# LOGICO-MATHEMATICAL AND EPISTEMOLOGICAL ANTINOMIES

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# CHAPTER I

## INTRODUCTION

Historical introduction to subject matter. The science of formal logic originated with the Greeks, as they were the first to collect and to analyze the principles which govern logical thinking and to formulate these principles into a theory. The leading exponent in the development of logical theory by the Greeks was Aristotle, who formulated the rules of class inference. The term "class" is used to refer to well-defined groups such as the class of human beings. For Aristotle, the fact that Socrates was a man was an instance of class membership--namely, Socrates was a member of the class of men.

Aristotle utilized the idea of class membership in formulating the rules of inference pertaining to the classical syllogism. He realized that from the premises "all humans are mortal" and "Socrates is human" the conclusion "Socrates is mortal" depended not upon the content of the statements but upon the form. With this discovery the development of symbolic logic as it is known today rested for over two thousand years. Kant became cognizant of this lack of development in logic when he observed that logic was the only science that had not made any progress since its beginnings.[1]

The modern logic of the past two hundred years grew not from philosophy but from mathematics. The first well-known mathematician who devoted his energies in this direction was Leibniz.[2] His methods were revolutionary, and he originated the usage of a type of notation which is the basis of the systems of logic and of set theory in existence today. Had Leibniz pursued his studies in the field with the diligence that he applied to differential calculus, he would have advanced the development of mathematical logic by one hundred fifty years. However, his work remained fragmentary and unknown during his life; writers of the nineteenth century had to collect his results from letters and unpublished manuscripts.

A sudden development of mathematical logic occurred during the middle of the nineteenth century when mathematicians like Boole and de Morgan began to set forth the principles of logic in symbolic notation similar to that which had proved beneficial in mathematics. Their work in the development of axiomatic theories was continued by

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[1]Hans Reichenbach, The Rise of Scientific Philosophy (Los Angeles: University of California Press, 1958), p. 218.

[2]Ibid.

mathematicians like Peano, Cantor, Schröder, Frege, Russell, and Whitehead. The results of the efforts of these men are well-known, and the systems and theories which they developed represent some of the most profound achievements by modern scholars. Thus, symbolic logic and axiomatic set theory became important, integral mathematical fields.

In the two areas mentioned above, the antinomies exist not only as interesting ideas, but also as a crucial part.

Definitions. Before this paper commences a more detailed account of the significance of the antinomies, a few definitions are appropriate. A contradiction will be defined as a statement form which has only false substitution instances, such as $(p \cdot \sim p)$. All logical mathematical symbols are defined on page 59 in the Appendix. An antinomy is a contradiction which results in the irreconcilability of seemingly necessary inferences or conclusions. If the antinomy arises in a specific theory--such as axiomatic set theory or mathematical logic--then it appears as the seemingly valid deduction of two contradictory statements from the axioms of the theory, even though the axioms of the theory appear to be consistent and the rules of inference valid.

The term paradox is often used synonymously with antinomy. In the opinion of this writer, however, such usage is misleading in that paradox is often used to denote an apparent rather than a real contradiction. Also, in the history of mathematics, the term "paradox" has often been applied to puzzling conclusions for which there are logical explanations. Therefore, throughout this paper the term paradox will be avoided except in those instances in which it appears to have a special historical significance.

Significance of the antinomies. The significance of the antinomies in an axiomatic theory reveals itself in various ways. Obvious, of course, is the fact that any contradiction destroys the usefulness of an axiomatic system; for from a contradiction all theorems can be derived, and the system becomes trivial. It is usually the case that apparent contradictions are the result of inconsistent axioms or faulty rules of inference. Such antinomies are trivial due to the insignificance of the systems from which they were derived.

The antinomies considered in this paper do not seem to be of a trivial nature, for they appear at the core of mathematics and logic. These antinomies, which were first discovered at the turn of the century, constitute a third major crisis in the history of mathematics.[3] This crisis occurred during a period when mathematicians were asserting that mathematics had finally approached a state of perfection.[4] In 1900, Henri Poincare', during an address before the Second International Congress of Mathematicians, stated this theme in the following manner:

> Today there remain in analysis only integers and finite or infinite systems of integers....Mathematics has been arithmetized....We may say today that absolute rigor has been obtained.[5]

Ironically, at the same time that Poincare' made the above claim, the discovery was made that the infinite system of integers, which was but a part of set theory, was resting upon something other than a totally rigorous foundation. Five years previously, Cantor had discovered an antinomy in his set theory; but he did not publish this discovery. Two years later, Burali-Forti rediscovered the same antinomy. Though

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[3]The first crisis occurred in the fifth century B.C. only a short time after the development of geometry as a rigorous, deductive science. This crisis centered around two seemingly paradoxical discoveries: a. the discovery of the existence of irrational numbers; and b. the paradoxes of the Eleatic school--commonly known as Zenon's paradoxes--which attempted to prove the non-constructibility of finite magnitudes out of infinitely small parts. The second crisis occurred at the beginning of the nineteenth century when it became increasingly clear that the basis of the calculus as utilized in the seventeenth and eighteenth centuries was insecure.

[4]Abraham A. Fraenkel and Yehoshua Bar-Hillel, Foundations of Set Theory (Amsterdam: North-Holland Publishing Company, 1958), p. 14-15.

[5]*Ibid.*, p. 15.

neither Cantor nor Burali-Forti were able to offer an immediate solution to the antinomy, their discovery did not at first cause too much concern among members of their school. This antinomy had emerged in a technical region of set theory and was concerned with the idea of well-ordered sets. It was hoped that a slight revision of the proofs of the theorems belonging to this region would rectify the situation.[6]

The optimism of Cantor and of his followers regarding this antinomy was not to be realized, however; for in 1902 Bertrand Russell shocked the philosophical and mathematical world with the publication of the discovery of an antinomy which was inherent in the foundations of logic and set theory. Russell's antinomy could be symbolized and derived from the calculus of mathematical logic. While Russell's antinomy was not the first to appear, it was apparently the first antinomy to be discovered at such a basic level.

> ...never before had an antinomy arisen at such an elementary level, involving so strongly the most fundamental notions of the two most "exact" sciences, logic and mathematics.[7]

Russell's antinomy had a disturbing effect upon scholars whose particular work was in the field of

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[6] Ibid., pp. 1-2.

[7] Ibid., p. 2.

foundations. Dedekind was, at that time, working on an essay concerning the nature and the purpose of numbers. In this work he had based number theory on the membership relation and had utilized the notion of a set in the Cantorian sense for the proof of the existence of an infinite set. Upon learning of Russell's antinomy, he stopped the immediate publication of his work, the rudiments of which he felt were destroyed.[8]

Frege experienced a similar impact from the discovery of Russell's antinomy. He had spent many years doing research in the bases of arithmetic and was finishing his work when Russell wrote to him about his discovery. In the very first sentence of the appendix in Grundgesetze der Arithmetik, Frege admitted that one of the foundations of his work had been badly shaken by Russell's antinomy.

The mathematical community in general, however, refused at first to place a great deal of importance upon the emergence of the antinomies in set theory and logic. It was believed that the antinomies were but a technical part of a highly specialized region which had little bearing upon the basis of mathematics proper. Literature in the field indicates,

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[8] F. P. Ramsey, The Foundations of Mathematics (New Jersey: Littlefield, Adams and Company, 1960), p. 80.

however, that such attitudes displayed more wishful thinking than critical observation. The work of Russell, Frege, Peano, and Cantor had opened new insights into the foundations of mathematics where logic and set theory were an indispensable part. It would seem that if contradictions arose in these areas and that steps were necessary to alleviate them, it would be preferable to formulate precautions rather than to disregard these contradictions. This attitude was not always the prevalent one, however, as indicated by Fraenkel's observations:

> The very fact that one continued to speak of paradoxes, or antinomies, rather than of contradictions serves as an indication that deep in their heart most modern mathematicians did not want to be expelled from the paradise into which Cantor's discoveries had led them.[9]

The choice left to the classical mathematician was not a pleasant one. He could point to the progress made in analysis, geometry, and algebra; but he was forced either to maintain naive faith in the essential soundness of these disciplines or to admit that the logic in their foundations was not free from certain contradictions. The choice was certainly an exclusive one since each point of view was the antithesis of the other. The psychological effect of this

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[9]Fraenkel and Bar-Hillel, op. cit., p. 4.

dilemma on the modern mathematician is expressed by Weyl in an article in the American Mathematical Monthly.

> We are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world today, we have our "crises." We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe," and has been a constant drain on the enthusiasm and determination with which I pursued my research work.[10]

The positive, rather than the negative, significance of the antinomies is expressed by the work of logicians and mathematicians in formulating their logical structure and in then attempting to take precautions within a theory to avoid them. The discovery, classification, and analysis of the various antinomies reveals that they were not merely a "puzzle pastime," but were a significant problem in the foundations of mathematics.

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[10]Herman Weyl, "Mathematics and Logic," American Mathematical Monthly, Vol. 53 (1946), 2-13.

# CHAPTER II

## HISTORICAL ACCOUNT OF THE DISCOVERY OF ANTINOMIES

*Paradox of Epimenides*. The history of logic reveals that the Megarian school of logic formulated the first semantical antinomy. The Megarian, or "dialectical" school, was founded by Euclid of Megara, a pupil of Socrates, around 400 B. C. The Megarians devoted much attention to various fallacies and paradoxes--some of which were concerned with the problem of the continuum, while others were problems in verbal reasoning. One of these paradoxes, the "Liar," has considerable logical interest and has been estensively studied by logicians for centuries.[1]

Although the antinomy known as the Liar paradox is classified as an epistemological antinomy (the subject matter of Chapter IV of this paper), it is discussed at this time because of its historical significance. The literature concerning this particular antinomy is quite extensive; thus, only the more important versions of the Liar paradox will be examined.

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[1] I. M. Bochenski, *A History of Formal Logic* (Notre Dame, Indiana: University of Notre Dame Press, 1961), pp. 107-109; 130-132.

The Liar paradox is often called the Paradox of Epimenides after a Greek scholar living at the beginning of the sixth century. However, it is doubtful that any actual connection between Epimenides and the formulation of the antinomy exists. The consensus of opinion by historians in the field of logic seems to be that Eubulides, a member of the Megarian school, was the first to formulate the Liar paradox. Evidence supporting this opinion can be found in the fact that a form of the Liar paradox appears in Aristotle's Sophistic Refutations, which appeared in 330 B. C. at the same time that Eubulides was active in his work.

An exact formulation of the Liar paradox by Eubulides no longer exists, but the extensive literature in this area provides many versions of it. The form of the more ancient versions appears uninteresting, but their basic idea not only puzzles but also challenges the mind. The ancient versions can be adequately summarized by the following list, which divides them into four categories:

I.

If you say that you lie, and in this say true, do you lie or speak the truth?

If I lie and say that I lie, do I lie or speak the truth?

II.

If you say that you lie, and say true, you lie; but you say that you lie, and you speak the truth; therefore, you lie.

If you lie and in that say true, you lie.

III.

If I say that I lie, and (in so saying) lie; therefore, I speak the truth.

Lying, I utter the true speech, that I lie.

IV.

If it is true, it is false; if it is false, it is true.

Whoso says "lie," lies and speaks the truth as the truth at the same time.[2]

These four categories are really possible interpretations of the problem posed in analyzing the simple proposition "I am lying." The first group simply states the question: Is the proposition made by the Liar, i.e., "I am lying," true or false? Those versions in the second category conclude that it is true, while those in the third group contend that it is false. The conclusion in the fourth group is that the proposition is both true and false.

A modern version of the Liar paradox is stated as follows: Assume that John Doe utters on December 1, 1963, the following English sentence and then says nothing else all day: "The only sentence uttered by John Doe on December 1, 1963, is false." Since the sentence uttered by John Doe is a declarative proposition, one is entitled to inquire whether it is true or false. Reflection soon leads to the conclusion that the sentence is true if and only if it is false.

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[2] Ibid., pp. 131-132.

Another modern version of the Liar paradox is the following sentence, which was formulated by J. Lukasiewica: "The sentence printed on page 13, line 3 of this paper is not true." (The above statement will be symbolized by the letter S in the following discussion.) At first glance proposition S appears harmless, and it seems proper to inquire about its truth value. Now, the statement "A is B" is true if and only if A is B; therefore, it seems quite apparent that proposition S is true if and only if the sentence printed on page 13, line 3 of this chapter is not true. However, if one counts to line 3 of page 13 of this paper, he will find that the sentence printed there is identical to S. Thus, one is led to the contradiction $S \leftrightarrow \sim S$.[3]

The question which then naturally arises is: What is the source of these contradictions? In the latter example, the source of the contradiction lies in the attempt to formulate the truth conditions for the statements of a language within that language itself.

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[3]Irving M. Copi, Symbolic Logic (New York: The Macmillan Company, 1954), p. 188.

After reflection on these two examples, one might conclude concerning the Liar paradox that the trouble appears to lie in the fact that the statement refers to itself and that, while it is not the self-reference alone which is the source of the contradiction, it is the nature or type of self-reference involved.

Russell presented a more sophisticated analysis of the Liar paradox in the following manner: When a man says, "I am lying," his statement may be interpreted as being equivalent to "There is a proposition which I am affirming and which is false." All statements that "There is so-and-so" may be regarded as denying that the opposite is always true; thus, "I am lying" becomes: "It is not true of all propositions that I am not affirming them or they are true." This is perhaps stated more clearly: "It is not true for all propositions P that if I affirm P, P is true." The paradox results from regarding this statement as affirming a proposition, which must therefore come within the scope of the statement.

As Russell emphasizes, one is forced to conclude that the idea of "all propositions" is not a legitimate one, for if the notion of "all propositions" is accepted, then there must be propositions which are about all propositions and yet cannot, without contradiction, be included among the propositions about which they are concerned. However, one might define a totality of propositions. Once this totality is spoken of in terms of "all propositions," other propositions are generated which must lie outside of the defined totality, or a contradiction is present. It does not help to enlarge the defined totality, for the same problem arises; hence, "all propositions" must be a meaningless phrase.[4]

_Cantor and Burali-Forti's Discoveries_. The end of the nineteenth century marked the emergence of renewed interest and study in antinomies. Previous to that time, the Liar paradox had been the only authenic antinomy to be discovered, and it was disregarded by many logicians as a plaything of semantics. Now there appeared a whole series of antinomies, some of which were logical rather than epistemological.

Between 1895 and 1897 C. Burali-Forti and G. Cantor independently stated the first logical antinomy, which concerned the set of all ordinal numbers. The Burali-Forti antinomy is a logical antinomy, the basic formulation of which is included at this point to illustrate the significance to mathematics and logic of this type of antinomy.

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[4]Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," _American Journal of Mathematics_, XXX (1908), p. 224.

The Burali-Forti contradiction may be stated as follows:

> The following three theorems can be proven in the classical theory of ordinal numbers developed by Cantor:
>
> 1. Every well-ordered series has an ordinal number.
> 2. The series of ordinals up to and including a given ordinal number, say $\alpha_1$, has an ordinal number $\alpha_1 + 1$.
> 3. The series of all ordinal numbers is well-ordered and hence, by (2) has an ordinal number, $\Omega$ say.
>
> But from (2) the series of all ordinals including $\Omega$ has ordinal number $\Omega + 1$, which is greater than $\Omega$. Hence, $\Omega$ cannot be the ordinal number of all ordinal numbers.[5]

Ramsey Classification. In 1925 Ramsey made the fundamental classification of antinomies into two distinct types--the logico-mathematical and the epistemological. In a paper entitled The Foundations of Mathematics, he observed that while Russell and Whitehead had gone to considerable effort to analyze and circumvent the antinomies in their monumental work Principia Mathematica, they had neglected to distinguish between the two fundamental types of antinomies. Ramsey classified the better-known antinomies into two groups, which he simply calls A and B:

> A. (1) The class of all classes which are not members themselves. (Russell's antinomy)
>
> (2) The relation between two relations when one does not have itself to the other.

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[5]Max Black, The Nature of Mathematics (London: Routledge and Kegan Paul, 1958), p. 99.

(3) Burali-Forti's contradiction of the greatest ordinal.

B. (4) "I am lying." (Liar paradox)
(5) The least integer not nameable in fewer than nineteen syllables.
(6) The least indefinable ordinal.
(7) Richard's Contradiction.
(8) Weyl's contradiction about heterological. (This antinomy was first formulated by Kurt Grelling.)[6]

The principle according to which Ramsey classified the antinomies into two groups is of fundamental importance. Group A consists of antinomies that are strictly logical or mathematical in nature--that is, antinomies which would occur in a mathematical or logical system if no precautions were taken to avoid them. "They involve only logical or mathematical terms such as class and number, and show that there must be something wrong with our logic or mathematics."[7]

Those antinomies in Group B are not strictly logical or mathematical and cannot be stated in logical terms alone. Each of the antinomies in this group contains a reference, usually a self-reference, to thought, language, or symbolism. It is for this reason that antinomies of this type are often termed epistemological. If their contradictory nature is

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[6] F. P. Ramsey, The Foundations of Mathematics (Paterson, New Jersey: Littlefield, Adams, and Company, 1960), p. 20.

[7] Ibid.

entirely due to faulty ideas in language, then, as Ramsey points out, they would not be relevant to logic or mathematics. However, such an attitude, while held by such mathematicians as Peano and Poincare, is far from satisfactory; for several of the antinomies in this latter group involve both mathematical and linguistic ideas. Several of them, notably Grelling's antinomy, can be formulated and symbolized in symbolic logic and the actual contradiction derived. Therefore, it would seem to be a severe neglect rather than an irrelevance which would lead to the dismissal of type B antinomies in a study such as this one.

Summary. This chapter has attempted to present an historical account of the development and discovery of antinomies. Included in this account were the Liar paradox, Cantor's and Burali-Forti's discoveries, and Ramsey's classification of antinomies. Chapters III and IV will deal with antinomies according to Ramsey's classification--logicomathematical and epistemological, respectively.

## CHAPTER III

## LOGICO-MATHEMATICAL PARADOXES

Russell's Paradox. Russell's antinomy is undoubtably the best-known and is certainly one of the more important mathematical antinomies. It was discovered by Russell in 1901 during his efforts to prove a particular version of the axiom of infinity. The axiom of infinity can be stated in the form of the following assumption: If $\alpha$ be any transfinite cardinal number, there is at least one set containing $\alpha$ elements. For example, with reference to natural numbers, the axiom would state that there exists a set containing 0 and containing the successor of each of its elements. In other words the axiom of infinity postulates the existence of sets which contain an infinite number of elements.

When the axiom of infinity was first formulated, Russell supposed that there must exist a proof for it. He based this supposition upon the following line of reasoning: Assume that the number of elements in a certain set is *n*; *n* may even be 0. Now, form a new set in the following manner: from the set containing *n* elements, form all its subsets; form a set from these subsets and form all the subsets of this set; etc. The final set will contain a number of terms equivalent to $n + 2^n + 2^{2^n} + 2^{2^{2^n}} + \ldots$ ad. inf. Therefore, taking all kinds of objects together and not confining attention to any one kind of object in a particular set, it is possible to form an infinite class from even a finite n of individuals. It would thus seem that there is no need for an axiom of infinity.

The fallacy involved in the above line of reasoning is quite subtle and is not an easy one of avoid. Russell labeled the fallacy a "confusion of types," and it was his analysis of this fallacy that led to the discovery of his antinomy. The first form of his contradiction was related to the theorem that the number of subsets of a given set is always greater than the number of elements in the given set, from which it can be inferred that there is no greatest cardinal number. However, if a set is formed from a set of all countable objects in a manner similar to the one described in the preceding paragraph--by combining the elements of the set, subsets of the set, subsets of the set of subsets, etc. --then a set is formed of which its own sub-sets would be elements. With respect to this particular set, which might be labeled the set of all sets, there would not be more subsets than elements. "The class consisting of all objects than can be counted, of whatever sort, must if there be such a class, have a cardinal number which is the greatest possible."[1]

Russell's most famous antinomy, the one commonly referred to as Russell's Paradox, resulted from his analysis of the contradictory conclusions of the above argument. Russell states:

> When I first came upon this contradiction, in the year 1901, I attempted to discover some flaw in Cantor's proof that there is no greatest cardinal.... Applying this proof to the supposed class of all imaginable objects, I was led to a new and simpler contradiction....[2]

A somewhat simple version of Russell's Paradox is the following: Consider a set A which contains as elements the numbers 1, 2, 3, 4, and itself. Of course, an immediate objection may be raised against such a set as being self-referent, fictitious, or meaningless. In part, these would be valid objections; but for the sake of the argument, they shall temporarily be disregarded. While any set so defined tends to arouse some logical suspicion, it does not immediately impress upon the untrained mind its contradictory nature. The contradiction arises during the analysis of the implications of the existence of such a set.

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[1] Bertrand Russell, <u>Introduction to Mathematical Philosophy</u> (New York: The Macmillan Company, 1938), p. 136.

[2] <u>Ibid</u>.

All sets can be divided into two general but exhaustive classifications--those which contain themselves as members and those which do not. These two classifications also define two specific sets: (1) a set which shall be designated C, which is the set containing all sets containing themselves as members; and (2) a set which shall be designated D, which is the set containing all sets which do not contain themselves as members. Since this classification is exhaustive, it must apply to sets C and D. The question arises as to the classification of set D.

Assume that D is a member of C. Since C contains only those sets which contain themselves as members, it follows that D must contain itself as a member. However, D was defined as a set containing only sets which do not contain themselves as members. Therefore, it follows that D does not contain itself as a member and must be a member of D, not C. However, if D is a member of D, then D contains itself as a member, which implies that D must be a member of C, not D.

<u>Structure of Russell's Paradox in symbolic logic</u>. The structure of the above contradiction can be symbolized in the following manner: If sets C and D are defined as previously stated, then the following implications are derived:

(1) $C = \{x\colon \ x \text{ is a set}, x \in x\}$ Def.

(2) $D = \{y\colon \ y \text{ is a set}, y \notin y\}$ Def.

(3) $(D \in D) \vee (D \in C)$ Def.

(4) $(D \in C) \rightarrow (D \in D)$ from Def. 1

(5) $(D \in D) \rightarrow (D \notin D) \rightarrow (D \in C)$ Def. 2,3. Disjunctive Syllogism

The definition of the classification leads to the statement:

(6) $(D \in C) \leftrightarrow \sim (D \in D)$

(7) $(D \in D) \rightarrow \sim (D \in D)$ 5,6 Hypothetical Syllogism

(8) $\sim (D \in D) \rightarrow (D \in D)$ 6,4 Hypothetical Syllogism

Statements 7 and 8 together imply that $\sim (D \in D) \leftrightarrow (D \in D)$, which is indeed a devastating contradiction.

The fallacy in Russell's antinomy lies in the supposition that there exist sets which contain themselves as members. This supposition results in what Russell labels "impure" classes and will later be seen to violate the theory of types.

> ...,classes are logical fictions, and a statement which appears to be about a class will only be significant if it is capable of translation into a form in which no mention is made of the class. This places a limitation upon the ways in which what are nominally, though not really, names for classes can occur significantly: a sentence or set of symbols in which such pseudo-names occur in wrong ways is not false, but strictly devoid of meaning. The supposition that a class is, or that it is not, a member of itself is meaningless in just this way.[3]

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[3] *Ibid.*, p. 137.

In simple axiomatic set theory, the same type of conclusion is reached--namely, that sets cannot be produced by simply uttering words: there are certain inherent restrictions which must be carefully observed. For example, consider the axiom of specification: To every set A and to every condition S(x), there corresponds a set B, whose elements are exactly those elements x of A for which S(x) holds.[4] It is customary to indicate set $B = \{x | x \in A \text{ and } S(x)\}$. Let the condition S(x) be in particular the statement $x \notin x$. Thus, whatever set A consists of, $B = \{x | x \in A \text{ and } x \notin x\}$ and (1) $(y \in B) \leftrightarrow (y \in A) \cdot (y \notin y)$. Now, is $B \in A$? Assume that it is. It is certainly true that $(B \in B) \vee (B \notin B)$. If $B \in B$, then (1) yields that $B \notin B$, which is a contradiction. If $B \notin B$, then (1) and the assumption $B \in A$ yields $B \in B$, which again is a contradiction. The conclusion is that $B \in A$ is impossible since its assumption leads to a contradiction. It is significant that nothing whatsoever was specified about the elements of set A, but in spite of this it was proven that there exists something, set B, which is not in A. In other words, nothing contains everything, which would certainly prohibit the concept of set of all sets.

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[4]Paul R. Halmos, Naive Set Theory (New York: D. Van Nostrand Company, Inc., 1960), p. 6.

Impredicable Paradox. Another interesting logico-mathematical antinomy, also constructed by Russell, is the Impredicable Paradox. In the logic of relations, it is seen that properties can be attributed to relations and other properties as well as to individuals. For example, the property of sincere might itself be said to have the property of being desirable. The question naturally arises: Does there exist a property P which itself has the property P? The property of being abstract seems itself to be abstract, while the property "green" is certainly not green. The property "old" is certainly old, as there have been old things since prehistoric times, while the property "new" is certainly not new.

By definition any property which can be predicated of itself will be said to be a predicable property. In other words predicable is a property which belongs to all those and only those properties which can be truly predicated of themselves. In contrast, any property which cannot be predicated of itself will be said to be impredicable. Being impredicable then is a property which belongs to all those and only those properties which cannot be truly predicated of themselves. As was the case with Russell's paradox, the classification made is exhaustive: every property must be either predicable or impredicable.

How shall the property impredicable be classified? If it is assumed that impredicable is predicable, which means the property it has applies to itself, then it follows that impredicable must be impredicable. However, if the property impredicable is itself impredicable, then by definition it has the property it represents, which means that it is predicable. Thus, the rather startling conclusion is reached that if impredicable is predicable, then impredicable is impredicable; and that if impredicable is impredicable, then impredicable is predicable.

<u>Structure of Impredicable Paradox in symbolic logic</u>. If impredicable is abbreviated "ipr" and predicable "pr," and if by A(f) it is meant that A has property f, then the contradiction can be derived from the two statements:

(1) $\text{ipr}(\text{pr}) \longrightarrow \text{ipr}(\text{ipr})$

(2) $\text{ipr}(\text{ipr}) \longrightarrow \text{ipr}(\text{pr})$

Because of the definition of the classification, statement 1 is equivalent to:

(3) $(\text{ipr}(\text{pr})) \rightarrow \sim(\text{ipr}(\text{pr}))$

and statement 2 is equivalent to:

(4) $(\text{ipr}(\text{ipr})) \rightarrow \sim(\text{ipr}(\text{ipr}))$

By the use of the definition of material implication, statements 3 and 4 are equivalent to:

(5) $\sim(\text{ipr}(\text{pr}))$

(6) $\sim(\text{ipr}(\text{ipr}))$

Statements 5 and 6 form a definite contradiction since they state that impredicable is neither predicable nor impredicable.

The contradiction can be derived even more clearly by symbolizing the property of being impredicable as "I" and by defining it formally as:

(7) IF = df ∽FF (F is a property variable)

The following statement is a logical consequence of statement 7.

(8) (F) (IF ≡ ∽FF)

If statement 8 is instantiated with respect to "I" itself according to the principle of Universal Instantiation, then it yields:

(9) II ≡ ∽II

Statement 9 is again an explicit contradiction.

The source of this contradiction lies in the formation of two exhaustive categories, predicable and impredicable, and in the allowance of the possibility that a property can be predicated of itself. The latter is again a violation of the theory of types, which would say that it does not make sense either to affirm or to deny of any property that it belongs to itself. Such expressions as ipr(ipr) and ∽ipr(ipr) must be dismissed as meaningless.

## CHAPTER IV

## EPISTEMOLOGICAL PARADOXES

The epistemological antinomies constitute a group of antinomies in which the self-references appear in different forms than did the self-references of the logical antinomies. In the epistemological antinomies the self-reference is stated with respect to linguistic expressions--that is, these antinomies are dependent upon some important reference to words. Because these antinomies are involved with terms other than those which are strictly logical or mathematical, several mathematicians have sought to dismiss them. These mathematicians take the view that the fault lies not with logic or mathematics but with a faulty language. For example, Peano dismissed Richards' antinomy because he felt that it was not pertinent to mathematics, but was strictly a problem in linguistics.[1]

However, such a view disregards the similarity between the two types of antinomies. They both involve the idea of self-reference and the formation of classes which contain themselves as members. Several of the antinomies involve both logical and semantical terms, and the distinction

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[1] F. P. Ramsey, The Foundations of Mathematics (Paterson, New Jersey: Littlefield, Adams, and Company, 1960), p. 21.

between epistemological and logical antinomies will be seen to hinge not so much on their content, but in the method of their resolution.

> The only solution which has ever been given, that in *Principia Mathematica*, definitely attributed the contradictions to bad logic, and it is up to opponents of this view to show clearly the fault in what Peano called linguistics, but what I should prefer to call epistemology, to which these contradictions are due.[2]

The Liar paradox, which has already been examined in Chapter II, is probably the least complex of the epistemological antinomies. However, it does exemplify the dependence of these antinomies upon a certain phrase or linguistic expression. More involved versions of semantical antinomies are presented in this chapter.

K. Grelling Antinomy. In Chapter III the antinomy termed the impredicable paradox was discussed. It should be recalled that the crux of that antinomy was the division of all properties into two exhaustive classes--namely, those properties which apply to themselves and those which do not. Instead of asking whether a property applies to itself, one might ask whether the name of a property has the property under consideration. Such a question is the starting point for an epistemological antinomy which is credited to Kurt Grelling and which is usually known as the Grelling Paradox.

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[2] *Ibid.*

In most instances the name of a property will not have the property which it denotes. For instance, the name "heavy" is not heavy; "long" is not a long word; and "French" is not a French word. Some words, however, do designate the property exemplified by the word: "English" is an English word; and "short" is certainly a short word. All properties whose names have the property they denote will be designated "autological." The term "heterological" will be used to designate the property possessed by words which designate properties not exemplified by themselves. Thus, the words "heavy," "long," and "French" are heterological, while "English" and "short" are autological.

The Grelling Paradox is a result of the word "heterological." How should it be classified--heterological or autological? If it is assumed that "heterological" is heterological, it is obvious that this is similar to saying that "short" is short and that "heterological" has the property it denotes and is, therefore, autological. On the other hand, if "heterological" is assumed to be autological, this is equivalent to saying that it has the property designated by "heterological"--that is, "heterological" is heterological. The contradiction is quite apparent: if "heterological" is heterological, then it is not; and if "heterological" is not heterological, then it is.

Grelling's antinomy can be derived in a more formal way as follows: Let "Des" designate the name relation--that is, "s Des F" is equivalent to "s designates the property F," where F is a property variable and s is a name variable. Thus, a heterological word is defined symbolically as:

$$\text{Het}(s) \leftrightarrow (\exists F) \quad (s \text{ Des } F \equiv F \cdot \sim F(s))$$

A literal translation of the previous statement would be as follows: A word s is heterological if and only if there exists a property F such that s designating F is equivalent to F and s does not have property F. In the case where heterological is assumed to be heterological, we have:

(1) $\text{Het}(\text{"Het"}) \leftrightarrow (\exists F) \quad (\text{"Het" Des } F \equiv F \cdot \sim F(\text{"Het"}))$

This follows from the definition by replacing the word variable s by the word "Heterological."

(2) $\text{Het}(\text{"Het"}) \leftrightarrow (\text{"Het" Des Het} \equiv \text{Het} \cdot \sim \text{Het}(\text{"Het"}))$

This follows from (1) by the existential instantiation of the property variable F by the property Heterological.

(3) Het ("Het") The assumption originally made.

(4) $\text{Het} \cdot \sim \text{Het}(\text{"Het"})$ This follows from (3) and (2) by Modus Ponens, equivalence, and simplification.

(5) $\sim \text{Het}(\text{"Het"})$ This follows from (4) by simplification.

The identical contradiction can be derived in a similar manner by assuming $\sim \text{Het}(\text{"Het"})$. Thus the contradiction $\text{Het}(\text{"Het"}) \leftrightarrow \sim \text{Het}(\text{"Het"})$ is shown to follow from the nature of the definition of the property heterological.

Richard's Antinomy. Richard's Antinomy is a classical example of an epistemological antinomy. The contradiction arises from the apparent definition of a denumerable set which is not denumerable. This definition is accomplished in the following manner: The first task is to write all possible arrangements of the English alphabet, first two letters at a time, then three at a time, etc., arranging each group in alphabetical order. These arrangements may contain the same letter repeated several times; thus, they are arrangements with repetition.

Whatever whole number "n" may be, every arrangement of the twenty-six letters of the English alphabet n at a time will be found in the constructed arrangement. Since everything that can be written with a finite number of words is an arrangement of letters, everything that can be written will be found in the arrangement.

Some of these arrangements will be definitions of numbers, since numbers are defined by means of words. The next step is to eliminate from the arrangements all combinations which do not define numbers. Let $u_1$ be the first number defined by an arrangement, $u_2$ the second, etc. Thus, all the numbers defined by a finite number of words have been arranged in a determined order. Call this set "D."

The contradiction arises when we define the following number, "N," by use of a finite number of words. If "a" is the n-th decimal of the n-th number in the set D, then the number N is formed by having zero for its integral part and a + 1 (or 0 if a=9) for its n-th decimal part. For example, if set D consisted of the arrangement of words defining the numbers:

$u_1$ : $.a_{11}\ a_{12}\ a_{13} \cdots a_{1n}$

$u_2$ : $.a_{21}\ a_{22}\ a_{23} \cdots a_{2n}$

. . . . . . . . . . . . . .

$u_n$ : $.a_{n1}\ a_{n2}\ a_{n3} \cdots a_{nn}$

. . . . . . . . . . . . . .

then the number N would be constructed by beginning in the upper left-hand corner. Its first decimal place would be $a_1 = a_{11} + 1$; the second decimal place would be $a_2 = a_{22} + 1$, ..., $a_n = a_{nn} + 1$, ..., etc.

The number N does not belong to the set D. If it were the n-th number of the set D, its n-th figure would be the n-th decimal figure of that number, which it is not. However, the number N was defined by a finite number of words and ought therefore to belong to the set D.

<u>Berry's Paradox</u>. Whole numbers can be divided in various ways to form two mutually exclusive classes according to whether the number of syllables in which they are expressed in a given language is $\leq$ n or $>$ n, where n is a natural number. If, for instance, n=19, then there must exist a natural number

which is the smallest number not expressable in fewer than 19 syllables. If the language is English, then the number will be 111,777, which is seen by the following:

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| one | hun | dred | and | el | ev | en | thou | sand | sev | en | hun | dred | and | sev | en | ty | sev | en |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |

However, the number 111,777 can also be defined as follows:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| the | least | in | teg | er | not | name | a | ble | in | few | er | than | nine | teen | syl | la | bles |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

The contradiction is reached that the smallest natural number not nameable in fewer than 19 syllables is found to be nameable in 18 syllables.

The semantical versions of the antinomies appear even more convincing to the mind than do the logical antinomies. For instance, an intuitive objection might be made against Russell's antinomy on the grounds that it is meaningless to talk about sets that contain themselves as members; but with regard to Grelling's paradox, it certainly seems plausible to designate the word "short" as short and the word "long" as not long.

However, the important distinction between the two types of antinomies lies not in their respective plausibility, but in two major distinctions. The first, which has already been mentioned, is in their formulation. The logical antinomies involve only logical or mathematical terms such as "set," "class," or "number" and would occur in a logical or mathematical system. The epistemological antinomies, however, are not purely logical, for they are dependent upon a reference to linguistic expressions. The second important distinction between the two groups of antinomies is that the solution or resolution of the two categories must be sought along different lines.

In the next chapter it will be shown that the logical antinomies may be avoided by a theory of types, but that stronger or more complex methods are needed to avoid the epistemological antinomies.

CHAPTER V

METHODS OF AVOIDING ANTINOMIES

In the preceding two chapters, it was seen that the antinomies fall into two classifications: the logical antinomies, which involve only logical or mathematical terms as "class" and "number"; and the epistemological antinomies, which contain an important reference to thought, language, or symbolism. Common to all of the antinomies thus far considered is the fact that something was asserted about a collection of objects and then additional objects were introduced which appeared to belong and at the same time not to belong to the collection under consideration.

It would appear to be a simple conclusion that if the antinomies are to be avoided, the collections which lead to the contradictions must be regarded as illegitimate totalities. One method of avoiding antinomies would be to prohibit the formation of such totalities. However, if this prohibition is to be employed with every individual collection as each one is found to lead to a contradiction, it would be an endless task, as new antinomies would always arise. The same argument applies to any attempt to refute the antinomies one by one. What is obviously needed is a method which will serve to prohibit, or at least to circumvent, all of the antinomies. This circumvention can be done, provided that the illegitimate totalities can be shown to have some peculiarity in common.

According to Russell and Whitehead such a common peculiarity exists; they maintain that the antinomies arise from the assumption that a set of objects can contain as members objects which can only be defined by the set as a whole. The restriction against such sets could be formulated in the following manner: Whatever is defined by <u>all</u> of a particular set must not be a member of that set. This restriction can be stated in a slightly different manner known as Russell and Whitehead's "vicious circle principle," which is "If, provided a certain collection had a total, it would have a member only definable in terms of that total, then the said collection has no total."

However, in empbying such a principle, care should be taken not to make it too prohibitive. In certain instances it is extremely convenient or even necessary to be able to speak of "all propositions," "all relations," "all classes," etc. What is needed is a method of limiting the totalities so that what remains will not lead to a contradiction but will still provide the necessary concepts for a logical system. These considerations led Russell and Whitehead to the construction of the Simple Theory of Logical Types.

<u>Simple Theory of Logical Types</u>. The main features of the Simple Theory of Logical Types are the following: According to Russell and Whitehead, entities are divided into a hierarchy of different logical types, the lowest of which consists of all individuals, the next of all properties of individuals, the next of all properties of properties of individuals, etc. Relations and their properties are also assigned different hierarchies in a similar manner.

The essential feature of the Theory of Types is not only the division of all entities into different logical types, but also the principle that any property which may significantly be predicated of an entity of one logical type cannot significantly be predicated of any entity of another logical type. In many instances the Theory of Types appears intuitively obvious. For example, an individual object may be green in color, but it is certainly nonsensical to assert that a property is green in color. Also, a property may have many instances, but to claim that an individual object has many instances is again nonsensical.

The Simple Theory of Types is seen to rule out the occurance of the logical antinomies in the following manner: The Theory of Types classifies certain expressions as meaningless--for example, according to the Theory of Types, it is meaningless to affirm that a set contains itself as a member. A set must be regarded as being of a higher type than its members; therefore, a set cannot be said to contain itself as a member. It should be emphasized that an expression like "a set which contains itself as a member" is not false but is meaningless, for if such statements were false, then their negations would be true, and the logical antinomies would still be derivable.

For example, reconsider the Impredicable Paradox, which was presented in Chapter III as an example of a logical antinomy. Briefly, it should be remembered that "predicable" is a property which belongs to all those properties which can be predicated of themselves and "impredicable" is a property which belongs to all properties which cannot be predicated of themselves. The contradiction was derived by three statements symbolizing the argument:

(1) $IF =_{df} \backsim FF$ (I = impredicable, and F is a property variable)

(2) $(F)\ (IF = \backsim FF)$

(3) $II = \backsim II$

Even if $\backsim FF$ is considered a false statement, the contradiction would still arise in the form of the statement $II = \backsim II$. Consequently, such expressions as FF and $\backsim FF$ must be regarded as meaningless. It follows that no such property as "impredicable" can be defined, and the antinomy no longer exists. A similar situation prevails with Russell's Paradox. Such expressions as "sets which contain themselves as members" and "sets which do not contain themselves as members" are seen to violate the Simple Theory of Types and must be dismissed as meaningless.

The Simple Theory of Types, however, does not suffice to eliminate the epistemological antinomies. Words, or sentences, are physical things and cannot be said to be of a higher type; thus, in the Grelling antinomy--where a predicate is said to pertain to the name expressing it, i.e., "short" is a short word--such a conception does not violate the Simple Theory of Types. However, it was seen that such a conception does lead to a contradiction; so additional methods must be formulated to avoid the epistemological antinomies. Two methods present themselves for consideration: the Ramified Theory of Types and Levels of Language Theory.

Ramified Theory of Types. As stated previously in the Simple Theory of Types, all entities are divided into different logical types, the first type containing all individuals, the second containing all properties of individuals (designated by functions of individuals), the third type containing all properties of properties of individuals (designated by functions of functions of individuals), etc. If small letters of the alphabet are used to designate individuals and capital letters are used to designate properties, the Simple Theory of Types can be represented as follows:

Type 1: a, b, c, ...

Type 2: $F_2$, $G_2$, $H_2$, ...

Type 3: $F_3$, $G_3$, $H_3$, ...

Type 4: $F_4$, $G_4$, $H_4$, ...

According to the Theory of Types, only a function of Type 2 can be applied to an individual; and only a function of Type 3 can be applied to a function of Type 2, etc.

The Ramified Theory of Types divides each of the types above level 1 into a further hierarchy, resulting in all the functions of Type 2 being classified into what are called different orders. This classification is accomplished in the following manner: Propositional functions of Type 2 which either contain no quantifiers or contain quantifiers on only individual variables are classified as first order functions in Type 2. For example, $F_2 (x) \vee G_2 (y)$ and $(x) [H_1 (x) \rightarrow p]$ are classified as first order functions, which are designated ${}^1F_2$, ${}^1G_2$, ${}^1H_2$, .... Second order functions of Type 2 are those which contain quantifiers on only first order functions --for example, $({}^1H_2) [{}^1H_2(y) \rightarrow {}^1H_2(b)]$. Similarly second order functions are designated by ${}^2F_2$, ${}^2G_2$, ${}^2H_2$, .... In general an $n^{th}$ order function of Type 2 will contain quantifiers on functions of order n-1, but no quantifiers on functions of order m, where $m \geq n$.

A simplified version of the Ramified Theory of Types can be represented as follows:

Type 2 $^{1}F_2$, $^{1}G_2$, $^{1}H_2$,... $^{2}F_2$, $^{2}G_2$, $^{2}H_2$,... $^{3}F_2$, $^{3}G_2$, $^{3}H_2$,...

Type 3 $^{1}F_3$, $^{1}G_3$, $^{1}H_3$,... $^{2}F_3$, $^{2}G_3$, $^{2}H_3$,... $^{3}F_3$, $^{3}G_3$, $^{3}H_3$,...

Type 4 $^{1}F_4$, $^{1}G_4$, $^{1}H_4$,... $^{2}F_4$, $^{2}G_4$, $^{2}H_4$,... $^{3}F_4$, $^{3}G_4$, $^{3}H_4$,...

Just as the Simple Theory of Types prohibits expressions pertaining to the totality of all properties or functions and makes it necessary to distinguish carefully the type of the function that is predicated of an individual or function, so the Ramified Theory of Types prohibits the use of expression involving the totality of all functions or properties of a given type. Thus, in the Ramified Theory of Types, it is not permissible to state that an orange has all the good qualities of an apple. The correct statement would be that an orange has all the good first order qualities of an apple. The difference in the symbolic structure of the two statements is as follows:

$$(F_2) \quad \{[G_3 (F_2) \cdot F_2 (a)] \rightarrow \quad F_2 (o)\}$$

$$(^{1}F_2) \quad \{[^{1}G_3 (^{1}F_2) \cdot {}^{1}F_2 (a)] \rightarrow {}^{1}F_2 (o)\}$$

The Ramified Theory of Types prevents the occurance of the contradiction in the Grelling antinomy. In the derivation of the contradiction of the Grelling antinomy presented in Chapter IV, the step from (1) to (2) is not allowed since the function Het is of a higher order than the function variable "F" and may not be instantiated in its place.

The Ramified Theory of Types is quite complex in all of its ramifications, and it entails certain difficulties which should be mentioned. One of these difficulties concerns the definition of the identity of individuals, which is usually given as: $(x = y) = df\ (F_2)\ [F_2(x) \leftrightarrow F_2(y)]$. This definition is fundamental to the logic of relations and forms the basis for all of the properties of the identity relation. However, in its usual form, this definition violates the Ramified Theory of Types. If its form is changed so that it does not violate the Ramified Theory of Types--that is, $(x = y) = df\ ({}^1F_2)\ [{}^1F_2(x) \leftrightarrow {}^1F_2(y)]$--then it is seen that x and y are identical if they have all of their first order properties in common, but there is the troubling possibility that they might have different higher order properties.

In specific areas of mathematics, the Ramified Theory of Types raises several difficulties which are nearly insurmountable. For example, certain existence theorems in analysis, such as that of the Least Upper Bound, cannot be proven if the restrictions of the Ramified Type Theory are imposed. Also the theory of the continuum cannot be adequately established within the framework of the Ramified Theory of Types. A similar difficulty is encountered in the principle of mathematical induction.

Axiom of Reducibility. To overcome these and other difficulties encountered with the Ramified Theory of Types, Russell and Whitehead introduced what they termed the Axiom of Reducibility. This axiom states that to any function of any order and any type, there corresponds a formally equivalent first order function of the same type. With the use of this axiom, the identity relation can be defined in terms of first order functions; and many of the difficulties mentioned in the preceding paragraph are said to be overcome. Russell's own argument for the acceptance of the Axiom of Reducibility is a pragmatic one caused by the difficulties encountered with the Theory of Types.

> The reason for accepting any axiom, as for accepting any other proposition, is always largely inductive, namely that many other propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it.
>
> The axiom of reducibility is introduced in order to legitamize a great mass of reasoning in which prima facie, we are concerned with such notions as "all properties of _a_" or "all _a_-functions" and in which, nevertheless, it seems scarcely possible to suspect any substantial error.[1]

The question arises as to whether there are less prohibitive methods of avoiding the antinomies than the Theory of Types. It should be made clear that the fundamental reason for the introduction of the rules given by the Theory of Types is that they exclude contradictions. It would, of

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[1] Bertrand Russell, Principia Mathematica (first edition; Cambridge, Cambridge University Press, 1910), I, pp. 55-56.

course, be more convenient if a method or theory could be found which was less complex and less prohibitive than the various theories of types which have been formulated to date.

The construction of such weaker rules which would still avoid the antinomies is still an open problem. Russell himself readily states a similar opinion:

> ...the theory of types emphatically does not belong to the finished and certain part of our subject; much of the theory is still inchoate, confused and obscure. But the need of some doctrine of types is less doubtful than the precise form the doctrine should take.[2]

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[2]Bertrand Russell, Introduction to Mathematical Philosophy (New York: The Macmillan Company, 1938), p. 135.

## CHAPTER VI

## SUMMARY OF THE STUDY

The existence of the antinomies indicates that logic, at least as it was taken intuitively in the nineteenth century, is inadequate as a final criterion of mathematical proof. Thus, the absolute consistency of classical mathematics, which was thought to have been obtained by the arithmetization of analysis, turns into a false optimism. However, the existence of antinomies does not imply that mathematics is in great danger of collapse. As was mentioned in Chapter I, there have been mathematical crises before, i.e. the discovery of irrational numbers and the paradoxes of the Eleatic school and the realization that the basis of the calculus used in the seventeenth and eighteenth centuries was insecure. In each instance mathematics has withstood the challenge and has actually become a more profound science because of the experience. The third crisis, the period from the discovery of the first logical antinomy to the present time, has had a disturbing influence on a few prominent mathematicians; yet, most mathematicians have not concerned themselves with the problem of the antinomies or with problems in the foundations of mathematics. Perhaps this situation is as it should be, for the fruitful results of the working mathematicians might cease if all competent mathematicians concerned themselves mainly with foundational problems.

However, the problem of the antinomies is still an important one and is still unresolved. Although a great amount of literature is devoted to the exposition of the various antinomies and although a number of methods of avoiding them have been offered, there is at present no one explanation which is universally accepted. The majority of the mathematical logicians concur that the present practice of attaching some form of type theory onto a logistic system is painfully inadequate for the following reasons. First, the attachment of a theory of types complicates the system because all the versions of the theory of types, simple or ramified, are themselves quite complex. Second, in many instances the restrictions of type theory prove too prohibitive in that they destroy the framework of essential mathematics.

There are essentially three major approaches to the problem of the antinomies: the Logistic, the Intuitionistic, and the Formalistic. In all three approaches, the position taken with regard to the significance or solution of the antinomies is a direct application of a philosophical viewpoint concerning the nature of the foundations of mathematics.

<u>Logistic approach</u>. The Logistic approach to the problem of the antinomies is simply a necessary byproduct of the Logistic approach to the foundations of mathematics. The Logistic approach stems from a fundamental distinction made by adherents of a philosophy known as Logical Positivism. This distinction divides all true propositions into two categories: (1) synthetic statements, which are empirical truths; and (2) analytic statements, which are logical and mathematical truths. In the category of analytic statements, there are theoretically five possible beliefs concerning the relationship between the set of logical truths and the set of mathematical truths. These beliefs are: the two sets are identical, the set of mathematical truths is a proper subset of the set of logical truths, the set of logical truths is a proper subset of the set of mathematical truths, the two sets are disjoint, or the two sets intersect.

The Logistic thesis is that the set of mathematical truths is a proper subset of the set of logical truths. This idea can be expressed by the following formulation: All specific mathematical terms are capable of being defined in terms of a logical vocabulary, and the proof of a mathematical theorem can be given using only the axioms and rules of inference of logic.

The first attempt to reduce the propositions of mathematics to propositions of logic was undertaken by Frege.

He was able to formulate the concepts associated with the system of natural numbers in propositions involving logical terms. Frege provided the stimulus for the monumental undertaking of Russell and Whitehead in writing the Principia Mathematica, in which they attempted to show that the whole of classical mathematics could be derived from the first and second order propositional calculus. However, as has been previously discussed in this paper, Russell become acutely aware that antinomies could be derived from the axioms of logic unless precautions were taken. The approach of the Logicists was to develop some form of a theory of types to avoid the derivation of the antinomies. However, the attachment of a type theory to a system of logic, along with such axioms as the axiom of infinity and the axiom of reducibility, made it apparent that the approach of the Logicists was not as "logical" as they had originally intended it to be. The Logicists were finally left with the necessity of reformulating the foundations of logic so that the antinomies would not be derivable, while at the same time eliminating the need for questionable axioms and a theory of types. Such a reformulation is still an unresolved problem for this school.

Intuitionist approach. As was mentioned in the preceding discussion, Logicism views the antinomies as evidence that something is wrong with certain mathematical methods;

more specifically, the Logicists hold the view that if mathematics is ultimately based upon logic, then the logic itself must be reformed to provide an antinomy-free basis for mathematics. The attitude of the Intuitionists is far more radical than that of any other school. The Intuitionists maintain that the concept of infinity and its implications have not received the correct "treatment" in modern mathematics. The view taken by the Intuitionists is not, as is sometimes erroneously believed, that a concept of infinity has no place in mathematics, but rather that mathematicians have tended to treat infinity with methods created for finite domains.

According to the Intuitionists the emergence of antinomies is but a secondary symptom, caused by the unstableness of more important branches of mathematics. For example, the Intuitionists explain the emergence of the antinomies in set theory by pointing out that set theory makes abundant and unlimited use of the concept of infinity. Thus, according to the Intuitionists, the challenge of the antinomies can be met only by reforming mathematics as a whole; this reform would automatically exclude not only the antinomies actually found to be present but also any other conceivable antinomy. The results of the Intuitionists and Neo-Intuitionists in reformulating the foundations of mathematics are quite metaphysical and bear a great similarity to the ideas advanced by the French philosopher, Henri Bergson--namely, that the real world is known only through primordial intuition. The Intuitionists offer no "solution as such" to the antinomies. They either disregard the antinomies as a meaningless group of words without any constructive validity; or they ignore the problem entirely, stating simply that in the correct interpretation of mathematics, the antinomies will not arise. Either view is, of course, quite unsatisfactory to anyone who takes a more positivistic approach to the problem of the antinomies.

Formalist approach. The proponents of the Logistic approach to the foundations of mathematics found themselves in need of a proof of the consistency of their systems, especially with the various theories of types attached. The classical method of providing such a proof--that is, the exhibition of a model taken from a theory whose consistency was not in doubt--would not suffice. It was not that the classical methods of displaying such a model were not valid, for Beltrami in 1868 had proven that certain non-Euclidean geometries were consistent, relative to Euclidean geometry, by constructing a model for them within Euclidean geometry. Also by 1899 Hilbert had shown that Euclidean geometry was consistent, relative to the real number system, by constructing a model of Euclidean geometry within real number theory. However, in view of the antinomies, none of the classical models appeared satisfactory; a different approach was needed.

It was Hilbert who formulated the desired approach, which was that it must be shown that the classical mathematical proof procedures are strong enough to derive the whole of classical mathematics from a "suitable" set of axioms, but that the antinomies will not at the same time be derivable. Hilbert's fundamental assumption was that classical mathematics was basically sound. Hilbert proposed to establish the consistency of classical mathematics by examining the language in which it was expressed.

> This language was to be formulated so completely and so precisely that its reasonings could be regarded as derivations according to precisely stated rules--rules which were mechanical in the sense that the correctness of their application could be seen by inspection of the symbols themselves as concrete physical objects, without regard to any meaning which they might or might not have.[1]

Hilbert's approach, labeled Formalism, was to make these formalized reasonings the subject of a new method of mathematical investigation which he called metamathematics. In examining the language of mathematics in a metalanguage, Hilbert sought to allow only those methods which he felt were absolutely certain in the metalanguage. He hoped to establish

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[1]Haskell B. Curry, Foundations of Mathematical Logic (New York: McGraw-Hill Book Company, Inc., 1963), p. 11.

the consistency of classical mathematics by such means. Hilbert's dream was dealt a shattering blow in 1931, when Kurt Gödel published a proof alluding to the fact that the consistency of a system powerful enough to construct classical mathematics could not be established by means which could be formalized in the theory itself--that is, either the theory is inconsistent, or it is inadequate to formalize any proof of its own consistency.

Conclusions. In the final analysis, the problem of the antinomies must be directly related to some philosophical view concerning the foundations of mathematics and logic. The problem of the antinomies is directly related to some of the "unresolved questions" in mathematics. For instance, the question whether mathematical propositions are all analytic statements or whether there exists a synthetic apriori is a constantly recurring problem in the philosophy of mathematics. The demonstration that there exist non-Euclidean geometries which could be constructed with a relative consistency to Euclidean geometry at first appeared to swing the evidence in favor of a Logistic, analytic view of mathematics. Then the emergence of Gödel's Proof, which showed that any system strong enough to derive theorems of classical mathematics must contain certain propositions which are undecidable in the system, seemed to add support to the viewpoint that there can and do exist mathematical propositions which are synthetic apriori.

However, in either instance one is left with a rather unsatisfactory choice. The Logistic or Formalist approaches, while eliminating disturbing metaphysics from mathematics, suffer from their insistence on "absolute consistency," perhaps an admirable goal, but one which has so far been unobtainable. The Intuitionist attitude is extremely unpalatable to those who are not of its "belief," for its reliance upon primordial intuition makes communication in any written or oral language difficult. Also, in the opinion of the writer, the Intuitionist viewpoint introduces into mathematics metaphysical obscurities which are a detriment to a logical system.

Perhaps there is no one "correct" view of mathematics, and, in turn, of the antinomies. It may be that absolute rigor is a delusion and that the best that can be hoped for is a pragmatic approach that recognizes that in mathematics we have only relative truth. In any case it certainly does not detract from the beauty, the usefulness, and the intellectual appeal of mathematics to recognize that there remain unresolved problems within its structure.

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# APPENDIX

## LOGICAL SYMBOLS USED IN PAPER

| | |
|---|---|
| $\cdot$ | and |
| $\vee$ | or, inclusive |
| $\sim$ | negation |
| $=$ | identity |
| $\longrightarrow$ | material implication |
| $\longleftrightarrow$ | material equivalence (if and only if) |
| $(x)$ | universal quantification |
| $F$ | property variable |
| $\exists(x)$ | existential quantification |