# NEGATIVE BASE NUMERATION SYSTEMS

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CHAPTER I

INTRODUCTION

Numeration systems with bases other than ten are discussed quite frequently in mathematics books and periodicals. These discussions are usually limited to positive bases and seem to be written for the purpose of increasing the reader's understanding of the base ten numeration system or digital computers. Two articles in the February, 1963, issue of The Mathematics Teacher discussed negative base numeration systems. These articles made this writer wonder why negative bases are not discussed more often. The purpose of this paper is to investigate negative base numeration systems and if possible show that they can also be used.

In this paper "-B" will be used to refer to a negative base in general and "B" will refer to a positive base in general. When a numeral or expression is written in a base other than ten this is indicated in parentheses to the lower right of the numeral or expression. The letters "T" and "E" are used as the numerals for ten and eleven, respectively, in the base negative twelve. Examples in this paper are in base negative seven, negative ten, and negative twelve. The terms radix point and radiximal are used in other bases like decimal point and decimal are used in base ten. The terms base and radix are used synonymously.

The first item discussed in this paper is change of base. Several methods are discussed for changing positive base numerals to negative base numerals and conversely. This is followed by a discussion of the fundamental operations in negative bases and a method of checking the results. The paper is concluded with a comparison of positive and negative base numeration systems.

CHAPTER II

CHANGE OF BASE IN NUMERATION SYSTEMS

Three methods are discussed in this chapter for changing numerals from a positive base numeral to a negative base numeral and conversely. They are: a detailed analysis of the place value of the various digits in a numeral, repeated division, and polynomial expansion. The order relation of negative base numerals is also discussed. The chapter is concluded with a brief discussion of fractions and radiximals in negative bases.

Before discussing change of base in numeration systems a few elementary facts about numeration systems and signed numbers are reviewed. These facts are used in this chapter without being reviewed in detail each time.

The place value of the first digit to the left of the radix point is the base to the zero power or units. The place value of the second digit to the left of the radix point is the base to the first power. In general, the kth digit to the left of the radix point has the place value of the base to the k-1 power. The value of the numeral $A_kA_{k-1}\cdots A_1A_0$ is $A_kB^k + A_{k-1}B^{k-1} + \ldots + A_0B^0$.

When the base is a positive number, the number represented by any digit is larger than the number represented by all the digits to the right of this digit. For example, in the numeral 2453, 2000 is larger than 453, and 400 is larger than 53. This is true with any positive base.

A negative number used as a factor an even number of times is positive; a negative number used as a factor an odd number of times is negative. When a negative number is used for a base, the odd and even numbered digits to the left of the radix point represent positive and negative numbers, respectively. However, the absolute value of the number represented by a digit is larger than the number represented by the digits that follow it. For example, $2000_{(-10)} < 453_{(-10)}$ but $|2000_{(-10)}| > 453_{(-10)}$. The number of digits in a negative base numeral determines whether the number is positive or negative. The negative base numeral is positive or negative if the number of digits is odd or even, respectively. The minus sign is not needed to represent negative numbers.

The largest number that can be expressed with a certain number of placeholders in base -B is the numeral with B-1 as the odd numbered placeholders and 0 as the even numbered placeholders to the left of the radix point. The largest number that can be expressed with five placeholders in base negative ten is $90909_{(-10)}$. The smallest number that can be expressed with a certain number of placeholders in base -B is the numeral with B-1 as the even numbered placeholders and 0 as the odd numbered placeholders.

To express a number in a negative base it is necessary to determine the number of digits needed or the largest exponent to which the base must be raised to express the number. The largest exponent needed to express the positive integer, N, as a numeral in base -B is the number 2n where n is the smallest integer such that $\sum_{x=0}^{n}(B-1)(-B)^{2x} \geq N$. It will be a 2n+1 digit numeral. The largest exponent needed to express the negative integer, -N, as a numeral in base -B is the number 2n+1 where n is the smallest integer such that $\sum_{x=0}^{n}(B-1)(-B)^{2x+1} \leq -N$, and it will be a 2n+2 digit numeral.

Next, each digit must be large enough and yet as small as possible. If a digit that represents a positive value is made larger than necessary, it will be impossible to make the base -B numeral small enough with the following digits that represent negative values. The digit that has the positive place value $(-B)^{2n}$ will be the smallest $C_{2n}$ such that $\cdots+C_{2n+1}(-B)^{2n+1}+C_{2n}(-B)^{2n}+\sum_{x=0}^{n-1}(B-1)(-B)^{2x}$ is greater than or equal to the number that is being changed to base -B. If a > b, the summation from a to b is defined to equal zero.

Likewise, if a digit that represents a negative value is larger than necessary, it will be impossible to make the base -B numeral large enough with the digits that follow and represent positive values. The digit that has the negative place value $(-B)^{2n+1}$ will be the smallest $C_{2n+1}$ such that $\cdots+C_{2n+2}(-B)^{2n+2}+C_{2n+1}(-B)^{2n+1}+\sum_{x=1}^{n}(B-1)(-B)^{2x-1}$ is less than or equal to the number that is being changed to base -B.

To illustrate, it will be shown below how 1964 is written in base negative twelve and how -1492 is written in base negative seven. Appendix I has the powers from zero through ten of the negative bases used as examples in this paper.

Example 1: Change 1964 to a base negative twelve numeral.

The largest exponent needed will be the smallest 2n such that $\sum_{x=0}^{n} 11(-12)^{2x} \geq 1964$. The terms of the summation are 11, 1595, and 229,691 as n has the values 0, 1, and 2, respectively. The largest exponent required to express 1964 in base negative twelve is 4 and it will be a 5 digit numeral.

The first digit has the place value $(-12)^4$ and is the smallest $C_4$ such that $\cdots+0(-B)^5+C_4(-12)^4+\sum_{x=0}^{1} 11(-12)^{2x}$ is greater than or equal to 1964. This simplifies to the inequality $20{,}736\ C_4+1595 \geq 1964$. One satisfies this inequality. The second digit is the smallest $C_3$ such that $1(-12)^4+C_3(-12)^3+\sum_{x=1}^{1} 11(-12)^{2x-1} \leq 1964$, that is,

20,736-1728 $C_3$-132 $\leq$ 1964. The second digit is eleven or E. The smallest $C_2$ for which
$1(-12)^4+11(-12)^3+C_2(-12)^2+\sum_{x=0}^{0} 11(-12)^{2x} \geq 1964$ is two. Therefore, $C_2=2$. The next digit is the smallest $C_1$ for which $1(-12)^4+11(-12)^3+2(-12)^2+C_1(-12) \leq 1964$. Five is the value of $C_1$. The final digit is the $C_0$ for which
$1(-12)^4+11(-12)^3+2(-12)^2+5(-12)+C_0=1964$. This digit is 8. Therefore, $1964=1E258_{(-12)}$.

Example 2: Change -1492 to a base negative seven numeral.

The largest exponent needed will be the smallest 2n+1 such that $\sum_{x=0}^{n} 6(-7)^{2x+1} \leq -1492$. As n has the values 0 and 1, the terms of the summation are -42 and -2100, respectively. Thus, the largest exponent needed is 3 and -1492 will be a 4 digit numeral in base negative seven.

The first digit is the smallest $C_3$ such that $C_3(-7)^3+\sum_{x=1}^{1} 6(-7)^{2x-1} \leq -1492$. The value of $C_3$ that satisfies this condition is 5. The second digit is the smallest $C_2$ such that $5(-7)^3+C_2(-7)^2+\sum_{x=0}^{0} 6(-7)^{2x} \geq -1492$, that is, $-1715+49\ C_2+6 \geq -1492$. This simplifies to $49\ C_2 \geq 217$ and $C_2$ is 5. The third digit is the smallest $C_1$ such that $5(-7)^3+5(-7)^2+C_1(-7) \leq -1492$, that is, $-7\ C_1 \leq -22$. The smallest $C_1$ that satisfies this condition is 4. The last digit is $C_0$ such that

$5(-7)^3+5(-7)^2+4(-7)+c_0=-1492$. This digit is 6 and $-1492=5546_{(-7)}$.

Repeated division is often used to change from one positive base to another positive base. Does this method also work to change a numeral from a positive base to a negative base and conversely? Yes, although this method may be more difficult to understand than the method of polynomial expansion which is the topic to be discussed after repeated division.

To change the base B' numeral to a base B" numeral begin by dividing the B' numeral by B". The division is performed in base B'. The quotient is made large enough (or small enough if it is negative) so that the remainder is a digit used in the base B" numeration system. The quotient is the number of groups of $(B'')^1$ and the remainder is the number of $(B'')^0$ left over. This remainder is the first digit to the left of the radix point in base B". Next, this first quotient is divided by B" with the remainder again being a digit in the base B" numeration system. The second quotient is the number of groups of $(B'')^2$ and the remainder is the number of $(B'')^1$ left over. This remainder is the second digit to the left of the radix point. In general, the quotient of the kth division is the number of groups of $(B'')^k$ and the kth remainder is the number of $(B'')^{k-1}$ left over. The kth remainder is the kth digit to the left of the radix point. This process is repeated until the quotient is also a digit in the base $B^n$ numeration system. The numeral in base $B^n$ is the last quotient followed by the remainders in reverse order. The following example uses repeated division to change a positive base numeral to a negative base numeral.

Example 3: Change 1964 to a base negative seven numeral.

-7 |1964

-7 |-280 groups of $(-7)^1$ and 4 $(-7)^0$ left over

-7 |40 groups of $(-7)^2$ and 0 $(-7)^1$ left over

-7 |-5 groups of $(-7)^3$ and 5 $(-7)^2$ left over

1 group of $(-7)^4$ and 2 $(-7)^3$ left over

The base negative seven numeral for 1964 is $12504_{(-7)}$.

In a place value numeration system the base B numeral $A_n A_{n-1} A_{n-2} \cdots A_1 A_0$ can be written as a polynomial $A_n B^n + A_{n-1} B^{n-1} + A_{n-2} B^{n-2} + \cdots + A_1 B + A_0$. The method of polynomial expansion is perhaps the easiest method to use to change a numeral from another base to base ten. This method is used in the next example.

Example 4: Change $2176_{(-10)}$ to a base ten numeral.

The polynomial expansion of $2176_{(-10)}$ is $\left[2(10)^3+1(10)^2+7(10)^1+6(10)^0\right]_{(-10)}$. Rewriting the polynomial in base ten numerals and performing the operations in base ten gives

$2(-10)^3+1(-10)^2+7(-10)^1+6(-10)^0=-2000+100-70+6=1964$.

Thus, $2176_{(-10)}=1964$.

Why is it so easy to change a numeral from another base to base ten by expanding a polynomial? Is it possible to use this method to change from base ten to another base? The answer to the first question is that we know how to perform operations in base ten. This method can also be used to change from base ten to another base if one is able to perform operations in the base to which one is changing.

There are three things necessary to change a numeral from base B' to base B". First the base B' numeral must be written as a polynomial in base B'. Then the numerals must be converted to base B" notation. To complete the change, the operations are performed in base B".

The next example shows how to change a numeral from one negative base to another negative base. It will be necessary to change the negative base numeral to a base ten numeral and the base ten numeral to the other negative base numeral.

Example 5: Change $1302651_{(-7)}$ to a base negative twelve numeral.

The polynomial expansion of $1302651_{(-7)}$ is

$$\left[1(10)^6+3(10)^5+0(10)^4+2(10)^3+6(10)^2+5(10)^1+1(10)^0\right]_{(-7)}.$$

Changing the numerals to base ten numerals and performing the operations gives $1(-7)^6+3(-7)^5+0(-7)^4+2(-7)^3+6(-7)^2+5(-7)^1$ $+1(-7)^0=117,649-50,421+0-686+294-35+1=66,802$.

Changing 66,802 to a base negative twelve numeral by repeated division gives:

-12 )66802
-12 )-5566 R 10
-12 )464 R 2
-12 )-38 R 8
4 R 10

The base negative twelve numeral for 66,802 and for $1302651_{(-7)}$ is $4T82T_{(-12)}$.

Which of two numerals in a negative base represents the larger number? In a positive base the more digits the numeral has and the larger the digits are the larger the number it represents. This is not the case for numerals using a negative base.

Numerals in a negative base numeration system with an odd or even number of digits represent positive or negative numbers, respectively. For example, the numeral $624_{(-B)}$ is positive and represents a larger number than $546213_{(-B)}$ which is negative. To compare the size of two numbers, first count the number of digits. A numeral with an odd number of digits represents a larger number than a numeral with an even number of digits.

If both numerals have an odd number of digits, the one with the more digits represents the larger number. If both numerals have an even number of digits the one with the fewer digits represents the larger number. For example, $41352_{(-B)} > 624_{(-B)}$ while $25_{(-B)} > 546213_{(-B)}$.

In case both numerals have the same number of digits, compare the digits one by one starting at the left until the two numerals differ in a digit. If these digits represent positive numbers the numeral with the larger digit represents the larger number. If these digits represent negative numbers the numeral with the smaller digit represents the larger number. The order relation of the four numerals $546213_{(-B)}$, $542631_{(-B)}$, $546313_{(-B)}$, and $543162_{(-B)}$ is $542631_{(-B)} > 543162_{(-B)} > 546313_{(-B)} > 546213_{(-B)}$.

In base ten the fraction negative one-half may be written as -(1/2), -1/2, or 1/-2. To change a base ten fraction to a negative base, change the base ten numerator and denominator to the negative base numerator and denominator. In a negative base numeration system the negative fractions look quite different depending on whether the numerator or denominator is negative. For example, negative one-half is written either as $(1/15)_{(-7)}$ or $(16/2)_{(-7)}$ in base negative seven. Since the minus sign is not needed to express negative numbers, negative one-half is not written as -(1/2) in a negative base.

The methods mentioned previously for changing base ten numerals to negative base numerals are not all applicable when the numerals contain decimals. Repeated division cannot be used because the remainder is not a digit from the negative base numeration system. Polynomial expansion is difficult because of the many divisions that need to be performed if there are many digits to the right of the decimal. For every digit to the right of the decimal the exponent is a negative integer which means that there will be a division to perform in the polynomial expansion. As long as the number is an integer, there are only non-negative exponents in the polynomial expansion and hence only multiplications to perform. The change can also be made by a detailed analysis of the place value of the various digits in the numeral.

An alternative method would be to express the decimal as a fraction (if it is rational), express the fraction in the negative base, and change the fraction to a radiximal by division in the negative base. The advantage this has over the polynomial expansion is the single division in the negative base.

In base ten a numeral represents a rational number if and only if it is a repeating infinite decimal. Is this true for other bases, particularly for a negative base? The proof for base ten can be modified a little so that it is applicable in any base. The modified proof of this theorem follows.

Theorem: A numeral in any base, B, represents a rational number if and only if it is a repeating infinite radiximal.

Proof: If $|a/b| \geq 1$ then $a/b=Y+p/q$ where Y is an integer and $p/q < 1$. It is sufficient to consider the rational number $x=p/q$ between 0 and 1 where $(p,q)=1$. Let $B=b_1^{\alpha}\cdot b_2^{\beta}\cdots$ be the prime factorization of the base B.

If all the factors of q are also factors of B then $q=b_1^{\alpha_1}\cdot b_2^{\beta_1}\cdots$. Let n be the maximum of all the exponents, $\alpha_1, \beta_1, \cdots$. Then $B^n x$ is an integer and $x=\cdot a_1 a_2 a_3 \cdots a_n$ with $a_k=0$ if $k > n$.

If $(q,B)=1$ then there exists a v such that $B^v \equiv 1 \pmod{q}$.[1] There also exists an integer m such that $B^v=qm+1$. Then $B^v x=\frac{B^v p}{q}=\frac{(qm+1)p}{q}=mp+\frac{p}{q}=mp+x$ and $x=\frac{mp}{B^v}+\frac{x}{B^v}$. Let u be the number of digits in the product mp. The first v-u digits of the radiximal for p/q are zero and the next u digits are the digits in the product mp. Since this process of constructing the radiximal repeats itself indefinitely, these first v digits also repeat indefinitely.

If some factors of q are also factors of B then $q=b_1^{\alpha_1}\cdot b_2^{\beta_1}\cdots R$ where $(R,B)=1$. There exists a w such that

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[1] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers (London, Great Britain: Oxford University Press, 1938), p. 71.

$B^w \equiv 1 \pmod{R}$ and an integer s such that $B^w = sR+1$. Let n again be the maximum of the exponents $\alpha_1$, $\beta_1$, .... Then $B^n x = \frac{p'}{R} = X + \frac{P}{R}$ where p', X, and P are integers such that $0 < X < B^n$, $0 < P < R$, and $(P,R)=1$. Let $A_1 A_2 \cdots A_t$ be the digits in the integer X. From the previous paragraph P/R is a repeating infinite radiximal. Then $B^n x = A_1 A_2 \cdots A_t . \overline{a_1 a_2 \cdots a_w}$ and $x = .c_1 c_2 \cdots c_n \overline{a_1 a_2 \cdots a_w}$ where the first n-t digits are zero.[2]

Let $C = A_0 . A_1 \cdots A_k D_1 \cdots D_p$ be a numeral in base B where the A's and D's represent digits in the radiximal expansion of C. Multiplying by $B^k$ gives $B^k C = A_0 A_1 \cdots A_k . \overline{D_1 \cdots D_p}$ and the repeating part begins immediately after the radix point. Next, multiplying by $B^p$ gives $B^{k+p} C = A_0 A_1 \cdots A_k D_1 \cdots D_p . \overline{D_1 \cdots D_p}$. Subtracting gives $B^{k+p} C - B^k C = A_0 A_1 \cdots A_k D_1 \cdots D_p - A_0 A_1 \cdots A_k$. Dividing by $B^{k+p} - B^k$ gives the rational expression for C which is

$$\frac{A_0 A_1 \cdots A_k D_1 \cdots D_p - A_0 A_1 \cdots A_k}{B^{k+p} - B^k}.$$[3]

In base ten irrational numbers can only be approximated by rational numbers or decimals. From the previous theorem

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[2] Ibid., pp. 110-111.

[3] C. B. Allendoerfer and C. O. Oakley, Fundamentals of Freshman Mathematics (New York: McGraw-Hill Book Company, 1959), p. 40.

it follows that the same is true in negative base numerals. Special symbols such as the radical sign and special numerals such as $\pi$ and e also need to be used to represent irrational numbers in negative bases.

# CHAPTER III

## OPERATIONS IN NEGATIVE BASE NUMERATION SYSTEMS

The fundamental operations are discussed in this chapter. As the operations are performed it is desirable to check and verify the results. One method of checking is to change each numeral to base ten and perform the operation in base ten. It is more convenient to check the results without changing to another base and performing the operations again.

In base ten a common method of checking is casting out nines. This method of checking works because the number and the sum of the digits have the same remainder when divided by nine. In other positive base numeration systems the method of checking is casting out base-minus-one. Casting out base-minus-one gives the remainder when the sum of the digits is divided by base-minus-one. This remainder is the same as the remainder when the number itself is divided by base-minus-one. Another way of stating this is that the sum of the digits of the numeral subtracted from the number is congruent to zero modulo base-minus-one.

Is $346_{(-7)} - [3_{(-7)} + 4_{(-7)} + 6_{(-7)}] \equiv 0 \pmod{-8}$?

The method of polynomial expansion gives $346_{(-7)} = 3(-7)^2 + 4(-7)^1 + 6(-7)^0 = 147 - 28 + 6 = 125$. The sum of the digits

$3_{(-7)}+4_{(-7)}+6_{(-7)}=3+4+6=13$. The difference of 125 and 13 is 112 which is congruent to zero modulo -8. Does this just happen to be true for this particular case or is this true for all cases? The following theorem shows that this is true for all cases in any base, positive or negative.

Theorem: A number, N, and S, the sum of the digits in the numeral that represent the number, are congruent modulo base-minus-one, that is, $N-S\equiv 0 \pmod{B-1}$.

Proof: Let N be a number in base B and let S be the sum of the digits in the base B numeral. Then

$N=a_nB^n+a_{n-1}B^{n-1}+\cdots+a_1B^1+a_0B^0$ and $S=a_n+a_{n-1}+\cdots+a_1+a_0$.

Subtracting S from N, regrouping terms, and factoring gives:

$$\begin{aligned} N-S &= a_nB^n+a_{n-1}B^{n-1}+\cdots+a_1B^1+a_0B^0-a_n-a_{n-1}-\cdots-a_1-a_0 \\ &= a_nB^n-a_n+a_{n-1}B^{n-1}-a_{n-1}+\cdots+a_1B^1-a_1+a_0B^0-a_0 \\ &= a_n(B^n-1)+a_{n-1}(B^{n-1}-1)+\cdots+a_1(B-1). \end{aligned}$$

Since $B^x-1$ is divisible by B-1 for any integer $x > 0$, each term of the above expression is divisible by B-1. Thus, N-S is divisible by B-1 and $N-S\equiv 0 \pmod{B-1}$. This completes the proof of the theorem.

If $M\equiv N \pmod{B}$ and $P\equiv Q \pmod{B}$, then $M\pm P\equiv N\pm Q \pmod{B}$ and $M\cdot P\equiv N\cdot Q \pmod{B}$. This is the basis for checking addition, subtraction, and multiplication by casting out nines or base-minus-one if a base other than ten is used. Checking a result by casting out base-minus-one is not proof that the answer is right. It is merely a strong indication that the answer is right. The following example shows this.

Example 6: Multiply 57 by 23 without placing the second partial product correctly as follows:

```
  5 7
  2 3
-----
1 7 1
1 1 4
-----
2 8 5
```

Checking the result gives $5+7=12\equiv3 \pmod 9$; $2+3=5\equiv5 \pmod 9$; $2+8+5=15\equiv6 \pmod 9$. Since $3\cdot5=15\equiv6 \pmod 9$ the result checks but it certainly is not correct!

Operations can be performed in base -7, -10, or -12 by referring to the addition and multiplication tables found in the appendix.

Example 7: Consider the problem of adding $346_{(-7)}$ and $213_{(-7)}$. The work is as follows:

```
.
  .
    .
    1 6
      1 6
        1 6
          1 6
            1 6
            3 4 6(-7)
            2 1 3(-7)
--------------------
 . . . 0 0 5 4 2(-7)
```

From here on 16 is carried and 0 is the next digit in the sum in each step. Checking the result gives $3+4+6=13\equiv5 \pmod{-8}$; $2+1+3=6\equiv6 \pmod{-8}$; $5+4+2=11\equiv3 \pmod{-8}$. The result checks because $5+6=11\equiv 3$ (mod $-8$).

Example 8: Next consider the problem of adding $57T9_{(-12)}$ and $3E58_{(-12)}$. The work is as follows:

```
   .
     .
       .
       1 E
         1 E
           1 E
             1 T
               1 T
                 1 E
                 5 7 T 9(-12)
                 3 E 5 8(-12)
   ------------------------
   . . . 0 0 7 5 2 5(-12)
```

Again the carrying process continues indefinitely. Checking this result gives $5+7+10+9=31\equiv 5$ (mod $-13$); $3+11+5+8=27\equiv 1$ (mod $-13$); $7+5+2+5=19\equiv 6$ (mod $-13$). This checks because $5+1=6\equiv 6$ (mod $-13$).

In examples 7 and 8 the carrying process goes on indefinitely. Consider example 7 again to see if this can be overcome. The problem is:

```
3 4 6(-7)
2 1 3(-7)
-----
```

The sum of six and three is nine and it is necessary to regroup the nine since nine cannot be represented with one digit in base negative seven. How shall this be done? The digits in the column to the left represent negative-sevens. Nine is equal to how many negative-sevens? It is equal to negative one negative-seven plus two. Instead of carrying 16 to the next two columns and adding as was done the first time, carry 1 to the next column and subtract. The work is as follows:

```
  1
3 4 6
     (-7)
2 1 3
     (-7)
-----
5 4 2
     (-7)
```

Take another look at example 8. Nine plus eight are seventeen which is equal to a negative one negative-twelve and five. After the first step the problem is as follows:

```
    1
5 7 T 9
       (-12)
3 E 5 8
       (-12)
-------
      5
```

Now, ten plus five are fifteen and when the one is subtracted this leaves fourteen negative-twelves. If twelve of these negative-twelves are regrouped, it is again necessary to subtract one in the previous column. This gives:

```
  1
5 7 T 9
       (-12)
3 E 5 8
       (-12)
-------
    2 5
```

Seven plus eleven minus one are seventeen and again subtracting one in the previous column gives five as the next digit in the sum as follows:

```
1
5 7 T 9
       (-12)
3 E 5 8
       (-12)
-------
  5 2 5
```

In the final step five plus three minus one are seven and this gives the same result as before:

$$\begin{array}{r} 5\ 7\ T\ 9_{(-12)} \\ 3\ E\ 5\ 8_{(-12)} \\ \hline 7\ 5\ 2\ 5_{(-12)} \end{array}$$

Observe that adding $16_{(-7)}$ and subtracting $1_{(-7)}$ gives the same result in base negative seven while adding $1E_{(-12)}$ and subtracting $1_{(-12)}$ give the same result in base negative twelve. This is very easily explained by noticing that these are the additive inverses of each other. Adding a number and subtracting the additive inverse of the number give the same result. The method of subtracting the additive inverse will be used from here on.

How is subtraction different when regrouping is necessary? Consider the following subtraction problem:

Example 9: Subtract $354_{(-7)}$ from $521_{(-7)}$.

$$\begin{array}{r} 5\ 2\ 1_{(-7)} \\ 3\ 5\ 4_{(-7)} \\ \hline \end{array}$$

Since 4 is greater than 1, regrouping is necessary. The one represents units and the two represents negative-sevens. After regrouping there are three negative-sevens and eight ones. Four from eight is four as follows:

$$\begin{array}{r} 5\ \overset{3}{\not{2}}\,{}^{16}1_{(-7)} \\ 3\ 5\ 4_{(-7)} \\ \hline 4 \end{array}$$

In the next step 5 is greater than 3 and regrouping is again necessary. Now there are five groups of $(-7)^2$ and three groups of $(-7)^1$ in the minuend. After regrouping this becomes six groups of $(-7)^2$ and ten groups of $(-7)^1$. Five from ten is five and three from six is three as follows:

```
 6  16  3
 5   2 16 1(-7)
 3   5    4(-7)
 ------------
 3   5    4(-7)
```

Checking gives 5+2+1=8≡0 (mod -8) and 3+5+4=12≡4 (mod -8). The result checks because 0-4=-4≡4 (mod -8).

In general, when regrouping is necessary in subtraction the previous digit in the minuend is increased by one. This is true because

$a_n(-B)^n+a_{n-1}(-B)^{n-1}=(a_n+1)(-B)^n+(a_{n-1}+B)(-B)^{n-1}$ for any negative base $-B$. For example,

$5(-7)^4+3(-7)^3=6(-7)^4+10(-7)^3$ and $7(-12)^9+4(-12)^8=8(-12)^9+16(-12)^8$.

Multiplication can be performed quite easily in a negative base if the carrying or regrouping procedure has been learned. Before doing a problem in base negative twelve recall that $1E_{(-12)}$ and $1_{(-12)}$, $1T_{(-12)}$ and $2_{(-12)}$, $19_{(-12)}$ and $3_{(-12)}$, ..., $12_{(-12)}$ and $T_{(-12)}$, and $11_{(-12)}$ and $E_{(-12)}$ are additive inverses. When regrouping is necessary the additive inverse will be subtracted.

Example 10: Consider the problem of multiplying $3476_{(-12)}$ by $8E7_{(-12)}$. Multiplying by seven in the first step gives:

```
     2 3 3
     3 4 7 6(-12)
       8 E 7(-12)
 ___________
 1 E 7 1 T 6
```

In the next step multiplying by E gives:

```
       3 6 5
       3 4 7 6(-12)
         8 E 7(-12)
 _____________
   1 E 7 1 T 6
 1 T 6 2 0 6
```

Multiplying by 8 and adding the partial products gives the result:

```
         2 4 4
         3 4 7 6(-12)
           8 E 7(-12)
 _______________
     1 E 7 1 T 6
   1 T 6 2 0 6
 1 E T 4 4 0
 _______________
 1 E 8 8 1 0 4 6(-12)
```

Checking the result gives $3+4+7+6=20\equiv 7 \pmod{-13}$; $8+11+7=26\equiv 0 \pmod{-13}$; $1+11+8+8+1+0+4+6=39\equiv 0 \pmod{-13}$. The result checks because $7\cdot 0=0\equiv 0 \pmod{-13}$.

Long division is more complicated with negative base numerals than are the other fundamental operations. In division with positive base numerals each digit of the quotient is made large enough so that the remainder in that step is positive but less than the divisor. This procedure will be followed in the next example with negative base numerals.

Example 11: Divide $576_{(-10)}$ by $3_{(-10)}$.

$$\begin{array}{r l} & \phantom{)}1 \\ 3_{(-10)} & \overline{)\,5\ 7\ 6_{(-10)}} \\ & \phantom{)}\underline{3} \\ & \phantom{)}2\ 7 \end{array}$$

In the next step a negative number is divided by a positive number which gives a negative number. This means that instead of getting one digit for the next term in the quotient there must be an even number of digits. This is impossible when using a place value numeration system.

The quotient in each step is to be positive or zero so it can be represented by one digit. If the next digit in the quotient will not be zero the remainder needs to be positive when the divisor is negative and the remainder needs to be negative when the divisor is positive. If this is done and the next digit of the dividend is brought down the partial dividend will be positive when the divisor is positive and negative when the divisor is negative. The absolute value of the remainder needs to be less than the absolute value of the divisor so that the next term in the quotient will be one digit instead of three.

The absolute value of the partial dividend may be larger than the absolute value of the divisor and the partial dividend and divisor may not both be positive or both be negative when the above procedure is followed. In this case some of the previous digits in the quotient need to be changed. This happens when some of the digits in the quotient are zero as example 12 illustrates.

Now consider the problem of dividing $576_{(-10)}$ by $3_{(-10)}$ again. The successive steps are:

```
            2
3(-10) ) 5 7 6(-10)
         6
        1 9
```

```
            2 6
3(-10) ) 5 7 6(-10)
         6
        1 9 7
        1 9 8
          1 9
```

```
            2 6 6(-10)
3(-10) ) 5 7 6(-10)
         6
        1 9 7
        1 9 8
          1 9 6
          1 9 8
            1 8
```

Thus, $576_{(-10)} \div 3_{(-10)} = 266_{(-10)}$ with a remainder of $18_{(-10)}$ or $266 \frac{18}{3}{}_{(-10)}$. Since the divisor and the dividend are both positive a positive remainder may be preferred. In that case the last digit in the quotient needs to be decreased by one. Then the last step of the problem is as follows:

```
            2 6 5(-10)
3(-10) ) 5 7 6(-10)
         6
        1 9 7
        1 9 8
          1 9 6
          1 9 5
              1
```

Then $576_{(-10)} \div 3_{(-10)} = 265_{(-10)}$ with a remainder of $1_{(-10)}$ or $265\frac{1}{3}_{(-10)}$.

Checking this result gives $2+6+5=13\equiv 2 \pmod{-11}$; $3\equiv 3 \pmod{-11}$; $1\equiv 1 \pmod{-11}$; $5+7+6=18\equiv 7 \pmod{-11}$. The result checks because $2\cdot 3+1=7\equiv 7 \pmod{-11}$.

To show that $266\frac{18}{3}_{(-10)}$ also checks, $2+6+6=14\equiv 3 \pmod{-11}$; $3\equiv 3 \pmod{-11}$; $18_{(-10)}=-2\equiv 9 \pmod{-11}$; $5+7+6=18\equiv 7 \pmod{-11}$. This result also checks because $3\cdot 3+9=18\equiv 7 \pmod{-11}$.

Example 12: Divide $EE3_{(-12)}$ by $5_{(-12)}$.

```
              2 0 3(-12)
5(-12) ) E E 3(-12)
         T
       -------
         1 E 3
         1 E 3
       -------
```

Since $2+0+3=5\equiv 5 \pmod{-13}$, $5\equiv 5 \pmod{-13}$, $11+11+3=25\equiv 12 \pmod{-13}$, and $5\cdot 5=25\equiv 12 \pmod{-13}$ the result checks.

Consider the following example in base negative seven with a three digit divisor.

Example 13: Divide $514263_{(-7)}$ by $254_{(-7)}$.

```
                   4 3 2
254(-7) ) 5 1 4 2 6 3(-7)
          6 4 2
          -----
            4 2 2
            4 0 5
            -----
              3 4 6
              3 2 1
              -----
                2 5 3
```

Since the dividend will have an even number of digits and be negative, a negative remainder may be preferred. The last digit in the quotient must then be a one. The problem is completed as follows:

```
                4 3 2 1(-7)
254(-7) ) 5 1 4 2 6 3(-7)
          6 4 2
            4 2 2
            4 0 5
              3 4 6
              3 2 1
                2 5 3
                2 5 4
                  1 6
```

Checking this result gives 4+3+2+1=10≡2 (mod -8); 2+5+4=11≡3 (mod -8); 1+6=7≡7 (mod -8); 5+1+4+2+6+3=21≡5 (mod -8). Since 2·3+7=13≡5 (mod -8) the result checks.

A final long division example is given in base negative twelve.

Example 14: Divide 3519595T8$_{(-12)}$ by 3E74$_{(-12)}$.

```
                            2 T 7 3 9 5(-12)
3 E 7 4(-12) )   3 5 1 9 5 9 5 T 8(-12)
                 5 9 2 8
               1 E 9 E 1 5
               1 E T 9 7 4
                 1 E 3 6 1 9
                 1 E 3 2 E 4
                       5 2 5 5
                       7 8 8 0
                     1 E 7 9 5 T
                     1 E 8 T 0 0
                           E 5 T 8
                           E 5 T 8
```

Checking this result gives 2+10+7+3+9+5=36≡10 (mod -13); 3+11+7+4=25≡12 (mod -13); 3+5+1+9+5+9+10+8=55≡3 (mod -13). Since 10·12=120≡3 (mod -13) the result checks.

Negative base arithmetic can be used to change numerals from one base to another as the following two examples show.

Example 15: Change $562_{(-12)}$ to a base ten numeral. Repeated division gives:

```
              1 7 5(-12)                                      6(-12)
T(-12) ) 5 6 2(-12)        and        T(-12) ) 1 7 5(-12)
             T                                     1 7 0
         1 7 6                                         5
         1 7 T
           1 8 2
           1 8 2
               0
```

Thus, $562_{(-12)}=650$.

Example 16: Change $12504_{(-7)}$ to base negative twelve.

The polynomial expansion of $12504_{(-7)}$ is $\left[1(10)^4+2(10)^3+5(10)^2+0(10)^1+4(10)^0\right]_{(-7)}$. Rewriting the polynomial in base negative twelve numerals and performing the operations in negative twelve gives

$\left[1(15)^4+2(15)^3+5(15)^2+0(15)^1+4(15)^0\right]_{(-12)}=$

$\left[1E541+18TT+245+0+4\right]_{(-12)}=1E258_{(-12)}$. Examples 1 and 3 show that $12504_{(-7)}$ and $1E258_{(-12)}$ are both equal to 1964.

The iterative method can be used to find the square root of a number to any number of significant figures. This method is perhaps the easiest to understand.

Let $x_1$ be the first approximation of the square root of a number N and let $q_1$ be the quotient when N is divided by $x_1$. The square root of N is $x_1$ if $x_1=q_1$. If $x_1 \neq q_1$ the square root of N is between $x_1$ and $q_1$. A good second approximation is $x_2=\frac{x_1+q_1}{2}$. This process is continued until $x_n$ has the desired number of significant figures. In general, $x_n=\frac{x_{n-1}+q_{n-1}}{2}$.

Example 17: Find the square root of $350_{(-10)}$ to six significant figures.

If $x_1=195_{(-10)}$ the following division shows that $q_1=197.3_{(-10)}$.

```
                  1 9 7.3(-10)
1 9 5(-10) )    3 5 0(-10)
             1 9 5
               2 8 5
               2 7 5
                 1 0 0
                 1 0 5
                   1 5 0
                   1 6 5
                       5
```

Now $x_2=\frac{195_{(-10)}+197.3_{(-10)}}{2_{(-10)}}=\frac{172.3_{(-10)}}{2_{(-10)}}=196.25_{(-10)}$

Dividing $350_{(-10)}$ by $196.25_{(-10)}$ gives:

```
                        1 9 6.3 8 8(-10)
1 9 6 2 5(-10) |     3 5 0 0 0(-10)
                 1 9 6 2 5
                   2 7 4 5 0
                   2 6 3 4 5
                     1 1 2 5 0
                     1 1 6 9 0
                       1 7 6 0 0
                       1 6 8 5 5
                         2 9 6 5 0
                         2 8 7 2 0
                           2 9 3 0 0
                           2 8 7 2 0
                             1 7 8 0
```

The third approximation is $x_3=196.229_{(-10)}$, the arithmetic mean of $196.25_{(-10)}$ and $196.388_{(-10)}$. The next division follows:

```
                              1 9 6.2 2 9 9(-10)
1 9 6 2 2 9(-10) |     3 5 0 0 0 0(-10)
                   1 9 6 2 2 9
                     2 7 4 9 1 0
                     2 6 3 7 0 1
                       1 1 2 2 9 0
                       1 1 5 2 7 4
                         1 7 0 3 6 0
                         1 7 2 4 3 8
                           1 9 9 4 2 0
                           1 7 2 4 3 8
                             2 7 0 0 2 0
                             2 6 3 7 0 1
                               2 8 3 3 9 0
                               2 6 3 7 0 1
                                 2 1 5 0 9
```

Thus, the square root of $350_{(-10)}$ is $196.229_{(-10)}$ to six significant figures. The fourth approximation is $x_4=196.2295_{(-10)}$. Changing this to base ten by polynomial expansion gives $1(-10)^2+9(-10)^1+6(-10)^0+2(-10)^{-1}+2(-10)^{-2}+9(-10)^{-3}+5(-10)^{-4}=100-90+6-.2+.02-.009+.0005=15.8115$.

Checking this in a table of square roots shows that this is correct to five significant figures in base ten.

The problem of finding least common multiples and greatest common divisors of a set of numbers is basically a problem of prime factorization. Since there is no formula known for finding the prime numbers this may be a difficult task.

The first step in factoring a certain number is finding how many times two is a factor. Then three, five, seven, and the other primes may be tried until the number is factored completely.

The question that may arise is, "How are the primes represented in a particular base?" This can be answered in two ways. First, a list of primes in base ten may be changed to the negative base in question.

The second method makes use of the sieve of Eratosthenes. The largest possible prime factor of a number is less than or equal to the square root of the number. All the possible prime factors can be found by writing the numbers from two to the square root of the number in the negative base and then using the sieve of Eratosthenes.

Example 18: Find the least common multiple and the greatest common divisor of $16665_{(-7)}$, $16797_{(-7)}$, and $15150_{(-7)}$.

The first ten primes written in base negative seven are: $2_{(-7)}$, $3_{(-7)}$, $5_{(-7)}$, $160_{(-7)}$, $164_{(-7)}$, $166_{(-7)}$, $153_{(-7)}$, $155_{(-7)}$, $142_{(-7)}$, and $131_{(-7)}$.

By dividing $16665_{(-7)}$ by two, three, and five it is found that $16665_{(-7)} = \left[2^3 \cdot 3 \cdot 5^2\right]_{(-7)}$. It is also found that $16797_{(-7)} = \left[2 \cdot 3^2 \cdot 5 \cdot 160\right]_{(-7)}$ and $15150_{(-7)} = \left[2^2 \cdot 5^2 \cdot 160\right]_{(-7)}$. Then the least common multiple is $\left[2^3 \cdot 3^2 \cdot 5^2 \cdot 160\right]_{(-7)}$ and the greatest common divisor is $\left[2 \cdot 5\right]_{(-7)}$.

In base ten there are numerous tests for divisibility. If much work is done in a certain negative base it would be desirable to find tests for divisibility. The chapter is concluded with two theorems which give tests for divisibility by two in any base.

Theorem: If an odd base is used: a number is divisible by two if and only if the sum of the digits in the numeral is even.

Proof: Since an odd number raised to any power is odd every place value of an odd base numeral is odd. The product of one odd and one even number is even and each even digit represents an even number. The product of two odd numbers is odd and each odd digit represents an odd number. Therefore, the sum of the digits and the number that the numeral represents will both be even or both be odd.

Theorem: If an even number is used for the base: a number is divisible by two if and only if the last digit in the numeral is even.

Proof: Since an even number raised to any positive integral power is even the place value of all but the last digit is even. All but the last digit represent an even number. The numeral represents an even number if and only if the last digit is even.

# CHAPTER IV

## COMPARISON OF NEGATIVE AND POSITIVE BASE NUMERATION SYSTEMS

Similarities and differences in positive and negative base numeration systems are discussed in this chapter. An attempt is made to analyze the differences and point out advantages of each.

The concept of positive numbers comes earlier in a child's development than negative numbers. The positive base numeration system follows logically from counting objects and regrouping them when the number used as the base is reached. In a positive base if we have N times the base to the nth power, the digit N is n+1 places to the left of the radix point in the numeral. Since this is not always true in negative bases, writing and giving the value of positive base numerals is easier than negative base numerals. For these reasons a child could likely understand the positive base numeration system at an earlier age.

Numerals written in a negative base follow a different but just as definite a pattern as in a positive base. An individual who has noticed this pattern should be able to write the numerals as far as desired in a negative base. It might still be impossible for the individual to write a certain numeral without starting with a smaller numeral and writing all numerals up to the one in question. For example, in order to write twenty-five in a certain negative base it might be necessary to write the numerals starting with one and continuing up to twenty-five.

The basic addition and multiplication facts and the regrouping process could be learned in a negative base as well as in a positive base. The fundamental operations can then be performed as easily and accurately with negative bases as with positive bases. The disadvantage of negative bases is that the regrouping procedure is not as easy to understand as in positive bases.

As has been mentioned previously, the minus sign is not needed to represent negative numbers when a negative base is used. If the number of symbols is to be kept to a minimum the negative base has the advantage.

It takes 2n digits to represent each number from $B^{2n-1}$ to $B^{2n}-1$ in the positive base, B. It takes 2n+1 digits to represent each of these numbers in the negative base, -B. If the number of digits used to represent various numbers is to be kept to a minimum, a positive base numeration system is preferred.

The primary advantage of positive base numeration systems is that they are easier to learn and understand. The most significant thing about negative base numeration systems is that the minus sign is not needed and that algorithms can be developed for the fundamental operations.

# CHAPTER V

## SUMMARY

Negative base numerals can be used to represent the numbers as well as positive base numerals. Algorithms have been developed for the fundamental operations with negative base numerals. A convenient method of checking results in any base was shown to be "casting out base-minus-one".

Most articles on other numeration systems are to increase the reader's understanding of the base ten numeration system or the operation of the digital computer. A conjecture regarding the limited amount of written information on negative bases is that they contribute little to either of the two items mentioned.

It has been noticed that the minus sign is not needed to express negative numbers in a negative base. At present this does not seem to be an advantage but merely a novelty. Perhaps someone will find an application where this is an advantage.

A suggestion for further study is the development of logarithms in a negative base numeration system. The number used as the base in the numeration system could not be used as the base for the logarithms since it is negative.

Fractions, radiximals, and irrational numbers in negative bases could be studied in more detail. These topics have been mentioned only briefly in this paper.

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APPENDIX

TABLE I

POWERS OF -7, -10, AND -12

| n | $(-7)^n$ | $(-10)^n$ | $(-12)^n$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | -7 | -10 | -12 |
| 2 | 49 | 100 | 144 |
| 3 | -343 | -1000 | -1728 |
| 4 | 2401 | 10000 | 20736 |
| 5 | -16807 | -100000 | -248832 |
| 6 | 117649 | 1000000 | 2985984 |
| 7 | -823543 | -10000000 | -35831808 |
| 8 | 5764801 | 100000000 | 429981696 |
| 9 | -40353607 | -1000000000 | -5159780352 |
| 10 | 282475249 | 10000000000 | 61917364224 |

TABLE II

ADDITION TABLE FOR BASE -7

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 2 | 3 | 4 | | | | |
| 3 | 3 | 4 | 5 | 6 | | | |
| 4 | 4 | 5 | 6 | 160 | 161 | | |
| 5 | 5 | 6 | 160 | 161 | 162 | 163 | |
| 6 | 6 | 160 | 161 | 162 | 163 | 164 | 165 |

The larger number is found in the left-hand column. All numerals in the table are written in base -7.

TABLE III

MULTIPLICATION TABLE FOR BASE-7

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 2 | 4 | | | | |
| 3 | 0 | 3 | 6 | 162 | | | |
| 4 | 0 | 4 | 161 | 165 | 152 | | |
| 5 | 0 | 5 | 163 | 151 | 156 | 144 | |
| 6 | 0 | 6 | 165 | 154 | 143 | 132 | 121 |

The larger number is found in the left-hand column. All numerals in the table are written in base -7.

TABLE IV

ADDITION TABLE FOR BASE -10

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | |
| 2 | 2 | 3 | 4 | | | | | | | |
| 3 | 3 | 4 | 5 | 6 | | | | | | |
| 4 | 4 | 5 | 6 | 7 | 8 | | | | | |
| 5 | 5 | 6 | 7 | 8 | 9 | 190 | | | | |
| 6 | 6 | 7 | 8 | 9 | 190 | 191 | 192 | | | |
| 7 | 7 | 8 | 9 | 190 | 191 | 192 | 193 | 194 | | |
| 8 | 8 | 9 | 190 | 191 | 192 | 193 | 194 | 195 | 196 | |
| 9 | 9 | 190 | 191 | 192 | 193 | 194 | 195 | 196 | 197 | 198 |

The larger number is found in the left-hand column. All numerals in the table are written in base -10.

TABLE V

MULTIPLICATION TABLE FOR BASE -10

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | |
| 2 | 0 | 2 | 4 | | | | | | | |
| 3 | 0 | 3 | 6 | 9 | | | | | | |
| 4 | 0 | 4 | 8 | 192 | 196 | | | | | |
| 5 | 0 | 5 | 190 | 195 | 180 | 185 | | | | |
| 6 | 0 | 6 | 192 | 198 | 184 | 170 | 176 | | | |
| 7 | 0 | 7 | 194 | 181 | 188 | 175 | 162 | 169 | | |
| 8 | 0 | 8 | 196 | 184 | 172 | 160 | 168 | 156 | 144 | |
| 9 | 0 | 9 | 198 | 187 | 176 | 165 | 154 | 143 | 132 | 121 |

The larger number is found in the left-hand column. All numerals in the table are written in base -10.

TABLE VI

ADDITION TABLE FOR BASE -12

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | T | E |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | |
| 2 | 2 | 3 | 4 | | | | | | | | | |
| 3 | 3 | 4 | 5 | 6 | | | | | | | | |
| 4 | 4 | 5 | 6 | 7 | 8 | | | | | | | |
| 5 | 5 | 6 | 7 | 8 | 9 | T | | | | | | |
| 6 | 6 | 7 | 8 | 9 | T | E | 1E0 | | | | | |
| 7 | 7 | 8 | 9 | T | E | 1E0 | 1E1 | 1E2 | | | | |
| 8 | 8 | 9 | T | E | 1E0 | 1E1 | 1E2 | 1E3 | 1E4 | | | |
| 9 | 9 | T | E | 1E0 | 1E1 | 1E2 | 1E3 | 1E4 | 1E5 | 1E6 | | |
| T | T | E | 1E0 | 1#1 | 1E2 | 1E3 | 1E4 | 1E5 | 1E6 | 1E7 | 1E8 | |
| E | E | 1E0 | 1E1 | 1E2 | 1E3 | 1E4 | 1E5 | 1E6 | 1E7 | 1E8 | 1E9 | 1ET |

The larger number is found in the left-hand column. All numerals in the table are written in base -12.

TABLE VII

MULTIPLICATION TABLE FOR BASE -12

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | T | E |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | | |
| 2 | 0 | 2 | 4 | | | | | | | | | |
| 3 | 0 | 3 | 6 | 9 | | | | | | | | |
| 4 | 0 | 4 | 8 | 1E0 | 1E4 | | | | | | | |
| 5 | 0 | 5 | T | 1E3 | 1E8 | 1T1 | | | | | | |
| 6 | 0 | 6 | 1E0 | 1E6 | 1T0 | 1T6 | 190 | | | | | |
| 7 | 0 | 7 | 1E2 | 1E9 | 1T4 | 1T# | 196 | 181 | | | | |
| 8 | 0 | 8 | 1E4 | 1T0 | 1T8 | 194 | 180 | 188 | 174 | | | |
| 9 | 0 | 9 | 1E6 | 1T3 | 190 | 199 | 186 | 173 | 160 | 169 | | |
| T | 0 | T | 1E8 | 1T6 | 194 | 182 | 170 | 17T | 168 | 156 | 144 | |
| E | 0 | E | 1ET | 1T9 | 198 | 187 | 176 | 165 | 154 | 143 | 132 | 121 |

The larger number is found in the left-hand column. All numerals in the table are written in base -12.