A COMPARISON OF PARABOLIC, ELLIPTIC, AND HYPERBOLIC GEOMETRIES AS SUBGROUPS OF ANALYTICAL PROJECTIVE GEOMETRY

A THESIS

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1. Introduction. Geometry was originally concerned with measurement of line segments, angles, and other figures on a plane. Gradually, the meaning of the word geometry was extended to include the study of lines and planes in the ordinary space of solids, and the study of spaces based upon systems of coordinates, where points are represented by ordered sets of numbers (coordinates) and lines are represented by sets of points whose coordinates satisfy linear equations.[1] Recently it has been extended to include the study of abstract spaces in which points, lines, and planes may be represented in many ways. In this thesis a geometry will be considered as a set of points and lines and a group of transformations under which some property is left invariant. It is a deductive science using both analytic and synthetic methods of representation.

1.2 Statement of the problem. Any linear geometric transformation can be represented by a matrix. In this thesis the analytic transformations of parabolic, Euclidean, hyperbolic and elliptic geometry are derived from defining invariants showing the conditions placed on the matrix of the transformation. A comparison is then made of the resulting conditions and implications of these conditions.

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[1] B. E. Meserve, Fundamental Concepts of Geometry (Cambridge, Massachusetts: Addison-Wesley Publishing Company, 1955), p. 1.

1.3. Importance of the problem. The algebraic representation of points, lines, and transformations of geometry often makes proofs of theorems simpler and the mathematical concept involved easier to visualize. When the analytical methods become more involved, a second method of expression and the shorter notation has certain advantages. With parabolic, Euclidean, elliptic, and hyperbolic geometry each represented on the same coordinate system, a comparison quickly emphasizes the similarities and differences of the geometries.

In modern living, with the everyday use of electronic computers and the advent of space exploration and navigation, the use of non-Euclidean geometries, along with Euclidean geometry, and their representation on a coordinate system, is becoming increasingly important.

1.4 Undefined terms and relations. The undefined terms used in this thesis are: 1) set,

2) points, denoted by capital letters P, Q, R . . . ,

3) lines, denoted by small letters p, q, r . . . ,

4) planes, denoted by small Greek letters λ, π, . . . ,

5) incidence, a symmetric relation between points and lines such that:

i) if P is incident with p, then p is incident with P,

ii) if p is incident with P, then P is incident with p,

iii) two distinct points are together incident with exactly one line,

iv) two distinct lines are together incident with exactly one point,

v) there are four distinct points such that no three of them are incident with the same line. "On" may be used as a synonym for "incident."

In this thesis the discussion shall be limited to the study of points and lines in the space of the projective plane.

1.5. Definition of terms. A Projective plane is the set of points and lines satisfying the conditions of incidence.

The unique line incident with two points is called the Join of the two points.

The unique point incident with two distinct lines is called the Intersection of the two lines.

Two or more lines incident with the same point are called Concurrent.

Two or more points incident with the same line are called Collinear.

A Figure is a set of points and lines.

A Projective transformation is a one-to-one correspondence between two figures in the projective plane such that incidence is preserved.

A Collineation is a projective transformation which maps points into points and lines into lines.

A Correlation is a projective transformation which maps points into lines and lines into points.

A point is Self-Conjugate with respect to a correlation if it is on its own transform.

A Polarity is a correlation which satisfies the condition that X is on the line corresponding to Y if and only if Y is on the line corresponding to X, for all X, $Y \in K$, where K is the set of all points of the space.

A Conic is the non-empty set of self-conjugate points with respect to some polarity.

The Identity Transformation is a transformation in which each point maps into itself.

An Involution is a projective transformation, not the identity transformation, such that its square is the identity transformation.

A Geometry consists of an ordered pair (K,R) of sets such that:

1. K is the set of points and R is the set of lines,
2. Every line is a set of points,
3. Every line contains at least two points,
4. Two distinct points determine a unique line,

and a group of transformations under which certain properties are left invariant.

1.6. Organization of thesis. Chapter II provides a general picture of the development of geometry from the earliest beginnings to its present state. In Chapter III a coordinate system is developed with which to compare the various geometries. Chapter IV discusses the Conic, and Chapter V shows the derivation of the conditions placed on the transformations of each geometry. In Chapter VI a summary is made of the comparisons, and the results and conclusions of the study are stated.

CHAPTER II

A BRIEF HISTORY OF PROJECTIVE GEOMETRY

2.1. Introduction. A better understanding of a subject is obtained by a knowledge of the development of the subject. By an over-all view of a subject and an inspection of the interesting points in the evolution of the subject, methods of learning and techniques of problem solving are suggested. The errors and successes of previous mathematicians are studied and utilized in further development of the subject. Projective geometry is a fairly recent development of geometry and results from a generalization of the previous geometries. This brief history is divided into five periods. The first period deals with primitive Egyptian and Babylonian geometry. The second period presents the early Greek geometry and the axiomatic approach. The next period shows the discovery of non-Euclidean geometry. The fourth period presents analytical geometry, and in the fifth period projective geometry is generalized from the previous geometries.

2.2. Egyptian and Babylonian geometry (4000 - 600 B. C.). Early geometry developed as a result of man's effort to construct a set of logical rules to correlate data obtained from observation and measurement. Tablets dating back beyond 2000 B. C. indicate the Babylonians and Egyptians were employing some of the fundamental geometric concepts. This geometry originally was an empirical method used for measuring area of rectangles. They probably had formulas for finding areas of right triangles, trapezoids with a right angle at the base, the volume of rectangular parallelopipeds, and the right prism with trapezoidal or circular base. They also knew the altitude from the vertex of an isosceles triangle bisects the base, that corresponding sides of similar right triangles are proportional, that the angle inscribed in a semicircle is a right angle, and the general formula for the area of a triangle.

2.3. Early Greek geometry (600 B.C. - 300 A.D.). About 600 B.C. the Greek culture was becoming an important factor in the ancient world. The early Greeks made the first recognizable progress in the study of geometry as a science independent of its practical applications.[2] The deductive feature, the fundamental characteristic of mathematics, was developed. Thales was the first known individual to whom mathematical discoveries were associated.[3] He is credited with a number of elementary discoveries in geometry.

From the time of Thales (about 600 B.C.) to the time of Euclid, a great deal of progress was made in geometry. Some of the more prominent names associated with this early Greek geometry were: Thales, Theaetetus, Proculus, Hippocrates, Pythagoras, Hippias, Eudemus, Menaechemus, Hipparchus, Theodorus, Eudoxus, and Euclid. One of their greatest contributions was the development of the axiomatic method. From the accumulated material, Euclid compiled his Elements, the most remarkable textbook ever written; one which despite a number of grave imperfections

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[2] H. E. Wolfe, Introduction to Non-Euclidean Geometry (New York: The Dryden Press, Inc., 1945), p. 1.

[3] Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, Inc., 1953), p. 52.

has served as a model for scientific treatises for over two thousand years.[4] The Elements contains thirteen books which include plane geometry, the theory of proportions, the theory of numbers, the theory of incommensurables, and solid geometry.

Next in the development of geometry was that of higher geometry, or the geometry of curves other than the circle and straight line, and of surfaces other than the sphere and plane. Much of this work was largely due to the discoveries of Archimedes, and Apollonius in works on conic sections, and the "Mathematical Collection" of Pappus.

2.4. The axiomatic approach of the early Greeks (300 B.C. - 300 A.D.). The axiomatic approach to geometry taken by the early Greeks, and which is the method in use today, consistscof a minimum of undefined terms and axioms, and a maximum of defined terms and theorems. The axioms must be consistent, and should be complete, independent, categorical, and fertile. A set of axioms is consistent if no contradictions can be deduced from the set. A set of axioms is complete if of any two contradictory statements involving terms of the system, at least one statement can be proved in the system. A set of axioms is said to be independent if no axiom can be deduced from the others. A set of axioms is categorical if there is essentially only one system for which its axioms are valid, that is, any two systems which satisfy the axioms are isomorphic. For a set of axioms to be fertile, at least one theorem can be deduced from them. A definition must: (1) name the concept being defined,

[4] Meserve, Op. Cit., p. 221.

(2) give the distinguishing characteristics of the concept being defined, (3) be concise (i.e., contains no superfluous information), (4) contain no new elements or relations, (5) be reversible.

2.5. The discovery of non-Euclidean geometry (19th Century). The attempts to deduce Euclid's fifth postulate as a result of the other Euclidean postulates led to the discovery of non-Euclidean geometry. These attempts persisted until the 19th Century when hyperbolic geometry was discovered independently by Gauss, Bolyai, and Lobachevsky. Gauss, however, did not publish his work, and credit for the discovery is given to Lobachevsky and Bolyai. Bolyai wrote an appendix for his father's treatise on geometry, which gave an account of his (the younger Bolyai) investigations. Later elliptic geometry was discovered by Riemann. Today, many have the idea that a geometry other than that of Euclid is the best model for our universe.

2.6. Analytical geometry invented (17th Century). Another appendix to a book that was of incomparably greater significance than the book itself was the first treatise on analytic geometry, which formed an appendix to Discours de la Methode written by the French philosopher-mathematician, Rene Descartes (1595-1650).[5] Descartes visualized all algebra expressions as numbers which were the measures of geometric objects instead of as the geometric objects, and found equations representing

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[5] Leonard M. Blumenthal, A Modern View of Geometry (San Francisco: W. H. Freeman and Company, 1961), p. 54.

several curves. This union of algebra and geometry made possible the establishment of a coordinate system by assuming that the points of a line are in a one-to-one correspondence with the numbers of the real number system, and that the space coordinatized had all the Euclidean properties. Thus the geometry is consistent if the real number system is consistent.

2.7. The development of projective geometry (19th Century). During the Renaissance, medieval painters, in their desire to paint realistically, worked to find a mathematical method to depict the three-dimensional world on a two-dimensional canvas. Since these painters were also architects, engineers, and the best mathematicians of the 15th Century, they were very successful in the task. The key to three-dimensional representation was found in what is known as the principle of projection and section.[6] The theorems which arose from this work led to the development of a more general geometry, projective geometry. Desargues and Pascal produced theorems which are fundamental in the development of projective geometry. These theorems show that there are significant properties common to sections of any projection of a given figure. Desargues and Pascal visualized the conic sections as projections of circles and discovered other properties of conics.

Klein and Cayley then showed that parabolic, elliptic, and hyperbolic geometries can be derived as special cases of projective geometry. Poncelet wrote the first text

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[6] James R. Newman, The World of Mathematics (New York: Simon and Schuster, 1956), I, p. 623.

on projective geometry. He considered ideal points (intersections of parallel lines) and developed the concept of duality. Plücker introduced a new type of coordinate system in the projective plane.

2.8. Summary. Projective geometry, as we know it today, is a result of evolution of geometry over a period of approximately four thousand years from simple practical methods of measurement to a highly developed abstract science. Projective geometry incorporates both the synthetic, deductive methods of the early Greeks and the algebraic approach introduced by Descartes with application of the techniques of algebra and calculus and recent discoveries of mathematical methods.

CHAPTER III

A COORDINATE SYSTEM

3.1. Introduction. A coordinate system is to be developed for the points on projective lines and planes. It will be based on the geometric properties of the projective plane. The set of points on the projective line shall be isomorphic to the extended real number system. This isomorphism makes it possible to use the real numbers as coordinates of points of the projective line. The purpose of the coordinate system is:

1. To identify the points on the line.
2. To establish a coordinate system for the projective plane.
3. To obtain and describe properties of the geometry.

3.2. Summation Convention. For brevity of notation the summation convention will be used in the following discussion. Whenever the same letter is used as a subscript twice in a term it will be understood to mean the sum of such terms where the subscript of summation is the repeated subscript. For example:

$$A_{ij}X_j, i, j = 1, 2, 3 \text{ means } \sum_{j=1}^{3} A_{ij}X_j, \text{ for } i = 1, 2, 3.$$

3.3. Method of representation. A point is represented by a 3 X 1 column vector (X_i), $i = 1, 2, 3$, (e.g., $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$) called the homogeneous coordinates of the point, where:

i) $(kX_i) = (X_i)$ when k is a real number and $k \neq 0$,

ii) there is no point corresponding to $(X_i) = (0)$.

A line is represented by a 1 X 3 row vector $\lceil u_k \rceil$, k = 1, 2, 3, called the homogeneous coordinates of the line such that:

i) $\lceil ku_k \rceil = \lceil u_k \rceil$ when k is a real number and $k \neq 0$,

ii) there is no line corresponding to $\lceil u_k \rceil = \lceil 0 \rceil$.

In this discussion parentheses shall be used in the symbol representing a point and square brackets shall be used in the symbol representing a line. Note that (X_i) (e.g., (1,0,0)), refers to a column vector which denotes a point. $\lceil u_i \rceil$ (e.g., $\lceil 1,0,0 \rceil$), refers to a row vector which denotes a line. A point is on a line if and only if their inner product is zero, $\lceil u_i \rceil (X_i) = 0$.

A projective transformation is represented by a non-singular 3 X 3 matrix (A_{ij}), i, j = 1, 2, 3, $|A_{ij}| \neq 0$, such that:

1) a projective transformation of point (X_j) to point (Y_i) is represented $(A_{ij})(X_j) = (Y_i)$;

2) a projective transformation of line $\lceil u_i \rceil$ to line $\lceil v_j \rceil$ is represented $(A_{ij}) \lceil u_i \rceil^t = (v_j)^t = \lceil v_j \rceil$;

3) a projective transformation of point (X_j) to line $\lceil u_i \rceil$ is represented $(A_{ij})(X_j) = (u_j)^t = \lceil u_i \rceil$;

4) a projective transformation of line $\lceil u_i \rceil$ to point (X_j) is represented $(A_{ij}) \lceil u_i \rceil^t = (X_j)$.

The equivalence of these with the definitions in Chapter I is shown in many standard texts. (e.g., Meserve, B. E., Fundamental Concepts of Geometry, Ch. 4).

Two points (X_i) and (Y_i), $i = 1, 2, 3$, are on the line $\lceil x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1 \rceil$.

Two lines $\lceil u_i \rceil$ and $\lceil v_i \rceil$, $i = 1, 2, 3$, intersect at the point

$$\begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

Three points (X_i), (Y_i) and (Z_i), $i = 1, 2, 3$, are collinear if and only if

$$\begin{vmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \\ x_3 y_3 z_3 \end{vmatrix} = 0.$$

Three lines $\lceil u_i \rceil$, $\lceil v_i \rceil$ and $\lceil w_i \rceil$, $i = 1, 2, 3$, are concurrent if and only if

$$\begin{vmatrix} u_1 u_2 u_3 \\ v_1 v_2 v_3 \\ w_1 w_2 w_3 \end{vmatrix} = 0.$$

A correlation (A_{ij}) is a polarity if and only if condition (i) implies (ii) and conversely;

i) $(A_{ij}X_j)^t = \lceil u_i \rceil$, $i, j = 1, 2, 3$, the line which is the transform of the point (X_j) contains the point (Y_i), i.e., $u_i Y_i = A_{ij}X_jY_i = 0$.

ii) $(A_{ij}Y_j)^t = \lceil u_i \rceil$, $i, j = 1, 2, 3$, the line which is the transform of the point (Y_j) contains the point (X_i), i.e., $u_i X_i = A_{ij} Y_j X_i = 0$.

Line $\lceil u_i \rceil$ is called the polar of the point (X_j) with respect to the polarity. Point (X_j) is called the pole of the line $\lceil u_i \rceil$ with respect to the polarity. This is true for all points if and only if $A_{ij} = A_{ji}$, that is the matrix of the transformation is symmetric.

Theorem: A correlation (A_{ij}) is a polarity if and only if $A_{ij} = A_{ji}$.

Proof: A point $X = (X_j)$ is transformed into the line $\lceil u_i \rceil$ by the correlation (A_{ij});

$(A_{ij}X_j)^t = \lceil u_i \rceil$, $i, j = 1, 2, 3$. If $Y = (Y_i)$ is a point on line $\lceil u_i \rceil$ then $\lceil u_i \rceil(Y_i) = 0$, i.e.,

$(A_{ij}X_j)^t(Y_i) = 0$, then $(X_j)^t(A_{ij})^t(Y_i) = 0$, $i, j = 1, 2, 3$.

If a point $Y = (Y_l)$ is transformed into the line $\lceil u_k \rceil$ by the correlation (A_{kl});

$(A_{kl}Y_l)^t = \lceil u_k \rceil$, $k, l = 1, 2, 3$. If $X = (X_k)$ is a point on line $\lceil u_k \rceil$ then $\lceil u_k \rceil(X_k) = 0$, i.e.,

$(A_{kl}Y_l)^t(X_k) = 0$, then $(Y_l)^t(A_{kl})^t(X_k) = 0$, $k, l = 1, 2, 3$.

In order that the correlation $(A_{ij}) = (A_{kl})$ be a polarity

$(X_j)^t(A_{ij})^t(Y_l) = 0 \Leftrightarrow (Y_l)^t(A_{kl})^t(X_k) = 0$ since

$\lceil (Y_l)^t(A_{kl})^t(X_k) \rceil^t = \lceil 0 \rceil^t$,

$(X_j)^t(A_{ij})^t(Y_l) = (X_k)^t(A_{kl})(Y_l) = 0$ hence

$(A_{ij})^t = (A_{kl})$, and $k = j$, and $l = i$ therefore

$(A_{ij}) = (A_{ji})$. Q.E.D.

A point, (X_j), is self-conjugate if $(A_{ij}X_j)^t = \lceil u_i \rceil$ and $u_i X_i = (A_{ij}X_j)^t X_i = 0$, i.e., $A_{11}x_1^2 + 2A_{12}x_1x_2 + 2A_{13}x_1x_3 + A_{22}x_2^2 + 2A_{23}x_2x_3 + A_{33}x_3^2 = 0$.

The polarity is called elliptic when all the self-conjugate points with respect to the polarity are not real.

The polarity is called hyperbolic when the self-conjugate points with respect to the polarity are real.

A collineation g is a projective transformation of the points of a projective plane to themselves satisfying:

1) g is one-to-one onto,

2) if points A, B, C are collinear, so also are the points g(A), g(B), g(C).

3.4. Coordinatizing the plane. In a projective plane π arbitrarily select:

1) any point of π and denote the point (0, 0, 1) and refer to it as the origin;

2) any three lines on (0, 0, 1), one line to be labeled [0, 1, 0] and called the x line, another to be labeled [1, 0, 0] and called the y line, the third to be labeled [1, -1, 0] and called the unit line;

3) any point on the unit line different from (0, 0, 1) label it (1, 1, 1) and refer to it as the unit point;

4) a line, distinct from [0, 1, 0], [1, 0, 0], and [1, -1, 0], and not on any point yet chosen and label it [0, 0, 1].

The line [0, 0, 1] shall be called the ideal line and all points of the form $(X_1, X_2, 0)$ shall be called ideal points. Other lines and points are called ordinary lines and points.

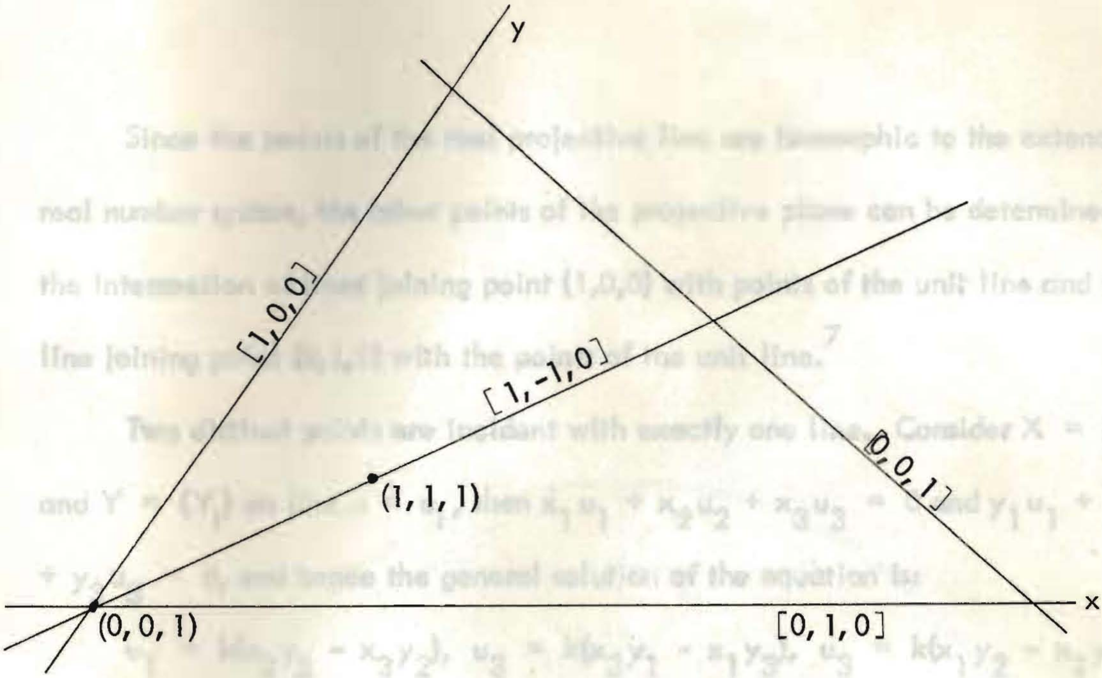

Figure 1

The Coordinate Axes

Since the points of the plane are represented by the homogeneous coordinates x_i, $i = 1, 2, 3$, and $(X_i) = (kX_i)$, $k \neq 0$, any point in which the coordinate $x_3 = 0$ may be represented by $(x_1\ x_2, 1)$ by multiplying (x_i) by $\frac{1}{x_3}$. The coordinate x_1 corresponds to the x line, the coordinate x_2 corresponds to the y line. Any point on the unit line has $x_1 = x_2$.

The intersection of [0,1,0] and [0,0,1] is (1,0,0).

The intersection of [1,0,0] and [0,0,1] is (0,1,0).

The intersection of [1,-1,0] and [0,0,1] is (1,1,0).

The join of (1,0,0) and (1,1,1) is [0,-1,1].

The join of (0,1,0) and (1,1,1) is [1,0,-1].

Since the points of the real projective line are isomorphic to the extended real number system, the other points of the projective plane can be determined by the intersection of lines joining point (1,0,0) with points of the unit line and the line joining point (0,1,0) with the points of the unit line.[7]

Two distinct points are incident with exactly one line. Consider $X = (X_i)$ and $Y = (Y_i)$ on line $u = u_i$, then $x_1u_1 + x_2u_2 + x_3u_3 = 0$ and $y_1u_1 + y_2u_2 + y_3u_3 = 0$, and hence the general solution of the equation is:

$$u_1 = k(x_2y_3 - x_3y_2),\ u_3 = k(x_3y_1 - x_1y_3),\ u_3 = k(x_1y_2 - x_2y_1).$$

u_1, u_2, u_3 not all equal to zero, otherwise $X = Y$. Therefore there is a unique line incident with two distinct points X and Y. Similarly two distinct lines p and q are incident with one point $X = (X_i)$ such that:

$$x_1 = k(p_2q_3 - p_3q_2),\ x_2 = k(p_3q_1 - p_1q_3),\ x_3 = k(p_1q_2 - p_2q_1).$$

The projective plane can be represented as in Figure 2.

3.5. Summary. In this Chapter representations for the points and lines of a projective plane are chosen. The algebraic representations of relations such as transformations, self-conjugate, collinear, and copuntal are shown along with the representation of point, line, and polarity.

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[7] Meserve, Op. cit., pp. 86-89.

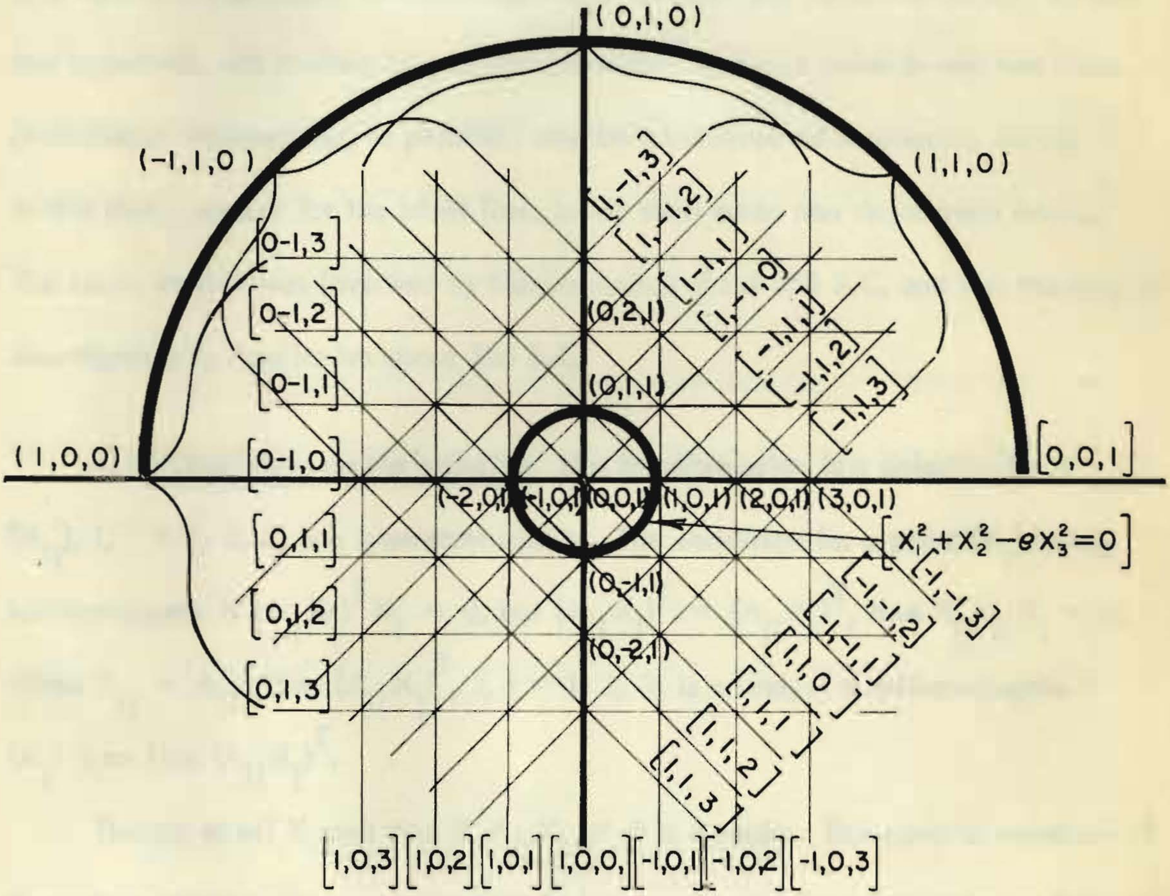

Figure 2

The Projective Plane

CHAPTER IV

CONICS

4.1. Introduction. A conic in geometry is usually defined as a plane section of a right circular cone. The nondegenerate conics, i.e., parabola, circle, ellipse and hyperbola, are studied in analytic geometry. A single point or any two lines (coincident, intersecting, or parallel) may be considered as degenerate conics. In this study, except for the ideal line, conic shall mean non degenerate conic.[8] The conic section was invented by Menaechemus about 350 B.C. and was thoroughly investigated by Apollonius about 225 B.C.

4.2. Conditions on the polarity. If a transformation is a polarity, then (A_{ij}), $i, j = 1, 2, 3$, is a symmetric matrix. The condition for a point (X_i) being self-conjugate is $(A_{ij}X_j)^T X_i = 0$, but $(A_{ij}X_j)^T = (A_{ji}X_i)^T$, thus $X_j A_{ij} X_i = 0$. When $A_{ij} = A_{ji}$, then $(A_{ij}X_j)^T$, $i, j = 1, 2, 3$, is a line, x is self-conjugate if (X_j) is on line $(A_{ij}X_j)^T$.

The set of all X such that $X_j A_{ij} X_i = 0$ is a conic. This general equation of a nondegenerate conic may be written in homogeneous point coordinates as follows:

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_3 + Ex_2x_3 + Fx_3^2 = 0 \quad \text{where} \quad \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} \neq 0.$$

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[8] Meserve, Op. cit., p. 63.

The condition that a conic be nondegenerate is identical to the condition that the matrix of the polarity, for which the points are self-conjugate, be nonsingular. When the general equation of a conic in homogeneous coordinates is $Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_3 + Ex_2x_3 + Fx_3^2 = 0$, the conic will meet the ideal line $x_3 = 0$ in ideal points $(x_1, x_2, 0)$ whose coordinates satisfy the equation $Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0$. Thus the number of real points of intersection is the same as the number of real solutions to the quadratic equation. From the theory of quadratic equations, $Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0$ will have two distinct, real roots, one real root, or no real root according as $B^2 - 4AC \gtreqless 0$.[9]

4.3. Classification of conics. A conic is defined to be a hyperbola, a parabola, or an ellipse accordingly as it contains two distinct, one distinct, or no real ideal points.

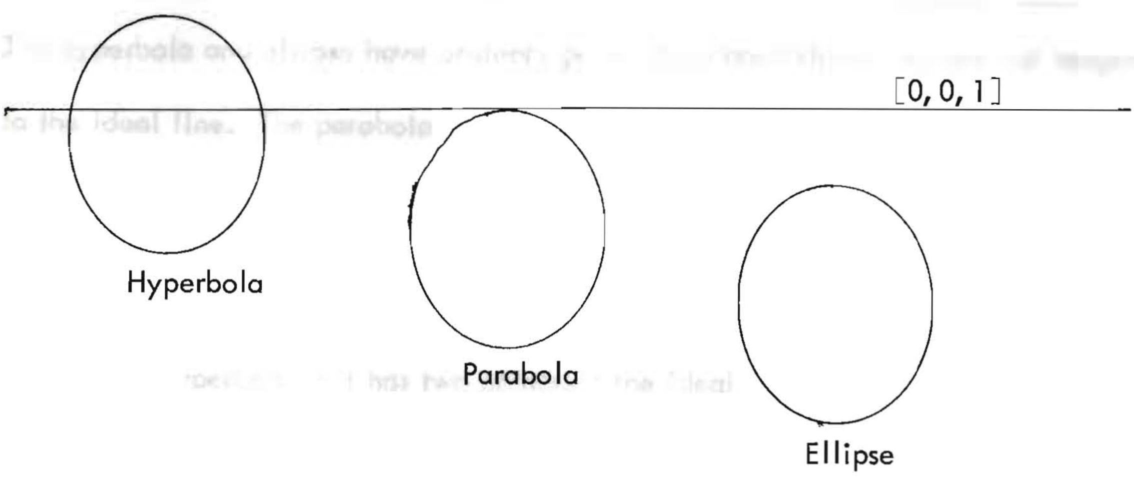

Figure 3

Ideal Points of the Conic

[9] C. F. Adler, Modern Geometry: An Integrated First Course (New York: McGraw-Hill Book Company, Inc., 1958), p. 173.

If a line p is the polar of the point P with respect to a polarity, then p is said to be the polar of P with respect to the conic that consists of the self-conjugate points of the polarity.

4.4. Special points. If p is the polar of a point P with respect to a conic, then the following statements define the special terms which they contain:

1) If p intersects the conic in exactly one point, p is tangent to the conic at P.
2) If p intersects the conic in two points, P is an exterior point of the conic.
3) If p does not intersect the conic, P is an interior point of the conic.
4) If P is an ideal point, p is a diameter of the conic.
5) If p is the ideal line, P is the center of the conic.

A conic that has an ordinary point as its center is called a central conic. The hyperbola and ellipse have ordinary points as centers since they are not tangent to the ideal line. The parabola has an ideal point as center.

4.5. Coordinates of the ideal points of a conic. The conic $Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0$, is,

1) a hyperbola if it has two points on the ideal line, (i.e., $B^2 - 4AC > 0$)
2) a parabola if it has one point on the ideal line, (i.e., $B^2 - 4AC = 0$)
3) an ellipse if it has no points on the ideal line, (i.e., $B^2 - 4AC < 0$)

The coordinates of the ideal points of each conic are derived from the possible conics as follows:

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0,$$

$$\frac{x_1}{x_2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \text{ where } x_2 \neq 0,$$

and

$$\frac{x_2}{x_1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}, \text{ where } x_1 \neq 0.$$

Hence for:

Case 1) $B^2 - 4AC > 0$,

The two ideal points on the conic are $(-B \pm \sqrt{B^2 - 4AC}\, k, 2Ak, 0)$ and $(2Ck, -B \pm \sqrt{B^2 - 4AC}\, k, 0)$.

Case 2) $B^2 - 4AC = 0$,

The ideal point on the conic is $(-Bk, 2Ak, 0)$ or $(2Ck, -Bk, 0)$.

Case 3) $B^2 - 4AC < 0$,

The ideal points on the conic $(-B \pm \sqrt{B^2 - 4AC}\, k, 2Ak, 0)$ and $(2Ck, -B \pm \sqrt{B^2 - 4AC}\, k, 0)$ have imaginary coordinates and therefore no real ideal points exist for this case.

4.6. <u>The center of a conic.</u> The center of a conic is the intersection of the transforms of two ideal points. Taking the polarity of the general conic

$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_3 + Ex_2x_3 + Fx_3^2 = 0$, to be

$$(M) = \begin{pmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{pmatrix}$$

and applying this transformation to two ideal points, (1,0,0) and (0,1,0), we have as follows:

$$\left((M) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)^t = \ulcorner 2A, B, D \urcorner \text{ and } \left((M) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)^t = \ulcorner B_1, 2C_1, E \urcorner$$

The center of the conic then would be the intersection of $\ulcorner 2A, B, D \urcorner$ and $\ulcorner B, 2C, E \urcorner$, which is

$$\begin{pmatrix} 2CD - BE \\ 2AE - BD \\ B^2 - 4AC \end{pmatrix}$$

and if the center is ordinary would be $\left(\frac{2CD - BE}{B^2 - 4AC}, \frac{2AE - BD}{B^2 - 4AC}, 1 \right)^t$. The center of a conic is the origin (0,0,1) if and only if $D = E = 0$.

4.7. <u>The absolute conic.</u> Select a polarity $f = (A_{ij})$, $i, j = 1, 2, 3$, such that $f(1,0,0) = \ulcorner 1,0,0 \urcorner$ and $f(0,1,0) = \ulcorner 0,1,0 \urcorner$ which would imply $A_{12} = A_{13} = A_{23} = 0$ and $A_{11} A_{22} A_{33} \neq 0$ and hence the polarity f may be written $\ulcorner A_{11}x_1, A_{22}x_2, A_{33}x_3 \urcorner$ and without loss of generality it can be assumed A_{11} and A_{22} 0. Therefore the polarity can be represented as $\ulcorner A^2x_1, B^2x_2, C^2ex_3 \urcorner$.

$(A^2B^2C^2 \neq 0, e^2 = 1)$. Then in order for a point to be on its own transform under this polarity, the condition $A^2x_1^2 + B^2x_2^2 + C^2ex_3^2 = 0$, which can be written $x_1^2 + x_2^2 + ex_3^2 = 0$, $e^2 = 1$, without loss of generality, must exist.[10] By a change of coordinates the polarity $\lceil A^2x_1, B^2x_2, C^2ex_3 \rceil$ may be written $\lceil x_1, x_2, ex_3 \rceil$, $(e^2 = 1)$. This polarity will be called the absolute polarity. The set of points satisfying the condition $x_1^2 + x_2^2 + ex_3^2 = 0$, under the polarity $\lceil x_1, x_2, ex_3 \rceil$, $(e^2 = 1)$ will be called the absolute conic or is sometimes called the ideal conic.

4.8. Summary. In this chapter the place of a conic in plane geometry and the idea of a conic in projective geometry is presented. The conditions placed on a transformation by a conic are developed and the conic is classified according to its ideal points. Other special points are defined and demonstrated. A particular conic is selected and called the ideal conic.

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[10] Meserve, Op. cit., p. 270.

CHAPTER V

A COMPARISON OF SEVERAL GEOMETRIES

5.1. Introduction. In this chapter several geometries will be defined and the conditions placed on the matrices of their transformations will be derived. A comparison of the conditions on the matrices will be made along with the invariant properties of these geometries.

5.2. Definition of geometries. Each geometry in this study is a subgeometry of projective geometry and will be defined by its invariant properties under projective transformations.

Projective geometry. The projective plane has been defined and the points and lines identified. Points on the ideal line [0, 0, 1] are called ideal points, all others are called ordinary points. In the projective geometry; two distinct points determine a unique line, every line contains at least two points and two distinct lines determine a unique point. The group of transformations of projective geometry is precisely the set of nonsingular transformations. Each of the other geometries discussed in this study is a subgeometry of projective geometry, and includes the properties of projective geometry along with its own properties.

Parabolic geometry. The group of transformations of parabolic geometry leave pairs of points with respect to the absolute involution invariant. Ideal points of parabolic geometry are the ideal points of the ideal line [0, 0, 1], all other points are ordinary.

Euclidean geometry. Euclidean geometry is a subgeometry of parabolic geometry. The group of transformations of Euclidean geometry leave the paris of points with respect to the absolute involution invariant and the absolute value of the determinant of the transformation matrix is equal to one. Euclidean geometry is defined on the projective plane with the ideal line removed. Hence there are no ideal points, all points and lines are ordinary.

Hyperbolic geometry. In the group of transformations of hyperbolic geometry the absolute conic, ($x_1^2 + x_2^2 + ex_3^2 = 0, e = -1$) is invariant. Real points of the absolute conic, ($x_1^2 + x_2^2 + ex_3^2 = 0, e = -1$), are the ideal points of hyperbolic geometry. Points inside the conic (i.e., $x_1^2 + x_2^2 + ex_3^2 < 0, e = -1$) are called ordinary points.[11] Points outside the conic (i.e., $x_1^2 + x_2^2 + ex_3^2 > 0, e = -1$) are called ultra-ideal points.[12]

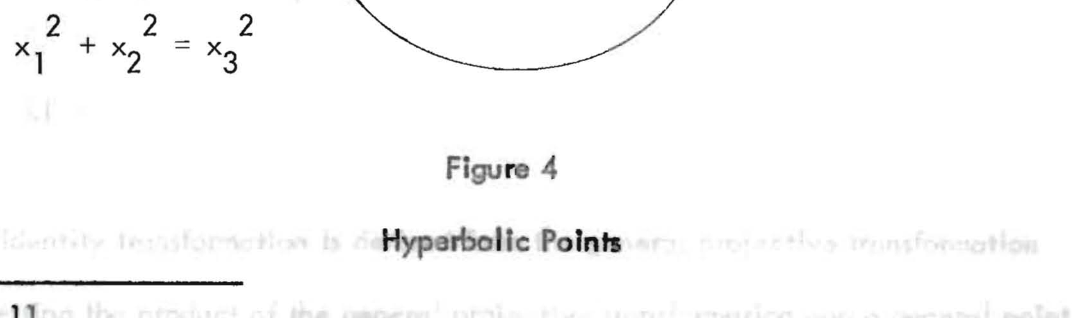

Figure 4

Hyperbolic Points

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11 Meserve, Op. Cit., p. 270.

12 Ibid.

Elliptic geometry. The group of transformations of elliptic geometry leave the absolute conic ($x_1^2 + x_2^2 + ex_3^2 = 0, e = 1$) invariant. The ideal points of elliptic geometry are the real points on the absolute conic ($x_1^2 + x_2^2 + ex_3^2 = 0$, $e = 1$). Ordinary points are points in which ($x_1^2 + x_2^2 + ex_3^2 > 0, e = 1$). Hence in elliptic geometry all real points are ordinary.

5.3. Derivation of the transformations. The projective transformation is restricted or specialized by placing conditions on the elements of the matrix of the projective transformation in order to obtain special transformationsor transformationsof less general geometries. Properties of the special geometries which remain invariant under the transformations of that geometry determine the conditions to be placed on the matrix of the projective transformation.

Projective transformation. A projective transformation is represented by a 3 X 3 matrix if and only if the matrix is nonsingular.

$$(A_{ij}),\ i, j = 1, 2, 3, |A_{ij}| \neq 0.$$

Identity transformation. The identity projective transformation leaves all points fixed and is represented by the matrix

$$kI = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

The identity transformation is derived from the general projective transformation by setting the product of the general projective transformation and a general point equal to a multiple of the general point and solving the resulting equations for the elements of the transformation matrix. The product of the general projective transformation and a general point is represented as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} kx_1 \\ kx_2 \\ kx_3 \end{pmatrix} \quad \text{where } (x_i) = (kx_i),$$

and $k \neq 0$.

The corresponding system of equations in homogeneous coordinates is:

$(a_{11} - k)x_1 + a_{12}x_2 + a_{13}x_3 = 0,$

$a_{21}x_1 + (a_{22} - k)x_2 + a_{23}x_3 = 0,$

$a_{31}x_1 + a_{32}x_2 + (a_{33} - k)x_3 = 0,$ which if true for all x_i, then

$a_{11} = k,$

$a_{22} = k,$

$a_{33} = k,$ and

$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.$

Therefore the resulting matrix may be written:

$$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = kI.$$

Inverse transformation. The inverse of a transformation A is the transformation A^{-1} if and only if $AA^{-1} = I = A^{-1}A$. No transformation has more than one inverse. If A inverse is A^{-1} and if B is another transformation such that $AB = I$, then $B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}I = A^{-1}$.

<u>The absolute involution.</u> An arbitrary involution on the ideal line, under which points of the ideal line form pairs is selected and called the absolute involution. This particular involution will be denoted I^{∞} and defined on the homogeneous coordinates of an ideal point as follows:

$$I\infty \begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} = \begin{pmatrix} kX_2 \\ -kX_1 \\ 0 \end{pmatrix}.$$

The matrix representation of the transformation for this involution is derived from the general projective transformation by solving the system of equations resulting from placing the invariant properties of the absolute involution on the product of the general projective transformation and a general ideal point as follows:

$$(A_{ij}) \begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} = \begin{pmatrix} kx_2 \\ -kx_1 \\ 0 \end{pmatrix}, \text{ where } i, j = 1, 2, 3, x_i = kx_i \text{ and } k \neq 0.$$

The corresponding system of equation is:

$$a_{11}x_1 + (a_{12} - k)x_2 = 0,$$
$$(a_{21} + k)x_1 + a_{22}x_2 = 0,$$
$$a_{31}x_1 + a_{32}x_2 = 0, \text{ for all } x_i.$$

Therefore, $a_{12} = k$,

$a_{21} = -k$,

$a_{31} = a_{32} = 0$, and the involution is represented by the matrix

$$\begin{pmatrix} 0 & k & 0 \\ -k & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Where by multiplying by $1/a_{33}$, a_{33} can be made equal to 1. The points

$$\begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} kX_2 \\ -kX_1 \\ 0 \end{pmatrix}$$

are a pair under the absolute involution.

<u>Parabolic transformation</u>. The parabolic transformation is derived from the general projective transformation (A_{ij}), $ij = 1, 2, 3$, $|A_{ij}| \neq 0$, by restricting the matrix so that the points that correspond to each other with respect to the absolute involution will be corresponding points with respect to the transformation. Since the absolute involution preserves pairs of ideal points the matrix must have the condition that whenever X is an ideal point f(X) is also an ideal point.

$$(A_{ij}) \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{pmatrix}$$

Therefore $a_{31}x_1 + a_{32}x_2 = 0$, for all x_1, x_2, hence $a_{31} = a_{32} = 0$. The resulting transformation matrix would be of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

where a_{23} can be made equal to 1 by multiplying by $1/a_{33}$.

If the points

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

are a pair under the absolute involution then under the parabolic transformation

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11}x_2 - a_{12}x_1 \\ a_{21}x_2 - a_{22}x_1 \\ 0 \end{pmatrix}$$

are a pair under the absolute involution.

These points form a pair of the absolute involution if and only if there exists a number $k \neq 0$ such that for all x_1 and x_2:

$$a_{11}x_1 + a_{12}x_2 = k(a_{21}x_2 - a_{22}x_1)$$

$$a_{21}x_1 + a_{22}x_2 = -k(a_{11}x_2 - a_{12}x_1)$$

that is

$$(a_{11} + ka_{22})x_1 + (a_{12} - ka_{21})x_2 = 0$$

$$(a_{21} - ka_{12})x_1 + (a_{22} + ka_{11})x_2 = 0$$

Which if true for all x_1 and x_2 then

$a_{11} + ka_{22} = 0$ and $a_{22} + ka_{11} = 0$, which implies

$a_{11}^2 = a_{22}^2$ and $k^2 = 1$, similarly

$a_{12}^2 = a_{21}^2$ and $a_{11}a_{12} + a_{21}a_{22} = 0.$

The square of an involution is the identity, therefore:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} & a_{11}a_{13} + a_{12}a_{23} + a_{13} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 & a_{21}a_{13} + a_{22}a_{23} + a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence,

$$a_{11}^2 + a_{12}a_{21} = 1,$$

$$a_{21}a_{12} + a_{22}^2 = 1,$$

$$a_{21}a_{11} + a_{22}a_{21} = 0,$$

and $a_{11}a_{12} + a_{12}a_{22} = 0.$

If $a_{11}a_{12}a_{21}a_{22} \neq 0,$

Then, $a_{11}^2 + a_{12}a_{21} = a_{21}a_{12} + a_{22}^2$, and $a_{11}^2 = a_{22}^2$.

Also, $a_{21}a_{11} + a_{22}a_{21} = a_{11}a_{12} + a_{12}a_{22}$,

or $a_{21}(a_{11} + a_{22}) = 0 = a_{12}(a_{11} + a_{22})$,

hence, $a_{11} + a_{22} = 0,$

or $a_{11} = -a_{22},$

and $a_{21} = a_{12}.$

Therefore, the parabolic matrix is:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Euclidean transformation. Euclidean transformations are derived as a special case of parabolic geometry in which the determinant of the transformation matrix,

$|A_{ij}| = \pm 1;\ i, j = 1, 2, 3.$ Hence from the general parabolic transformation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

the Euclidean transformation is derived by the condition: $a_{11}^2 + a_{12}^2 = \pm 1$.

Therefore, the Euclidean matrix is:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad |A_{ij}| = \pm 1.$$

<u>Hyperbolic and elliptic transformations.</u> The general projective transformation $(A_{ij})\ i, j = 1, 2, 3;\ |A_{ij}| \neq 0$ is specialized to obtain the transformations of hyperbolic and elliptic geometries in which the absolute conic is invariant. If the condition $x_1^2 + x_2^2 + ex_3^2 = 0$ is invariant, the geometry is:

1) elliptic if $e = +1$, or

2) hyperbolic if $e = -1$.

Given the general transformation $A = (A_{ij})\ i, j = 1, 2, 3$ and the general point $X = (x_j),\ j = 1, 2, 3;\ AX = \overline{X}$

$$(A_{ij})(x_j) = (\overline{x}_i), \text{ or } \sum_{ij=1}^{3} A_{ij} x_j = \overline{x}_i$$

A must be restricted so that

$$x_1^2 + x_2^2 + ex_3^2 = 0 \Leftrightarrow \overline{x}_1^2 + \overline{x}_2^2 + e\overline{x}_3^2 = 0$$

Now

$$\bar{x}_i^{\,2} = \sum_{i,j=1}^{3} (A_{ij}\, x_j)^2$$

$$= \sum_{i=1}^{3} A_{i1} x_1 A_{i1} x_1 + \sum_{i=1}^{3} A_{i1} x_1 A_{i2} x_2 + \sum_{i=1}^{3} A_{i1} x_1 A_{i3} x_3$$

$$+ \sum_{i=1}^{3} A_{i2} x_2 A_{i1} x_1 + \sum_{i=1}^{3} A_{i2} x_2 A_{i2} x_2 + \sum_{i=1}^{3} A_{i2} x_2 A_{i3} x_3$$

$$+ \sum_{i=1}^{3} A_{i3} x_3 A_{i1} x_1 + \sum_{i=1}^{3} A_{i3} x_3 A_{i2} x_2 + \sum_{i=1}^{3} A_{i3} x_3 A_{i3} x_3 .$$

Then $\bar{x}_1^{\,2} + \bar{x}_2^{\,2} + e\bar{x}_3^{\,2} = 0$, implies

$$\sum_{i=1}^{3} A_{i1}^{\,2} x_1^{\,2} + 2 \sum_{i=1}^{3} A_{i1} A_{i2} x_1 x_2 + 2e \sum_{i=1}^{3} A_{i1} A_{i3} x_1 x_3$$

$$+ \sum_{i=1}^{3} A_{i2}^{\,2} x_2^{\,2} + 2e \sum_{i=1}^{3} A_{i2} A_{i3} x_2 x_3 + 2 \sum_{i=1}^{3} A_{i3}^{\,2} x_3^{\,2} = 0.$$

For the hyperbolic case where $e = -1$, substituting $x_3^{\,2} = x_1^{\,2} + x_2^{\,2}$, the following conditions result

$$x_1^2(a_{13}^2 + a_{23}^2 + a_{11}^2 + a_{21}^2 - a_{33}^2 - a_{31}^2)$$

$$+ x_2(a_{13}^2 + a_{23}^2 + a_{12}^2 + a_{22}^2 - a_{33}^2 - a_{32}^2)$$

$$\pm 2x_1\sqrt{x_1^2 + x_2^2}\ (a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33})$$

$$\pm 2x_2\sqrt{x_1^2 + x_2^2}\ (a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33})$$

$$+ 2x_1x_2(a_{11}a_{12} + a_{21}a_{22} - a_{31}a_{32}) = 0.$$

Therefore:

$$a_{13}^2 + a_{23}^2 + a_{11}^2 + a_{21}^2 - a_{33}^2 - a_{31}^2 = 0$$

$$a_{13}^2 + a_{23}^2 + a_{12}^2 + a_{22}^2 - a_{33}^2 - a_{32}^2 = 0$$

$$a_{13}a_{11} + a_{23}a_{21} - a_{33}a_{31} = 0$$

$$a_{13}a_{12} + a_{23}a_{22} - a_{33}a_{32} = 0$$

$$a_{11}a_{12} + a_{21}a_{22} - a_{31}a_{32} = 0$$

and hence:

$$(a_{33} + a_{31})^2 = (a_{11} + a_{13})^2 + (a_{21} + a_{23})^2$$

$$(a_{32} + a_{33})^2 = (a_{12} + a_{13})^2 + (a_{22} + a_{23})^2$$

$$(a_{31} + a_{32})^2 = (a_{11} + a_{12})^2 + (a_{21} + a_{22})^2$$

For the elliptic case where $e = 1$, to have $x_1^2 + x_2^2 + ex_3^2 = 0$ at least one coordinate of the point must be complex. With suitable change of coordinates it can be made to be x_3. Then $x_3^2 < 0$ and $-x_3^2 > 0$.

Substituting $x_3^2 = -(x_1^2 + x_2^2)$ the following conditions result:

$$x_1^2(a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{13}^2 - a_{23}^2 - a_{33}^2)$$

$$+ x_2^2(a_{12}^2 + a_{22}^2 + a_{32}^2 - a_{13}^2 - a_{23}^2 - a_{33}^2)$$

$$\pm 2ix_{11}\sqrt{x_{11}^2 + x_{21}^2}\,(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})$$

$$\pm 2ix_2\sqrt{x_{11}^2 + x_{21}^2}\,(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})$$

$$+ 2x_1x_2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}) = 0.$$

Therefore:

$$a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{13}^2 - a_{23}^2 - a_{33}^2 = 0$$

$$a_{12}^2 + a_{22}^2 + a_{32}^2 - a_{13}^2 - a_{23}^2 - a_{33}^2 = 0$$

$$a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0$$

$$a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$$

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0$$

and hence:

$$(a_{11} - ia_{13})^2 + (a_{21} - ia_{23})^2 = (a_{33} + ia_{31})^2$$

$$(a_{12} - ia_{13})^2 + (a_{22} - ia_{23})^2 = (a_{33} + ia_{32})^2$$

$$(a_{11} - ia_{12})^2 + (a_{21} - ia_{22})^2 = (a_{32} + ia_{31})^2$$

Now if $a_{13} = a_{23} = a_{32} = a_{31} = 0$, the same transformation will apply to both hyperbolic and elliptic geometry. A transformation satisfying the condition is:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad |A_{ij}| \neq 0.$$

5.4. <u>A comparison</u>. Parabolic, Euclidean, hyperbolic, and elliptic geometries will now be compared with respect to certain figures of the projective plane.

<u>Triangle</u>. A <u>triangle</u> is an ordered set of 3 noncollinear points. The points (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) are noncollinear if and only if:

$$m(xyz) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

<u>Measure</u>. This determinant m (xyz) is defined as the measure of the triangle. The measure is positive or negative depending on the order in which the vertices

are named. The area of a triangle is equal to 1/2 its measure. The areas of other figures shall be determined by dividing the figure into triangles. A transformation preserves measure if and only if the determinant of the transformation $= \pm 1$. A comparison of the transformations of parabolic, Euclidean, hyperbolic and elliptic geometries:

Parabolic	Euclidean	Hyperbolic and Elliptic
$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\lvert A_{ij}\rvert = a_{11}^2 + a_{12}^2 \neq 0$	$\lvert A_{ij}\rvert = 1$	$\lvert A_{ij}\rvert \neq 0$

shows the determinants of Euclidean transformations to be 1, hence they preserve area and are referred to as rigid motions. The parabolic transformations can increase or decrease area and are called similarities.

Lines. $[0,0,1]$ is the ideal line of parabolic geometry. Euclidean geometry has no ideal line. The ideal conic $(x_1^2 + x_2^2 + ex_3^2 = 0)$ is the ideal line of hyperbolic geometry where $e = -1$ and the ideal line of elliptic when $e = 1$. A point on the ideal line does not separate the ideal line into two segments. A point on an ordinary line in Euclidean or hyperbolic geometry separates the line into two segments, in parabolic or elliptic it does not. A point may or may not separate an ultra-ideal line into two segments.

Parallel. Two lines are said to be parallel if they have an ideal point in common.

Nonintersecting. Two lines are said to be nonintersecting if they do not have an ordinary point or an ideal point in common, or if they have an ultra-ideal point in common.

Intersecting. Two lines are intersecting if they have an ordinary point in common. Lines in parabolic geometry are either intersecting or parallel. In Euclidean geometry lines are either intersecting or nonintersecting. Euclidean nonintersecting lines are called parallel. In hyperbolic geometry lines can be intersecting, nonintersecting or parallel. In elliptic geometry all lines are intersecting. In parabolic geometry there is only one ideal point on each line therefore through a point not on a line there can be one and only one line parallel to an ordinary given line. This is also true in Euclidean geometry, however in hyperbolic geometry with two ideal points on every ordinary line there are exactly two lines through a point not on that line parallel to the given line. A line intersecting one of two parallel lines in parabolic geometry or Euclidean geometry must intersect the other. In hyperbolic geometry it may or may not intersect the other.

Perpendicular lines. Two ordinary lines l_1 and l_2 are perpendicular if and only if the ideal point of l_1 and the ideal point of l_2 form a pair under the absolute involution.

If l_1 is an ordinary line, p is an ordinary point. There is one and only one line l_2 through p perpendicular to l_1. Hence in parabolic geometry and Euclidean geometry two lines perpendicular to the same line are parallel, in hyperbolic geometry they are nonintersecting and in elliptic geometry they are intersecting. All lines perpendicular to a given line in hyperbolic geometry have an ultra-ideal point in common.

Points. In parabolic geometry points are either ordinary or ideal. In Euclidean geometry and elliptic geometry all points are ordinary. In hyperbolic geometry points are ordinary, ideal or ultra-ideal.

Ideal line of the projective plane. An inspection of the transformations of parabolic, Euclidean, hyperbolic, and elliptic geometries shows $a_{31} = a_{32} = 0$, $a_{33} = 1$, and therefore all four leave the ideal line of the projective plane invariant.

The absolute involution. The four transformations also leave the points that correspond to each other with respect to the absolute involution invariant.

5.5. Summary. In this chapter four subgeometries of projective geometry; parabolic geometry, Euclidean geometry, hyperbolic geometry, and elliptic geometry are defined and the conditions on their transformations are derived. The geometries are then compared according to ideal and ordinary points, ideal and ordinary lines, parallel lines, perpendicular lines, intersecting and nonintersecting lines and invariance of area and invariance of the line $[0,0,1]$, and the absolute involution.

CHAPTER VI

CONCLUSION

6.1. Introduction. This thesis is a study of parabolic geometry, Euclidean geometry, hyperbolic geometry, and elliptic geometry as subgeometries of projective geometry. The geometries are identified by their algebraic properties and compared on the projective plane.

In Chapter I the problem is outlined and terms defined. Chapter II presents a brief history of the development of geometry. The history is divided into four general periods. The first period includes the time from nearly 4000 B.C. to 600 B.C. and shows man's early use of geometry in measuring. The second is one in which geometry is made into a rigorous deductive science; rules are set up and geometry is placed on a sound logical basis. The third period from about the fourth century A.D. to the nineteenth century is characterized by attempts to prove Euclid's fifth postulate and culminating in the discovery of non-Euclidean geometry. During this period analytic geometry is also discovered. These discoveries lead to a renewed interest in geometry and in recent years geometry has become organized and classified under the more general geometry, projective geometry. In Chapter III the projective plane is coordinatized and the points and lines of the plane are identified. Chapter IV is a description of conics. The general conic is derived, and special points are defined. In Chapter V the geometries are defined and their transformations derived. The geometries are then compared and their similarities and differences noted.

6.2. Results. The transformations of parabolic geometry, Euclidean geometry, hyperbolic geometry, and elliptic geometry all leave the ideal line [0,0,1] of the projective plane invariant, which is not invariant in the more general projective geometry. By specializing the transformations so that pairs of points that correspond to each other with respect to the absolute involution will be corresponding points with respect to the transformations in parabolic geometry, Euclidean geometry, hyperbolic geometry and elliptic geometry, all four geometries can be represented by the same general transformation. In this case the differences in the geometries depend on the selection of the ideal line for each geometry and the statements involving ideal points.

6.3. Suggestions for further study. In this study the projective plane is defined for real points. Since the ideal conic of elliptic geometry involves imaginary numbers, further research into the idea of complex coordinates on the projective plane and transformations involving complex points is indicated. This thesis compares the general transformations of four particular subgeometries of projective geometry. There are several other subgroups of projective transformations which could be given closer investigation. A metric comparison of these geometries could also be made by defining the distance concept.

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