, if this claim be granted, what does it mean to say such sets exist? This is the question responsible for the uneasiness among many mathematicians regarding the axiom. Most mathematicians have held one or the other of two opinions concerning the Axiom of Choice; a brief examination of these attitudes is given in the next few pages.

One group of mathematicians, which will be called the 'constructivists,' emphatically rejects the axiom to the extent that it asserts the existence of sets which cannot be precisely defined. The constructivist typically professes to be unable to understand the meaning of the word "existence" apart from definition;

this group considers the statement 'the set A exists' to be devoid of content unless supported by an accompanying propositional function to define the set A. Most of the constructivists find the claim of the Axiom of Choice and that of the opening statement of Genesis to be of the same general character.

The main argument offered by constructivists for their viewpoint is a powerful one and has considerable intellectual appeal to nearly all mathematicians. By taking the statements 'A exists' and 'A is well-defined' to be equivalent, the unpleasantly vague concept of "existence" is given a precise and determinate meaning.

The second major group of mathematicians, which will be called the 'non-constructivists,' typically accepts the axiom, although even here there are wide variations in the scope of the axiom which is considered admissible. Thus some mathematicians accept the axiom only when dealing with countable collections of sets, while others hold that its use is to be restricted to those cases in which the successive choices are dependent on those previously made (in this case an application of the axiom vaguely resembles a recursive definition, but in fact is not).

Possibly the majority of the non-constructivists find themselves uncomfortably close to pleading guilty to the "theology" accusation leveled at them by the constructivist camp, but they feel that rejection of the axiom would be a catastrophe for mathematics. For example, they point out that if the axiom were rejected, it seems certain that essentially all of topology would be lost. This would mean the immediate end of modern general analysis because of its heavy dependence on the topology of metric spaces. While modern algebra would not receive such stunning blows, it appears that much of the generality of important theorems in this area would be lost.[1]

Those who accept the Axiom of Choice are usually pragmatists or idealists (in the technical philosophical sense). The pragmatist accepts the axiom because so many interesting and useful consequences can be deduced from its use, and also because many of these same consequences can be obtained without it. The idealists accept the axiom because they are prepared to conceive of an object as 'existing' even though they cannot ascertain just what it is. The idealist approach is frequently rather metaphysical; Gödel appears to endorse this position when he writes:

> Classes and concepts may . . . be conceived as real objects . . . existing independently of our definitions and constructions. It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.[2]

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[1]For a discussion of these and related matters, see J. Rosser, Logic for Mathematicians (New York: McGraw-Hill Book Co., 1953), p. 510-512.

[2]K. Gödel, "Russell's Mathematical Logic," The Philosophy of Bertrand Russell (New York: Tudor Publishing Co., 1951), p. 137.

From the above discussion the author hopes to have made clear that the divergent attitudes of mathematicians toward the Axiom of Choice are produced by irreconcilable differences of opinion on a matter of ontology. If this be true, the question of whether or not the Axiom of Choice is 'legitimate' cannot be resolved within the domain of mathematics. It is entirely possible that future generations of mathematicians will regard the current arguments over the Axiom of Choice in much the same way as present day mathematicians regard the ancient controversy over Euclid's fifth postulate.

CHAPTER III

MATHEMATICAL PRELIMINARIES

This chapter is divided into two sections. The first of these sets forth the definitions and notational conventions concerning general set theory which are used in this paper, while the second is specifically concerned with the proof of those theorems about ordinal numbers which are needed in Chapter IV. Notational conventions are not well-established in set theory; the symbolism adopted in this paper is that which seems simplest to the author.

Definitions and Notation. Throughout this paper the symbolism $A \subseteq B$ will indicate that A is a subset of B; the notation $A \subset B$ will be reserved for the case in which A is a proper subset of B.

Suppose that A is a family of sets. By the union of A will be meant the set of those elements x such that $x \in y$ for some $y \in A$; the union of A is written $\bigcup A$. The collection of those elements x such that $x \in y$ for all $y \in A$ is called the intersection of A, and is denoted by $\bigcap A$. The notation $A - B$ is used to indicate the set of elements in A which are not in B.

A relation is a set of ordered pairs; if R is a relation, $(x, y) \in R$ and $x \, R \, y$ will be taken to mean precisely the same thing. The domain of a relation R, denoted by dom (R), is the set of all left components of the ordered pairs in R; analogously, the range of R is the set of all right components of elements of R, and is written range (R).

A relation F is said to be a mapping iff, for each $x \in$ dom (F), there is precisely one element y for which $(x, y) \in F$. If F is a mapping, the fact that $(x, y) \in F$ is expressed by the notation $y = F(x)$. F is said to be a mapping of A into B, written $F: A \longrightarrow B$, iff A = dom (F) and range (F) $\subseteq$ B. If it happens that range (F) = B, the mapping is said to be onto B. A mapping F is called one-to-one provided F(x) and F(y) are distinct for all distinct elements x and y of dom (F). If F is a one-to-one mapping, the set of all ordered pairs (y, x) such that $(x, y) \in F$ is a mapping called the inverse of F. The inverse of F is denoted by F^{-1}. If F is a mapping of A into B, and if E is a subset of A, then the collection of all F(x) with $x \in E$ is said to be the image of E under F, written F(E).

Let A be a family of sets indexed by a set I. The set of all mappings F of I into $\bigcup A$ such that $F(k) \in X_k$ for all $k \in I$ is called the Cartesian Product of A, and is denoted by πA.

Two sets A and B are equipollent, written $A \sim B$, iff there exists a one-to-one mapping of A onto B. The notation $A \precsim B$ is used to indicate that A is equipollent with some subset of B. Lastly, $A \prec B$ is written when $A \precsim B$ is true and $A \sim B$ is false.

A set S is said to be partially-ordered by a relation $\leq$ iff the following properties hold for all elements of S: (1) $x \leq x$,

(2) if $x \leq y$ and $y \leq x$, then $x = y$, and (3) if $x \leq y$ and $y \leq z$, then $x \leq z$. If S is partially-ordered by $\leq$, the notation $x < y$ is used to indicate that x and y are distinct elements of S such that $x \leq y$. In this case x is said to be *smaller* than y and y *larger* than x. The set of all those elements of S which are smaller than a given element x of S is called the *strict initial segment* of x; the strict initial segment of x together with x itself is called the *weak initial segment* of x.

Suppose A is partially-ordered by $\leq$ and that B is a subset of A. An element $u \in A$ is said to be an *upper bound* of B in A provided $x \leq u$ for all $x \in B$. An element $s \in B$ is the *smallest* element of B iff s is smaller than any other element of B. An element $m \in B$ is said to be a *maximal element* of B iff no element of B is larger than m.

If A and B are partially-ordered by relations $\leq$ and $\leq^*$, respectively, then the partially-ordered sets are said to be *similar* iff there exists a one-to-one mapping F of A onto B such that $F(a) \leq^* F(b)$ whenever $a \leq b$.

If S is partially-ordered by $\leq$ and either $x \leq y$ or $y \leq x$ holds for all x and y in S, then S is said to be *totally-ordered* by the relation. A totally-ordered subset B of a partially-ordered set S is called a *chain* of S. In case S is partially-ordered by the subset relation, a chain of S is sometimes called a *nest* of S. A *well-ordered* set is a totally-ordered set such that every non-empty subset has a smallest element.

Suppose A is a family of sets partially-ordered by set inclusion, and let F be a mapping of A into itself. A subcollection B of A is said to be a <u>tower</u> of A relative to F provided the following conditions are satisfied: (1) B contains the empty set, (2) F(x) is an element of B whenever x is, and (3) the union of every nest of B is an element of B.

A set C is said to be <u>complete</u> iff every element of C is also a subset of C. The set C is <u>connected</u> provided either $x \in y$ or $y \in x$ holds for any distinct elements x and y of C. A set C is called an <u>ordinal number</u> provided C is both complete and connected. If A and B are ordinal numbers, the notation $A < B$ will mean that $A \in B$. Finally, the symbolism $A \leq B$ will indicate that either $A = B$ or $A < B$ holds.

A set S is said to be W*-ordered by a relation $<$ iff the following conditions are satisfied for all elements of S: (1) if x and y are distinct, then either $x < y$ or $y < x$, (2) not $x < x$, (3) if $x < y$ and $y < z$, then $x < z$, and (4) every non-empty subset of S has a smallest element.

<u>Ordinal Numbers</u>. As previously stated, this paper is not concerned with the axiomatic structure of set theory. However, it is necessary to state one axiom of Zermelo set theory, the <u>Axiom of Regularity</u>: If A is any non-empty set, there exists an element B of A such that $A \cap B = \emptyset$.[1]

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[1] P. Suppes, <u>Axiomatic Set Theory</u> (New York: D. Van Nostrand Company, Inc., 1960), p. 53.

Using this axiom it is easy to prove several lemmas which are quite useful in the theory of ordinal numbers.

LEMMA 1. If $A_1, A_2, \ldots, A_n$ are sets, it is false that $A_1 \in A_2 \in \ldots \in A_n \in A_1$. In particular, $A \in A$ is false.

PROOF. By way of contradiction, suppose that $A_1, A_2, \ldots, A_n$ are sets which violate the conclusion of the theorem. Define S to be the set $\{A_1, A_2, \ldots, A_n\}$. Clearly $X \cap S \neq \emptyset$ for any X in S; this contradicts the Axiom of Regularity.

LEMMA 2. Let S be W*-ordered by $<$, and suppose B is a proper subset of S. There exists an element x in S such that B is the strict initial segment of x iff, for all elements x and y of S, if $y \in B$ and $z < y$, then $z \in B$.

PROOF. Suppose that B is the strict initial segment of x, and assume both $z < y$ and $y \in B$. Then $y < x$, so $z < x$. This means that $z \in B$.

Assume that the second condition holds. Since S - B is non-empty, this set has a smallest element, say x. Let I(x) denote the strict initial segment of x; it will be shown that $I(x) = B$.

Let $y \in I(x)$. Then $y < x$, so y is not an element of S - B. Thus $y \in B$, which implies $I(x) \subseteq B$. Next suppose that $y \in B$. In this case x and y are necessarily distinct, so either $x < y$ or $y < x$ is true. The hypothesis $x < y$ implies $x \in B$, a contradiction;

thus $y < x$ holds, which means $y \in I(x)$. This proves that $B \subseteq I(x)$.

THEOREM 0.1. If S is an ordinal number, the membership relation is a W*-ordering of S.

PROOF. Let x and y be distinct elements of S. Since S is complete, either $x \in y$ or $y \in x$ must hold.

Lemma 1 shows that $x \in x$ is false for all elements x in S.

Suppose $x \in y$ and $y \in z$, where $x, y, z \in S$. Lemma 1 implies that x and z are distinct, so either $x \in z$ or $z \in x$ must hold. The hypothesis that $z \in x$ implies $x \in y \in z \in x$, contradicting Lemma 1, so $x \in z$.

Lastly, assume A is a non-empty subset of S. By the Axiom of Regularity there exists $x \in A$ such that $x \cap A = \phi$. The element x is the smallest element of A. For suppose $y \in A$, where y is distinct from x. From the assumption that $y \in x$ the contradiction $y \in x \cap A$ is obtained, so y is not an element in x.

THEOREM 0.2. If A is a complete proper subset of an ordinal S, then A is an element of S.

PROOF. Assume the condition of the theorem, and suppose both $z \in y$ and $y \in A$ hold. Since A is complete, y must be a subset of A; but then z is an element of A. From Lemma 2 and Theorem 0.1 it is seen that A is the strict initial segment of some element x in S, so necessarily $A = x$. Since $x \in S$, the conclusion $A \in S$ is immediate.

THEOREM 0.3. If S and T are ordinals, then S is a proper subset of T iff S is an element of T.

PROOF. If S is a proper subset of T, then S is an element of T by Theorem 0.2.

Suppose S is an element of T, where T is an ordinal. It is not possible that S = T, since this would imply $S \in S$, contradicting Lemma 1. This, together with the fact that T is complete, shows that S is a proper subset of T.

THEOREM 0.4. Every element of an ordinal is itself an ordinal.

PROOF. Suppose S is an element of an ordinal T. Since T is complete, S is a subset of T.

Assume that x and y are distinct elements of S. Then x and y are distinct elements of the ordinal T, so either $x \in y$ or $y \in x$ holds; this shows S is connected.

Suppose that $y \in S$. If $x \in y$, then $x \in S$ by Theorem 0.1, so y is a subset of S. This means that S is complete.

THEOREM 0.5. Every non-empty set C of ordinals has a smallest element, namely the intersection of C.

PROOF. Let x and y be distinct elements of $\cap C$, and let z be an element of C. Then x and y must both be elements of the ordinal z, so either $x \in y$ or $y \in x$ must hold; this shows that $\cap C$ is connected.

Next assume that $x \in \cap C$. Then x is an element of every member of C, so x is a subset of every element of C. Thus x is a subset of $\cap C$. This shows that $\cap C$ is complete, and hence an ordinal.

By way of contradiction, suppose $\cap C$ is not an element of C. Since $\cap C$ is certainly a subset of every element of C, this would show that $\cap C$ is a proper subset of every element of C. But then Theorem 0.3 implies that $\cap C$ is an element of every member of C, so that the conclusion $\cap C \in \cap C$ results; this contradicts Lemma 1.

The fact that $\cap C$ is the smallest ordinal in C is now quite obvious.

THEOREM 0.6. The Trichotomy Law is valid for ordinal numbers.

PROOF. It must be shown that, for any two ordinal numbers S and T, exactly one of the following is true: (1) $S = T$, (2) $S < T$, and (3) $T < S$.

Suppose S and T are distinct ordinals. Theorem 0.5 shows that $S \cap T$ is an element of the set $\{S, T\}$. It may be assumed that $S \cap T = S$; that is, assume that S is a subset of T. Since S is distinct from T, S is an element of T by Theorem 0.3. This means that $S < T$. It need only be shown that no two of the three possible relations between S and T can hold simultaneously.

If $S = T$, then the hypothesis that $S \in T$ implies $S \in S$, contradicting Lemma 1. Similarly, it is impossible that $T \in S$.

If $S \in T$, then $S = T$ is false by the above argument. Also, it cannot happen that $T \in S$, since this contradicts Lemma 1.

THEOREM 0.7. Every set of ordinals is well-ordered by the relation $\leq$.

PROOF. Let S be a set of ordinals. For any ordinal $A \in S$, the fact that $A = A$ shows that $A \leq A$. Suppose that A and B are elements of S such that $A \leq B$ and $B \leq A$; Theorem 0.6 then establishes that $A = B$. Assume that $A, B, C \in S$ and that both of $A \leq B$ and $B \leq C$ hold. Obviously $A \leq C$ if any two of these are equal, so suppose they are distinct. Then $A < B$ and $B < C$ are both true, so $A \in B$ and $B \in C$. Theorem 0.3 then implies $A \in C$; that is, $A \leq C$ holds. It has been shown that S is partially-ordered by $\leq$.

Theorem 0.6 shows the ordering is a total-ordering, while Theorem 0.5 proves it to be a well-ordering.

THEOREM 0.8. If C is a set of ordinals, then $\bigcup C$ is an ordinal. Furthermore, no ordinal in C is larger than $\bigcup C$.

PROOF. It need only be shown that $\bigcup C$ is an ordinal; the last assertion of the theorem is then obvious.

Let x be an element in $\bigcup C$. Then $x \in y$ for some $y \in C$, so x is a subset of y. This implies that x is a subset of $\bigcup C$, so the set is complete.

Suppose x and y are distinct elements of $\bigcup C$. Then there are ordinals A and B in C such that $x \in A$ and $y \in B$. By Theorem 0.6 it can be assumed that $A \leq B$. Theorem 0.3 and Theorem 0.6 now show that A is a subset of B, so x and y are distinct elements of B. B is an ordinal, so either $x \in y$ or $y \in x$ must hold. This shows $\bigcup C$ to be connected.

THEOREM 0.9. There is no largest ordinal; in particular, if A is an ordinal, then $A \cup \{A\}$ is an ordinal and $A < A \cup \{A\}$.

PROOF. Suppose that $x \in A \cup \{A\}$. If $x = A$, then certainly x is a subset of $A \cup \{A\}$. If $x \neq A$, it must be so that $x \in A$. In this case Theorem 0.4 proves that x is an ordinal. Theorem 0.3 shows that x is a subset of A, so certainly x is a subset of $A \cup \{A\}$. It has been established that $A \cup \{A\}$ is complete.

Assume x and y are distinct elements of $A \cup \{A\}$. If both x and y are elements of the ordinal A, then either $x \in y$ or $y \in x$ holds. The only other possibility may be taken as $y = A$ and $x \in A$. Then $x \in y$ is trivial. This shows $A \cup \{A\}$is connected.

That $A < A \cup \{A\}$ is trivial, since $A \in A \cup \{A\}$.

THEOREM 0.10. There is no set which contains all ordinal numbers.

PROOF. Let C be an arbitrary set of ordinals. Theorem 0.8 proves that $\bigcup C$ is an ordinal at least as large as any ordinal in C. Define A to be the set $(\bigcup C) \cup \{\bigcup C\}$; A is an ordinal larger than any than any element of C by Theorem 0.9. Since A cannot be larger than itself, it is impossible that A be an element of the set C. Thus C does not contain all ordinal numbers.

In conclusion, the comforting fact that ordinal numbers do exist should be noted; for example, the empty set is an ordinal number. Theorem 0.9 supplies a method for constructing more inspiring examples of ordinals from this one given ordinal.

CHAPTER IV

THE AXIOM OF CHOICE

In this chapter the author has stated twelve commonly encountered formulations of the Axiom of Choice and shown their mutual equivalence. The reader will find the various formulations of the axiom in the first section; in each case the full title of the formulation is given, along with an abbreviation of that title which will be used in the sequel. The second section is exclusively devoted to the equivalence proofs.

Formulations of the Axiom.

1. Choice Function Formulation -- CF.

   Let A be any set of non-empty sets. Then there exists a mapping $F\colon A \longrightarrow \bigcup A$ such that $F(X) \in X$ for all $X \in A$.

2. Cartesian Product Formulation -- CP.

   The Cartesian Product of any non-empty collection of non-empty sets is non-empty.

3. Relation Formulation -- RL.

   Every relation includes as a subset a function with the same domain.

4. <u>Intersection Formulation</u> -- INT.

Let C be a collection of pairwise disjoint non-empty sets. Then there exists a set X such that $X \cap A$ has precisely one element for each $A \in C$.

5. <u>Well-Ordering Principle</u> -- WOP.

If A is a set, there exists a relation which well-orders A.

6. <u>Numeration Theorem</u> -- NUM.

If A is a set, there exists an ordinal number S such that A is equipollent with S.

7. <u>Trichotomy Law for Cardinals</u> -- TRI.

If A and B are sets, exactly one of the following relations is true:

(i) $A \prec B$

(ii) $B \prec A$

(iii) $A \sim B$.

8. <u>Zorn's Lemma</u> -- ZL.

Let A be a non-empty partially-ordered set. If every chain of A has an upper bound in A, then A has a maximal element.

9. <u>Maximal Principle 1</u> -- MP1.

Let B be a chain of a partially-ordered set A. Then there exists a maximal chain C of A such that $B \subseteq C$.

10. Maximal Principle 2 -- MP2.

Every partially-ordered set has a maximal chain.

11. Maximal Principle 3 -- MP3.

Let B be a nest of a set A. Then there exists a maximal nest C of A such that $B \subseteq C$.

12. Maximal Principle 4 -- MP4.

Every set has a maximal nest.

Equivalence Proofs. The first three theorems proven below are devoted to showing the equivalence of the first four formulations of the preceding section.

THEOREM 1. CF $\longleftrightarrow$ CP.

PROOF. Let A be a non-empty set of non-empty sets, and assume CF. It may be assumed that A is indexed by a set I. By CF there exists a mapping F such that $F(X_k) \in X_k$ for each $k \in I$. Define a mapping $G: I \longrightarrow \cup A$ by the formula $G(k) = F(X_k)$. Then $G \in \pi A$, so πA is non-empty.

Assume that A is a set of non-empty sets, and suppose CP holds. As before, suppose that A is indexed by a set I. By CP there exists a mapping G such that $G(k) \in X_k$ for each $k \in I$; define a mapping F for each $k \in I$ by the formula $F(X_k) = G(k)$. Clearly F is a mapping of the type required in CF.

THEOREM 2. CF $\longleftrightarrow$ RL.

PROOF. Let R be a relation and assume CF. For each element b in dom (R), define S_b to be the set of all x such that $(b, x) \in R$. Let A be the set of all such S_b. By CF there exists a mapping F such that $F(S_b) \in S_b$ for each b in dom (R). Obviously the mapping F satisfies the claim of RL.

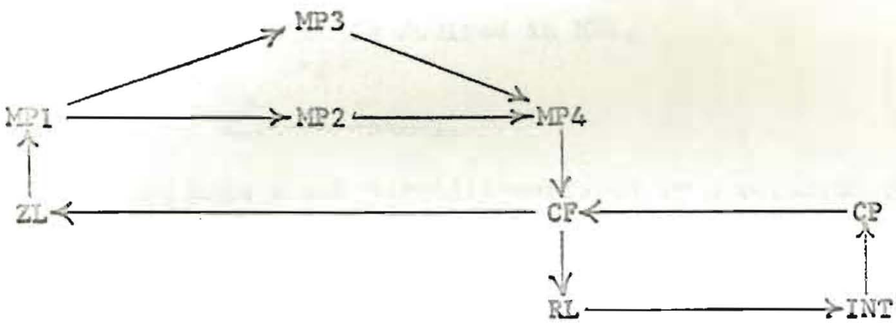
Let A be a collection of non-empty sets, and assume RL. Define S to be the set of all ordered pairs (X, x) such that $x \in X \in A$; note that dom (S) = A. By RL there exists a mapping F such that dom (F) = A and $F \subseteq S$. Clearly F satisfies the assertion of CF.

THEOREM 3. CF $\longleftrightarrow$ INT.

PROOF. Let C be a collection of non-empty pairwise disjoint sets, and assume CF. By CF there exists a mapping F such that $F(A) \in A$ for each A in C. Let X be the set F(C). It is an easy matter to show $X \cap A$ is a singleton for each A in C.

Let A be a collection of non-empty sets indexed by a set I, and assume INT. With each set X_k in A associate a set X_k^* defined as follows: X_k^* is the set of ordered pairs (x, k) with x in X_k. Then let A* be the set of all such X_k^*. Since A* is a set of pairwise disjoint non-empty sets, INT can be used to infer the existence of a set C* such that $C^* \cap X_k^*$ is a singleton for each $k \in I$. Define a mapping F as follows: for each $k \in I$, let $F(X_k)$ be the unique element x in X_k for which $(x, k) \in C^* \cap X_k^*$. Clearly the mapping F satisfies the claim of CF.

The next seven theorems, together with those already proven, suffice to establish the following implication scheme:



This will show that the first four and last five formulations of the Axiom of Choice are equivalent.

THEOREM 4. ZL $\longrightarrow$ MP1.

PROOF. Let B be a chain of a set A partially-ordered by a relation $\leqslant$, and assume ZL.

Define S to be the set of all chains of A having B as a subset; S is not empty, since B is an element of S. Partially-order S by set inclusion and suppose T is any nest of S. The set $\bigcup T$ is a subset of A and, consequently, is partially-ordered by $\leqslant$; obviously $\bigcup T$ contains as a subset every element of T. To prove that $\bigcup T$ is an upper bound of T in S, it need only be shown that $\bigcup T$ is totally-ordered by $\leqslant$. Thus let x and y be elements of $\bigcup T$. There exist elements F and G in T such that $x \in F$ and $y \in G$. Since T is a nest, it may be assumed that F is a subset of G. Then x and y are elements of the chain G, so either $x \leqslant y$ or $y \leqslant x$ must hold. This proves that $\bigcup T$ is an upper bound of T in S. By ZL we conclude that S has a maximal element, say C. Clearly C is the maximal chain containing B which is desired in MP1.

THEOREM 5. MP1 $\longrightarrow$ MP2.

PROOF. Let A be a set partially-ordered by a relation $\leqslant$, and assume MP1. The empty set is a subset of A which is also a chain of A. By MP1 there exists a maximal chain C of A containing ϕ as a subset. Since every chain of A has ϕ as a subset, C is the maximal chain whose existence is asserted by MP2.

THEOREM 6. MP1 $\longrightarrow$ MP3.

PROOF. MP3 is a special case of MP1 in which the partial-ordering is taken to be set inclusion.

THEOREM 7. MP3 $\longrightarrow$ MP4.

PROOF. This is a special case of Theorem 5.

THEOREM 8. MP2 $\longrightarrow$ MP4.

PROOF. As in Theorem 6, it need merely be noted that MP4 is a special case of MP2.

THEOREM 9. MP4 $\longrightarrow$ CF.

PROOF. Let A be a collection of non-empty sets, and assume MP4. Define S to be the set of all mappings f such that dom (f) $\subseteq$ A and, for each x in dom(f), $f(x) \in x$. S is non-empty, since ϕ is an element in S. By MP4 there exists a maximal nest of S, say N. It will be shown that there exists F in N such that dom (F) = A. Clearly such a mapping F is of the type required in CF.

Define F to be $\bigcup N$, and suppose both of (x, y) and (x, z) are elements of F. It will be proven that y = z, which allows the conclusion that F is a mapping.

There exist elements f and g in N such that $(x, y) \in f$ and $(x, z) \in g$. Since N is a nest, it may be assumed that $g \subseteq f$. Then (x, y) and (x, z) are both elements of the mapping f, so y = z.

It is clear that dom (F) is a subset of A; it must be shown that dom (F) = A. By way of contradiction, suppose that dom (F) is a proper subset of A. Note that F is an element of N, since otherwise $N \cup \{F\}$ is a nest of S properly containing N. This would contradict the assumed maximality of N. Let x be an element of A which is not in dom (F). Since x is non-empty, there exists an element y in x. Define G to be the mapping $F \cup \{(x, y)\}$. By construction, G is an element of S which properly contains F as a subset. But now $N \cup \{G\}$ is a nest of S which properly contains the nest N; this is a contradiction.

THEOREM 10. CF ⟶ ZL.

PROOF. Let A be a non-empty set partially-ordered by $\leq$ such that every chain of A has an upper bound in A. Assume CF. The first step will be to show that it is sufficient to prove the following result:

(1) Let X be a non-empty collection of sets partially-ordered by set inclusion. Then X has a maximal element provided:

(i) each subset of an element of X is in X, and

(ii) the union of each nest of X is an element of X.

For define X to be the set of all chains of A, and let X be partially-ordered by set inclusion. It is easy to prove the conditions of (1) are then satisfied, so X has a maximal element, say M. If A has no maximal element, then there is an upper bound b in A of M (since M is a chain of A). But this would mean that $M \cup \{b\}$ is a chain of A, which contradicts the maximality of M. Thus it is sufficient to prove the theorem given in (1).

By CF there exists a mapping F which chooses an element from each non-empty subset of $\bigcup X$. For each element A of X, define the set A^* to be the set of elements x of $\bigcup X$ for which $A \cup \{x\} \in X$. Define a mapping G for each A in X as follows: $G(A) = A$ if $A^* = A$; otherwise, $G(A) = A \cup \{F(A^* - A)\}$. It is clear that an element .. of X is maximal iff $G(A) = A$. It is, therefore, sufficient to prove that a set A exists such that $G(A) = A$ to establish the theorem.

It is easy to show that X is a tower of X relative to G and that the intersection of a non-empty collection of towers of X is again a tower of X. Define T to be the intersection of all towers of X; obviously T is the smallest tower of X. If T can be proven to be a nest of X, then $\bigcup T$ is an element of T by definition of a tower.

But then, again applying the definition of a tower, it is seen that $G(\bigcup T)$ is in T. This means that $G(\bigcup T) \subseteq \bigcup T$. But the definition of G shows that $\bigcup T \subseteq G(\bigcup T)$, so $G(\bigcup T) = \bigcup T$. This shows that $\bigcup T$ is a maximal element of X. Thus it is seen that:

(2) The proof is complete if T is a nest of X.

Certainly T is partially-ordered by the set inclusion relation on X; thus it need only be shown that, for any elements A and B in T, either $A \subseteq B$ or $B \subseteq A$. This will establish (2). An element A in T will be called <u>comparable</u> provided $A \subseteq B$ or $B \subseteq A$ holds for all B in T. There are comparable elements in T, since ϕ is one. Define W to be the collection of all comparable elements of T. If it can be shown that W is a tower of X, then (2) is proven, since then $T \subseteq W$ (T is the smallest tower of X). Obviously ϕ is an element of W, and it is easily shown that the union of every nest of W is an element of W. Then W is a tower of X provided $G(A) \in W$ whenever $A \in W$, and this will prove T is a nest of X. Hence:

(3) The proof is complete if $G(A) \in W$ for all $A \in W$.

Let A in W be given. Define U to be the collection of elements B in T such that either $B \subseteq A$ or $G(A) \subseteq B$. If U can be shown to be a tower, then U = T; for $U \subseteq T$ by definition, and $T \subseteq U$ since T is the smallest tower. Supposing this, let $B \in T$. Then, if $B \subseteq A$, it is clear that $B \subseteq G(A)$, since $A \subseteq G(A)$. If not $B \subseteq A$, then $G(A) \subseteq B$, since B $\in$ U. This means G(A) is comparable, so G(A) $\in$ W. Thus the proof is finished if U is a tower. Obviously U contains the empty set; it is easy to prove that the union of any nest of U is an element of U. This clearly shows that:

(4) The proof is complete if G(B) $\in$ U for all B $\in$ U.

As before, A is the given element of W. Let B be any element of U. Three cases must be considered.

Suppose that B is a proper subset of A. Since G(B) can have no more than one element not in B, it is clear that G(B) is a subset of A. But B is also in T, so G(B) $\in$ T. Thus G(B) is in U.

If B = A, then G(B) = G(A). A is an element of T, so G(A) is in T. Thus G(B) is a member of T. This shows that G(B) is in U.

The last possibility is that A is a proper subset of B. In this case G(A) is a subset of B, so G(A) is a subset of G(B). But B is in T, so G(B) is an element of T. These facts prove that G(B) is in U.

The last paragraphs prove (4) holds, so the proof of the theorem is complete.

The final four theorems are devoted to establishing the implication scheme given below:

ZL ⟶ TRI ⟶ NUM ⟶ WOP ⟶ CF.

These theorems, together with those already proven, suffice to show the equivalence of all twelve formulations.

THEOREM 11. ZL $\longrightarrow$ TRI.

PROOF. Assume ZL. Let A and B be any two sets. If either A or B is empty, TRI is a trivial conclusion, so suppose A and B are non-empty.

Suppose that $B \prec A$. By definition, the case $A \sim B$ is then impossible. The hypothesis that $A \prec B$ contradicts the Schroder-Bernstein theorem.[1]

As a consequence of the preceeding paragraph, it may be assumed that $B \prec A$ is false. Precisely stated, this means that one of the following must hold: (1) there is no one-to-one mapping of B into A, and (2) there exists a one-to-one mapping of A onto B. Define S to be the collection of all one-to-one mappings F such that dom $(F) \subseteq A$ and range $(F) \subseteq B$. Since A and B are non-empty, S is non-empty. Partially-order S by set inclusion. It is a trivial matter to show that the hypothesis of ZL is satisfied by the partially-ordered set S, so S has a maximal element, say F. Then F is a maximal one-to-one mapping of some subset of A into B.

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[1] P. Halmos, Naive Set Theory (New York: D. Van Nostrand Company, Inc., 1960), p. 88.

If range (F) = B, then F^{-1} is a one-to-one mapping of B into A, so B $\precsim$ A. Since B $\prec$ A is false by hypothesis, the conclusion A $\sim$ B follows.

The only other possibility is that range (F) is a proper subset of B. Let y be an element in B - range (F). It is easy to see that dom (F) = A, since otherwise there exists an element x in A - dom (F); this allows the construction of a function F $\cup$ $\{(x, y)\}$ which is one-to-one on a subset of A and properly contains F, a contradiction. Thus F is a one-to-one mapping of A into a proper subset of B. But this means that A $\precsim$ B. Since not both of A $\prec$ B and A $\sim$ B can hold, the proof is complete.

THEOREM 12. TRI $\longrightarrow$ NUM.

PROOF. Let A be a set and define H(A) to be the collection of all ordinals which are equipollent to some subset of A. Assume TRI.

The first step will be to prove that H(A) is an ordinal. That H(A) is connected is obvious from Theorem 0.6 of Chapter III. Suppose x is an element of H(A), and let y be any element of x. By Theorem 0.3 of Chapter III, y is a subset of x. Since x is equipollent to some subset of A, the same is true of y. Thus y is an element of H(A). It has been shown that every element of x is an element of H(A), so x is a subset of H(A); that is, H(A) is complete. This shows H(A) is an ordinal.

If it is assumed that H(A) is equipollent with some subset of A, a contradiction results. For it has been proven that H(A) is an ordinal, so this hypothesis would imply H(A) is a member of itself; Lemma 1 of Chapter III shows this to be impossible. Since $H(A) \precsim A$ is false, TRI implies that $A \prec H(A)$ is true. Thus A is equipollent with some subset of H(A). But every subset of a well-ordered set is either similar to that set or to a strict initial segment of it.[1] Thus it is clear that A is either equipollent to H(A) or an initial segment of H(A), say I(B). But I(B) = B, so A is equipollent to B or H(A); in either case, A is equipollent to some ordinal.

THEOREM 13. NUM ⟶ WOP.

PROOF. Let A be a set and assume NUM. By NUM there exists a one-to-one mapping F of A onto S for some ordinal number S.

Define an ordering relation R on A as follows: for any elements x and y in A, xRy iff $F(x) \leq F(y)$. Obviously R is a total-ordering for A.

Let C be any non-empty subset of A. Then F(C) is a non-empty set of ordinals by Theorem 0.4 of Chapter III. Theorem 0.5 of Chapter III insures that F(C) has a smallest element, say E. It is easy to prove that the element $F^{-1}(E)$ of A is the smallest element of A. Thus A is well-ordered by the relation R.

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[1] Halmos, *op. cit.*, p. 73.

THEOREM 14. WOP ⟶ CF.

PROOF. Let A be any set of non-empty sets, and assume WOP. By WOP there exists a relation $\leqslant$ which well-orders the set $\cup A$. Every element X of A is a subset of $\cup A$; thus there exists a unique smallest element of each member of A. Define a mapping F for each element X of A by the following method: F(X) is the smallest element of X. Clearly the mapping F chooses one element from each member of A; this is precisely the claim of CF.

CHAPTER V

APPLICATIONS

The first four formulations in Chapter IV have an undeniable intuitive appeal for most mathematicians. For this reason, particularly in the area of mathematical analysis, one finds many tacit usages of the Axiom of Choice. Thus Apostol[1] instructs his readers to:

> . . . Let $\{x_n\}$ be a sequence whose terms are distinct points of S . . . where S is an infinite set in E_n. . . such that $x_n \neq a$ and $\lim x_n = a$.

Apostol's readers must, therefore, be prepared to pick a countable subset from S; since S is an infinite set, it is considered 'obvious' that countable subsets of S do exist. It will shortly be seen that the only known proof of this fact utilizes the Axiom of Choice. No doubt Apostol was aware of this, but does not indicate so to his readers.

The example above is quite typical of the tacit usages of the axiom to be found in the literature of mathematics. In many instances

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[1] T. Apostol, Mathematical Analysis (Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1957), p. 66. That the example was chosen from this text is not to be construed as a criticism of Apostol; his text is written with great care.

the use of the axiom is so thoroughly concealed by plausible verbal reasoning that only the most careful reader is likely to notice its employment. This does not necessarily imply any lack of rigor or care in such proofs, but merely indicates that the axiom is being accepted and used without comment in common mathematics, just as logic is so assumed.

The remainder of this chapter is concerned with illustrating the proper use of the Axiom of Choice in the proof of some common mathematical theorems. In each case the theorems are, to the best of the author's knowledge, incapable of proof without the aid of the axiom.

EXAMPLE I

A set X is said to be finite iff X is empty or equipollent with the set $\{1, 2, \ldots, n\}$ for some integer n. The set X is infinite iff X is not finite. This definition of 'infinite' sets is somewhat unsatisfactory in the sense that it requires a preliminary development of the natural numbers. Thus a non-numerical definition is desirable, one of the more common being:

> A set X is Dedekind infinite iff X is equipollent with some proper subset of the sex X.[1]

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[1] P. Suppes, Axiomatic Set Theory (New York: D. Van Nostrand Company, Inc., 1960), p. 152.

The proof that these definitions are equivalent utilizes the following theorem, which is not known to be provable without the Axiom of Choice.[1]

THEOREM: If X is any infinite set, then X has a countable subset.

PROOF. Since X is non-empty, choose $x_1 \in X$. Let F be a choice function for the collection of non-empty subsets of X. Recursively define for $n \geq 1$, $x_n + 1 = F(X - \{x_1, x_2, \ldots, x_n\})$, and note that, since X is not finite, this can be done for every natural number n.

The set $\{x_1, x_2, \ldots, x_n, \ldots\}$ is the required countable subset.

EXAMPLE II

Using the Bolzano-Weierstrass theorem, it is not difficult to characterize the countably compact subsets of the real line as precisely those sets which are closed and bounded. Consequently, the content of the celebrated Heine-Borel theorem for the real line is that countable compactness and compactness are equivalent. The generalization of this result holds in any second axiom Hausdorff space. The interesting point here is that the Heine-Borel theorem

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[1] *Ibid.*

can be proven without the Axiom of Choice,[1] but the generalization cannot, to the best of our knowledge, be so proved.[2] The Heine-Borel theorem is one example of a comforting class of theorems obtainable with or without the Axiom of Choice.

THEOREM: Let (X, T) be a second axiom Hausdorff space. A subset of X is countably compact iff the subset is compact.

PROOF. (The 'only if' portion of the proof illustrates the use of the axiom, so this is the only part proven here.)

Let $\{B_n: n = 1, 2, \ldots\}$ be a countable base for the topology T and suppose that $C = \{O_k: k \in I\}$ is any open covering of the countably compact set A. It will be shown that a finite subcovering of A can be found among the open sets of C. A standard theorem for Hausdorff spaces[3] asserts that this can be done provided a countable subcovering of A can be extracted from C.

1J. Rosser, Logic for Mathematicians (New York: McGraw-Hill Book Company, Inc., 1953), p. 452-454.

2Ibid.

3W. Pervin, Foundations of General Topology (New York: Academic Press, 1960), p. 71.

Define S to be the collection of all integers k such that $B_k \subseteq O_i$ for some $i \in I$. Define E to be the collection $\{C_k: k \in S\}$, where C_k is the set of all those $O_i \in C$ for which $B_k \subseteq O_i$; note that C_k is non-empty for $k \in S$. Let F be a choice function for the set E and denote $F(C_k)$ by G_k. Then, for $k \in S$, it is seen that $B_k \subseteq G_k$ and $G_k \in C$.

Suppose that $x \in A$. Then $x \in O_k$ for some $k \in I$. There exists a natural number $n \in S$ such that $x \in B_n \subseteq O_k$; this shows that $x \in G_n$. Thus the collection $\{G_n: n = 1, 2, \ldots\}$ is the promised countable subcovering of A.

EXAMPLE III

It was remarked in Chapter I that to reject the Axiom of Choice would mean the loss of considerable generality in modern algebra. The following theorem of algebra (and that of the next example) is an illustration of one which, to the best of the author's knowledge, is not known to be provable without the Axiom of Choice.

THEOREM: If V is a vector space over a field F and A is a linearly independent subset of V, there exists a basis X for V such that $A \subseteq X$.[1]

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[1]For finite dimensional spaces this theorem can be proven without the Axiom of Choice. For such a proof, see P. Halmos, Finite Dimensional Vector Spaces (New York: D. Van Nostrand Co., Inc., 1958), p. 11.

PROOF. Suppose A is a linearly independent subset of V. Let S be the collection of all linearly independent subsets of V which contain A as a subset; this collection is non-empty since it contains A itself. S may be partially-ordered by set inclusion. Suppose T is any chain of S, and let $C = \bigcup T$. It will be shown that C is an upper bound of T in S; Zorn's lemma may then be applied to the partially-ordered set S.

Assume that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$, where $x_1, x_2, \ldots, x_n \in C$ and $a_1, a_2, \ldots, a_n \in F$. There exist sets $A_1, A_2, \ldots, A_n$ in T such that $x_k \in A_k$ for $k = 1, 2, \ldots, n$; but T is a chain, so, for some j, it is the case that $x_1, x_2, \ldots, x_n \in A_j$. Now A_j is linearly independent, so $a_1 = a_2 = \ldots = a_n = 0$. Hence C is linearly independent. Obviously $G \subseteq C$ for all $G \in T$, so C is an upper bound of T in S; by Zorn's lemma it is seen that S has a maximal element, say X.

It has been shown that there exists a maximal linearly independent subset of V which contains A; that is, there exists a basis X which contains A.

EXAMPLE IV

Given a field F, the set of all polynomials having coefficients in F will be denoted by $F[x]$. By a 'root field' for a polynomial $f(x)$ of $F[x]$ will be meant a minimal extension field F^* of F such that $f(x)$ is factorable into linear factors having coefficients in F^*. A well-known theorem of modern algebra[1] states that, if F is any given field and $f(x) \in F[x]$, there exists a root field of $f(x)$ which is unique to within isomorphism.

An extension field K of a field F is called 'algebraically complete over F' provided every polynomial in $F[x]$ can be factored into linear factors having coefficients in K. The famous "Fundamental Theorem of Algebra" insures that the field of complex numbers is algebraically complete over the reals. A partial generalization of this result will be proven.

THEOREM: Every field has an algebraically complete extension field.[2]

PROOF. Let F be a field and suppose $F[x] = \{f_i(x): i \in I\}$. By the Well-Ordering Principle the index set I can be considered to be well-ordered by a relation $\leqslant$. Let 'o' denote the first element of I and 'F_o' denote the root field of $f_o(x)$. For $k \neq o$, define F_k to be the root field of $f_k(x)$ over the field $\bigcup\{F_i: i < k\}$.[3] Lastly, define K to be the field $\bigcup\{F_k: k \in I\}$.

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[1] G. Birkhoff and S. MacLane, A Survey of Modern Algebra (New York: The Macmillan Company, 1953), p. 428.

[2] This theorem can be proven for finite fields without the use of the Axiom of Choice. For such a proof, see Birkhoff and MacLane, op. cit., p. 428.

[3] It is easy to show that the union of an ascending chain of fields is a field under the obvious definition of addition and multiplication. Using transfinite induction on the well-ordered set I, there is no difficulty in proving that F_k is defined for all $k \in I$.

Since K is an extension field of F, it need only be shown that K is algebraically complete over F. Thus suppose that $f_k(x) \in F[x]$. Now F_k is the root field of $f_k(x)$ over the field $\bigcup\{F_i: i < k\}$ and $F \subseteq \bigcup\{F_i: i < k\}$; thus $f_k(x)$ is factorable into linear factors having coefficients in F_k. Since $F_k \subseteq K$, the proof is finished.

CHAPTER VI

THE CURRENT STATUS OF THE AXIOM OF CHOICE

Recent Developments. One of the classic open problems of set theory is the question of the independence of the Axiom of Choice with respect to the remaining axioms of Zermelo set theory. Suppes[1] remarks that the likelihood of its independence is very high. As previously stated, the source of interest in this problem lies in the non-constructive character of the axiom.

The fear that the use of the Axiom of Choice might lead to some contradiction is definitely without foundation in the following sense. Gödel[2] has shown that if the axioms of Zermelo set theory are consistent without the Axiom of Choice, then they are consistent if the axiom is added to them. In other words, it is quite as 'safe' to use the axiom as to reject it. On the other hand, application of the axiom can lead to some extremely unintuitive results, possibly the most notorious of which is the Banach-Tarski paradox.[3] A special

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[1] P. Suppes, Axiomatic Set Theory (New York: D. Van Nostrand Co., 1960), p. 250.

[2] K. Gödel, "The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis," Proc. of the National Academy of Sciences, U. S. A., Vol. 24 (1938), p. 556-557.

[3] A. Tarski and S. Banach, "Sur la décomposition des ensembles de points en parties respectivement congruentes," Fundamenta Mathematicae, Vol. 6 (1924), p. 244-277.

case of the theorem proved by Banach and Tarski is that a sphere of unit radius can be decomposed into a finite number of parts which can be reassembled to form two spheres, each having unit radius.

Research in recent years has uncovered implications which hold between the Axiom of Choice and statements in other areas of mathematics which bear no discernible relationship to any of the formulations given in this paper. Sierpinski proved[1] that the Generalized Continuum Hypothesis (asserting that, if A is any infinite set, there is no set B such that $A \prec B \prec 2^A$) implies the Axiom of Choice, while Kelley has established[2] that the Tychonoff Product theorem of topology is equivalent to the axiom.

Mathematical Analysis Without the Axiom of Choice. A brief discussion is given here on the question of how much of classical analysis can be developed if the Axiom of Choice is not assumed. The discussion given is essentially an abridged version of one by Rosser.[3]

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[1] W. Sierpinski, "L'hypothèse généralisée du continu et l'axiome du choix," Fundamenta Mathematicae, Vol. 34 (1947), p. 1-5.

[2] J. Kelley, "The Tychonoff Product Theorem Implies the Axiom of Choice," Fundamenta Mathematicae, Vol. 37 (1950), p. 75-76.

[3] J. Rosser, Logic for Mathematicians (New York: McGraw-Hill Book Co., 1953), p. 510-512.

Starting with the axioms of Zermelo set theory, one can proceed with Halmos[1] to the proof of the Peano Postulates for the natural numbers. From here Landau's popular little book[2] can be consulted for a rigorous development of all the properties of the real and complex number systems; no use of the Axiom of Choice need be made to this point.

With an eye to obtaining theorems such as the Cauchy Integral Theorem, next proceed to a careful analytical treatment of Euclidean geometry by the usual Cartesian constructions. In this way the common geometric results can be obtained without difficulty. Since complex function theory requires the Jordan Curve Theorem for polygons, turn to Courant and Robbins[3] for a proof of this result which does not require the Axiom of Choice.

Using the background outlined above, consult Hardy[4] for a rigorous development of the calculus. Occasionally Hardy does make

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[1] P. Halmos, Naive Set Theory (New York: Van Nostrand, 1960), p. 46-47.

[2] E. Landau, Grundlagen der Analysis (New York: Chelsea Publishing Company, 1946).

[3] R. Courant and H. Robbins, What is Mathematics? (New York: Oxford University Press, 1941).

[4] G. Hardy, A Course of Pure Mathematics (New York: The Macmillan Company, 1947).

use of the Axiom of Choice, but in each case the proofs can be rewritten so that the 'choices' are made from the field of rational numbers; since the rationals are well-ordered, the axiom can be avoided in each case.

After completing Hardy's text, see Titchmarsh[1] for a careful development of considerable portions of complex function theory. There are occasional uses of the Axiom of Choice, but they can all be circumvented in the manner previously described.

It appears that the first unavoidable use of the Axiom of Choice in Titchmarsh's text occurs in the proof of the first fundamental theorem of Lebesgue measure.[2] Rosser remarks:

> . . . Here one picks an open set O_n for each set E_n of a sequence of measurable sets. There seems to be no way to specify O_n uniquely, so that the proof fails unless one is permitted to use the denumerable axiom of choice. We know of no other proof of the theorem which will proceed without the denumerable axiom of choice.

In addition to the example cited, there are several other instances in which the axiom appears to be used in an essential way.[3]

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[1] E. Titchmarsh, The Theory of Functions (New York: Oxford University Press, 1939).

[2] Titchmarsh, op. cit., p. 326.

[3] Titchmarsh, op. cit., p. 329 and p. 369.

Evaluating these difficulties, Rosser states:

> . . . It is thus open to grave doubt that one can develop the theory of Lebesgue measure without use of the denumerable act of choice.

If this opinion is true of real and complex analysis, it is beyond doubt that no appreciable portion of modern analysis could possibly be developed without the axiom. In short, it is virtually certain that the Axiom of Choice will not be banished from mathematics in the foreseeable future.

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