SEMI-OPEN SETS

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This Thesis is dedicated to Jessie Lee, Barbara and Virginia.

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William James Maurice Thomas

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CHAPTER I

INTRODUCTION

The purpose of this report is to investigate the properties of semi-open sets in topological spaces and to attempt to develop a topology based on the semi-open sets.

I. THE PROBLEM

Very little has been done on semi-open sets. This may be due to their similarity to open and closed sets in some cases or to their lack of closure under the operation of intersection.

Norman Levin[1] has developed a few properties of semi-open sets and has defined semi-continuity in terms of them but has not developed a topology on them. This report is an attempt to carry his development further.

II. ORGANIZATION

The rest of the introduction is concerned with definitions of terms to be used later. Chapter II develops some of the general topological properties for semi-open sets in topological spaces. Chapter III is the summary.

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[1]N. Levin, "Semi-Open Sets and Semi-Continuity in Topological Spaces," American Mathematical Monthly, LXX (1963), 36-40.

Standard set notation is used in this report with explanations of unusual symbols where needed.

This report is intended to be read by graduate and undergraduate students who have had introductory courses in topology and advanced calculus.

III. DEFINITIONS OF TERMS USED

Definition 1. A <u>topological space</u> $(X,\mathcal{T})$ is a set X of points and a family $\mathcal{T}$ of subsets of X which satisfies the following axioms:

[0.1] The union of any number of members of $\mathcal{T}$ is a member of $\mathcal{T}$. $(\varphi \in \mathcal{T})$

[0.2] The intersection of any finite number of members of $\mathcal{T}$ is a member of $\mathcal{T}$. $(X \in \mathcal{T})$

Definition 2. The family $\mathcal{T}$ is called a <u>topology</u> for X.

Definition 3. The members of $\mathcal{T}$ are called <u>open sets</u> in this topology.

Note that all sets are not necessarily open sets as there may be other subsets of X than the members of $\mathcal{T}$.

Example 1. Let X be the set of real numbers and $\mathcal{T}$ be the family of open intervals about each point. This satisfies the axioms and thus is a topology for the real numbers. This is called the <u>usual</u> topology for the real numbers.

Definition 4. A point x is called a <u>limit or accumulation point</u> of a subset E iff every open set G containing x contains a point of E different from x; i.e., if $x \in G \in \mathcal{T}$ then $E \cap G - \{x\} \neq \varphi$. (E may not be an open set and x may not be contained in E.)

Example 2. The set of real numbers with the usual topology does have limit points, and in fact, every real number is a limit point of the set.

Definition 5. The set of all limit points of set E is called the <u>derived set</u> of E and is denoted by d(E).

Definition 6. A set is a <u>closed set</u> iff it contains all its limit points. A set F is a closed set iff $d(F) \subseteq F$.

A few of the properties of open and closed sets are given without proof. The theorems are basic and the proofs are given in the general topology books such as <u>Foundations of General Topology</u> by William J. Pervin.

Theorem 1. If $x \notin F$, where F is a closed subset of a topological space $(X, \mathcal{T})$, then there exists an open set G such that $x \in G \subseteq CF$, where CF denotes the complement of F.

Theorem 2. If F is a closed set, CF is an open set.

Theorem 3. A set is a closed subset of a topological space iff its complement is an open subset of the space.

Theorem 4. The family $\mathcal{F}$ of all closed subsets in a topological space has the following properties:

[C.1] The intersection of any number of members of $\mathcal{F}$ is a member of $\mathcal{F}$. ($X \in \mathcal{F}$)

[C.2] The union of any finite number of members of $\mathcal{F}$ is a member of $\mathcal{F}$. ($\varphi \in \mathcal{F}$)

Definition 7. A topology for X is defined as a family of closed sets, $\mathcal{F}$, satisfying [C.1] and [C.2].

Example 3. Let X be the set of real numbers and let $\mathcal{F}$ be the family of all unions of closed intervals of X. Then $\mathcal{F}$ is a topology of the reals by Definition 7.

Definition 8. The <u>closure</u> of a set E contained in $(X, \mathcal{F})$ is the intersection of all closed subsets of X containing E. It is denoted by c(E). By [C.1], c(E) is a closed set and so is the smallest closed set containing E. A set is closed iff it equals its own closure.

Definition 9. Two subsets A and B form a <u>separation</u> of a set E in a topological space, written $E = A|B$, iff E is the union of A and B, and they are nonempty, disjoint sets, neither of which contains a limit point of the other. The requirements that A and B be disjoint and neither contain a limit point of the other are combined in the formula $[A \cap c(B)] \cup [c(A) \cap B] = \varphi$.

Definition 10. A set is <u>connected</u> if it has no separation.

The following theorem is stated without proof.

Theorem 5. If C is a connected subset of a topological space $(X, \mathcal{T})$ which has a separation $X = A|B$, then either $C \subseteq A$ or $C \subseteq B$.

Connectedness and separatedness are topological properties. (Topological property will be defined in Definition 15 below.)

Definition 11. If x is a point of a subset E in a topological space then the union of all connected sets containing x and contained in E will be called the <u>component</u> of E corresponding to x.

Definition 12. A <u>neighborhood</u> of a point is any set which contains an open set containing the point.

Some of the properties of neighborhoods are:

(N.1) Every point of X is in at least one neighborhood and is contained in each of its neighborhoods.

(N.2) The intersection of any two neighborhoods of a point is a neighborhood of the point.

(N.3) Any set which contains a neighborhood of a point is itself a neighborhood of the point.

(N.4) If N is a neighborhood of a point x, then there exists a neighborhood N* such that N is a neighborhood of each point of N*.

These properties can also be used to define a topology for X.

Definition 13. The <u>interior</u> of a set E contained in $(X,\mathcal{J})$ is the union of all open sets contained in E. It is denoted by i(E) and is an open set by (0.1). A set E is open iff $E = i(E)$.

Definition 14. If $f;X \to X^*$ is one to one, onto, continuous, and maps open sets onto open sets, it is called a <u>homeomorphism</u>.

Definition 15. Any property that is preserved under a homeomorphism is a <u>topological property</u>.

Definition 16. A set A in $(X,\mathcal{J})$ will be called <u>semi-open</u> (written s.o.) iff there exists an open set E such that $E \subseteq A \subseteq cE$.

Theorem 6. A subset A in a topological space X is s.o. iff $A \subseteq c(i(A))$.[2]

Proof: Sufficiency. Let $A \subseteq c(i(A))$. Then for

$$E = i(A),\ E \subseteq A \subseteq cE.$$

Necessity. Let A be s.o. Then $E \subseteq A \subseteq cE$ for some open set E. But $E \subseteq i(A)$ and thus $cE \subseteq c(i(A))$.

Hence $A \subseteq cE \subseteq c(i(A))$.

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[2]Ibid., p. 36.

Note that with this definition, a semi-open set may be open or closed as the next three example show.

Example 4. Let X be the reals with the usual topology and let E be the set consisting of the open interval (0,1). Then cE is the set consisting of the closed interval [0,1]. Then if A is either one of the half open intervals (0,1] or [0,1), or if A is E or cE, then A satisfies the relation $E \subseteq A \subseteq cE$ and therefore, A is semi-open. In this example, (0,1) is an open set that is semi-open, [0,1] is a closed set that is semi-open and (0,1] and [0,1) are half open intervals that are semi-open sets which are neither open nor closed.

Example 5. Let X be the space of the reals with the usual topology and let $A = (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup \cdots \cup (\frac{1}{2^{m+1}}, \frac{1}{2^m}) \cup \cdots$ and $B = \{0\} \cup (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup \cdots \cup (\frac{1}{2^{m+1}}, \frac{1}{2^m}) \cup \cdots$ Here, cA is [0,1] and $A \subseteq B \subseteq cA$ shows that B is s.o. in X. In this example, B is neither open nor closed but is a semi-open set.[3]

Example 6. Let X be the Euclidean plane with the usual topology. Let E be set $\{(x,y) \mid a_1 < x < a_2,\ b_1 < y < b_2\}$. Then cE is the set $\{(x,y) \mid a_1 \le x \le a_2,\ b_1 \le y \le b_2\}$. Then if A is either E or cE, or if A is any subset of cE which contains E, then A is s.o. in X. The remarks in Example 4 apply here.

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[3]Ibid., p. 38.

Theorem 7. Let $\{A_\alpha\}_{\alpha\varepsilon\Delta}$ be a collection of s.o. sets in a topological space X. Then $\bigcup_{\alpha\varepsilon\Delta}A_\alpha$ is s.o. in X.[4]

Proof. Every A_α is a semi-open set. From Definition 16, for every $\alpha\varepsilon\Delta$, there exists an open set 0_α such that $0_\alpha \subseteq A_\alpha \subseteq c0_\alpha$. Then $\bigcup_{\alpha\varepsilon\Delta}0_\alpha \subseteq \bigcup_{\alpha\varepsilon\Delta}A_\alpha$

$\subseteq \bigcup_{\alpha\varepsilon\Delta}c0_\alpha \subseteq c\bigcup_{\alpha\varepsilon\Delta}0_\alpha$ and if $0 = \bigcup_{\alpha\varepsilon\Delta}0_\alpha$, $A = \bigcup_{\alpha\varepsilon\Delta}A_\alpha$, $0 \subseteq A \subseteq c0$ satisfies the definition of a s.o. set. Therefore, $A = \bigcup_{\alpha\varepsilon\Delta}A_\alpha$ is s.o. in X.

Theorem 8. Let A be s.o. in the topological space X and suppose $A \subseteq B \subseteq cA$. The B is s.o. in X.[5]

Proof. From Definition 16, there exists an open set 0 such that $0 \subseteq A \subseteq c0$. Then $0 \subseteq A \subseteq B \subseteq cA$. But $cA \subseteq c0$. Thus, $B \subseteq c0$ and $0 \subseteq B \subseteq c0$. Therefore, B is s.o. in X.

An open set is a subset of itself which is contained in its closure; this implies that if E is open in X, then E is s.o. in X. The converse is false as shown by Example 4. (a.)

Definition 17. S.O.(X) will denote the class of all s.o. sets in X.

It is not true that the components of semi-open sets are semi-open as shown by Example 5. In Example 5, B is s.o. and {0} is a component of B, but {0} is not s.o. in X.

In general, the complement of a semi-open set is not semi-open as shown by the following example.

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[4]Ibid., p. 36.

[5]Ibid.

Example 7. Let X be the subset [0,1] of the reals with the usual topology, and let S be the open interval (0,1) and cS the closed interval [0,1]. Let A be the half open interval (0,1]. Then A is semi-open in X and CA in X is {0}. But {0} is not semi-open.

In general, the intersection of two semi-open sets is not semi-open as shown by the following example.

Example 8. Let X be the reals with the usual topology, A be the closed interval [0,1], and B the closed interval [1,2]. A and B are semi-open as shown in Example 4, but $A \cap B = \{1\}$. {1} is not semi-open.

Definition 18. A subset E of a topological space X is called <u>nowhere dense</u> iff every nonempty open set in X contains a nonempty open set which is disjoint from E.

Lemma 1. Let O be open in X. Then cO-O is nowhere dense in X.

Proof. $cO = O \cup d(O)$ implies $(cO-O) \subseteq d(O)$, i.e., cO-O consists of limit points of O. Let G be any nonempty open set in X. Then there are three cases:

Case (1) $G \subseteq O$ implies $G \cap (cO-O) = \varphi$.

Case (2) $G \cap O = \varphi$. Since G contains no points of O, it can contain no limit points of O, and therefore $G \cap (cO-O) = \varphi$.

Case (3) $G \cap O \neq \varphi$, $G \nsubseteq O$. Since G and O are open sets, $G \cap O$ is a nonempty open subset of G and of O. Thus, $(cO-O) \cap (G \cap O) = \varphi$. Therefore, for any nonempty open set G, there exists a nonempty open subset of G that is disjoint from (cO-O).

Norman Levin in his article in the American Mathematical Monthly, defines semi-continuity as follows:

Definition 19. Let $f:X \to X^*$ be single valued (not necessarily continuous) where X and X* are topological spaces. Then $f:X \to X^*$ is termed semi-continuous iff for O* open in X*, then $f^{-1}(O^*) \in$ S.O.(X).

Continuity implies semi-continuity but not conversely.

The article then develops some of the properties of semi-continuity but does not show whether or not this definition is equivalent to the usual definitions of semi-continuity. In particular, it is not shown equivalent to upper or lower semi-continuity.

This report is concerned primarily with properties of semi-open sets other than semi-continuity, so this is the only consideration given here to this property.

CHAPTER II

SEMI-OPEN SETS

This chapter develops some of the properties of semi-open sets in topological spaces.

For the purposes of this report, when there exists a semi-open set E having an open subset C as in the definition above, C will be said to define the semi-open set E. As there may be many semi-open sets in c(C), C cannot be said to define a unique semi-open set, as shown in Example 4 of Chapter I. Also, the same semi-open set may be defined by more than one open set. An example which illustrates this will be given later (see Example 14).

The empty set does not define any semi-open sets except itself as $\varphi \subseteq c\,\varphi$ implies $\varphi \subseteq \varphi \subseteq c\,\varphi$ and shows that the empty set satisfies the definition of semi-open sets. A result of this is that no set of the reals containing only one point can be semi-open as the only open set it contains is the empty set.

Theorem 9. Let $A \subseteq B \subseteq cA$ where A is open and B is s.o. If B -A is nonempty, the points of B -A are limit points of A.

Proof. Since $B \subseteq cA$, $(B-A) \subseteq (cA-A) \subseteq d(A)$. Therefore, $(B-A) \subseteq d(A)$ and the theorem is proved.

Theorem 10. Let $A \subseteq B \subseteq cA$ where A is open and B is s.o. If B-A is nonempty, all of the limit points of B-A are contained in cA.

Proof. Let x be any limit point of B-A. Then any open set containing x must contain at least one point y of B-A distinct from x. But since any point of B-A is a limit point of A by Theorem 9, every open set containing x and y contains a point of A distinct from y. Therefore, x is a limit point of A and must be in cA.

Theorem 11. Let $A \in S.O.(X)$, let $f:X \to X^*$ be a continuous open mapping where X and X* are topological spaces. Then $f(A) \in S.O.(X^*)$.[6]

Proof. Let $A = O \cup B$ where O is open and $B \subseteq cO - O$. Then $f(O) \subseteq f(A) = (f(O) \cup f(B)) \subseteq (f(O) \cup f(cO)) \subseteq (f(O) \cup cf(O))$ $= cf(O)$; hence, $f(O) \subseteq f(A) \subseteq cf(O)$ and $f(O)$ is open in X* since $f:X \to X^*$ is open. Therefore, $f(A) \in S.O.(X^*)$.

This shows that semi-openness is a topological property. The theorem is not true if the mapping is not open as shown by Example 9 below.

Example 9. Let X and X* both be the space of reals and $f:X \to X^*$ as follows: $f(x) \equiv 1$ for all $x \in X$. Now if $A \in S.O.(X)$, $f(A) = \{1\} \subseteq X^*$, $A \neq \varphi$, but $\{1\}$ is not s.o. in X*.[7]

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[6]Ibid., p. 38.

[7]Ibid., p. 38.

Theorem 12. Let $\mathcal{T}$ be the class of open sets in a topological space X, then $\mathcal{T} \in$ S.O.(X).

Proof. $\mathcal{T} \in$ S.O.(X) follows from (a.) on page 8 and Definition 17.

Theorem 13. Let $A \in$ S.O.(X) where X is a topological space. Then $A = O \cup B$ where (1) $O \in \mathcal{T}$, the class of open sets in X, (2) $O \cap B = \varphi$ and (3) B is nowhere dense.[8]

Proof. From Definition 16 there exists a set O where $O \in \mathcal{T}$, the class of open sets in X such that $O \subseteq A \subseteq cO$. Then $A = O \cup (A-O)$. Let $B = A-O$, then $B \subseteq cO-O$ and this is nowhere dense by Lemma 1, and $A = O \cup B$. $O \cap B = \varphi$ follows from $B = A-O$.

This theorem shows that a semi-open set can always be expressed as the union of two disjoint sets, one open and the other nowhere dense. The converse of Theorem 13 is false, as shown by Example 10.

Example 10. Let X be the space of reals and $A = \{x | 0<x<1\} \cup \{2\}$. Then $A \notin$ S.O.(X) even though (1), (2) and (3) in Theorem 13 hold, i.e., $0 = (0,1) \in \mathcal{T}$, where $\mathcal{T}$ is the class of open sets in X, $(0,1) \cap \{2\} = \varphi$, and $B = \{2\}$ is nowhere dense. Here since $\{2\}$ has no limit points and is not a limit point of any open set contained in A, it is not in the closure of any open set contained in A. Therefore, A cannot be contained in the closure of any open set it contains.

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[8] Ibid., p. 37.

Example 11. Let X be the space of reals and $A = \{x|0<x<2\} \cup \{3\}$ and $B = \{x|1<x<4\}$. Then A is not semi-open but $A \cup B \in S.O.(X)$. Here the union of a set, A, that is not semi-open with a set B, that is s.o., results in a semi-open set.

Example 12. Let X be the space of reals and let $A = \{x|0<x<3\} \cup \{4\}$ and $B = \{x|2<x<5\} \cup \{1\}$, A and B are not semi-open, but $A \cup B \in S.O.(X)$.

Theorem 14. If X_1 and X_2 are topological spaces, then $(X_1)X(X_2)$ is the topological product. Let $A_1 \in S.O.(X_1)$ and $A_2 \in S.O.(X_2)$. Then $A_1 X A_2 \in S.O.[(X_1)X(X_2)]$.[9]

Proof. By Theorem 13, $A_i = O_i \cup B_i$ where O_i is open in X_i and $B_i \subseteq (c\ O_i - O_i)$ for $i = 1, 2$ and B_i is nowhere dense in X_i. Then $A_1 \times A_2 = (O_1 \cup B_1) \times (O_2 \cup B_2)$ and expanding, $A_1 \times A_2 = (O_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times O_2) \cup (B_1 \times B_2)$ but $O_1 \times O_2$ is open in $(X_1)X(X_2)$ and $(O_1 \times O_2) \cup (B_1 \times O_1) \cup (O_1 \times B_2) \cup (B_1 \times B_2) \subseteq [(c_{x_1} O_1) X (c_{x_2} O_2)] = c_{(x_1)X(x_2)}(O_1 \times O_2)$. Therefore, $O_1 \times O_2 \subseteq (A_1 \times A_2) \subseteq c_{(x_1)X(x_2)}(O_1 \times O_2)$ and from Definition 17, $A_1 \times A_2 \in S.O.(O_1 \times O_2)$.

It was shown in Example 8 that the intersection of semi-open sets need not be semi-open. Using Theorem 13, it is seen that if the intersection of semi-open sets is semi-open, it must consist of the disjoint union of an open set and a nowhere dense set. However, the converse of this is not true as shown by the next example.

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[9]Ibid., p. 39.

Example 13. Let $A = (1,2) \cup (3,5) \cup \{1, 2, 3, 5\}$ and $B = (0,1) \cup (2,4) \cup \{0, 1, 2, 4\}$. A and B are semi-open in X and each consists of the disjoint union of a nonempty open set and a nowhere dense set. $A \cap B = (3,4) \cup \{1, 2, 3, 4\}$ shows that the intersection of A and B consists of the disjoint union of a nonempty open set and a nowhere dense set, but the subset $\{1,2\}$ of $A \cap B$ is not in the closure of any open set contained in $A \cap B$.

Theorem 15. If C is an open connected set and $C \subseteq F \subseteq c(C)$, then E is s.o. and connected.

Proof. That E is s.o. follows from Definition 16. If E is not a connected set, it must have a separation $E = A|B$. By Theorem 5, C must be contained in A or contained in B. Without loss of generality, suppose $C \subseteq A$. Then it follows that $c(C) \subseteq c(A)$ and hence $(c(C) \cap B) \subseteq (c(A) \cap B) = \varphi$. But $B \subseteq E \subseteq c(C)$ and $c(C) \cap B = \varphi$, so $B = \varphi$ which contradicts the hypothesis that $E = A|B$. Therefore, E must be connected.

This shows that semi-open sets are connected when the open sets which defined them are connected. The converse of this is not true in general as shown by the following example.

Example 14. Let $A = (0,1) \cup (1,2)$, $E = (0,2)$, $c(A) = [0,2]$. A is an open set that is not connected, E is a semi-open set that is connected. Note that

$F = (0,1) \cup (1,2]$ is also a semi-open set defined by A but F is not connected. Finally, since A and E are both open sets and cE is s.o., it is seen that the same semi-open set can be defined by more than one open set, i.e., $E \subseteq cE \subseteq cE$ and $A \subseteq cE \subseteq cA$ where $cE = cA = [0,2]$.

Theorem 15 and Example 14 lead to the following definition.

Definition 20. If a semi-open set E is connected but an open set A which defined it is not connected, then A will be said to be <u>semi-connected</u>.

Theorem 16. If a semi-open set E is a separated set, any open set C which defined it is also separated.

Proof. Let $C \subseteq E \subseteq c(C)$ where C is open and E is s.o. with a separation $E = A|B$. Then $[A \cap c(B)] \cup$

$[c(A) \cap B] = \varphi$.

Assume C is not separated. Then $C \subseteq A$ or $C \subseteq B$. Now it was shown in Theorem 9 that E-C consists of limit points of C. Without loss of generality, let $C \subseteq A$. Since $B \subseteq (E-C)$ and $B \neq \varphi$, $C \subseteq A$ implies a limit point of C is a limit point of A and thus B contains at least one limit point of A, i.e., $c(A) \cap B \neq \varphi$. This contradicts the statement that A and B are separated.

Theorem 17. Semi-connectedness is a topological property.

Proof. Let $f: X \to X^*$ be a homeomorphism, where X and X^* are topological spaces. Let $C \subseteq E \subseteq c(E) \subseteq X$ where C is an open separated set and E is s.o. and connected, then C is semi-connected. Then $f(C) = C^*$ implies C^* is an open separated set. $f(E) = E^*$ implies that E^* is semi-open since semi-openness is a topological property by Theorem 11, and since connectedness is a topological property, E^* must be connected. Therefore, C^* is semi-connected since f is a homeomorphism and $C^* \subseteq E^* \subseteq c(E^*) \subseteq X^*$.

Let $(Y, \mathcal{F})$ be a topological space, and Y an infinite set where the members of $\mathcal{F}$ are the complements of finite sets and the empty set, i.e., $C(A) \in \mathcal{F}$ where A is finite. Since each set C(A) is open, each finite set A must be closed. Y will always be used to denote a space with this topology in this paper.

The entire set Y contains all points of Y and therefore all limit points of itself, and is thus open and closed.

Since all finite sets are closed sets, the only open set they contain is the empty set. The only semi-open set that the empty set defines is itself; thus, there are no finite semi-open sets in this topology.

Theorem 18. In the topological space Y every semi-open set is an open set.

Proof. Since any open set E, except φ, is infinite, every semi-open set defined by E is infinite. But every open set is the complement of a finite set and thus an infinite set.

Lemma 2. In the topological space Y every point of a finite set A is a limit point of its complement.

Proof. Let x be any point of A where A is finite. Then C(A) is an open set. Let E be any other open set which contains x. Then, since E is infinite and A is finite, E must contain at least one point of C(A) since $(E-A) \subseteq CA$.

Theorem 19. In the topological space Y, each open set defines at most, a finite number of semi-open sets.

Proof. Let A be a finite subset of Y. Then CA is an open set. Since every point of A is a limit point of CA by Lemma 2, the closure of CA is $c(CA) = CA \cup A$ (i.e., the entire space). Let E be any s.o. set such that $CA \subseteq E \subseteq Y$, $E = CA \cup K$, K is any subset of A. Since there are a finite number of points in A, there can be only a finite number of sets contained in Y and containing CA.

Theorem 20. In the topological space Y, if an open set is compact, each of its semi-open subsets is compact.

Proof. Let CA be a compact open subset of Y, and E a s.o. set of Y defined by CA. Now given any open covering, G, of E there exists a finite number of sets of G that cover CA since $CA \subseteq E$. E-CA is a finite subset of E and thus a finite number of elements of G which cover E-CA. Hence, E is covered by the union of the finite coverings of CA and E-CA; and the union of these two finite coverings is again finite. Therefore, E is compact.

Let X be a topological space with the discrete topology, i.e., $\mathscr{F}$ is the family of all subsets of X. Here, every set is an open set, and since every open set is s.o., all sets are s.o.

Let X be a topological space with the indiscrete topology. Here, $\mathscr{F}$ consists of φ and X itself. The closure of φ is φ and the closure of X is X; therefore, the only semi-open sets possible are φ and X, i.e., $\varphi \subseteq \varphi \subseteq c\varphi$ and $X \subseteq X \subseteq cX$.

Let X be a T_0 space. Then if there are two distinct points, x and y in X, there exists an open set G which contains one point but not the other. Without loss of generality, let $x \in G$ where G is open.

If y is not contained in any open set other than the entire space, or if y is contained in a nowhere dense set disjoint from G, then $G \cup y$ may be semi-open provided $y \in cG$. However, unless more is known about a space, the property of being T_0 is not sufficient to investigate semi-open sets.

Let X be a T_1 space. Then if there exist two distinct points x and y, there exist two open sets, one containing x, and the other containing y. (Any T_1 space is, of course, T_0).

The following theorem is stated without proof.

Theorem 21. In a T_1 space X, a point x is a limit point of a set E iff every open set containing x contains an infinite number of distinct points of E.

From Theorem 21, it follows that no finite set E in a T_1 space can have a limit point, as no open set containing such a limit point can possibly contain an infinite number of points of E. Therefore, every finite set is a closed set. Thus, it would seem that no finite set can define a semi-open set in a T_1 space unless the set is both open and closed in the topology of the space.

Let X be a T_2 space. Then for any two distinct points x and y there exist two disjoint open sets, one containing x and one containing y. Indeed, most of the spaces discussed in this report are T_2 spaces.

Since every T_2 space is also T_1, the statement above concerning finite semi-open sets in any T_1 space probably applies to T_2 spaces.

The reals with the usual topology are a T_2 space and have no finite semi-open sets since they have no finite open sets.

Infinite semi-open sets are possible in T_1 or T_2 spaces.

A brief survey of other types of spaces will now be presented.

Let X be a regular space. Then if F is a closed subset of X and x is a point of X not in F, there exist two disjoint open sets, one containing F and one containing x. The closed set F may contain semi-open sets. Since x is not in this closed set, it is not in the closure of any open set contained in F, and therefore, there is no semi-open set containing x which is defined by an open set contained in F. There may be a semi-open set which contains F or some points of F and x.

A T_3 space is a regular space that is also a T_1 space.

Let X be a normal space. Then if F_1 and F_2 are two disjoint closed subsets of X, there exist two disjoint open sets, one containing F_1 and the other containing F_2.

A T_4 space is a normal space that is also a T_1 space.

Let X be a completely normal space. Then if A and B are two separated subsets of X, there exist two disjoint open sets, one containing A and the other containing B.

A T_5 space is a completely normal space that is also a T_1 space.

Since T_3, T_4, and T_5 spaces are also T_1, the previous comment regarding finite semi-open sets in T_1 spaces could be made here. In the regular and normal spaces, semi-open sets may be contained in closed sets.

To investigate semi-open sets more fully in these spaces is beyond the scope of this report as these are specialized areas of study.

A semi-open set contains an open set. Therefore, any semi-open set B is a neighborhood of a point x if $x \in A \subseteq B \subseteq cA$ where A is an open set in the topological space. The semi-open set B satisfies all of the axioms for neighborhood in Definition 10, but since the intersections of semi-open sets need not be semi-open, not all neighborhoods are semi-open. Thus, there appears to be little value in applying the neighborhood concept to semi-open sets.

Finally, a topology in terms of semi-open sets is considered. Let X be a nonempty set of points and S be the family of semi-open sets of X such that the intersection of any finite number of elements of S is semi-open. (X,S) is a topological space as defined by Definition 1.

Example 15. Let X be the set of reals. Let S be the family of semi-open sets which are unions of sets of the form [a,b), (or (a,b]) where a and b are real numbers. This family of sets satisfies the axioms of Definition 1.

But if some intervals are closed on the left and others on the right, then some of the intersections may not be s.o. since they can be single points.

Example 16. Let X be the Euclidean plane. Let S be the family of semi-open sets which are unions of sets $\{(x,y) \mid a \leq x < b,\ c \leq y \leq d\}$. Then this family satisfies the axioms of Definition 1. If the interval $a \leq x < b$ were closed on the right instead of the left, or if the interval $c \leq y < d$ were closed above instead of below, the members of S would still satisfy the axioms. However, if some of the intervals $a \leq x < b$ were closed on the right and some on the left and/or if some of the intervals $c \leq y < d$ were closed above and some below, then the intersections would not necessarily be semi-open.

Example 17. Let X be the set of reals. Let S be the set of semi-infinite intervals of the form $(a, +\infty) = \{x \mid x > a\}$. This set satisfies the axioms of Definition 1.

Semi-open sets may be a base for a topology, but as the intersection of any two members of a base must be a union of members of the base, it is necessary to form the base of s.o. sets whose intersections are semi-open.

In conclusion, a topology formed from semi-open sets is possible, but since it would have to exclude those semi-open sets whose intersections are not semi-open, it may be impractical.

CHAPTER III

SUMMARY AND CONCLUSIONS

I. SUMMARY

In Chapter I, background information and definitions were provided to lead up to the concept of semi-open sets. Semi-open sets were defined, and several of their properties were presented.

If A is an open set, and $A \subseteq E \subseteq cA$, then E was defined to be semi-open and the following were demonstrated. Open sets are semi-open, but not all semi-open sets are open. The unions of semi-open sets are semi-open, but the intersections of semi-open sets may not be semi-open. The complement and the components of semi-open sets may not be semi-open. Examples were given to illustrate these properties.

Semi-continuity was defined in terms of semi-open sets, but this property was not investigated in this report.

In Chapter II, many properties of semi-open sets were investigated. When there existed a semi-open set contained in the closure of an open set, the open set was said to define the semi-open set. It was shown that the only semi-open set the empty set defines is the empty set itself. An open set does not define a unique semi-open set, nor is a semi-open set defined by a unique open set.

Semi-openness is a topological property, and a semi-open set can be represented as the disjoint union of an open set and a nowhere dense set, but not conversely.

Probably the most interesting property discussed, and certainly the most original, is that of semi-connectedness. If a connected open set defines a semi-open set, that semi-open set is connected; but when a separated open set E defines a connected semi-open set, then E is semi-connected. Semi-connectedness is a topological property.

Semi-open sets in several different topologies were investigated.

Neighborhoods were found to be of little interest here.

Topologies formed with families of semi-open sets were briefly examined and several examples were shown. These topologies had to be restricted to semi-open sets whose intersections are semi-open. To define a topology in terms of semi-open sets appears to be quite difficult and certainly is a problem for future research.

II. CONCLUSIONS

The principal conclusion of this report is that a semi-open topology is of little value as it must be restricted to semi-open sets whose intersections are semi-open.

Other properties studied may be investigated further, particularly semi-continuity, which was not studied, and compactness.

Semi-open sets in T_3, T_4, T_5, regular, normal and completely normal spaces were mentioned only briefly and may be investigated more fully.

The terminology that an open set "defines" a semi-open set is original in this report.

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