THE ANALYTICAL REPRESENTATION OF POINTS, LINES, PLANES, AND SPHERES ASSOCIATED WITH A TETRAHEDRON

A THESIS

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Rollin Quinn

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Recent years have witnessed a remarkable increase in interest in the study of modern pure geometry. This is especially true of that division of geometry usually designated as the "geometry of the triangle".

Because of the relative newness of this revival of interest, only a few books by American writers[1] are now available on the subject. However, an ever increasing amount of new material is being published in mathematical journals, especially the American Mathematical Monthly.[2]

For reasons that are not difficult to fathom, the developments in two dimensional geometry have far out-distanced those in three dimensions. To the writer this situation seems regrettable. This thesis is being written with the hope that it may contribute, in some measure, to the extension and development of solid geometry.

One may liken the writing of a thesis of this nature to the taking of a trip through unfamiliar regions. Vast preparedness must be made. Unforseen difficulties must be surmounted. Joy and sorrow, success and failure are but elements of the composite. In a voyage of discovery

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[1] Nathan Altshiller-Court, Modern Pure Solid Geometry (New York: Macmillan Company, 1935).

[2] The American Mathematical Monthly is a valuable source of new material in this field. This publication offers specific suggestions in articles dealing with pure geometry and in the department of Problems and Solutions.

it appears to the writer permissible to cut adrift from the usual vehicles of travel, and to proceed by any available route or by any available vehicle.

This was attempted recently, in two dimensional geometry by Charles Girod[5] who developed and collected analytical representations for points, lines, and circles associated with a triangle with the idea of providing a vehicle of discovery.

In the field of solid geometry an analytical vehicle seems even more necessary by virtue of the difficulty usually encountered in visualizing space configurations. Analytical expressions of points, lines, planes, and spheres yield geometrical interpretations without the necessity of a cumulative study. Collinearity and coplanarity of points are frequently obvious from their analytical expressions, even when the points have apparently little, if any, geometric connection.

It is the purpose of this thesis to provide an analytical foundation for the study of the geometry of the tetrahedron. It is hoped that the analytical framework herein provided, and summarized in the final chapter, will be of value in new studies or in expanding the study of this subject. The algebraic representations of points, lines, planes, and spheres given in this thesis may provide a useful and somewhat adequate foundation for further study of the geometry of the tetrahedron. The writer is fully aware of the fact that this study is by no means exhaustive, but presents it simply as an analytical guide to the searcher in the field of modern solid geometry.

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[5] Charles Girod, _The Analytical Representation of Points, Lines, and Circles Associated with a Triangle_ (Emporia: Unpublished thesis, Kansas State Teachers College, May, 1956).

CHAPTER II

QUADRIPLANAR COORDINATES

In order to provide an analytical representation of the configurations of space, it is most essential that a suitable coordinate system be adopted. The writer has chosen quadriplanar coordinates[1] (defined in next paragraph) as the simplest and the most satisfactory means of displaying both metrical and projective properties of figures associated with the tetrahedron. However, tetrahedral coordinates[2] are employed at various times where conditions suggest their use. All solutions in terms of tetrahedral coordinates are so indicated throughout this study.

It seems wise to indicate the essential features of the system of quadriplanar coordinates in order to avoid a confusion of terms or of implications.

Let A_1, A_2, A_3, A_4 be the points of intersection of four independent planes taken three at a time. The tetrahedron with vertices A_1, A_2, A_3, A_4, is called the fundamental tetrahedron. The areas of the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ are denoted by a_1, a_2, a_3, a_4, respectively. The length of the edges A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4 are denoted by a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}, respectively. Let the directed perpendicular distances of any point X from the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ be respectively x_1, x_2, x_3, x_4; then

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[1] C. Smith, <u>Solid Geometry</u> (London: Macmillan and Company, 1928), p. 164.

[2] Ibid., p. 164.

x_1, x_2, x_3, x_4, or numbers proportional to them are called quadriplanar coordinates of the point X referred to the tetrahedron $A_1A_2A_3A_4$. The distances, x_1, x_2, x_3, x_4, of the point X are so directed that they are all positive when X is in the interior of the tetrahedron. That is, the directed distance x_1 of a point X from a face of the tetrahedron is positive when X is on the same side of that face as is the vertex A_1. Three of these four actual distances, or merely the ratios of the four distances, are sufficient to determine the position of a point.

The four distances, x_1, x_2, x_3, x_4, are connected by the relation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 3\triangle$$

where $\triangle$ is the volume of the tetrahedron $A_1A_2A_3A_4$. If k is a common multiplier of the coordinates x_1, x_2, x_3, x_4 such that kx_1, kx_2, kx_3, kx_4 are the actual distances of the point X from the faces of the tetrahedron of reference, that is $\overline{x_1} = kx_1$, then

$$k(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 3\triangle$$

$$\text{or } k = \frac{3\triangle}{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4} = 0$$

The coordinates of the vertices, A_1, A_2, A_3, A_4, are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). The equations of the faces opposite these vertices are respectively $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$. The equations of the edges are as follows:

A_1A_2: $x_3 = 0$, $x_4 = 0$;

A_1A_3: $x_2 = 0$, $x_4 = 0$;

A_1A_4: $x_2 = 0$, $x_3 = 0$;

A_2A_3: $x_1 = 0$, $x_4 = 0$;

A_2A_4: $x_1 = 0$, $x_3 = 0$;

A_3A_4: $x_1 = 0$, $x_2 = 0$.

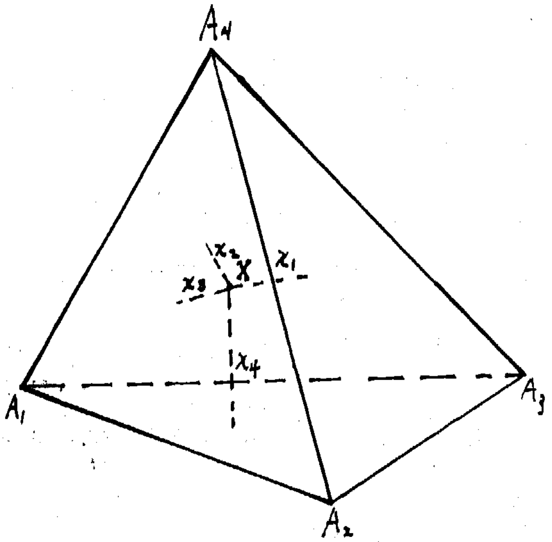

FIGURE 1

FUNDAMENTAL TETRAHEDRON

The equation $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0$ represents a plane which will be designated by (m) or the plane m. Its intersection with the faces of the tetrahedron are

$$x_1 = 0, \quad m_2x_2 + m_3x_3 + m_4x_4 = 0;$$

$$x_2 = 0, \quad m_1x_1 + m_3x_3 + m_4x_4 = 0;$$

$$x_3 = 0, \quad m_1x_1 + m_2x_2 + m_4x_4 = 0;$$

$$x_4 = 0, \quad m_1x_1 + m_2x_2 + m_3x_3 = 0.$$

The points of intersection of the plane (m) with the six edges are A_1A_2: $(m_2, -m_1, 0, 0)$, A_1A_3: $(m_3, 0, -m_1, 0)$, A_1A_4: $(m_4, 0, 0, -m_1)$, A_2A_3: $(0, m_3, -m_2, 0)$, A_2A_4: $(0, m_4, 0, -m_2)$, A_3A_4: $(0, 0, m_4, -m_3)$.

The equation of the ideal plane, or plane at infinity, is a_1x_1 +

$a_2x_2 + a_3x_3 + a_4x_4 = 0$

The equation of a plane which passes through three points X', X'', X''' is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \\ x_1''' & x_2''' & x_3''' & x_4''' \end{vmatrix} = 0.$$

The condition that four points, X', X'', X''', X'''', lie in a plane is

$$\begin{vmatrix} x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \\ x_1''' & x_2''' & x_3''' & x_4''' \\ x_1'''' & x_2'''' & x_3'''' & x_4'''' \end{vmatrix} = 0.$$

Any three planes not belonging to the same pencil intersect in a point. Let the equations of the given planes be

$$m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0,$$

$$n_1x_1 + n_2x_2 + n_3x_3 + n_4x_4 = 0,$$

$$p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 = 0.$$

These have the unique solution[3]

$$x_1 : x_2 : x_3 : x_4 = \begin{vmatrix} m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \\ p_2 & p_3 & p_4 \end{vmatrix} : -\begin{vmatrix} m_1 & m_3 & m_4 \\ n_1 & n_3 & n_4 \\ p_1 & p_3 & p_4 \end{vmatrix} : \begin{vmatrix} m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \\ p_1 & p_2 & p_4 \end{vmatrix} : -\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

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[3] F. S. Woods, Higher Geometry (New York: Ginn and Co., 1922), p. 196.

The condition that four planes k, l, m, n have a point in common is that the determinant of the associated coefficients be equal to zero.

$$\begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix} = 0$$

If the rank is three, the four planes form a bundle; if the rank is two, the four planes form a pencil; if the rank is one, the four planes are coincident.

Two finite planes m, n not coincident, are parallel if they form a pencil with the ideal plane. The condition for this is that the matrix

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{Vmatrix}$$ be of rank two.

The equations of the line of intersection of the planes m, n are $mx = 0$, $nx = 0$.

The equations of the line PQ may be written in the parametric form,

$$x_1 = h\,p_1 + kq_1\,,$$
$$x_2 = h\,p_2 + kq_2\,,$$
$$x_3 = h\,p_3 + kq_3\,,$$
$$x_4 = h\,p_4 + kq_4\,,$$

where h, k are parameters not both zero. They may also be written

$$\frac{x_1 - q_1}{p_1 - q_1} = \frac{x_2 - q_2}{p_2 - q_2} = \frac{x_3 - q_3}{p_3 - q_3} = \frac{x_4 - q_4}{p_4 - q_4}\,.$$

The ideal line in a plane m is the line of intersection of the plane and the ideal plane. Its equation is

$$m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0.$$

The ideal point on the line of intersection of $\sum mx = 0$, $\sum mx = 0$ is the point where a third plane n intersects the ideal line, namely,

$$x_1 : x_2 : x_3 : x_4 = \begin{vmatrix} a_2 & a_3 & a_4 \\ m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \end{vmatrix} : -\begin{vmatrix} a_3 & a_4 & a_1 \\ m_3 & m_4 & m_1 \\ n_3 & n_4 & n_1 \end{vmatrix} : \begin{vmatrix} a_4 & a_1 & a_2 \\ m_4 & m_1 & m_2 \\ n_4 & n_1 & n_2 \end{vmatrix} : -\begin{vmatrix} a_1 & a_2 & a_3 \\ m_4 & m_1 & m_2 \\ n_1 & n_2 & n_3 \end{vmatrix} .$$

The ideal point on the line PQ is

$$x = h\,p_i + kq_i \quad \text{where } h:k = -\sum aq : \sum a\,p$$

If P (p_1, p_2, p_3, p_4) (Figure 2) be any point, then the equations of a line[4] through P may be found. The coordinates of Q and X are respectively (q_1, q_2, q_3, q_4) and (x_1, x_2, x_3, x_4).

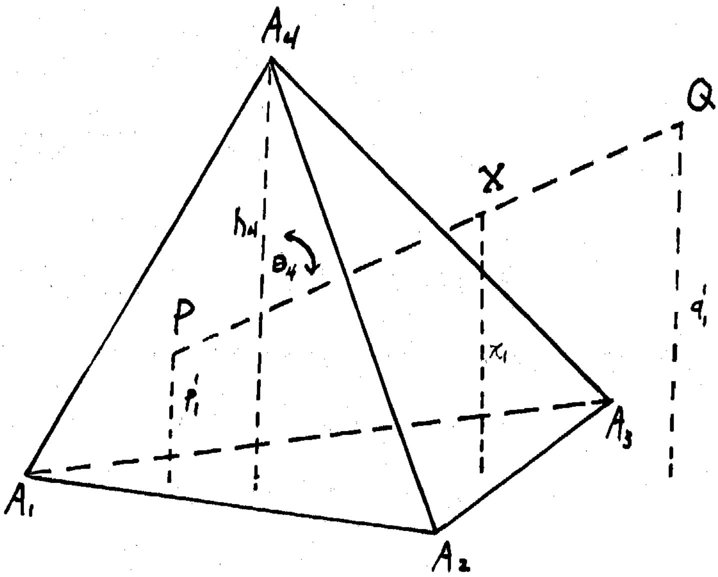

FIGURE 2

A LINE THROUGH A POINT P

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[4] C. Smith, Solid Geometry (London: Macmillan and Company, 1928), p. 167

PQ: $$\frac{x_1 - p_1}{q_1 - p_1} = \frac{x_2 - p_2}{q_2 - p_2} = \frac{x_3 - p_3}{q_3 - p_3} = \frac{x_4 - p_4}{q_4 - p_4}$$

If $\frac{PX}{XQ} = u$, then

$$\bar{x}_1 : \bar{x}_2 : \bar{x}_3 : \bar{x}_4 = \frac{\bar{p}_1 + u\bar{q}_1}{1 + u} : \frac{\bar{p}_2 + u\bar{q}_2}{1 + u} : \frac{\bar{p}_3 + u\bar{q}_3}{1 + u} : \frac{\bar{p}_4 + u\bar{q}_4}{1 + u}$$

or $x_i = p_i + uq_i$.

$\bar{p}_i = \triangle PA_jA_kA_l = 1/3\ a_i p_i'$, where primes are used to denote the distances in quadriplanar coordinates.

$\bar{x}_i = \triangle XA_jA_kA_l = 1/3\ a_i x_i$.

$\bar{x}_i - \bar{p}_i = 1/3\ a_i(x_i' - p_i') = 1/3\ a_i PX \cos\theta_i$, where θ_i denotes the angle between the line PQ and the altitude of the fundamental tetrahedron. Then

$$\frac{x_i\ p_i}{a_i \cos\theta_i} = \frac{PX}{3} = \frac{\delta}{3}, \text{ where } \delta = PX.$$

The equations of a line through the point P may be expressed as

$$\frac{x_1 - p_1}{a_1 \cos\theta_1} = \frac{x_2 - p_2}{a_2 \cos\theta_2} = \frac{x_3 - p_3}{a_3 \cos\theta_3} = \frac{x_4 - p_4}{a_4 \cos\theta_4} = \frac{\delta}{3}$$

Since the sum of the projections of four faces of the tetrahedron on a plane perpendicular to the line PQ is zero,

$$a_1 \cos\theta_1 + a_2 \cos\theta_2 + a_3 \cos\theta_3 + a_4 \cos\theta_4 = 0.$$

$$x_1 = p_1 + \frac{a_1 \cos\theta_1}{3}$$

$$x_2 = p_2 + \frac{a_2 \cos\theta_2}{3}$$

$$x_3 = p_3 + \frac{a_3 \cos\theta_3}{3}$$

$$x_4 = p_4 + \frac{a_4 \cos\theta_4}{3}$$

The parametric form of the equations of the line PQ are

$x_1 = p_1 + q_1 t,$

$x_2 = p_2 + q_2 t,$

$x_3 = p_3 + q_3 t,$

$x_4 = p_4 + q_4 t,$

When $t = 0$, $X \equiv P$. When $t = \infty$, $X \equiv Q$.

$$\Sigma a_i p_i + t \Sigma a_i q_i = 0, \quad t = \frac{-\Sigma a_i p_i}{a\, q}, \quad x_i = p_i - q_i \frac{\Sigma a_i p_i}{\Sigma a_i q_i},$$

or $x_i = p_i \Sigma a_i q - q_i \Sigma a_i p$ which is the ideal point on the line PQ in parametric form.

Any plane parallel to (m) is of the form $\Sigma mx - \lambda \Sigma x = 0$. The plane through P parallel to (m) is

$$\Sigma m_i x_i - \frac{\Sigma m_i p_i \Sigma x}{\Delta} = 0 \quad \text{or } \Sigma p_i \Sigma m_i x_i - \Sigma m_i p_i \Sigma x_i = 0, \text{ or}$$

$$\begin{vmatrix} \Sigma m_i x_i & \Sigma x_i \\ \Sigma m_i p_i & \Sigma p_i \end{vmatrix} = 0$$

, this may be written as follows:

$$(m_1 - \frac{p_1}{\Delta})x_1 - (m_2 - \frac{p_2}{\Delta})x_2 - (m_3 - \frac{p_3}{\Delta})x_3 - (m_4 - \frac{p_4}{\Delta})x_4 = 0.$$

By letting $M_{ij} = m_j - m_i$, the plane through P parallel to m may be written

$$\begin{aligned}
&x_1\{\qquad\qquad p_2(m_2-m_1) - p_3(m_3-m_1) - p_4(m_4-m_1)\} \\
&-x_2\{p_1(m_1-m_2) \qquad\qquad - p_3(m_3-m_2) - p_4(m_4-m_2)\} \\
&-x_3\{p_1(m_1-m_3) - p_2(m_2-m_3) \qquad\qquad - p_4(m_4-m_3)\} \\
&-x_4\{p_1(m_1-m_4) - p_2(m_2-m_4) - p_3(m_3-m_4) \qquad\qquad\} = 0.
\end{aligned}$$

This may be written as $\Sigma_i \Sigma_j x_i(p_j m_j - m_i \Sigma p_i) = 0.$

The points on the ideal line of a plane (m) are

plane $x_1 = 0$: $(0: \ a_3m_4 - a_4m_3 : a_4m_2 - a_2m_4 : a_2m_3 - a_3m_2)$,

plane $x_2 = 0$: $(a_3m_4 - a_4m_3 : 0 : a_4m_1 - a_1m_4 : a_1m_3 - a_3m_1)$,

plane $x_3 = 0$: $(a_4m_2 - a_2m_4 : a_1m_4 - a_4m_1 : 0 : a_2m_1 - a_1m_2)$,

plane $x_4 = 0$: $(a_2m_3 - a_3m_2 : a_3m_1 - a_1m_3 : a_1m_2 - a_2m_1 : 0)$.

The perpendiculars[5] from the angular points of the tetrahedron of reference on the plane whose equation is

$$m_1X_1 + m_2X_2 + m_3X_3 + m_4X_4 = 0$$

are proportional to m_1, m_2, m_3, m_4. (Figure 3)

The coordinates (tetrahedral coordinates) of M_{12} are $(-m_2, m_1, 0, 0)$, then $X_1 : X_2 = -m_2 : m_1$ since $m_1X_1 - m_2X_2 = 0$ or $\dfrac{X_1}{X_2} = \dfrac{-m_2}{m_1}$.

$$r_1 : r_2 = A_1M_{12} : - M_{12}A_2 = \triangle A_1M_{12}A_3 : - \triangle A_2M_{12}A_3$$

$$= \triangle A_4A_1M_{12}A_3 : \triangle A_4A_2M_{12}A_3$$

$$= X_2 : - X_1 .$$

Therefore $r_1 : r_2 = m_1 : m_2$, and similarly for the other distances r_3 and r_4, so then $r_1 : r_2 : r_3 : r_4 = m_1 : m_2 : m_3 : m_4$, and $\overline{r}_i = km_i$. If the actual distance, r_i of any vertex of the tetrahedron not lying in the face m in plane (m) be known, k may be determined by the relationship

$$k = \frac{\overline{r}_i}{m_i}.$$

The distance, then, of any point P from the plane (m) is given by the formula

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[5] C. Smith, Solid Geometry (London: Macmillan and Company, 1928), p. 186.

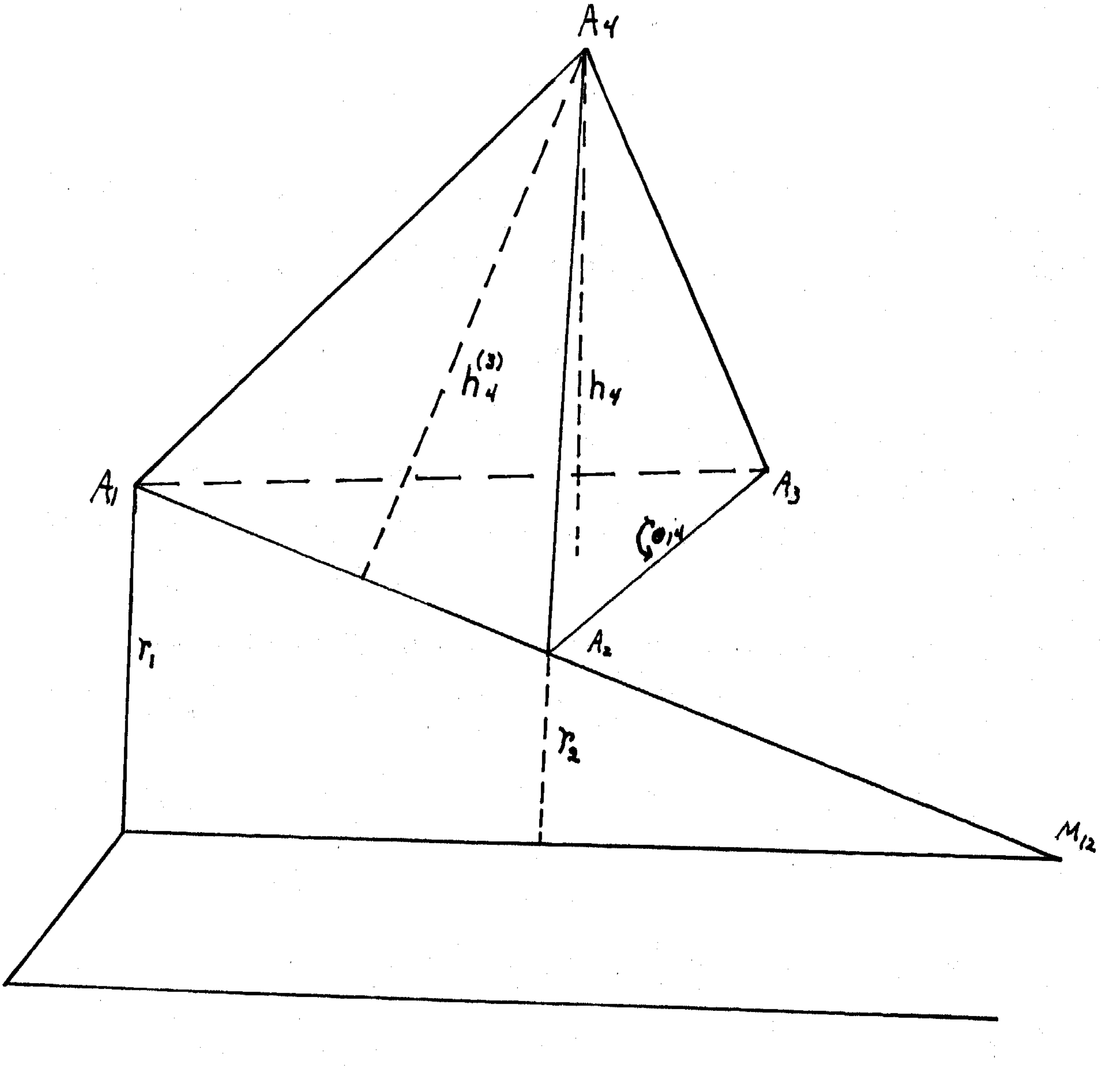

FIGURE 5

PERPENDICULARS AND ANGLES

$$\delta = k\,(m_1p_1 - m_2p_2 - m_3p_3 - m_4p_4) \text{ where } k = \frac{F_i}{m_i} = \frac{a_{ij}\sin(a_{ij}m)}{m_i - m_j}$$

$$\delta = \frac{\Sigma a_{ij}\sin(a_{ij}m)}{\Sigma m_i - \Sigma m_j}.$$

The general equation of a sphere[6] is

$$a_{12}^2a_1a_2x_1x_2 + a_{13}^2a_1a_3x_1x_3 + a_{14}^2a_1a_4x_1x_4 + a_{23}^2a_2a_3x_2x_3 + a_{24}^2a_2a_4x_2x_4$$
$$+ a_{34}^2a_3a_4x_3x_4 + (m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4)(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 0.$$

The equation of the circumsphere is

$a_{12}^2a_1a_2x_1x_2 + a_{13}^2a_1a_3x_1x_3 + a_{14}^2a_1a_4x_1x_4 + a_{23}^2a_2a_3x_2x_3 + a_{24}^2a_2a_4x_2x_4 + a_{34}^2a_3a_4x_3x_4 = 0.$ The equation of the ideal plane is $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0.$ The equation of the radical plane of the circumsphere and any sphere is $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0.$

The center of the general sphere is found by solving the equations $F_1 : F_2 : F_3 : F_4 = a_1 : a_2 : a_3 : a_4$, where

$$F_1 = 2a_1m_1x_1 + (a_{12}^2a_1a_2 + a_1m_2 + a_2m_1)x_2 + (a_{13}^2a_1a_3 + a_1m_3 + a_3m_1)x_3 + (a_{14}^2a_1a_4 + a_1m_4 + a_4m_1)x_4,$$

$$F_2 = (a_{12}^2a_1a_2 + a_1m_2 + a_2m_1)x_1 + 2a_2m_2x_2 + (a_{23}^2a_2a_3 + a_2m_3 + a_3m_2)x_3 + (a_{24}^2a_2a_4 + a_2m_4 + a_4m_2)x_4,$$

$$F_3 = (a_{13}^2a_1a_3 + a_1m_3 + a_3m_1)x_1 + (a_{23}^2a_2a_3 + a_2m_3 + a_3m_2)x_2 + 2a_3m_3x_3 + (a_{34}^2a_3a_4 + a_3m_4 + a_4m_3)x_4,$$

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[6] C. Smith, Solid Geometry (London: Macmillan and Company, 1928), p. 173.

$$F_4 = (a_{14}^2 a_1 a_4 + a_1 m_4 + a_4 m_1)x_1 + (a_{24}^2 a_2 a_4 + a_2 m_4 + a_4 m_2)x_2 +$$

$$(a_{34}^2 a_3 a_4 + a_3 m_4 + a_4 m_3)x_3 + 2 a_4 m_4 x_4.$$

All spheres pass through the absolute, the equation of which is

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 \\ a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + \\ a_{34}^2 a_3 a_4 x_3 x_4 + (m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4)(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 0. \end{cases}$$

CHAPTER III

POINTS, LINES, PLANES, AND SPHERES ASSOCIATED WITH A GIVEN POINT

Associated with any point in space are a number of points, lines, planes, and spheres having some significant geometrical connection with it. In this chapter the analytical representation of the most important will be presented.

<u>The pedal points associated with P.</u> Let $P(p_1, p_2, p_3, p_4)$ be any given point. The pedal points, P_1, P_2, P_3, P_4, (Figure 4) are the projections of the point P upon the faces of the tetrahedron of reference. The coordinates of the pedal points are obviously

$P_1:(0, p_2, p_3, p_4)$,

$P_2:(p_1, 0, p_3, p_4)$,

$P_3:(p_1, p_2, 0, p_4)$,

$P_4:(p_1, p_2, p_3, 0)$.

<u>The pedal tetrahedron.</u> The pedal tetrahedron is here defined as the tetrahedron having the points P_1, P_2, P_3, P_4 as vertices. The faces of the pedal tetrahedron $P_2P_3P_4$, $P_1P_3P_4$, $P_1P_2P_4$, and $P_1P_2P_3$ have the equations

$$-2p_2p_3p_4x_1 + p_1p_3p_4x_2 + p_1p_2p_4x_3 + p_1p_2p_3x_4 = 0,$$

$$p_2p_3p_4x_1 - 2p_1p_3p_4x_2 + p_1p_2p_4x_3 + p_1p_2p_3x_4 = 0,$$

$$p_2p_3p_4x_1 + p_1p_3p_4x_2 - 2p_1p_2p_4x_3 + p_1p_2p_3x_4 = 0,$$

$$p_2p_3p_4x_1 + p_1p_3p_4x_2 + p_1p_2p_4x_3 - 2p_1p_2p_3x_4 = 0.$$

<u>The pedal sphere.</u> The pedal sphere of P is the sphere which passes through the four pedal points P_1, P_2, P_3, P_4. By substituting the coordinates of P_1, P_2, P_3, P_4 in the general equation of a sphere

the following relations are obtained:

$$p_2m_2 + p_3m_3 + p_4m_4 = \frac{-a_{23}^2a_2a_3p_2p_3 + a_{24}^2a_2a_4p_2p_4 + a_{34}^2a_3a_4p_3p_4}{a_2p_2 + a_3p_3 + a_4p_4},$$

$$p_1m_1 + p_3m_3 + p_4m_4 = \frac{-a_{13}^2a_1a_3p_1p_3 + a_{14}^2a_1a_4p_1p_4 + a_{34}^2a_3a_4p_3p_4}{a_1p_1 + a_3p_3 + a_4p_4},$$

$$p_1m_1 + p_2m_2 + p_4m_4 = \frac{-a_{12}^2a_1a_2p_1p_2 + a_{14}^2a_1a_4p_1p_4 + a_{24}^2a_2a_4p_2p_4}{a_1p_1 + a_2p_2 + a_4p_4},$$

$$p_1m_1 + p_2m_2 + p_3m_3 = \frac{-a_{12}^2a_1a_2p_1p_2 + a_{13}^2a_1a_3p_1p_3 + a_{23}^2a_2a_3p_2p_3}{a_1p_1 + a_2p_2 + a_3p_3}.$$

Let

$$R_1 = \frac{a_{23}^2a_2a_3p_2p_3 + a_{34}^2a_3a_4p_3p_4 + a_{24}^2a_2a_4p_2p_4}{p_1(a_2p_2 + a_3p_3 + a_4p_4)},$$

$$R_2 = \frac{a_{34}^2a_3a_4p_3p_4 + a_{13}^2a_1a_3p_1p_3 + a_{14}^2a_1a_4p_1p_4}{p_2(a_1p_1 + a_3p_3 + a_4p_4)},$$

$$R_3 = \frac{a_{24}^2a_2a_4p_2p_4 + a_{14}^2a_1a_4p_1p_4 + a_{12}^2a_1a_2p_1p_2}{p_3(a_1p_1 + a_2p_2 + a_4p_4)},$$

$$R_4 = \frac{a_{23}^2a_2a_3p_2p_3 + a_{13}^2a_1a_3p_1p_3 + a_{12}^2a_1a_2p_1p_2}{p_4(a_1p_1 + a_2p_2 + a_3p_3)},$$

and let

$S = (R_1 + R_2 + R_3 + R_4)/3.$

Then the values of m_1, m_2, m_3, m_4 which determine the pedal sphere are $m_1 = R_1 - S$, $m_2 = R_2 - S$, $m_3 = R_3 - S$, $m_4 = R_4 - S$, and the equation of the pedal sphere is

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 + (R_1 x_1 + R_2 x_2 + R_3 x_3 + R_4 x_4)(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = S(x_1 + x_2 + x_3 + x_4)(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4).$$

Quadriplanar polar of P. Associated with any given point P are a number of other points which are the harmonic conjugate of P, or of the pedal points of P with respect to other pairs of points associated with the tetrahedron, and also the harmonic conjugates of these in turn. The relationships of these are indicated either in figure four, or by considering their coordinates, or by both. The harmonic conjugate of P with respect to A_4P_4 will be denoted by P^{iv}; the projection of P_4 on the edge A_1A_2 from the vertex A_3 is denoted by P_{34}. The harmonic conjugate of P_4 with respect to A_3P_{34} is denoted by P_4'''. The harmonic conjugate of P_{34} with respect to A_1A_2 is denoted by P_{34}'; similarly for other pairs of points.

The coordinates of these points are given in the accompanying list:

$P_{12}\ (0,\ 0,\ p_3,\ p_4)$,	$P_{12}'\ (0,\ 0,\ -p_3,\ p_4)$,
$P_{13}\ (0,\ p_2,\ 0,\ p_4)$,	$P_{13}'\ (0,\ -p_2,\ 0,\ p_4)$,
$P_{14}\ (0,\ p_2,\ p_3,\ 0)$,	$P_{14}'\ (0,\ -p_2,\ p_3,\ 0)$,
$P_{23}\ (p_1,\ 0,\ 0,\ p_4)$,	$P_{23}'\ (-p_1,\ 0,\ 0,\ p_4)$,
$P_{24}\ (p_1,\ 0,\ p_3,\ 0)$,	$P_{24}'\ (-p_1,\ 0,\ p_3,\ 0)$,
$P_{34}\ (p_1,\ p_2,\ 0,\ 0)$.	$P_{34}'\ (-p_1,\ p_2,\ 0,\ 0)$.

P' $(-p_1, p_2, p_3, p_4)$,

P'' $(p_1, -p_2, p_3, p_4)$,

P''' $(p_1, p_2, -p_3, p_4)$,

P^{iv} $(p_1, p_2, p_3, -p_4)$.

P_1 $(0, p_2, p_3, p_4)$,

P_2 $(p_1, 0, p_3, p_4)$,

P_3 $(p_1, p_2, 0, p_4)$,

P_4 $(p_1, p_2, p_3, 0)$.

P_1'' $(0, -p_2, -p_3, p_4)$,

P_1''' $(0, p_2, -p_3, p_4)$,

P_1^{iv} $(0, p_2, p_3, -p_4)$,

P_2' $(-p_1, 0, p_3, p_4)$,

P_2''' $(p_1, 0, -p_3, p_4)$,

P_2^{iv} $(p_1, 0, p_3, -p_4)$.

P_3' $(-p_1, p_2, 0, p_4)$,

P_3'' $(p_1, -p_2, 0, p_4)$,

P_3^{iv} $(p_1, p_2, 0, -p_4)$,

P_4' $(-p_1, p_2, p_3, 0)$,

P_4'' $(p_1, -p_2, p_3, 0)$,

P_4''' $(p_1, p_2, -p_3, 0)$.

The collinearity of sets of these points which lie in the same plane is well known.

The six points P'_{12}, P'_{13}, P'_{14}, P'_{23}, P'_{24}, P'_{34} are coplanar since the rank of the matrix of their coefficient is three. The plane of these points is called the quadriplanar polar plane[1] of P. Its equation is

$$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} = 0$$

[1] Nathan Altshiller-Court, Modern Pure Solid Geometry (New York: Macmillan Company, 1935), p. 235.

The equation may also be derived from the formula

$$\left(\frac{\partial f}{\partial x_1}\right)_p x_1 + \left(\frac{\partial f}{\partial x_2}\right)_p x_2 + \left(\frac{\partial f}{\partial x_3}\right)_p x_3 + \left(\frac{\partial f}{\partial x_4}\right)_p x_4 = 0.$$

In this case the surface, $f(x_1, x_2, x_3, x_4) = 0$, is the degenerate surface, $x_1x_2x_3x_4 = 0$, and $\left(\frac{\partial f}{\partial x_1}\right)_p = p_2p_3p_4$, $\left(\frac{\partial f}{\partial x_2}\right)_p = p_1p_3p_4$, $\left(\frac{\partial f}{\partial x_3}\right)_p = p_1p_2p_4$, $\left(\frac{\partial f}{\partial x_4}\right)_p = p_1p_2p_3$; so that the equation of the polar plane is

$$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} = 0.$$

The point P is called the pole of this plane[2]. Associated with any given plane (m) is a point which is its pole. If the equation of the plane $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0$, the coordinates of the pole are $\left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}, \frac{1}{m_4}\right)$.

Harmonic relationships. From the edges of the tetrahedron project planes through P. The equations of the planes determined by the point P and the six edges are:

A_1A_2P : $p_4x_3 - p_3x_4 = 0$, $\qquad$ A_2A_3P : $p_4x_1 - p_1x_4 = 0$,

A_1A_3P : $p_4x_2 - p_2x_4 = 0$, $\qquad$ A_2A_4P : $p_3x_1 - p_1x_3 = 0$,

A_1A_4P : $p_3x_2 - p_2x_3 = 0$, $\qquad$ A_3A_4P : $p_2x_1 - p_1x_2 = 0$.

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[2] Ibid, p. 231.

The plane $p_3x_2 - p_2x_3 = 0$ will cut the face $x_4 = 0$ in the line A_1P_{14}. The plane $p_3x_1 - p_1x_3 = 0$ will cut the face through the line A_2P_{24}. Any plane through $P_{14}P_{24}$ will intersect A_1A_2, extended, in the point P'_{34} $(-p_1, p_2, 0, 0)$ which is the harmonic conjugate of P_{34} relative to A_1 and A_2. The four planes determined by the edge A_3A_4 and the points A_1, P'_{34}, A_2, P_{34} form a harmonic set. The equation of the plane which is the harmonic conjugate of $p_2x_1 - p_1x_2 = 0$ with respect to $x_1 = 0$ and x_2 0 is $p_2x_1 + p_1x_2 = 0$. Similarly other pairs of conjugate planes associated with the point P are

$p_4x_1 - p_1x_4 = 0$ and $p_4x_1 + p_1x_4 = 0$,	(Conjugate with respect to $x_1 = 0$, $x_4 = 0$);
$p_3x_2 - p_2x_3 = 0$ and $p_3x_2 + p_2x_3 = 0$,	$x_2 = 0$, $x_3 = 0$);
$p_4x_3 - p_3x_4 = 0$ and $p_4x_3 + p_3x_4 = 0$,	$x_3 = 0$, $x_4 = 0$);
$p_3x_1 - p_1x_3 = 0$ and $p_3x_1 + p_1x_3 = 0$,	$x_1 = 0$, $x_3 = 0$);
$p_4x_2 - p_2x_4 = 0$ and $p_4x_2 + p_2x_4 = 0$,	$x_2 = 0$, $x_4 = 0$).

Polar with respect to circumsphere. The equation of the circumsphere is

$$a_{12}^2a_1a_2x_1x_2 + a_{13}^2a_1a_3x_1x_3 + a_{14}^2a_1a_4x_1x_4 + a_{23}^2a_2a_3x_2x_3 + a_{24}^2a_2a_4x_2x_4 + a_{34}^2a_3a_4x_3x_4 = 0.$$

In this case at the point $P(p_1, p_2, p_3, p_4)$ the partial derviatives with respect to x_1, x_2, x_3, x_4 are

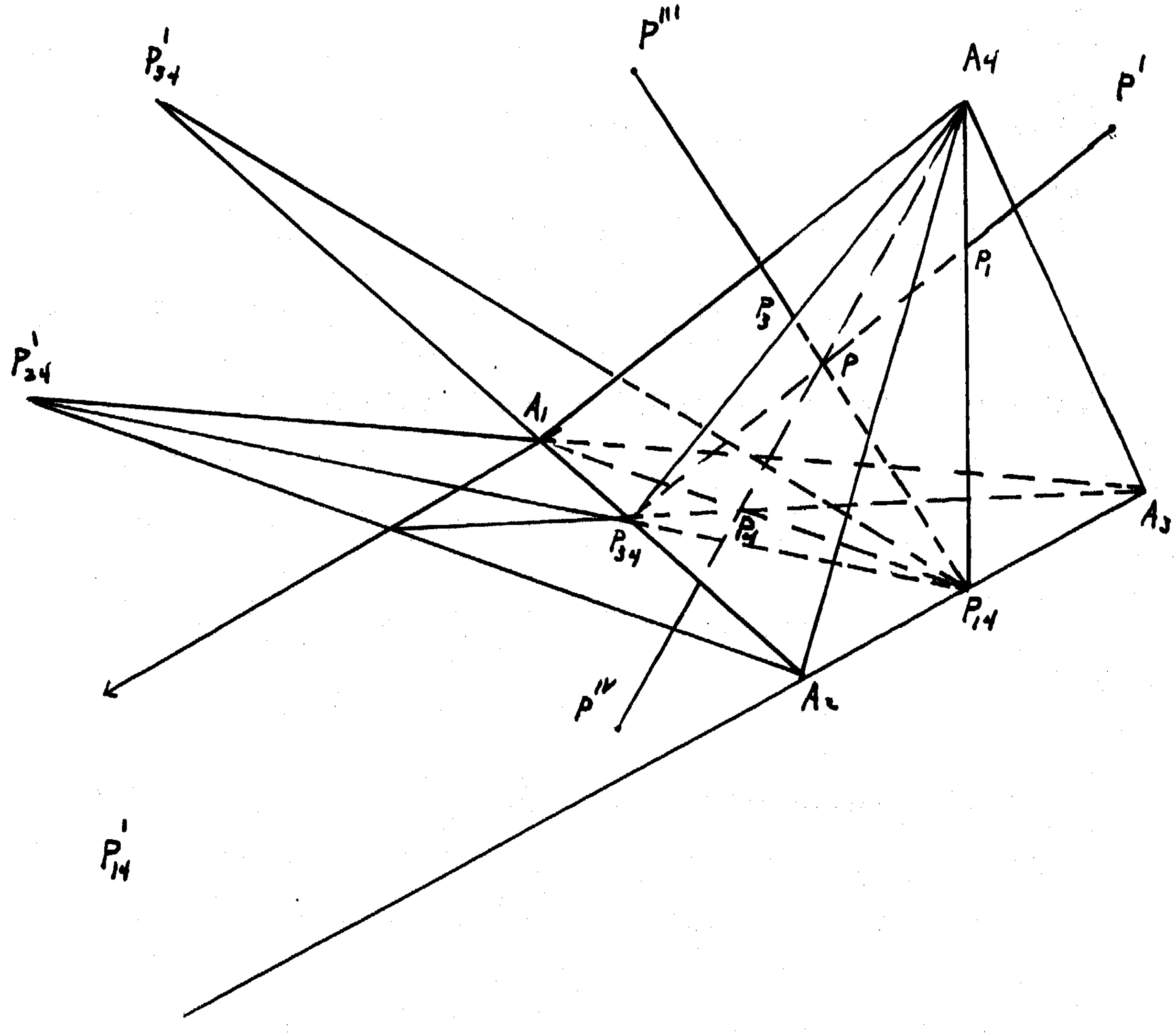

FIGURE 4

QUADRIPLANAR POLAR OF P, HARMONIC CONJUGATES OF P,

AND OF ASSOCIATED POINTS

$$F_1 = a_{13}^2 a_1 p_3 + a_{12}^2 a_1 a_2 p_2 + a_{14}^2 a_1 a_4 p_4,$$

$$F_2 = a_{23}^2 a_2 p_3 + a_{12}^2 a_1 a_2 p_2 + a_{24}^2 a_2 a_4 p_4,$$

$$F_3 = a_{23}^2 a_2 p_3 + a_{31}^2 a_3 a_1 p_1 + a_{34}^2 a_3 a_4 p_4,$$

$$F_4 = a_{14}^2 a_1 a_4 p_4 + a_{24} a_2 a_4 p_2 + a_{34} a_3 a_4 p_3,$$

and the equation of the polar P is

$$(a_{13}^2 a_1 p_3 + a_{12}^2 a_1 a_2 p_2 + a_{14} a_1 a_4 p_4)x_1 + (a_{23}^2 a_2 a_3 p_3 + a_{12}^2 a_1 a_2 p_1 + a_{24}^2 a_2 a_4 p_4)x_2$$
$$+ (a_{23}^2 a_2 a_3 p_3 + a_{31}^2 a_3 a_1 p_1 + a_{34} a_3 a_4 p_4)x_3 + (a_{14}^2 a_1 a_4 p_1 + a_{24}^2 a_2 a_4 p_2 + a_{34} a_3$$
$$a_4 p_3)x_3 = 0.$$

__Polar with respect to the general sphere.__ The general equation of a sphere is

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 +$$
$$a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 + (m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4)$$
$$(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.$$

Denote by F_1, F_2, F_3, F_4 the values of the partial derivatives of the left member with respect to x_1, x_2, x_3, x_4 at the point $P(p_1, p_2, p_3, p_4)$.

$$F_1 = 2\, a_1 m_1 p_1 + (a_{12}^2 a_1 a_2 + a_1 m_2 + a_2 m_1)p_2 +$$
$$(a_{13}^2 a_1 a_3 + a_1 m_3 + a_3 m_1)p_3 + (a_{14}^2 a_1 a_4 + a_1 m_4 + a_4 m_1)p_4,$$

$$F_2 = a_{12}^2 a_1 a_2 + a_1 m_2 + a_2 m_1)p_1 + 2\, a_2 m_2 p_2 +$$
$$(a_{23}^2 a_2 a_3 + a_2 m_3 + a_3 m_2)p_3 + (a_{24}^2 a_2 a_4 + a_2 m_4 + a_4 m_2)p_4,$$

$$F_3 = (a_{13}^2 a_1 a_3 + a_1 m_3 + a_3 m_1)p_1 + (a_{23}^2 a_2 a_3 + a_2 m_3 + a_3 m_2)p_2 +$$

$$(2\, a_3 m_3 p_3 + (a_{34} a_3 a_4 + a_3 m_4 + a_4 m_3)p_4,$$

$$F_4 = (a_{14}^2 a_1 a_4 + a_1 m_4 + a_4 m_1)p_1 + (a_{24}^2 a_2 a_4 + a_2 m_4 + a_4 m_2)p_2 +$$

$$(a_{34}^2 a_3 a_4 + a_3 m_4 + a_4 m_3)p_3 + 2a_4 m_4 p_4$$

The equation of the polar P is

$$F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4 = 0.$$

The center of the sphere is the point whose polar is the ideal plane. Identifying the equation of the polar of P with the equation $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$, it is found that P is the center of the sphere of the equations

$$F_1 : F_2 : F_3 : F_4 = a_1 : a_2 : a_3 : a_4$$

are satisfied.

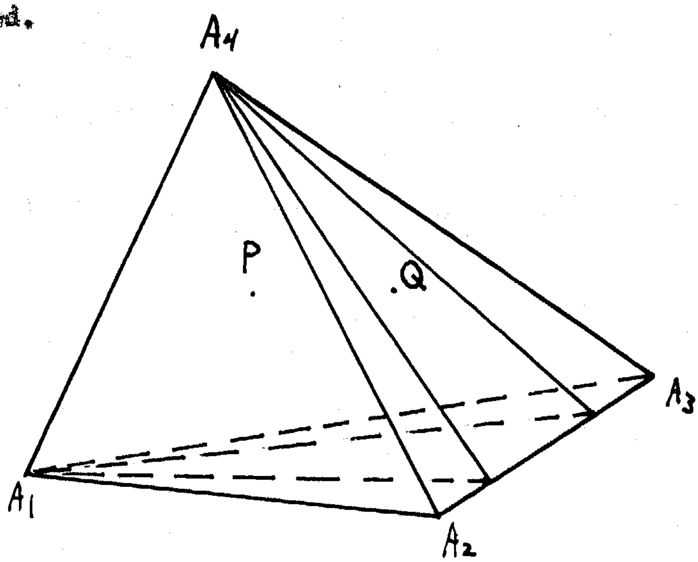

FIGURE 5

ISOGONAL CONJUGATES

Isogonal conjugates. If two planes through the vertex of a dihedral angle make equal angles with its faces, they are said to be "isogonal"[5] of "isogonal conjugates" (Figure 5). It is evident that two isogonally conjugate planes are symmetrical with regard to the bisector of the angle. The isogonal planes, with respect to the corresponding dihedral angles of a given tetrahedron, of the six planes passing through a given point and the six edges of the tetrahedron, have a point in common. Let P be the given point. The equations of the planes through the poing $P(p_1, p_2, p_3, p_4)$ and the edges A_1A_4, A_2A_4, A_3A_4, A_1A_2, A_1A_3, and A_2A_3 are respectively

$$p_3x_2 - p_2x_3 = 0,$$

$$p_3x_1 - p_1x_3 = 0,$$

$$p_2x_1 - p_1x_2 = 0,$$

$$p_4x_3 - p_3x_4 = 0,$$

$$p_4x_2 - p_2x_4 = 0,$$

$$p_4x_1 - p_1x_4 = 0.$$

Then the equations of the isogonal conjugates of the above planes are respectively,

$$p_2x_2 - p_3x_3 = 0,$$

$$p_1x_1 - p_3x_3 = 0,$$

$$p_1x_1 - p_2x_2 = 0,$$

$$p_3x_3 - p_4x_4 = 0,$$

$$p_2x_2 - p_4x_4 = 0,$$

[5] Nathan Altshiller-Court, Modern Pure Solid Geometry (New York: Macmillan Company, 1935), p. 240.

$p_1x_1 - p_4x_4 = 0.$

It is apparent that the six isogonally conjugate planes are concurrent in a point $Q(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4})$. This point Q is called the isogonal conjugate of P. These points are sometimes referred to as inverse points. Every point in space not lying in a face of the tetrahedron has an isogonal point with respect to the tetrahedron and only one. The distances of the points P and Q from the faces of the tetrahedron are inversely proportional, and, conversely, if the distances of P and Q from the faces of the tetrahedron are inversely proportional, the two points are isogonal with respect to the tetrahedron. The only self-conjugate points are the equicenters. If P is on the circumsphere, the isogonals of A_1A_4P and A_2A_4P will be parallel. It then follows that the isogonal conjugate of any point P on the circumcircle is at infinity.

<u>Isotomic conjugates</u>. Let $P(p_1, p_2, p_3, p_4)$ be any point and let $P_1, P_2, P_3, P_1', P_2', P_3'$ be the points where planes projected through P and containing an edge cuts the opposite edge. Let $Q_1, Q_2, Q_3, Q_1', Q_2', Q_3'$ (Figure 6) be points on the respective edges such that, considering directed segments on the edges, $A_1P_3 = Q_3A_2$, $A_2P_1 = Q_1A_3$, $A_3P_2 = Q_2A_1$. Then the plane through $A_1A_4Q_1$ divides the directed edge a_{23} in the same ratio in which $A_1A_4P_1$ divides a_{23}; or the ratios in which P_1 and Q_1 divide the edge a_{23} are reciprocals. Since the altitudes and bases of the tetrahedrons formed by the planes through $A_1A_4P_1$ and $A_1A_4Q_1$ are equal, the ratio of the volumes are reciprocals. Similar statements may be made about the other corresponding tetrahedrons formed by the other edges and the respective points on the opposite edges. It is clearly evident from this that

$$\frac{1}{a_1p_1} : \frac{1}{a_2p_2} : \frac{1}{a_3p_3} : \frac{1}{a_4p_4} = a_1q_1 : a_2q_2 : a_3q_3 : a_4q_4.$$

Then

$$q_1 : q_2 : q_3 : q_4 = \frac{1}{a_1^2p_1} : \frac{1}{a_2^2p_2} : \frac{1}{a_3^2p_3} : \frac{1}{a_4^2p_4} .$$

The coordinates of Q_1 are $(0,\ a_3^2p_3,\ a_2^2p_2,\ 0)$.

Similarly the coordinates of Q_2, Q_3, Q_1', Q_2', Q_3' are respectively

$(a_3^2p_3,\ 0,\ a_1^2p_1,\ 0)$,

$(a_2^2p_2,\ a_1^2p_1,\ 0,\ 0)$,

$(a_4^2p_4,\ 0,\ 0,\ a_1^2p_1)$,

$(0,\ a_4^2p_4,\ 0,\ a_2^2p_2)$,

$(0,\ 0,\ a_4^2p_4,\ a_3^2p_3)$.

The six planes $A_1A_4Q_1$, $A_2A_4Q_2$, $A_3A_4Q_3$, $A_2A_3Q_1'$, $A_1A_3Q_2'$, $A_1A_2Q_3'$ are concurrent in the point $Q\left(\frac{1}{a_1^2p_1},\ \frac{1}{a_2^2p_2},\ \frac{1}{a_3^2p_3},\ \frac{1}{a_4^2p_4}\right)$.

The point Q is called the isotomic conjugate of P.

The only isotomically self-conjugate points are the five points, M, M', M'', M''', M'''', namely, the median point and the four exmedian points.

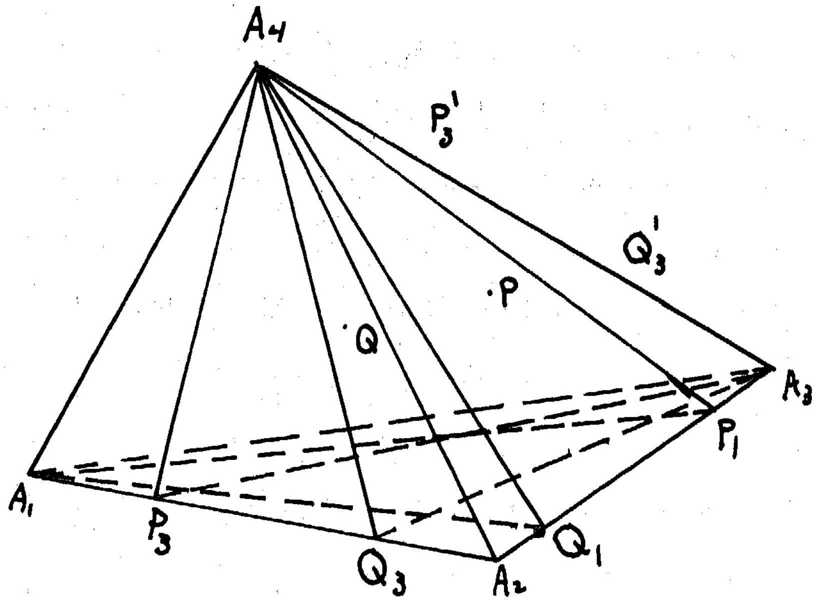

FIGURE 6

ISOTOMIC CONJUGATES

CHAPTER IV

SPECIAL POINTS AND GROUPS OF POINTS

This section and the three chapters following treat of several points, planes, and spheres of especial interest in connection with the study of the tetrahedron. Some of the geometric properties herein contained are well known and have been derived in other works dealing with solid geometry. However, much of the material is new, or at least, it has escaped the writer's knowledge in his search in this field. The presenting of these points, planes, and spheres in the analytical method is suggestive of what can be done with known facts as well as providing a means of proof for new ones. Cases arise wherein compactness has been forsaken in order to fulfill the purpose of the thesis; yet these forms are unique and furnish properties for further study.

The incenter and excenters. (A. C.-72)[1]. Since the incenter is equidistant from the faces of the tetrahedron of reference, its coordinates are (1, 1, 1, 1). A tetrahedron has seven excenters, three of which are located in the roofs and four in the truncs, all of which are equidistant from the faces of the tetrahedron of reference. The coordinates of I', I'', I''', the excenters located in the roofs, are (1, -1, -1, 1), (-1, 1, -1, 1), and (-1, -1, 1, 1), or (-1, 1, 1, -1), and (1, 1, -1, -1), where I' is associated with the roof A_1A_4 or A_2A_3.

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[1] A. C.-72, refers to Altshiller-Court's Modern Pure Geometry, page 72. This system of cross reference is used throughout Chapters IV, V, and VI.

I'' is associated with the roof A_2A_4 or A_1A_3, and I''' is associated with the roof A_3A_4 or A_1A_2. The coordinates of I_{a_1}, I_{a_2}, I_{a_3}, I_{a_4} are $(-1, 1, 1, 1)$, $(1, -1, 1, 1)$, $(1, 1, -1, 1)$, and $(1, 1, 1, -1)$ where I_{a_1} is the excenter associated with the face a_1, and similar locations for the other excenters.

<u>Median point and exmedian points</u>. (A. C.-51) (Figure 7) Let M denote the median point of the tetrahedron. Then M_1 is the median point of the face a_1. The coordinates of M_1 are $(0, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4})$. The median plane issued from the edge A_1A_2, that is, the plane $A_1A_2M_1$, has the equation $a_3x_3 - a_4x_4 = 0$. Similarly, the equations of the medians issued from A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4 are

$$a_2x_2 - a_4x_4 = 0,$$

$$a_2x_2 - a_3x_3 = 0,$$

$$a_1x_1 - a_4x_4 = 0,$$

$$a_1x_1 - x_3x_3 = 0,$$

$$a_1x_1 - a_2x_2 = 0.$$

It is evident that the coordinates of M, the point of intersection of the median planes, are

$$\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right).$$

A plane containing an edge of a tetrahedron and parallel to the opposite edge is referred to in this thesis as an exmedian plane or simply as an exmedian (Figure 7). The exmedians are the harmonic conjugates of the median planes with respect to the including faces. The point of concurrency of a median and three exmedians is called an exmedian point.

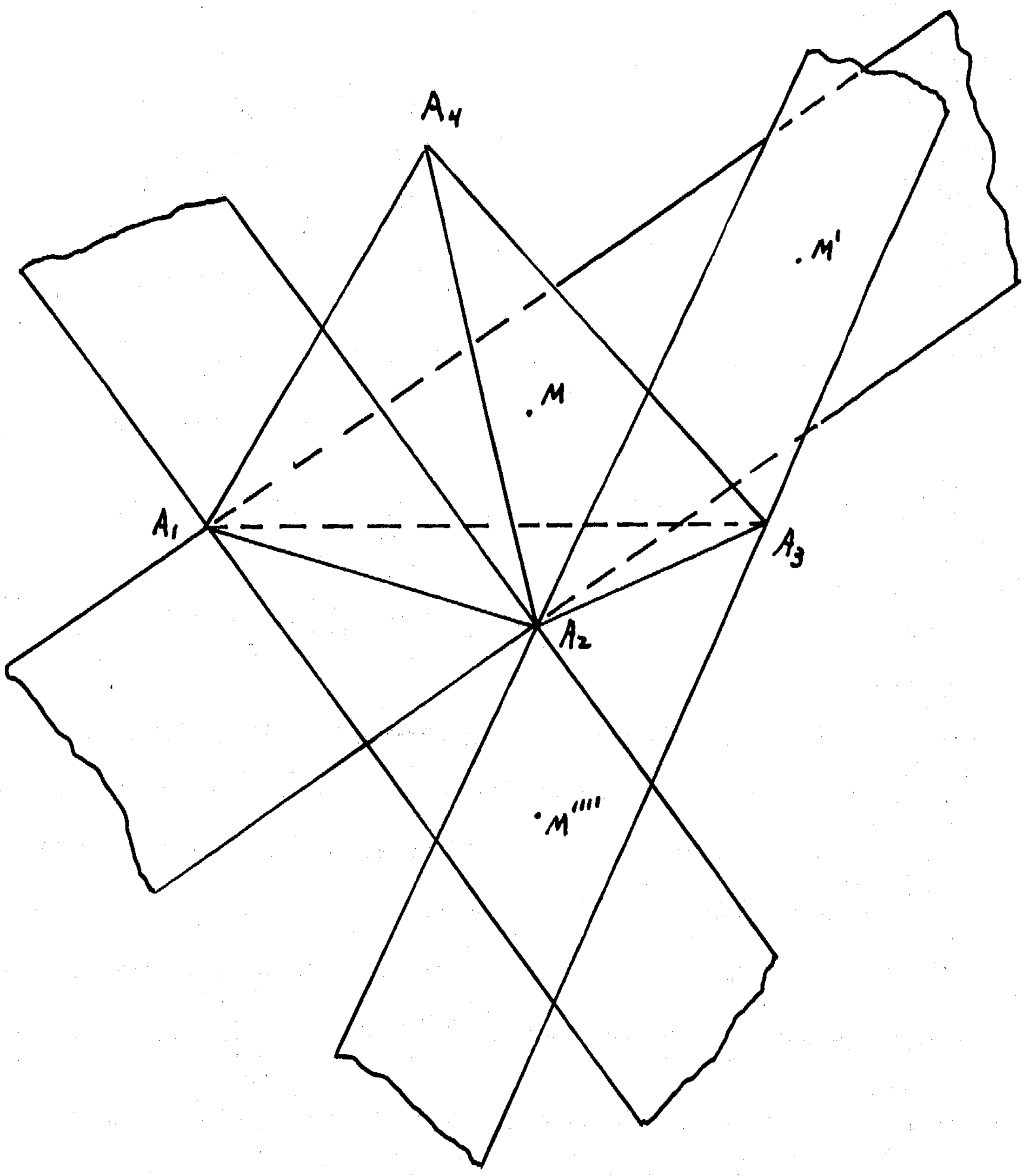

FIGURE 7

MEDIANS AND EXMEDIANS

The equations of the six exmedians are obviously

$$a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 + a_4x_4 = 0,$$

$$a_2x_2 + a_4x_4 = 0,$$

$$a_1x_1 + a_3x_3 = 0,$$

$$a_1x_1 + a_2x_2 = 0,$$

$$a_2x_2 + a_3x_3 = 0.$$

The coordinates of the exmedian points are

$$M'\left(\frac{-1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right), \quad M''\left(\frac{1}{a_1}, \frac{-1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right),$$

$$M'''\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{-1}{a_3}, \frac{1}{a_4}\right), \quad M''''\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{-1}{a_4}\right).$$

Symmedians and exsymmedians. The symmedian point, K, of a tetrahedron is the isogonal conjugate of the median point. A line through a vertex and the symmedian point is called a symmedian. A plane through an edge and the symmedian point is called a symmedian plane (Figure 8). Since the coordinates of the median point are $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right)$, the point K has coordinates (a_1, a_2, a_3, a_4). The symmedian point, then, is a point whose distances from the faces of the tetrahedron are proportional to the area of the faces. The equations of the symmedian planes are

$$a_2x_1 - a_1x_2 = 0,$$

$$a_3x_2 - a_2x_3 = 0,$$

$$a_3x_1 - a_1x_3 = 0,$$

$$a_4x_1 - a_1x_4 = 0,$$

$a_4x_2 - a_2x_4 = 0,$

$a_4x_3 - a_3x_4 = 0.$

The harmonic conjugate of the symmedian plane with respect to the including faces is referred to here as an exsymmedian plane (Figure 8). The equations of the exsymmedian planes are

$a_2x_1 + a_1x_2 = 0,$

$a_3x_2 + a_2x_3 = 0,$

$a_3x_1 + a_1x_3 = 0,$

$a_4x_1 + a_1x_4 = 0,$

$a_4x_2 + a_2x_4 = 0,$

$a_4x_3 + a_3x_4 = 0.$

A symmedian plane and three exsymmedian planes are concurrent in a point called an exsymmedian point. The four exsymmedian points are $K'(-a_1, a_2, a_3, a_4)$, $K''(a_1, -a_2, a_3, a_4)$, $K'''(a_1, a_2, -a_3, a_4)$, $K''''(a_1, a_2, a_3, -a_4)$.

<u>Circumcenter</u>. (A. C.-17; S.-186)[2] The equation of the circumsphere is

$$a_{12}^2a_1a_2x_1x_2 + a_{13}^2a_1a_3x_1x_3 + a_{14}^2a_1a_4x_1x_4 + a_{23}^2a_2a_3x_2x_3 + a_{24}^2a_2a_4x_2x_4 + a_{34}^2a_3a_4x_3x_4 = 0.$$

The center of the circumsphere is found by solving $F_1:F_2:F_3:F_4 = a_1:a_2:a_3:a_4$ where F_1, F_2, F_3, F_4 are the partial derivatives of x_1, x_2, x_3, x_4 with respect to the equation of the circumsphere.

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[2] S.-186, refers to Smith's <u>Solid Geometry</u>, page 186.

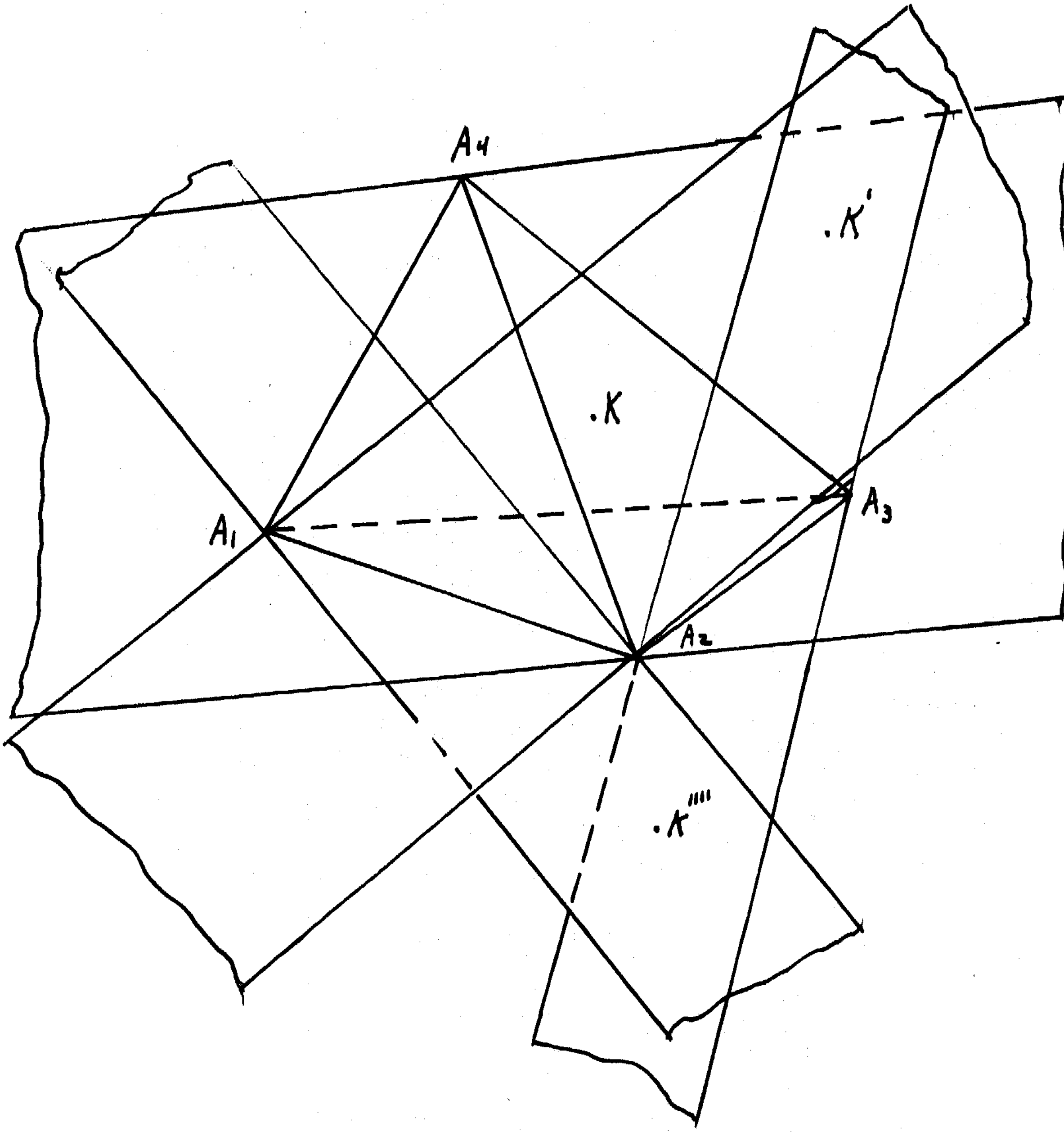

FIGURE 8

SYMMEDIANS AND EXSYMMEDIANS

$$\frac{\partial f}{\partial x_1} = \qquad a_{12}^2 x_2 + a_{13}^2 x_3 + a_{14}^2 x_4 = \rho,$$

$$\frac{\partial f}{\partial x_2} = a_{12}^2 x_1 + \qquad + a_{23}^2 x_3 + a_{24}^2 x_4 = \rho,$$

$$\frac{\partial f}{\partial x_3} = a_{13}^2 x_1 + a_{23}^2 x_2 \qquad + a_{34}^2 x_4 = \rho,$$

$$\frac{\partial f}{\partial x_4} = a_{14}^2 x_1 + a_{24}^2 x_2 + a_{34}^2 x_3 \qquad = \rho.$$

Since $F_1 : F_2 : F_3 : F_4 = a_1 : a_2 : a_3 : a_4$, then

$$x_1 : x_2 : x_3 : x_4 = \begin{vmatrix} 1 & a_{12}^2 & a_{13}^2 & a_{14}^2 \\ 1 & 0 & a_{23}^2 & a_{24}^2 \\ 1 & a_{23}^2 & 0 & a_{34}^2 \\ 1 & a_{24}^2 & a_{34}^2 & 0 \end{vmatrix} : \begin{vmatrix} 0 & 1 & a_{13}^2 & a_{14}^2 \\ a_{12}^2 & 1 & a_{23}^2 & a_{24}^2 \\ a_{13}^2 & 1 & 0 & a_{34}^2 \\ a_{14}^2 & 1 & a_{34}^2 & 0 \end{vmatrix} :$$

$$\begin{vmatrix} 0 & a_{12}^2 & 1 & a_{14}^2 \\ a_{12}^2 & 0 & 1 & a_{24}^2 \\ a_{13}^2 & a_{23}^2 & 1 & a_{34}^2 \\ a_{14} & a_{24}^2 & 1 & 0 \end{vmatrix} : \begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & 1 \\ a_{12}^2 & 0 & a_{23}^2 & 1 \\ a_{13}^2 & a_{23}^2 & 0 & 1 \\ a_{14}^2 & a_{24}^2 & a_{34}^2 & 1 \end{vmatrix}.$$

$$x_1 = 2\, b_{12} b_{13} b_{14} \left(\frac{\cos A_1^{(2)}}{b_{12}} + \frac{\cos A_1^{(3)}}{b_{13}} + \frac{\cos A_1^{(4)}}{b_{14}} \right) + a_{23} a_{24} a_{34}$$

$$-(b_{12}^2 + b_{13}^2 + b_{14}^2),$$

$$x_2 = 2\, b_{21} b_{23} b_{24}\left(\frac{\cos A_1^{(2)}}{b_{21}} + \frac{\cos A_1^{(3)}}{b_{23}} + \frac{\cos A_1^{(4)}}{b_{24}}\right) + a_{13} a_{14} a_{34}$$

$$-(b_{21}^2 + b_{23}^2 + b_{24}^2),$$

$$x_3 = 2 b_{31} b_{32} b_{34}\left(\frac{\cos A_3^{(1)}}{b_{31}} + \frac{\cos A_3^{(2)}}{b_{32}} + \frac{\cos A_3^{(4)}}{b_{34}}\right) + a_{12}^2 a_{14}^2 a_{24}^2$$

$$-(b_{31} + b_{32} + b_{34}),$$

$$x_4 = 2\, b_{41} b_{42} b_{43}\left(\frac{\cos A_4^{(1)}}{b_{41}} + \frac{\cos A_4^{(2)}}{b_{42}} + \frac{\cos A_4^{(3)}}{b_{43}}\right) + a_{12} a_{13} a_{23}$$

$$-(b_{41} + b_{42} + b_{43}).$$

Or by generalizing, the coordinates are

$$x_i = 2\, b_{ij} b_{ik} b_{il}\left(\frac{\cos A_i^{(j)}}{b_{ij}} + \frac{\cos A_i^{(k)}}{b_{ik}} + \frac{\cos A_i^{(l)}}{b_{il}}\right) + a_{jk} a_{jl} a_{kl}$$

$$-(b_{ij} + b_{ik} + b_{il}),$$

where $b_{ij} = a_{ij} a_{kl}^2$, $b_{ik} = a_{ik} a_{jl}^2$, $b_{il} = a_{il} a_{jk}^2$.

The center of the circumsphere may also be expressed in the following manner:

$$x_1 = -\, 2\, a_{23}^2 a_{24}^2 a_{34}^2 + a_{23}^2 A_{11} + a_{24}^2 A_{22} + a_{34}^2 A_{33},$$

$$x_2 = -\, 2\, a_{14}^2 a_{31}^2 a_{34}^2 + a_{14}^2 A_{11} + a_{31}^2 A_{22} + a_{34}^2 A_{33},$$

$$x_3 = -\, 2\, a_{14}^2 a_{24}^2 a_{12}^2 + a_{14}^2 A_{11} + a_{24}^2 A_{22} + a_{12}^2 A_{33},$$

$$x_4 = -\, 2\, a_{23}^2 a_{31}^2 a_{12}^2 + a_{23}^2 A_{11} + a_{31}^2 A_{22} + a_{12}^2 A_{33},$$

where $A_{11} = -a_{23}^2 a_{14}^2 + a_{31}^2 a_{24}^2 + a_{12}^2 a_{34}^2,$

$$A_{22} = a_{23}^2 a_{14}^2 - a_{31}^2 a_{24}^2 + a_{12}^2 a_{34}^2,$$

$$A_{33} = a_{23}^2 a_{14}^2 + a_{31}^2 a_{24}^2 - a_{12}^2 a_{34}^2.$$

or, denoting by Φ_i the angles of the auxiliary triangle whose sides are $a_{23}a_{14}$, $a_{31}a_{24}$, $a_{12}a_{34}$,

$$x_1 = a_{31}a_{12}a_{24}^2 a_{34}^2 \left(\cos\alpha_1^{(4)} - \cos\alpha_4^{(1)} \cos\Phi_1\right),$$

$$x_2 = a_{12}a_{23}a_{34}^2 a_{14}^2 \left(\cos\alpha_2^{(4)} - \cos\alpha_4^{(2)} \cos\Phi_2\right),$$

$$x_3 = a_{23}a_{31}a_{14}^2 a_{24}^2 \left(\cos\alpha_3^{(4)} - \cos\alpha_4^{(3)} \cos\Phi_3\right),$$

$$x_4 = a_{31}^2 a_{12}^2 a_{24}a_{34} \left(\cos\alpha_4^{(1)} - \cos\alpha_1^{(4)} \cos\Phi_1\right),$$

$$= a_{12}^2 a_{23}^2 a_{24}a_{34} \left(\cos\alpha_4^{(2)} - \cos\alpha_2^{(4)} \cos\Phi_2\right),$$

$$= a_{23}^2 a_{31}^2 a_{14}a_{24} \left(\cos\alpha_4^{(3)} - \cos\alpha_3^{(4)} \cos\Phi_3\right).$$

It is noted that

$$A_{ii} = 2\, a_{ij}a_{ik}a_{jl}a_{kl} \cos\Phi_i.$$

Monge Point. (A. C.-68) The six planes through the mid-points of the edges of a tetrahedron and perpendicular to the edges respectively opposite have a point in common, the Monge point (Figure 2). The equations of the planes through the mid-points and perpendicular to the opposite edges may be found by finding the ideal point on each perpendicular in the faces of the tetrahedron which contains the opposite edge. Then using these two points and the mid-point, the equation may be found by the standard method.

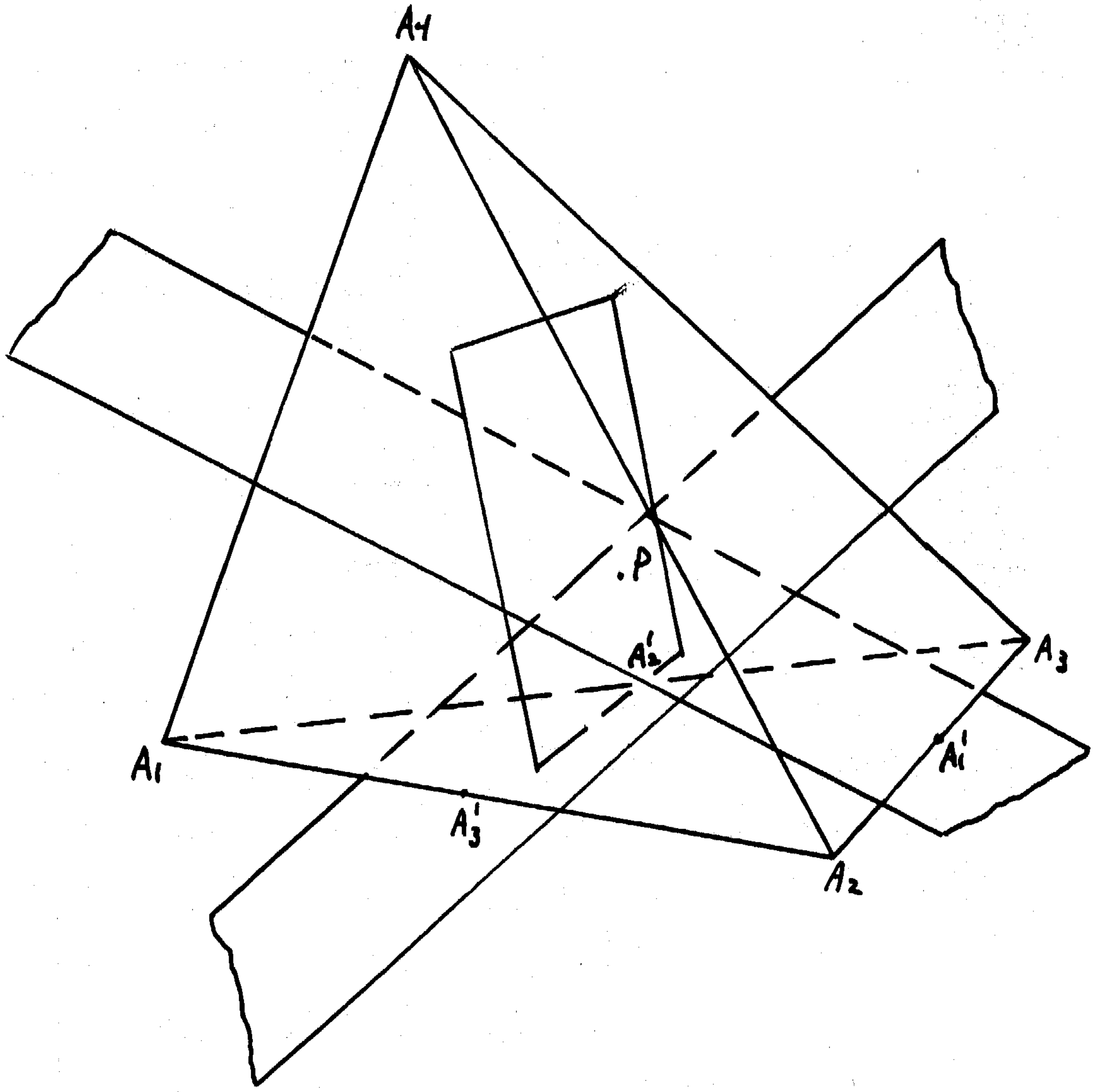

FIGURE 9

MONGE POINT

The ideal point in trilinear coordinates of $h_4^{(1)}$ is (0, −1, cos A_3, cos A_2). This may be changed to quadriplanar coordinates by the relationship $x_i = \frac{a_{jk}}{a_k} \int_i$ where $\int_i$ is one of the trilinear coordinate points.

In quadriplanar coordinates, the ideal point on $h_4^{(1)}$ is $\left(0, \frac{a_{43}\cos A_3^{(1)}}{a_2}, \frac{a_{42}\cos A_2^{(1)}}{a_3}, \frac{-a_{23}}{a_4}\right)$.

The ideal point on $h_1^{(4)}$ is $\left(\frac{-a_{23}}{a_1}, \frac{a_{13}\cos A_3^{(4)}}{a_2}, \frac{a_{12}\cos A_2^{(4)}}{a_3}, 0\right)$.

The coordinates of A_1'', the mid-point of A_1A_4, are $\left(\frac{1}{a_1}, 0, 0, \frac{1}{a_4}\right)$.

The required plane is parallel to both $h_4^{(1)}$ and $h_1^{(4)}$. It contains the two ideal points and the point A_1''.

In order to avoid the use of fractions, tetrahedral coordinates are employed here. The relationship between tetrahedral and quadriplanar coordinates is $X_i = a_i x_i$, where X_i represents the tetrahedral coordinates.

The equation of the plane in tetrahedral coordinates is

$$\begin{vmatrix} X_1 & X_2 & X_3 & X_4 \\ 1 & 0 & 0 & 1 \\ 0 & a_{43}\cos A_3^{(1)} & a_{42}\cos A_2^{(1)} & -a_{23} \\ -a_{23} & a_{13}\cos A_3^{(4)} & a_{12}\cos A_2^{(4)} & 0 \end{vmatrix} = 0,$$

or $(-a_{13}\cos A_3^{(4)} + a_{43}\cos A_2^{(1)})X_1 + (a_{13}\cos A_3^{(4)} + a_{43}\cos A_3^{(1)} -2a\ \ )X +$

$$(\ a_{13}\cos A_3^{(4)} + a_{32}\cos A_3^{(1)})X_3 + (a_{13}\cos A_3^{(4)} - a_{43}\cos A_3^{(1)})X_4 = 0.$$

The equation of the plane through A_2'', the mid-point of A_2A_4, and parallel to the altitudes $h_2^{(4)}$ and $h_4^{(2)}$ is

$$(-a_{21}\cos A_1^{(4)} - a_{41}\cos A_1^{(2)})X_1 + (a_{21}\cos A_1^{(4)} - a_{41}\cos A_1^{(2)})X_2 + (-a_{21}\cos A_1^{(4)} - a_{41}\cos A_1^{(2)} + 2\,a_{13})X_3 + (-a_{21}\cos A_1^{(4)} + a_{41}\cos A_2^{(4)})X_4 = 0.$$

The plane through A_3'', the mid-point of A_3A_4, and perpendicular to A_1A_2 is parallel to the altitudes $h_4^{(3)}$ and $h_3^{(4)}$. The equation of this plane is

$$(a_{41}\cos A_1^{(3)} + a_{31}\cos A_1^{(4)})X_1 + (a_{41}\cos A_1^{(3)} + a_{31}\cos A_1^{(4)} - 2\,a_{13})X_2 + (a_{41}\cos A_1^{(3)} - a_{13}\cos A_1^{(4)})X_3 + (-a_{41}\cos A_1^{(3)} + a_{31}\cos A_1^{(4)})X_4 = 0.$$

The plane through A_1', the mid-point of A_2A_3, and perpendicular to A_1A_4 is parallel to the altitudes $h_2^{(3)}$ and $h_3^{(2)}$. The equation of the plane is

$$(a_{21}\cos A_1^{(3)} + a_{31}\cos A_1^{(2)})X_1 + (-a_{21}\cos A_1^{(3)} + a_{31}\cos A_1^{(2)})X_2 + (a_{21}\cos A_1^{(3)} - a_{31}\cos A_1^{(2)})X_3 + (a_{21}\cos A_1^{(3)} + a_{31}\cos A_1^{(2)} - 2\,a_{14})X_4 = 0.$$

The plane through A_2', the mid-point of A_1A_3, and perpendicular to A_2A_4 is parallel to the altitudes $h_1^{(3)}$ and $h_3^{(1)}$. The equation of this plane is

$$(-a_{14}\cos A_4^{(3)} + a_{34}\cos A_4^{(1)})X_1 + (a_{14}\cos A_4^{(3)} + a_{34}\cos A_4^{(1)} - 2\,a_{24})X_2 + (a_{14}\cos A_4^{(3)} - a_{34}\cos A_4^{(1)})X_3 + (a_{14}\cos A_4^{(3)} + a_{34}\cos A_4^{(1)})X_4 = 0.$$

The plane through A_3', the mid-point of A_1A_2, and perpendicular to A_3A_4 is parallel to the altitudes $h_1^{(2)}$ and $h_2^{(1)}$. The equation of the plane is

$$(-a_{14}\cos A_4^{(2)} + a_{24}\cos A_4^{(1)})X_1 + (a_{14}\cos A_4^{(2)} - a_{24}\cos A_4^{(1)})X_2 + (a_{14}\cos A_4^{(2)} + a_{24}\cos A_4^{(1)} - 2a_{34})X_3 + (a_{14}\cos A_4^{(2)} + a_{24}\cos A_4^{(1)})X_4 = 0.$$

Since three planes in the same pencil intersect in a point, the Monge point may be found by solving simultaneously the equations of any three of the six planes.

Let m_1, m_2, m_3, m_4 represent the coefficients of X_1, X_2, X_3, X_4, respectively, in the equation of the first Monge plane of the series. Likewise let n_1, n_2, n_3, n_4 represent the coefficients of X_1, X_2, X_3, X_4 in the equation of the second plane, and let p_1, p_2, p_3, p_4 represent the coefficients of X_1, X_2, X_3, X_4 in the equations of the third plane. Then by formula

$$X_1 : X_2 : X_3 : X_4 = \begin{vmatrix} m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \\ p_2 & p_3 & p_4 \end{vmatrix} : - \begin{vmatrix} m_3 & m_4 & m_1 \\ n_3 & n_4 & n_1 \\ p_3 & p_4 & p_1 \end{vmatrix} : \begin{vmatrix} m_4 & m_1 & m_2 \\ n_4 & n_1 & n_2 \\ p_4 & p_1 & p_2 \end{vmatrix} : - \begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{vmatrix},$$

and

$$x_1 : x_2 : x_3 : x_4 = \frac{X_1}{a_1} : \frac{X_2}{a_2} : \frac{X_3}{a_3} : \frac{X_4}{a_4}$$ which is the desired solution

or the coordinates of the Monge point.

<u>Ideal points on altitudes of tetrahedron of reference.</u> (Figure 3)

$$h_1 = \frac{3\,\Delta}{a_1},\ h_2 = \frac{3\,\Delta}{a_2},\ h_3 = \frac{3\,\Delta}{a_3},\ h_4 = \frac{3\,\Delta}{a_4}.$$

$\overline{O}_1 = h \cos \theta_{14}$	$x = \cos \theta_{14}$
$\overline{O}_2 = h \cos \theta_{24}$	$x = \cos \theta_{24}$
$\overline{O}_3 = h \cos \theta_{34}$	$x = \cos \theta_{34}$
$\overline{O}_4 = 0$	$x = u$

$a_1 \cos \theta_{14} + a_2 \cos \theta_{24} + a_3 \cos \theta_{34} + a_4 u = 0$; $u = -1$. The ideal point on the altitude h_4 is

$(\cos \theta_{14}, \cos \theta_{24}, \cos \theta_{34}, -1)$. Since the coordinates of the incenter are (1, 1, 1, 1), the equations of $I\ H_4,\infty$, the line through the incenter and the ideal point of the altitude h_4 are as follows:

$x_1 = \lambda + \cos \theta_{14}$

$x_2 = \lambda + \cos \theta_{24}$

$x_3 = \lambda + \cos \theta_{34}$

$x_4 = \lambda - 1$

$x_4 = 0$ if $\lambda = 1$. Therefore the coordinates of I_{a_4} are

$(1 + \cos \theta_{14}, 1 + \cos \theta_{24}, 1 + \cos \theta_{34}, 0)$.

The eq uations of $A_4 I_{a_4}$ are

$x_1 : x_2 : x_3 : x_4 = (1 + \cos \theta_{14}) : (1 + \cos \theta_{24}) : (1 + \cos \theta_{34}) : \lambda$.

The equations of h_1, h_2, h_3 are respectively

$x_1 = u$	$x_1 = \cos \theta_{12}$	$x_1 = \cos \theta_{13}$
$x_2 = \cos \theta_{21}$	$xx_2 = u$	$x_2 = \cos \theta_{23}$
$x_3 = \cos \theta_{31}$	$x_3 = \cos \theta_{32}$	$x_3 = u$
$x_4 = \cos \theta_{41}$	$x_4 = \cos \theta_{42}$	$x_4 = \cos \theta_{43}$

The ideal points on h_1, h_2, and h_3 are respectively

$(-1, \cos \theta_{21}, \cos \theta_{31}, \cos \theta_{41})$,

$(\cos \theta_{12}, -1, \cos \theta_{32}, \cos \theta_{42})$,

$(\cos \theta_{13}, \cos \theta_{23}, -1, \cos \theta_{43})$.

The coordinates of I_{a_1}, I_{a_2}, I_{a_3}, I_{a_4}, repeating those given above for

I_{a_4}, are

$$I_{a_1}:(0,\ 1 + \cos \theta_{12},\ 1 + \cos \theta_{13},\ 1 + \cos \theta_{14}),$$

$$I_{a_2}:(1 + \cos \theta_{21} 0,\ 1 + \cos \theta_{23},\ 1 + \cos \theta_{24}),$$

$$I_{a_3}:(1 + \cos \theta_{31},\ 1 + \cos \theta_{32},\ 0,\ 1 + \cos \theta_{34}),$$

$$I_{a_4}:(1 + \cos \theta_{41},\ 1 + \cos \theta_{42},\ 1 + \cos \theta_{43},\ 0).$$

The equations of the lines from the vertices of the tetrahedron of reference to the feet of the corresponding perpendiculars issued from I are

$$A_1I_{a_1}:\ x_1:x_{22}:x_3:x_4 = \lambda : 1 + \cos \theta_{12} : 1 + \cos \theta_{13} : 1 + \cos \theta_{14};$$

$$A_2I_{a_2}:\ x_1:x_2:x_3:x_4 = 1 + \cos \theta_{12} : \lambda : 1 + \cos \theta_{23} : 1 + \cos \theta_{24};$$

$$A_3I_{a_3}:\ x_1:x_2:x_3:x_4 = 1 + \cos \theta_{13} : 1 + \cos \theta_{23} : \lambda : 1 + \cos \theta_{34};$$

$$A_4I_{a_4}:\ x_1:x_2:x_3:x_4 = 1 + \cos \theta_{14} : 1 + \cos \theta_{24} : 1 + \cos \theta_{34} : \lambda .$$

<u>Points of contact of the inscribed sphere</u>. (Figure 3) Since the points of contact of the inscribed sphere are the same points as the feet of the perpendiculars issued from I to the faces a they are respectively,

$$(0\ ,\ 1 + \cos \theta_{12},\ 1 + \cos \theta_{13},\ 1 + \cos \theta_{14}),$$

$$(1 + \cos \theta_{21},\ 0,\ 1 + \cos \theta_{23},\ 1 + \cos \theta_{24}),$$

$$(1 + \cos \theta_{31},\ 1 + \cos \theta_{32},\ 0,\ 1 + \cos 0_{34}),$$

$$(1 + \cos \theta_{41},\ 1 + \cos \theta_{42},\ 1 + \cos \theta_{43},\ 0).$$

CHAPTER V

SPECIAL PLANES AND GROUPS OF PLANES

This chapter consists of isolated planes and of planes so related as to be considered in groups. Associated with these planes are some points that could not be considered in the chapter of special points due to the fact that there is a dependency upon the intersection of special planes for a solution.

<u>Bimedian</u> (A. C.–48). A line joining the mid-points of two opposite edges of a tetrahedron is called a bimedian of the tetrahedron relative to the pair of edges considered. A tetrahedron has three bimedians (Figure 10). They will be designated m_{a_1}, m_{a_2}, m_{a_3}, the bimedian m_{a_1} being relative to the points A_1', A_1'', and similar notations for m_{a_2}, and m_{a_3}. The mid-points of two pairs of opposite edges of a tetrahedron are coplanar. As an example A_1A_4, A_2A_3 and A_1A_2, A_3A_4, two pairs of opposite edges, may be taken. The coordinates of A_1'', A_1', A_3', A_3', the mid-points of the respective edges given above, are

$$\left(\frac{1}{a_1}, 0, 0, \frac{1}{a_4}\right), \left(0, \frac{1}{a_2}, \frac{1}{a_3}, 0\right), \left(\frac{1}{a_1}, \frac{1}{a_2}, 0, 0\right), \text{ and } \left(0, 0, \frac{1}{a_3}, \frac{1}{a_4}\right).$$

The determinant

$$\begin{vmatrix} \frac{1}{a_1} & 0 & 0 & \frac{1}{a_4} \\ 0 & \frac{1}{a_2} & \frac{1}{a_3} & 0 \\ \frac{1}{a_1} & \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & \frac{1}{a_3} & \frac{1}{a_4} \end{vmatrix} = 0$$

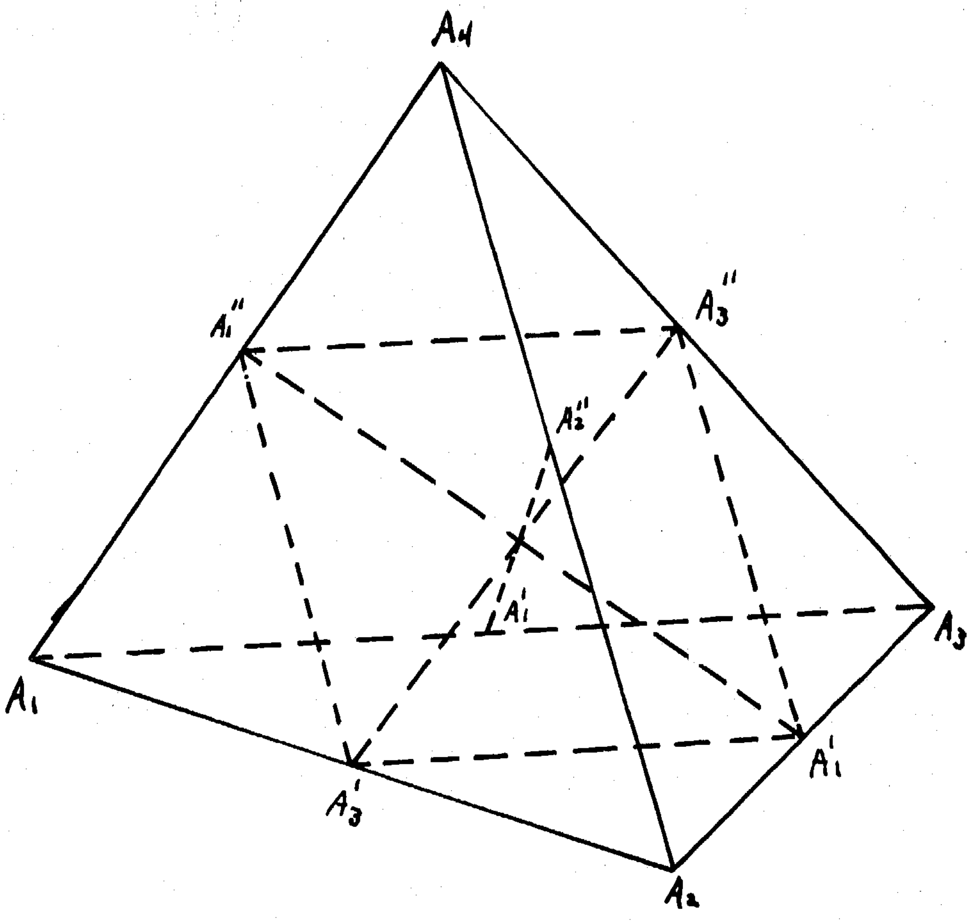

FIGURE 10

BIMEDIANS AND BIMEDIAN PLANES

This proves the coplanarity of the mid-points of the two pairs of opposite edges.

The equation of the plane is

$a_1x_1 - a_2x_2 + a_3x_3 - a_4x_4 = 0$. It contains the bimedians m_{a_1} and m_{a_3}.

A_2', A_3', A_2'', A_3'' and A_1'', A_2', A_1', A_2'' are two sets of coplanar points.

The equation of the plane determined by the first set is

$-a_1x_1 + a_2x_2 + a_3x_3 - a_4x_4 = 0$. It contains the bimedians m_{a_2} and m_{a_3}.

The equation of the plane determined by the last set is $a_1x_1 + a_2x_2 - a_3x_3 - a_4x_4 = 0$. It contains the bimedians m_{a_1} and m_{a_2}. To find the point of intersection on the three planes, one needs but to use the usual determinant method. This reduces to $x_1 : x_2 : x_3 : x_4 = \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3} : \frac{1}{a_4}$ which obviously are the coordinates of the centroid, M. This point of intersection is also common to the medians.

<u>Director</u> (A. C.-49). A plane parallel to two opposite edges of a tetrahedron is referred to as a directing plane or a director of the tetrahedron.

If a plane is passed through any point $P(p_1, p_2, p_3, p_4)$ in the tetrahedron and parallel to A_1A_3 and A_2A_4, the plane is determined that point and the ideal point on A_1A_3 and on A_2A_4. The equation of the plane is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ p_1 & p_2 & p_3 & p_4 \\ \frac{1}{a_1} & 0 & -\frac{1}{a_3} & 0 \\ 0 & \frac{1}{a_2} & 0 & -\frac{1}{a_4} \end{vmatrix} = 0,$$

or $(a_2p_2 + a_4p_4)(a_1x_1 + a_3x_3) = (a_1p_1 + a_3p_3)(a_2x_2 + a_4x_4)$.

The equation of the plane through the same point and parallel to A_2A_3, and A_1A_4 is

$$(a_1p_1 + a_4p_4)(a_2x_2 + a_3x_3) = (a_2p_2 + a_3p_3)(a_1x_1 + a_4x_4).$$

Likewise the equation of the plane through the same point and parallel to the remaining edges, A_1A_2 and A_3A_4, is $(a_3p_3 + a_4p_4)(a_1x_1 + a_2x_2) =$

$(a_1p_1 + a_2p_2)(a_3x_3 + a_4x_4)$.

When the point P is the centroid, the directing plane will contain the two bimedians relative to the two pairs of opposite edges not parallel to the director (Figure 10). From this it is obvious that the equations of the three directing planes through the centroid are

$$a_1x_1 - a_2x_2 + a_3x_3 - a_4x_4 = 0,$$

$$-a_1x_1 + a_2x_2 + a_3x_3 - a_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 - a_3x_3 - a_4x_4 = 0,$$

which agree with the equations of the planes found under the caption of bimedians.

It is plainly seen that the line of intersection of two directing planes contains the same ideal point as the bimedian of the two opposite edges not relative to the directing planes. Therefore, this line of intersection and the third bimedian are parallel.

Isoclinal planes. An isoclinal plane of a tetrahedron is one making equal angles with the edges of the trihedron (A. C.-32). An isoclinal plane of a trihedron cuts equal segments on the three edges of the trihedron (A. C.-34). Moreover, the isoclinal plane of the trihedron $A_4: A_1A_2A_3$ (Figure 11) which passes through A_4 contains the three external bisectors of the face angles of the trihedral angle at A_4. The equations of these bisectors are

$$x_1 = 0,\ a_2a_{24}x_2 + a_3a_{34}x_3 = 0;$$

$$x_2 = 0,\ a_1a_{14}x_1 + a_3a_{34}x_3 = 0;$$

$$x_3 = 0,\ a_1a_{14}x_1 + a_2a_{24}x_2 = 0.$$

The plane containing these lines is

$a_1a_{14}x_1 + a_2a_{24}x_2 + a_3a_{34}x_3 = 0$. This is the equation of the isoclinal plane of the trihedron $A_4: A_1A_2A_3$, passing through the vertex A_4. Isoclinal planes associated with the trihedron but not necessarily passing through A_4 have the equation

$$a_1a_{14}x_1 + a_2a_{24}x_2 + a_3a_{34}x_3 = r(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4).$$

The r, appearing in the equation, is the perpendicular distance of the vertex A_4 from the isoclinal plane associated with A_4.

The ideal point on the line B_2B_3 is

$$0:\ -\frac{a_{34}}{a_2}:\frac{a_{24}}{a_3}:\frac{a_{34}-a_{24}}{a_4},$$

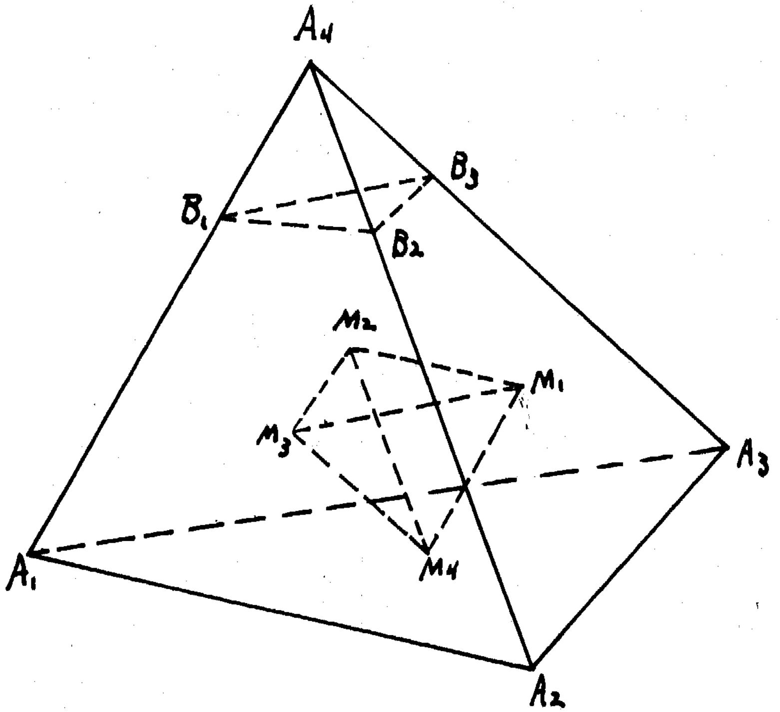

FIGURE 11

ISOCLINAL PLANES AND PLANES THROUGH THREE FACE CENTROIDS and the ideal line in the isoclinal plane associated with A_4 has the equation

$$x_1 : x_2 : x_3 : x_4 =$$

$$\frac{a_{34}}{a_1}\lambda : -\frac{a_{34}}{a_2}\lambda : \frac{(a_{24} - a_{14})}{a_3}\lambda : \frac{(a_{34} - a_{24}) + (a_{14} - a_{34})}{a_4}\lambda .$$

<u>Planes through three face centroids</u>. If a plane is passed through three face centroids, the plane will be parallel to the fourth face of the tetrahedron (Figure 11). The equation of the plane through the centroids in faces

a_2, a_3, a_4 is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ \frac{1}{a_1} & 0 & \frac{1}{a_3} & \frac{1}{a_4} \\ \frac{1}{a_1} & \frac{1}{a_2} & 0 & \frac{1}{a_4} \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & 0 \end{vmatrix} = 0,$$

or $-2\, a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$. It is parallel to a_1. The equations of the planes parallel to a_2, a_3, a_4 and through the face centroids of the other faces are

$$a_1x_1 - 2\, a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 - 2\, a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 + a_3x_3 - 2\, a_4x_4 = 0.$$

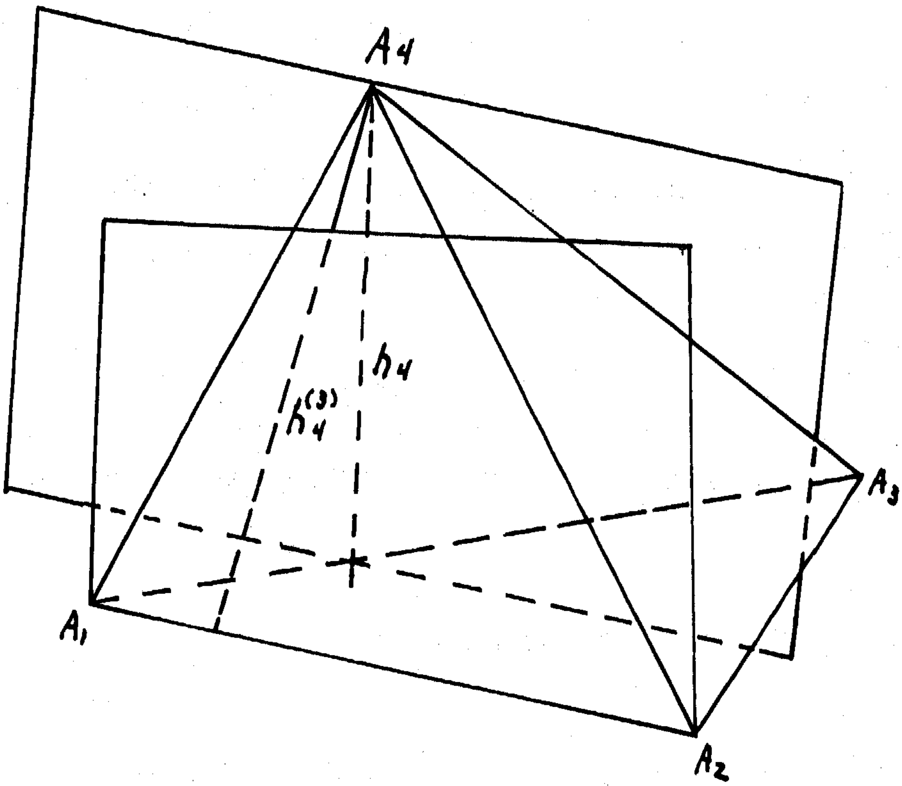

FIGURE 12

PERPENDICULAR PLANES

<u>Plane through an edge and perpendicular to a face containing the edge</u> (Figure 12). A plane containing the edge A_2A_3 and perpendicular to $x_4 = 0$ is of the form

$$m_1a_1x_1 + m_4a_4x_4 = 0.$$

The perpendicular distances of A_1, A_4 from the plane are proportional to the coefficients m_1, m_4. But if the plane is perpendicular to $x_4 = 0$, then $h_1^{(4)}$ is the actual distance of A_1 from the plane and $h_4^{(1)} \cos \theta_{14}$ is the actual distance of A_4 from the plane. The equation of the plane is then

$$h_1^{(4)} a_1x_1 + h_4^{(1)} \cos \theta_{14}a_4x_4 = 0.$$

By virtue of the identity $a_1h_1^{(4)} = a_4h_4^{(1)}$, this reduces to

$$x_1 + \cos \theta_{14}x_4 = 0.$$

The equations of similar planes through the other edges may be derived by a like manner.

<u>Plane through a vertex perpendicular to the opposite face and parallel to an edge in the opposite face</u> (Figure 12). If a plane is passed through the vertex A_4 so as to be perpendicular to the face $x_4 = 0$ and parallel to the edge A_2A_3, its equation will be

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4 \sec \theta_{14}x_1.$$

Similar planes through A_4 perpendicular to $x_4 = 0$ and parallel to the edges A_3A_1 and A_1A_2 are respectively

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4 \sec \theta_{24}x_2,$$

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4 \sec \theta_{34}x_3.$$

As a check the equations of the altitude A_4H_4 are

$$x_1 : x_2 : x_3 = \cos \theta_{14} = \cos \theta_{24} = \cos \theta_{34}.$$

Planes through mid-point of edges and parallel to faces (Figure 13). The equations of the planes through the mid-points of three concurrent edges, concurrent at A_1, A_2, A_3, A_4 are respectively,

$$a_1x_1 = a_2x_2 + a_3x_3 + a_4x_4,$$
$$a_2x_2 = a_1x_1 + a_3x_3 + a_4x_4,$$
$$a_3x_3 = a_1x_1 + a_2x_2 + a_4x_4,$$
$$a_4x_4 = a_1x_1 + a_2x_2 + a_3x_3.$$

Planes through vertices parallel to opposite faces (Figure 13). Any plane parallel to a given plane $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0$ is of the form $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + \lambda \ (x_1 - x_2 - x_3 - x_4) = 0$. Hence in tetrahedral coordinates the equation of the plane (S.-168) through any point $P(p_1, p_2, p_3, p_4)$ and parallel to $m_1X_1 + m_2X_2 + m_3X_3 + m_4X_4 = 0$ can be expressed in one of the following forms:

$\Sigma p_i \Sigma m_iX_i = \Sigma m_ip_i \Sigma X_i$, or $\Sigma p_i \Sigma m_iX_i - \Sigma m_ip_i \Sigma X_i = 0$, or

$$\begin{vmatrix} \Sigma p_i & \Sigma X_i \\ \Sigma m_ip_i & \Sigma m_iX_i \end{vmatrix} = 0$$

This in turn may be expressed as

$$\Sigma p_i(m_1X_1 + m_2X_2 + m_3X_3 + m_4X_4) = (m_1p_1 + m_2p_2 + m_3p_3 + m_4p_4)(X_1 + X_2 + X_3 + X_4).$$

Σp_i at the vertex $A_1(1, 0, 0, 0)$ is 1. The eq uation of the face opposite A_1 is $X_1 = 0$. The equation of the plane through A_1 and parallel

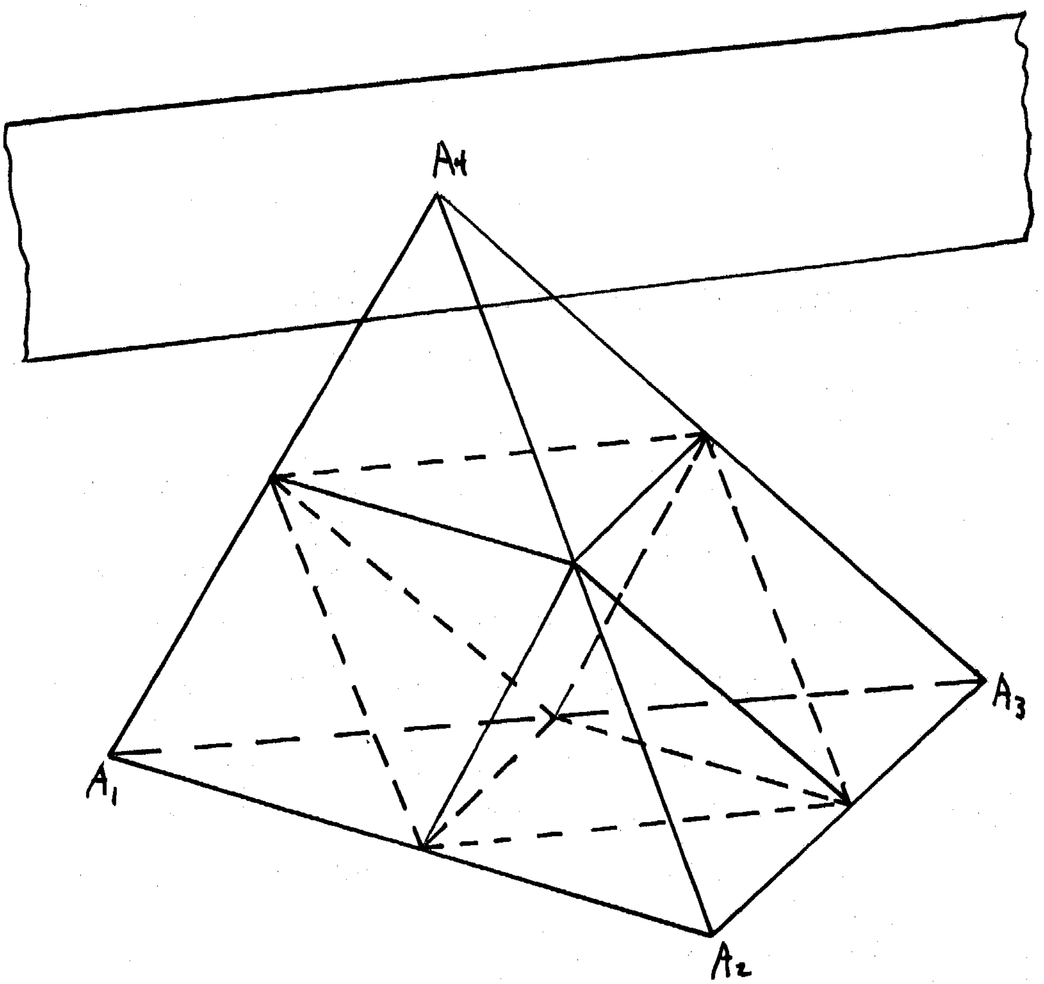

FIGURE 15

PARALLEL PLANES

to $X_1 = 0$ is $X_2 + X_3 + X_4 = 0$, or in quadriplanar coordinates it is $a_2x_2 + a_3x_3 + a_4x_4 = 0$. Similarly the equations of the planes through A_2, A_3, A_4 and parallel to their respective opposite faces are

$$a_1x_1 + a_3x_3 + a_4x_4 = 0,$$
$$a_1x_1 + a_2x_2 + a_4x_4 = 0,$$
$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Planes through vertices and parallel to median planes. The equation of the median plane containing the edge A_1A_2 and the median point of the tetrahedron is $a_3x_3 - a_4x_4 = 0$. Then the equations of the planes through A_3 and through A_4 and parallel to the median plane are $a_1x_1 + a_2x_2 + 2a_4x_4 = 0$ and $a_1x_1 + a_2x_2 + 2a_3x_3 = 0$.

The equation of the median plane containing the edge A_1A_3 is $a_2x_2 - a_4x_4 = 0$. The equations of the two planes through A_2 and A_4 and parallel to $a_2x_2 - a_4x_4 = 0$ are $a_1x_1 + a_3x_3 + 2a_4x_4 = 0$ and $a_1x_1 + 2a_2x_2 + a_3x_3 = 0$.

The equation of the median plane containing the edge A_2A_3 is $a_1x_1 - a_4x_4 = 0$. The equations of the parallel planes through A_1 and A_4 are $a_2x_2 + a_3x_3 + 2a_4x_4 = 0$ and $2a_1x_1 + a_2x_2 + a_3x_3 = 0$.

The equation of the median plane containing the edge A_1A_4 is $a_2x_2 - a_3x_3 = 0$. The equations of the parallel planes through A_2 and A_3 are $a_1x_1 + 2a_3x_3 + a_4x_4 = 0$ and $a_1x_1 + 2a_2x_2 + a_4x_4 = 0$.

The equation of the median plane containing the edge A_2A_4 is $a_1x_1 - a_3x_3 = 0$. The equations of the planes through A_1 and A_3 and parallel to $a_1x_1 - a_3x_3 = 0$ are $a_2x_2 + 2a_3x_3 + a_4x_4 = 0$ and $2a_1x_1 + a_2x_2 + a_4x_4 = 0$.

The equation of the median plane containing the edge A_3A_4 is $a_1x_1 - a_2x_2 = 0$. The equations of the parallel planes through A_1 and A_2

are $2a_2x_2 + a_3x_3 + a_4x_4 = 0$ and $2a_1x_1 + a_3x_3 + a_4x_4 = 0$.

<u>Bundle of planes through median point parallel to faces</u>. If a bundle of planes having the median point of the tetrahedron as their point of intersection has a plane parallel to each face, the planes must have the following equations:

$$3a_1x_1 = a_2x_2 + a_3x_3 + a_4x_4,$$

$$3a_2x_2 = a_1x_1 + a_3x_3 + a_4x_4,$$

$$3a_3x_3 = a_1x_1 + a_2x_2 + a_4x_4,$$

$$3a_4x_4 = a_1x_1 + a_2x_2 + a_3x_3.$$

These are parallel to a_1, a_2, a_3, a_4, respectively.

<u>Bundle of planes through isogonal conjugate of median point parallel to faces</u>. The equations of the planes through K, the isogonal conjugate of M, and parallel respectively to a_1, a_2, a_3, a_4 are

$$a_1(a_2^2 + a_3^2 + a_4^2)x_1 = a_1^2(a_2x_2 + a_3x_3 + a_4x_4),$$

$$a_2(a_1^2 + a_3^2 + a_4^2)x_2 = a_2^2(a_1x_1 + a_3x_3 + a_4x_4),$$

$$a_3(a_1^2 + a_2^2 + a_4^2)x_3 = a_3^2(a_1x_1 + a_2x_2 + a_4x_4),$$

$$a_4(a_1^2 + a_2^2 + a_3^2)x_4 = a_4^2(a_1x_1 + a_2x_2 + a_3x_3).$$

These may be written in the form

$$\frac{x_1}{a_1} = \frac{a_2x_2 + a_3x_3 + a_4x_4}{a_2^2 + a_3^2 + a_4^2},$$

$$\frac{x_2}{a_2} = \frac{a_1x_1 + a_3x_3 + a_4x_4}{a_1^2 + a_3^2 + a_4^2},$$

$$\frac{x_3}{a_3} = \frac{a_1x_1 + a_2x_2 + a_4x_4}{a_1^2 + a_2^2 + a_4^2},$$

$$\frac{x_4}{a_4} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1^2 + a_2^2 + a_3^2}.$$

<u>Planes through vertices tangent to circumsphere</u>. The equations of the planes through the vertices and tangent to the circumsphere may be found by substituting the points of contact in the equation of the polar with respect to the circumsphere. The equation of the polar of any point P is

$$(a_{12}^2a_1a_2p_2 + a_{13}^2a_1a_3p_3 + a_{14}^2a_1a_4p_4)x_1 + (a_{12}^2a_1a_2p_1 + a_{23}^2a_2a_3p_3 + a_{24}^2a_2a_4p_4)x_2 + (a_{13}^2a_1a_3p_1 + a_{23}^2a_2a_3p_2 + a_{34}^2a_3a_4p_4)x_3 + (a_{14}^2a_1a_4p_1 + a_{24}^2a_2a_4p_2 + a_{34}^2a_3a_4p_4)x_4 = 0.$$

The equations of the tangent planes at A_1, A_2, A_3, A_4 are respectively,

$$a_{12}^2a_1a_2x_2 + a_{13}^2a_1a_3x_3 + a_{14}^2a_1a_4x_4 = 0,$$

$$a_{12}^2a_1a_2x_1 + a_{23}^2a_2a_3x_3 + a_{24}^2a_2a_4x_4 = 0,$$

$$a_{13}^2a_1a_3x_1 + a_{23}^2a_2a_3x_2 + a_{34}^2a_3a_4x_4 = 0,$$

$$a_{14}^2a_1a_4x_1 + a_{24}^2a_2a_4x_2 + a_{34}^2a_3a_4x_3 = 0.$$

<u>Antiparallel sections</u> (A. C.-247). If a plane is passed through the edges A_1A_4, A_2A_4, A_3A_4 (Figure 14) so that the points of intersection

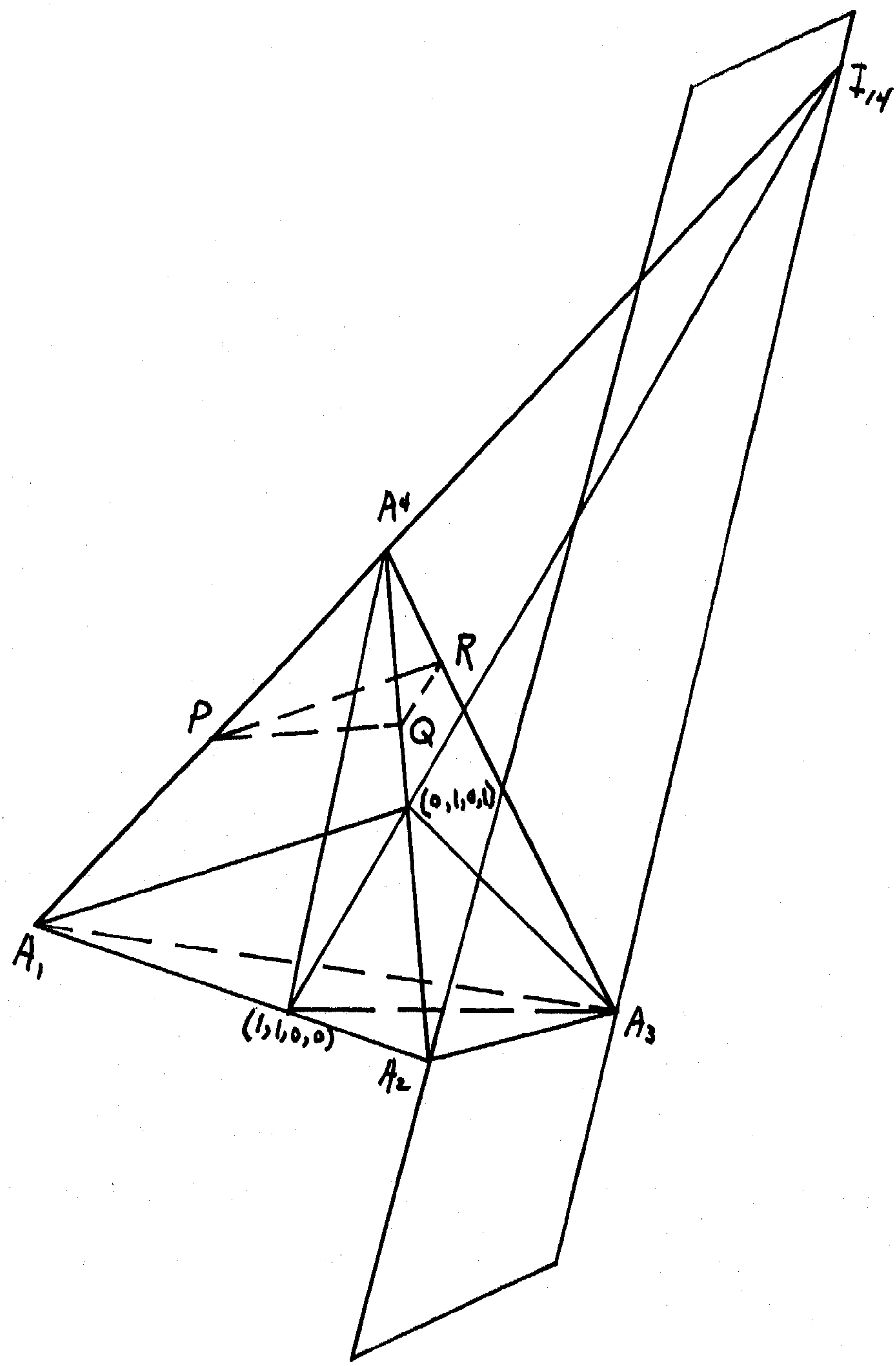

FIGURE 14

ANTIPARALLEL SECTION, INTERNAL AND EXTERNAL BISECTING PLANES are cospherical with the three remaining vertices of the tetrahedron, the tetrahedron cuts off on the plane what is known as an antiparallel section. The section PQR is antiparallel to $A_1A_2A_3$. This same section is parallel to the tangent plane to the circumsphere of the tetrahedron at the vertex A_4. The coordinates of P are $(p_1, 0, 0, p_4)$. Then the equation of any plane through P and parallel to the tangent plane at A_4 is $(p_1 + p_4)(a_{14}^2 X_1 + a_{24}^2 X_2 + a_{34}^2 X_3) = (a_{14}^2 p_1)(X_1 + X_2 + X_3 + X_4)$.

The equation of the tangent at A_4 is $a_{14}^2 a_1 x_1 + a_{24}^2 a_2 x_2 + a_{34}^2 a_3 x_3 = 0$. The two planes PQR and its parallel through A_4 should form a pencil with the ideal plane. The condition for this is for their matrix to be of rank two.

The symmedian points of the antiparallel sections relative to a given face of a tetrahedron lie on the line joining the symmedian point of the face considered to the opposite vertex of the tetrahedron (A. C.-248).

<u>Theorem</u> (A. C.-72). The external bisecting planes of the six dihedral angles of a tetrahedron meet the respective opposite edges in six coplanar points (Figure 14).

The equation of the external bisecting plane of θ_{14} is $x_1 + x_4 = 0$. The equation of the opposite edge, A_1A_4, is $x_2 = 0$, $x_3 = 0$. Solving these simultaneously, the point of intersection is found to be $I_{14}(1, 0, 0, -1)$. In a similar manner the other points of intersection are found to be $I_{24}(0, 1, 0, -1)$, $I_{34}(0, 0, 1, -1)$, $I_{23}(0, 1, -1, 0)$, $I_{13}(1, 0, -1, 0)$, $I_{12}(1, -1, 0, 0)$. Since three points determine a plane, any three of the above six points may be chosen to form the equation. If I_{14}, I_{24}, I_{13} are chosen, the equation of the plane is $x_1 + x_2 +$

$x_3 + x_4 = 0$. Then by substituting in the equation of the plane the coordinates of any of the six points, the theorem is readily verified. It may also be verified by making a four-rowed determinant from the coordinates of groups of any four points.

<u>Theorem</u> (A. C.-71). The point where an edge of a tetrahedron is met by the external bisecting plane of the opposite dihedral angle is collinear with the two points in which the two edges coplanar with the first are met by the internal bisecting planes of the dihedral angles respectively opposite these edges (Figure 14).

The coordinates of I_{14}, the point of intersection of the edge A_1A_4 and the external bisecting plane of the dihedral angle Θ_{23}, are $(1, 0, 0, -1)$. A_1A_2 and A_2A_4 are coplanar with the edge A_1A_4. The points of intersection of these edges and the respective internal bisecting planes of the opposite dihedral angles are $(1, 1, 0, 0)$ and $(0, 1, 0, 1)$. The determinant,

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix},$$

is identically zero, which proves the collinearity of the three points. The others may be proved in a similar manner.

<u>Cesaro's Theorem</u> (A. C.-72). The three points determined on the three coplanar edges of a tetrahedron by the external bisecting planes of the opposite dihedral angles are collinear (Figure 15). This line belongs to the plane determined by the three points in which the remaining three (concurrent) edges are met by the internal bisecting planes of the respectively opposite dihedral angles.

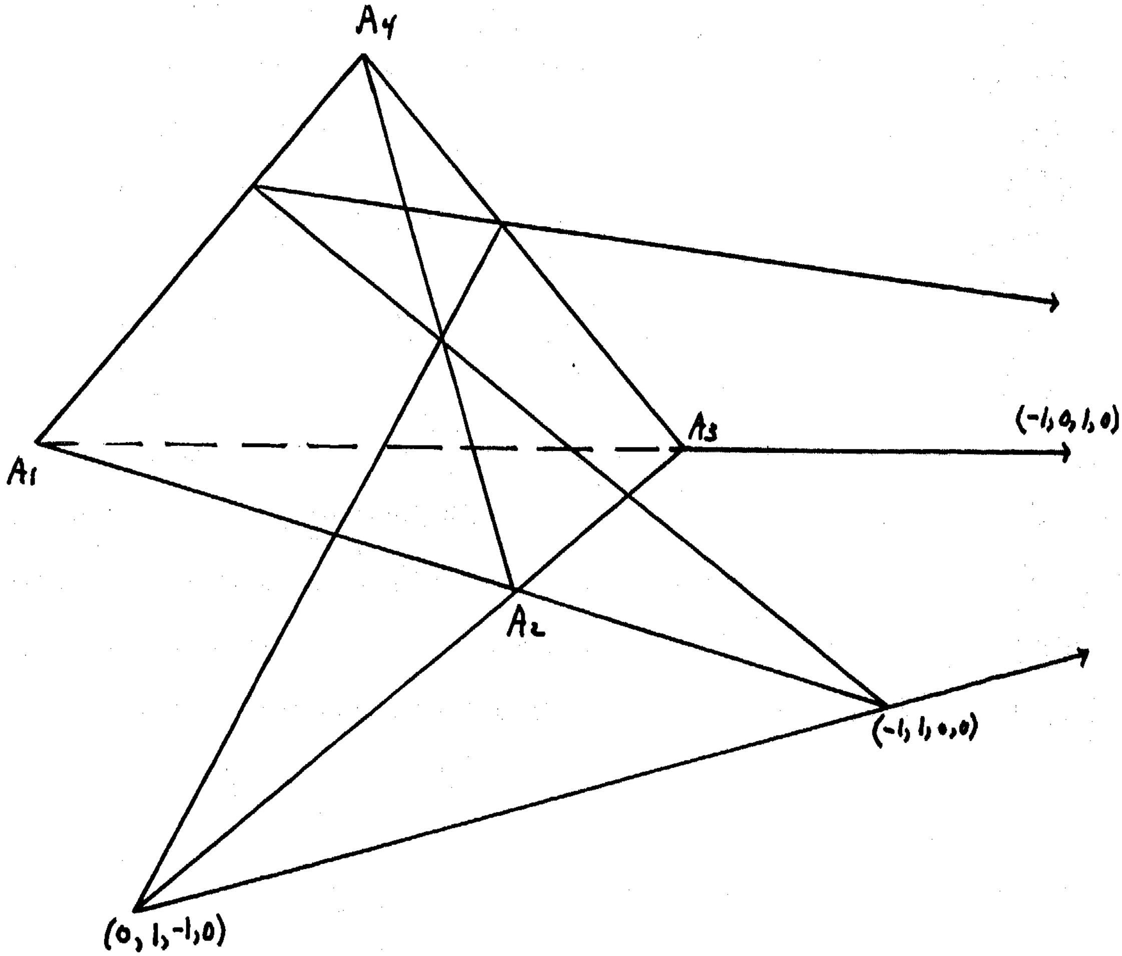

FIGURE 15

CESARO'S THEOREM

Take A_1A_2, A_1A_3, A_2A_3 as the three coplanar edges. The external bisecting plane of Θ_{12} intersects A_1A_2 at the point (-1, 1, 0, 0). The external bisecting plane of Θ_{13} intersects A_1A_3 at (-1, 0, 1, 0), while the external bisecting plane Θ_{23} intersects A_2A_3 at (0, 1, -1, 0). These three points are collinear for the determinant,

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{vmatrix}$$, is identically zero.

The equation of the internal bisecting plane of Θ_{34} is $x_3 - x_4 = 0$. It intersects the edge A_3A_4 at the point (0, 0, 1, 1). Likewise the internal bisecting planes of Θ_{24} and Θ_{14} intersect the edges A_2A_4 and A_1A_4 at the points (0, 1, 0, 1) and (1, 0, 0, 1). The equation of the plane which is determined by the points on the edges concurrent at A_4 is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0,$$

or $x_1 + x_2 + x_3 + x_4 = 0$.

This plane intersects the plane $x_4 = 0$ in the line determined by the three collinear points on the plane $x_4 = 0$, or the three coplanar edges A_1A_2, A_1A_3, A_2A_3. For by substituting the coordinates (-1, 1, 0, 0), (-1, 0, 1, 0), and (0, 1, -1, 0) in the above equation an identity of zero is obtained.

CHAPTER VI

SPECIAL TETRAHEDRONS

<u>Anticomplementary tetrahedron</u> (A. C.-55). If through the vertices of the tetrahedron of reference planes are drawn parallel to the opposite faces, the faces of the new tetrahedron will have for their centroids the vertices of the given tetrahedron (Figure 16).

The faces of the anticomplementary tetrahedron which pass through A_1, A_2, A_3, A_4, the vertices of the tetrahedron of reference, are respectively, $\underline{A_2A_3A_4}$, $\underline{A_1A_2A_4}$, $\underline{A_1A_3A_4}$, $\underline{A_1A_2A_3}$. The equations of these faces are, in the same order,

$$a_2x_2 + a_3x_3 + a_4x_4 = 0,$$
$$a_1x_1 + a_2x_2 + a_4x_4 = 0,$$
$$a_1x_1 + a_3x_3 + a_4x_4 = 0,$$
$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

The vertices of the anticomplementary tetrahedron are

$$\underline{A_1}\left(\frac{-2}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right), \underline{A_2}\left(\frac{1}{a_1}, \frac{-2}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}\right), \underline{A_3}\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{-2}{a_3}, \frac{1}{a_4}\right),$$

$$\underline{A_4}\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{-2}{a_4}\right).$$

The coordinates of the centroid in the face $\underline{A_2A_3A_4}$ are (1, 0, 0, 0). Likewise the coordinates of the centroids in the faces $\underline{A_1A_3A_4}$, $\underline{A_1A_2A_4}$, and $\underline{A_1A_2A_3}$ are (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). The centroid of the anticomplementary tetrahedron is evidently the incenter of the tetrahedron of reference. Its coordinates are

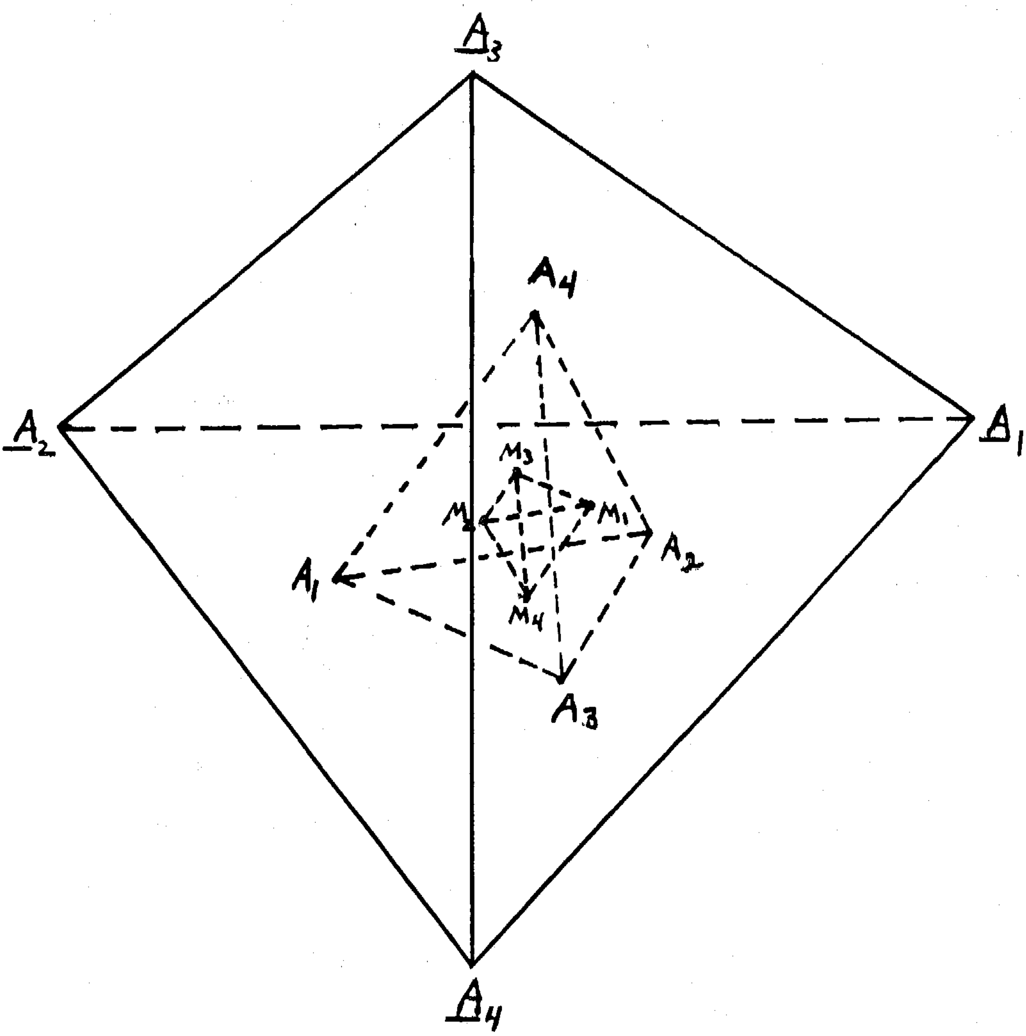

FIGURE 16

ANTICOMPLEMENTARY TETRAHEDRON, MEDIAL TETRAHEDRON

(1, 1, 1, 1). The symmedian point of the anticomplementary tetrahedron coincides with the centroid.

Medial tetrahedron (A. C.-53). If through the centroids of the faces of the tetrahedron of reference the planes determined by taking three centroids at a time are made to intersect, they form the medial tetrahedron (Figure 16). The vertices of the medial tetrahedron are

$$M_1(0, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}),\ M_2(\frac{1}{a_1}, 0, \frac{1}{a_3}, \frac{1}{a_4}),\ M_3(\frac{1}{a_1}, \frac{1}{a_2}, 0, \frac{1}{a_4}),\ M_4(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, 0).$$

The equations of the faces opposite M_1, M_2, M_3, M_4 are respectively

$$-2a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 - 2a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 - 2a_3x_3 + a_4x_4 = 0,$$

$$a_1x_1 + a_2x_2 + a_3x_3 - 2a_4x_4 = 0.$$

The coordinates of the centroid of the medial tetrahedron are obviously $(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_3^2}, \frac{1}{a_4^2})$.

Isodynamic tetrahedron (A. C.-276). The tetrahedron in which the three products of the three pairs of opposite edges are equal is referred to as an isodynamic tetrahedron (Figure 17).

The point $M(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4})$ is the median point of the tetrahedron. The point K, the Lemoine point of the tetrahedron, is the isogonal conjugate of M. The coordinates of K are (a_1, a_2, a_3, a_4). In an isodynamic tetrahedron each edge is met in the same point by the two symmedians of the two face angles of the tetrahedron opposite the edge considered.

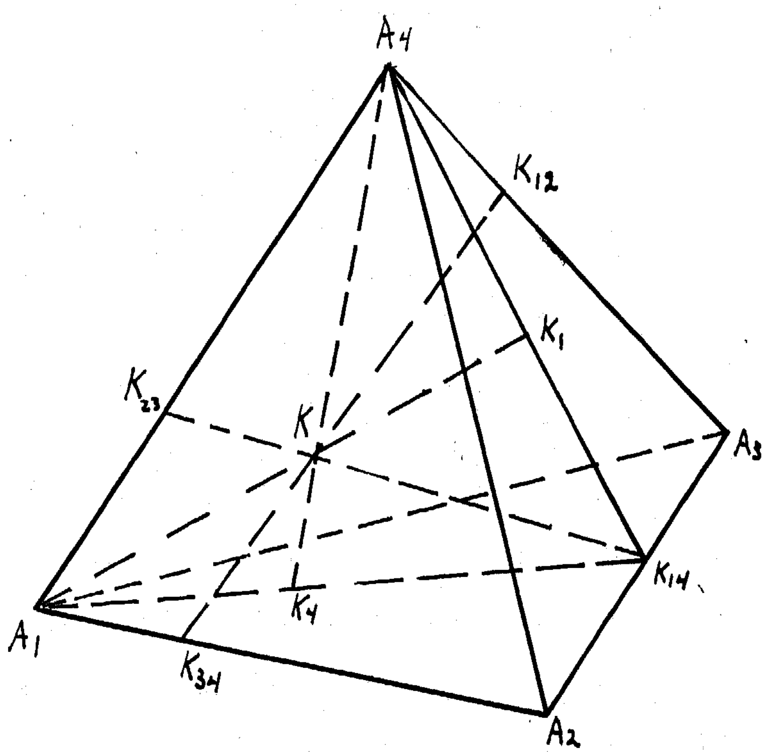

FIGURE 17

ISODYNAMIC TETRAHEDRON

The edge A_2A_3 is met by A_4K_1 at the point K_{14} whose coordinates are $(0, a_2, a_3, 0)$. The same edge is met by A_1K_4 at the point $(0, a_2, a_3, 0)$ which coincides with K_{14}. In a like manner the other edges are met at K_{12}, K_{13}, K_{23}, K_{24}, K_{34} by pairs of symmedians.

The equations of the symmedian planes, the planes determined by an edge and the symmedian point, are A_1A_4K: $a_3x_2 - a_2x_3 = 0$,

A_2A_4K: $a_3x_1 - a_1x_3 = 0$,

A_3A_4K: $a_2x_1 - a_1x_2 = 0$,

A_1A_2K: $a_4x_3 - a_3x_4 = 0$,

A_2A_3K: $a_4x_1 - a_1x_4 = 0$,

A_1A_3K: $a_4x_2 - a_2x_4 = 0$.

The four Lemoine axes of the four faces of an isodynamic tetrahedron are coplanar. The plane containing the four Lemoine axes of an isodynamic tetrahedron is called the Lemoine plane of the tetrahedron. The equation of the Lemoine plane is

$$a_2a_3a_4x_1 + a_1a_3a_4x_2 + a_1a_2a_4x_3 + a_1a_2a_3x_4 = 0$$

which may be expressed as

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} = 0.$$

It is obviously the tetrahedral polar plane of the point K, and conversely, K is the tetrahedral pole of the Lemoine plane (Figure 18).

The lines $K_{14}K_{23}$, $K_{12}K_{34}$, $K_{13}K_{24}$, joining the three pairs of points determined on the pairs of opposite edges of an isodynamic tetrahedron by the twelve symmedians of the faces of the tetrahedron pass through the Lemoine point of the tetrahedron. These lines are known as bisymmedians (Figure 17).

<u>Lemoine tetrahedron</u> (A. C.-128). Given the isodynamic tetrahedron $A_1A_2A_3A_4$ and its Lemoine point K, the harmonic conjugates K', K'', K''' of K with respect to the three pairs of opposite edges A_1A_4, A_2A_3; A_2A_4, A_1A_3; A_3A_4, A_1A_2, is referred to as the harmonic associates of K with respect to the tetrahedron (Figure 18).

The points K', K'', K''' lie in the harmonic plane of K with respect to $A_1A_2A_3A_4$, i. e., in the Lemoine plane of $A_1A_2A_3A_4$.

The tetrahedron $K\,K'K''K'''$ is referred to as the Lemoine tetrahedron of the isodynamic tetrahedron.

Since the following relationships

$$\left\{A_1'' \quad A_1' \;,\; K\,K_1'\right\} = -1,$$

$$\left\{A_3' \quad A_3'' \;,\; K\,K_3'''\right\} = -1,$$

$$\left\{A_2' \quad A_2'' \;,\; K\,K_2''\right\} = -1$$

are readily obtained, it is an easy matter to affix the proper signs to the coordinates of K_1', K_2'', K_3'''.

The coordinates of the Lemoine tetrahedron are $K(a_1, a_2, a_3, a_4)$, $K_1'(-a_1, a_2, a_3, -a_4)$, $K_2''(a_1, -a_2, a_3, -a_4)$, $K_3'''(-a_1, -a_2, a_3, a_4)$.

The equations of the faces of the Lemoine tetrahedron are

$$k \quad : \quad a_2a_3a_4x_1 + a_1a_3a_4x_2 + a_1a_2a_4x_3 + a_1a_2a_3x_4 = 0,$$

$$k_1' \quad : \quad a_2a_3a_4x_1 - a_1a_3a_4x_2 - a_1a_2a_4x_3 + a_1a_2a_3x_4 = 0,$$

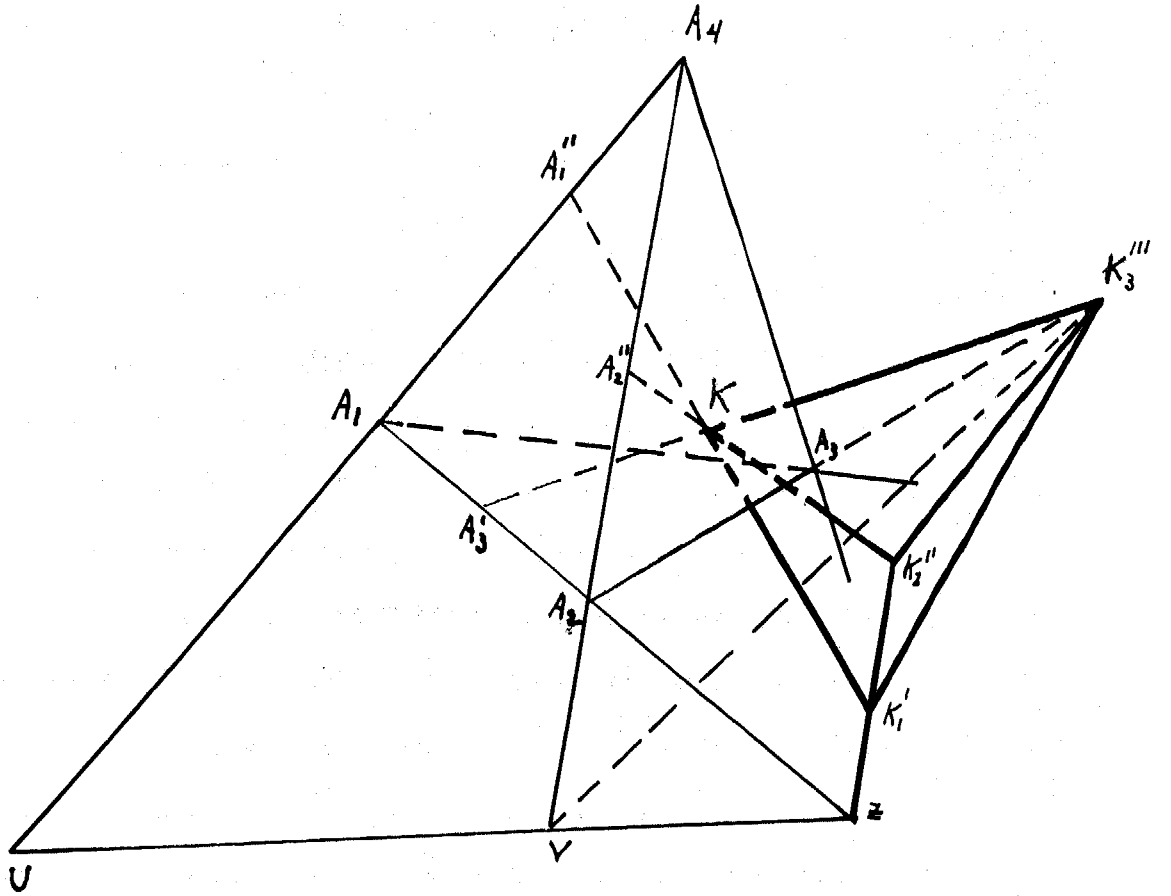

FIGURE 18

LEMOINE TETRAHEDRON, LEMOINE PLANE

$$K_2'' : \quad a_2a_3a_4x_1 - a_1a_3a_4x_2 + a_1a_2a_4x_3 - a_1a_2a_3x_4 = 0,$$

$$K_3''' : \quad a_2a_3a_4x_1 + a_1a_3a_4x_2 - a_1a_2a_4x_3 - a_1a_2a_3x_4 = 0.$$

The face K is opposite the vertex K, and similarly for the other notations.

The equation of k may be expressed as $\dfrac{x_1}{a_1} + \dfrac{x_2}{a_2} + \dfrac{x_3}{a_3} + \dfrac{x_4}{a_4} = 0$

which is recognised as being the equation of the Lemoine plane.

Isogonic tetrahedron (A. C.-290). If the lines joining the vertices of a tetrahedron to the points of contact of the opposite faces with the inscribed sphere of the tetrahedron are concurrent, the tetrahedron is said to be isogonal or isogonic (Figure 19).

The common point F of the four concurrent lines will be referred to as the Fermat point of the isogonic tetrahedron.

The tetrahedron determined by the points of contact of the faces of an isogonic tetrahedron with the inscribed sphere will be referred to as the Fermat tetrahedron of the isogonic tetrahedron. The Fermat tetrahedron is isodynamic since the lines from the vertices to the points of contact of the inscribed sphere and the face opposite the vertex meet in a point F. From definition the point F is the Lemoine point of the Fermat tetrahedron.

The equation of the face of the Fermat tetrahedron determined by the points of contact in faces $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 + \cos\theta_{12} & 0 & 1 + \cos\theta_{23} & 1 + \cos\theta_{24} \\ 1 + \cos\theta_{13} & 1 + \cos\theta_{23} & 0 & 1 + \cos\theta_{34} \\ 1 + \cos\theta_{14} & 1 + \cos\theta_{24} & 1 + \cos\theta_{34} & 0 \end{vmatrix} = 0,$$

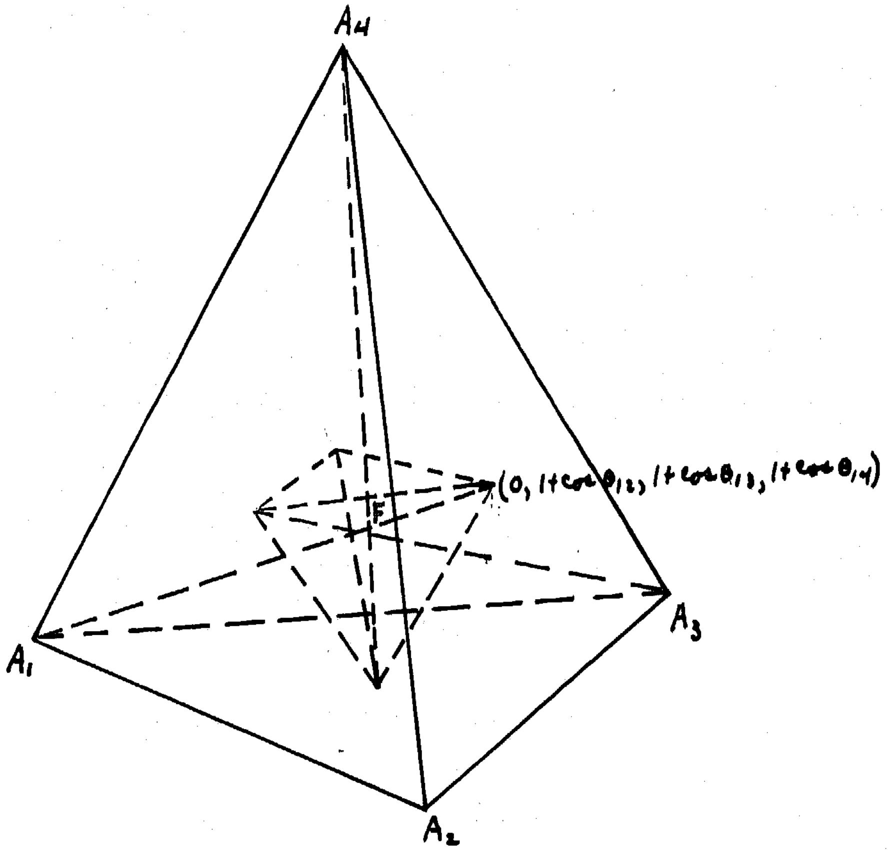

FIGURE 19

ISOGONIC TETRAHEDRON, FERMAT POINT, POINTS OF CONTACT

OF INSCRIBED SPHERE

or

$$\begin{vmatrix} A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 & x_2 & x_3 & x_4 \\ 2S & 0 & 1 + \cos\theta_{23} & 1 + \cos\theta_{24} \\ 2S & 1 + \cos\theta_{23} & 0 & 1 + \cos\theta_{34} \\ 2S & 1 + \cos\theta_{24} & 1 + \cos\theta_{34} & 0 \end{vmatrix} = 0.$$

The point F can be determined by finding the point of intersection of the three planes through the edge A_1A_2 and the point of contact in face $x_2 = 0$, A_2A_3 and the point of contact in face $x_3 = 0$, A_1A_3 and the point of contact in face $x_1 = 0$. The equations of these three planes are respectively

$$(1 + \cos\theta_{24})x_3 - (1 + \cos\theta_{23})x_4 = 0$$

$$(1 + \cos\theta_{34})x_1 - (1 + \cos\theta_{13})x_4 = 0$$

$$(1 + \cos\theta_{14})x_2 - (1 + \cos\theta_{12})x_4 = 0$$

The coordinates of the point of intersection of the three planes are

$$\begin{aligned}\big[&(1 + \cos\theta_{13})(1 + \cos\theta_{14})(1 + \cos\theta_{24}),\ (1 + \cos\theta_{12}) \\ &(1 + \cos\theta_{24})(1 + \cos\theta_{34}),\ (1 + \cos\theta_{14})(1 + \cos\theta_{23}) \\ &(1 + \cos\theta_{34}),\ (1 + \cos\theta_{14})(1 + \cos\theta_{24})(1 + \cos\theta_{34})\big].\end{aligned}$$

This point is the Fermat point.

<u>Isosceles tetrahedron</u> (A. C.-94). A tetrahedron in which each edge is equal to its opposite is referred to as an isosceles tetrahedron.

In an isosceles tetrahedron the incenter, circumcenter, Monge point, and the centroids coincide.

The coordinates of the centroid of the general tetrahedron were found to be $M(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4})$ while the coordinates of the incenter were I(1, 1, 1, 1). The faces of an isosceles tetrahedron are congruent. Then $a_1 : a_2 : a_3 : a_4 = 1 : 1 : 1 : 1$. The coordinates of the centroid then become (1, 1, 1, 1), identical to those of the incenter.

The coordinates of the circumcenter of the general tetrahedron are

$$x_1 = 2\, b_{ij} b_{ik} b_{il} \left\{ \frac{\cos A_i^{(j)}}{b_{ij}} + \frac{\cos A_i^{(k)}}{b_{ik}} + \frac{\cos A_i^{(l)}}{b_{il}} \right\} + a_{jk}^2 a_{jl}^2 a_{kl}^2 - \left\{ b_{ij}^2 + b_{ik}^2 + b_{il}^2 \right\}$$

where $b_{ij} = a_{ij} a_{kl}^2$. Since the faces are congruent all edges are equal and all angles are equal. The general equation reduces to

$$x_1 = 2\, a_{ij}^6 \,(\cos 180^\circ) + a_{ij}^6 - 3a_{ij}^6 = -2\, a_{ij}^6 + a_{ij}^6 - 3\, a_{ij}^6.$$

Then the coordinates of the circumcenter are

$$x_1 : x_2 : x_3 : x_4 = -4\, a_{12}^6 : -4\, a_{12}^6 : -4\, a_{12}^6 : -4\, a_{12}^6 = 1 : 1 : 1 : 1.$$

The coordinates of the Monge point of the isosceles tetrahedron are readily seen to coincide with the coordinates of the incenter when the equality of edges, faces, and angles are considered. Since the centroid, circumcenter, and Monge point coincide with the incenter, they must all coincide with each other.

CHAPTER VII

SPECIAL SPHERES

This chapter consists of material relative to the spheres most commonly found associated with the tetrahedron. Most of these spheres are determined by special points such as vertices or intersections of planes. Since four points determine a sphere, a multitude of spheres may be determined, however, this chapter deals with but a few. Illustrations or figures are left somewhat to the imagination of the reader. Occasionally parts of a figure are more illustrative than is the whole figure. Where a part suffices it is employed rather than the whole.

<u>Circumsphere</u>. (A. A.-24). The circumsphere is the sphere which passes through the vertices of the fundamental tetrahedron. From the coordinates of A_1, A_2, A_3, A_4 it is evident that $m_1 = 0$, $m_2 = 0$, $m_3 = 0$, $m_4 = 0$, and the equation of the circumsphere is

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 = 0$$

<u>Inscribed sphere</u> (A. C.-75). The general equation of a sphere is

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 + (m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4)(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.$$

From this it is found that

$$m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = -\frac{1}{3\Delta}(a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4).$$

$$
\begin{aligned}
0 \qquad\qquad + m_2(1+\cos\theta_{12}) + m_3(1+\cos\theta_{13}) + m_4(1+\cos\theta_{14}) &= -\tfrac{1}{3V}\Sigma_1,\\
m_1(1+\cos\theta_{21}) + 0 \qquad\qquad + m_3(1+\cos\theta_{23}) + m_4(1+\cos\theta_{24}) &= -\tfrac{1}{3V}\Sigma_2,\\
m_1(1+\cos\theta_{31}) + m_2(1+\cos\theta_{32}) + 0 \qquad\qquad + m_4(1+\cos\theta_{34}) &= -\tfrac{1}{3V}\Sigma_3,\\
m_1(1+\cos\theta_{41}) + m_2(1+\cos\theta_{42}) + m_3(1+\cos\theta_{43}) \qquad\qquad &= -\tfrac{1}{3V}\Sigma_4,
\end{aligned}
$$

where Σ_1 denotes the expression obtained by substituting the coordinates of I_{a_1} in the expression

$$a_{12}^2 a_1 a_2 x_1 x_2 + --- + a_{34}^2 a_3 a_4 x_3 x_4.$$

$$
m_1 = -\frac{1}{3V}\begin{vmatrix}
\Sigma_1 & 1+\cos\theta_{12} & 1+\cos\theta_{13} & 1+\cos\theta_{14}\\
\Sigma_2 & 0 & 1+\cos\theta_{23} & 1+\cos\theta_{24}\\
\Sigma_3 & 1+\cos\theta_{32} & 0 & 1+\cos\theta_{34}\\
\Sigma_4 & 1+\cos\theta_{42} & 1+\cos\theta_{43} & 0
\end{vmatrix} \div
$$

$$
\begin{vmatrix}
0 & 1+\cos\theta_{12} & 1+\cos\theta_{13} & 1+\cos\theta_{14}\\
1+\cos\theta_{21} & 0 & 1+\cos\theta_{23} & 1+\cos\theta_{24}\\
1+\cos\theta_{31} & 1+\cos\theta_{32} & 0 & 1+\cos\theta_{34}\\
1+\cos\theta_{41} & 1+\cos\theta_{42} & 1+\cos\theta_{43} & 0
\end{vmatrix},
$$

$$
\frac{6\,S\,Vm_1}{a_1} = -\begin{vmatrix}
\Sigma_1 & 1+\cos\theta_{12} & 1+\cos\theta_{13} & 1+\cos\theta_{14}\\
\Sigma_2 & 0 & 1+\cos\theta_{23} & 1+\cos\theta_{24}\\
\Sigma_3 & 1+\cos\theta_{32} & 0 & 1+\cos\theta_{34}\\
\Sigma_4 & 1+\cos\theta_{42} & 1+\cos\theta_{43} & 0
\end{vmatrix} \div
$$

$$\begin{vmatrix} 1 & 1 + \cos\theta_{12} & 1 + \cos\theta_{13} & 1 + \cos\theta_{14} \\ 1 & 0 & 1 + \cos\theta_{23} & 1 + \cos\theta_{24} \\ 1 & 1 + \cos\theta_{32} & 0 & 1 + \cos\theta_{34} \\ 1 & 1 + \cos\theta_{42} & 1 + \cos\theta_{43} & 0 \end{vmatrix},$$

where $\Sigma S \equiv a_1 + a_2 + a_3 + a_4$.

$$m_1 = -\frac{a_1}{6 S V} \left\{(\Sigma_1\theta_1 + \Sigma_2\theta_2 + \Sigma_3\theta_3 + \Sigma_4\theta_4)\right\} \div \left\{(\theta_1 + \theta_2 + \theta_3 + \theta_4)\right\},$$

where θ_i is the cofactor of the i th element in the first row of the determinant in which it is denominate.

Similar expressions exist for m_2, m_3, m_4. From the values of these, the expression for the inscribed sphere may be written.

<u>Isoclinal sphere</u>. The equation of the sphere through the vertex A_4 and through the points B_1, B_2, B_3 (Figure 11) equidistant from this vertex on the three edges concurrent at A_4 is found readily by noting the coordinates of these points:

$A_4(0, 0, 0, 1)$;

$B_1(r, 0, 0, a_{14} - r)$;

$B_2(0, r, 0, a_{24} - r)$;

$B_3(0, 0, r, a_{34} - r)$.

Hence

$m_1 = a_{14}\,(a_{14} - r)$,

$m_2 = a_{24}\,(a_{24} - r)$,

$m_3 = a_{34}\,(a_{34} - r)$,

$m_4 = 0$.

These values of m may be substituted in the general equation of a sphere, thereby rendering the equation of the isoclinal sphere.

<u>Twelve point sphere or medial sphere</u> (A. C.—231). When the projections of $P(p_1, p_2, p_3, p_4)$ coincide with the centroids of the faces of the tetrahedron of reference, the pedal sphere becomes the twelve point sphere of the tetrahedron. The sphere determined by the four centroids of the faces may be considered as a second analogue of the nine point circle of a triangle.

By substituting the coordinates of the centroids for p_1, p_2, p_3, p_4 in the values of m_1, m_2, m_3, m_4 as found in the equation of the pedal sphere, the new values of m_1, m_2, m_3, m_4 are as follows:

$$m_1 = \left(\frac{a_{23}^2 + a_{24}^2 + a_{34}^2 - 2\,a_{12}^2 - 2\,a_{13}^2 - 2\,a_{14}^2}{9}\right)a_1 =$$

$$-\frac{2\,a_1}{9}\left(a_{13}a_{14}\cos A_1^{(2)} + a_{12}a_{14}\cos A_1^{(3)} + a_{12}a_{13}\cos A_1^{(4)}\right),$$

$$m_2 = \left(\frac{a_{13}^2 + a_{14}^2 + a_{34}^2 - 2\,a_{12}^2 - 2\,a_{23}^2 - 2\,a_{24}^2}{9}\right)a =$$

$$-\frac{2\,a_2}{9}\left(a_{23}a_{24}\cos A_2^{(1)} + a_{21}a_{24}\cos A_2^{(3)} + a_{21}a_{23}\cos A_2^{(4)}\right),$$

$$m_3 = \left(\frac{a_{12}^2 + a_{14}^2 + a_{24}^2 - 2\,a_{13}^2 - 2\,a_{23}^2 - 2\,a_{43}^2}{9}\right)a_3 =$$

$$-\frac{2\,a_3}{9}\left(a_{32}a_{34}\cos A_3^{(1)} + a_{31}a_{34}\cos A_3^{(2)} + a_{31}a_{32}\cos A_3^{(4)}\right),$$

$$m_4 = \left(\frac{a_{12}^2 + a_{13}^2 + a_{23}^2 - 2\,a_{14}^2 - 2\,a_{24}^2 - 2\,a_{34}^2}{9}\right)a_4 =$$

$$-\frac{2\,a_4}{9}\left(a_{42}a_{43}\cos A_4^{(1)} + a_{41}a_{42}\cos A_4^{(2)} + a_{41}a_{43}\cos A_4^{(3)}\right).$$

Thus by substituting the values of m_1, m_2, m_3, m_4 as found above in the general equation of a sphere, the equation of the twelve point sphere may be secured.

<u>First twelve point sphere of an orthocentric tetrahedron</u> (A. C.-261). In case of an orthocentric tetrahedron the mid-points of the six edges and the six feet of the bialtitudes are twelve points on the same sphere, the center of which coincides with the centroid of the tetrahedron (Figure 20). This sphere is called the first twelve point sphere of the orthocentric tetrahedron. The bimedians are the radii of the sphere. The intersection of the sphere and each face is the nine-point circle of each face.

Since four points not in the same plane determine a sphere, the values of m_1, m_2, m_3, m_4 may be found from the coordinates of $A_1'\left(0, \frac{1}{a_2}, \frac{1}{a_3}, 0\right)$, $A_1''\left(\frac{1}{a_1}, 0, 0, \frac{1}{a_4}\right)$, $A_3''\left(0, 0, \frac{1}{a_3}, \frac{1}{a_4}\right)$, $A_2''\left(0, \frac{1}{a_2}, 0, \frac{1}{a_4}\right)$. These values may then be substituted in the general equation of a sphere.

$$\frac{m_2}{a_2} + \frac{m_3}{a_3} = -\frac{a_{23}^2}{2},$$

$$\frac{m_1}{a_1} + \frac{m_4}{a_4} = -\frac{a_{23}^2}{2},$$

$$\frac{m_3}{a_3} + \frac{m_4}{a_4} = -\frac{a_{34}^2}{2},$$

$$\frac{m_2}{a_2} + \frac{m_4}{a_4} = -\frac{a_{24}^2}{2},$$

$$m_1 = -\frac{a_1(-a_{23}^2 - a_{14}^2 + a_{34}^2 + a_{24}^2)}{4} = -(\cos A_4^{(1)} - a_{14}^2)a_1 a_{24} a_{34},$$

$$m_2 = -\frac{a_2(a_{23}^2 + a_{14}^2 - a_{34}^2 + a_{24}^2)}{4} = -(\cos A_2^{(1)} - a_{14})a_2 a_{23} a_{24},$$

$$m_3 = -\frac{a_3(a_{23}^2 + a_{34}^2 - a_{24}^2)}{4} = -(\cos A_3^{(1)}\ a_3 a_{23} a_{34}),$$

$$a_{12}^2 a_1 a_2 + a_1 m_2 + a_2 m_1 = 0,$$

$$a_{13}^2 a_1 a_3 + a_1 m_3 + a_3 m_1 = 0,$$

$$a_{14}^2 a_1 a_4 + a_1 m_4 + a_4 m_1 = 0,$$

$$a_{23}^2 a_2 a_3 + a_2 m_3 + a_3 m_2 = 0,$$

$$a_{24}^2 a_2 a_4 + a_2 m_4 + a_4 m_2 = 0,$$

$$a_{34}^2 a_3 a_4 + a_3 m_4 + a_4 m_3 = 0.$$

This system is satisfied by

$$m_1 = \frac{a_1(-a_{12}^2 + a_{23}^2 - a_{13}^2)}{2} = -a_1 a_{12} a_{13} \cos A_1^{(4)},$$

$$m_2 = \frac{a_2(-a_{12}^2 + a_{13}^2 - a_{23}^2)}{2} = -a_2 a_{12} a_{23} \cos A_2^{(4)},$$

$$m_3 = \frac{a_3(a_{12}^2 - a_{13}^2 - a_{23}^2)}{2} = -a_3 a_{13} a_{23} \cos A_3^{(4)},$$

$$m_4 = \frac{a_4(a_{12}^2 - a_{14}^2 - a_{24}^2)}{2} = -a_4 a_{14} a_{24} \cos A_4^{(3)}.$$

The equation of the polar sphere is

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 + (-a_1 a_{12} a_{13} \cos A_1^{(4)}\, x_1 - a_2 a_{12} a_{23} \cos A_2^{(4)} x_2 - a_3 a_{13} a_{23} \cos A_3^{(4)} - a_4 a_{14} a_{24} \cos A_4^{(3)})(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0,$$ or

$$a_1^2 a_{12} a_{13} \cos A_1^{(4)} x_1^2 + a_2^2 a_{12} a_{23} \cos A_2^{(4)} x_2^2 + a_3^2 a_{13} a_{23} \cos A_3^{(4)} x_3^2 + a_4^2 a_{24} a_{34} \cos A_4^{(3)} x_4 = 0.$$

The values of m_1, m_2, m_3, m_4 show that $a_{14}^2 + a_{23}^2 = a_{24}^2 + a_{13}^2 =$

$a_{34}^2 + a_{12}^2$ or that the tetrahedron is orthocentric. From this it follows that at any vertex the product of two edges times the cosine of the included angle is constant. That is, at the vertex A_4, $a_{24}a_{34}\cos A_4^{(1)} = a_{14}a_{34}\cos A_4^{(2)} = a_{14}a_{24}\cos A_4^{(3)}$, and similarly for the other vertices.

The following relationships may also be deduced:

$$\cos A_4^{(1)} : \cos A_4^{(2)} : \cos A_4^{(3)} = a_{14} : a_{24} : a_{34} ,$$

$$\cos A_3^{(1)} : \cos A_3^{(2)} : \cos A_3^{(4)} = a_{13} : a_{23} : a_{34} ,$$

$$\cos A_2^{(1)} : \cos A_2^{(3)} : \cos A_2^{(4)} = a_{12} : a_{23} : a_{24} ,$$

$$\cos A_1^{(2)} : \cos A_1^{(3)} : \cos A_1^{(4)} = a_{12} : a_{13} : a_{14} .$$

If $A_{lj}\cos A_k^{(i)} = a_{kl}\cos A_l^{(j)}$, this means that the projection of A_i and A_j upon the edge A_{kl} coincide, which indicates that the edge A_{ij} is perpendicular to the edge A_{kl}.

<u>Three spheres through a vertex and tangent to the opposite face at a vertex</u>. If three spheres are passed through a vertex of the tetrahedron of reference so that each is tangent to the face opposite, each at a different vertex, they will intersect in a point other than the first vertex (Figure 21).

The notation $\overline{A}_1$ which is used in this topic indicates that the sphere is tangent to the face a_1 at the vertex A_1. $\overline{A}_1, A_4; \overline{A}_2, A_4; \overline{A}_3, A_4$ is the notation used to indicate that the common vertex of the three spheres is A_4, while the point of tangency is that explained in the first part of this paragraph. Similar notations are used in making reference to other spheres and points of tangency.

Following is a table of the values of m_1, m_2, m_3, m_4 associated with the spheres through a vertex and tangent to the opposite face at a vertex:

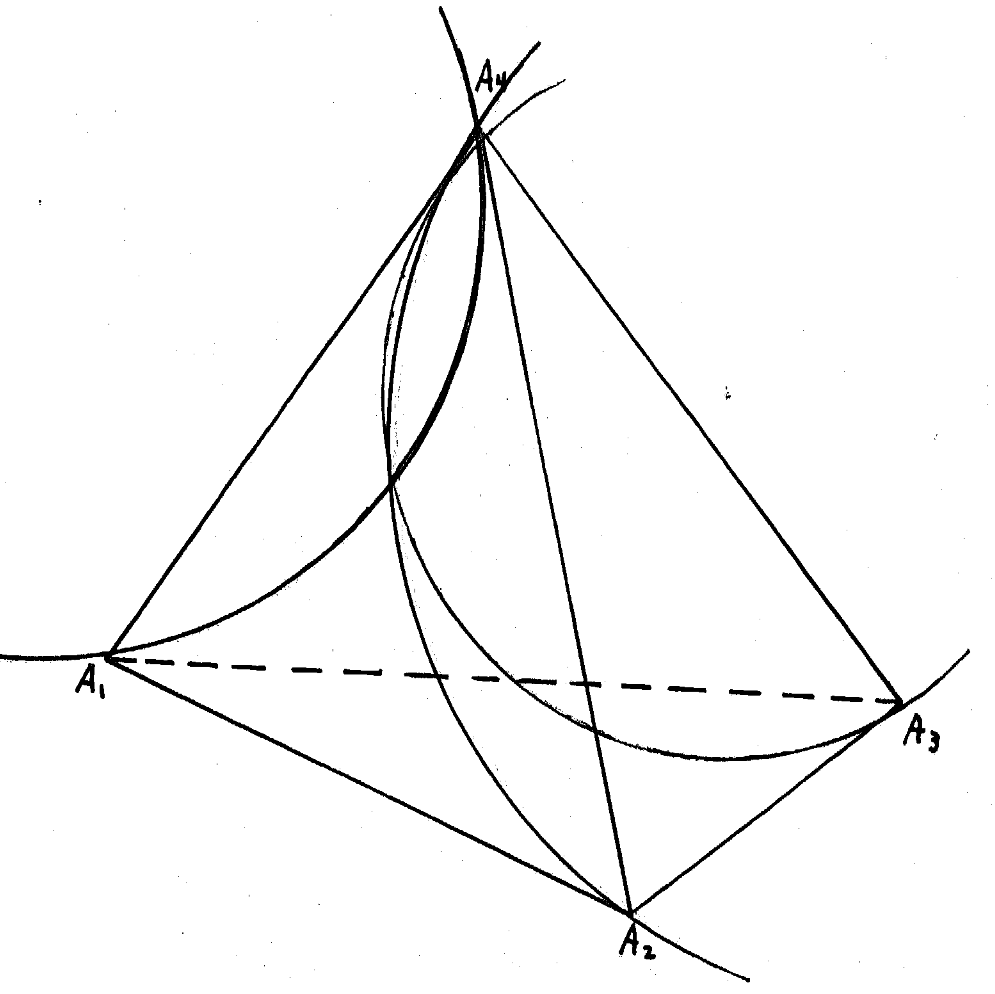

FIGURE 21

SPHERES THROUGH A VERTEX AND TANGENT TO

OPPOSITE FACE AT OTHER VERTICES

$\bar{A}_1, A_4$: $m_1 = 0$, $m_2 = -a_{12}^2 a_2$, $m_3 = -a_{13}^2 a_3$, $m_4 = 0$,

$\bar{A}_2, A_4$: $m_1 = -a_{21}^2 a_1$, $m_2 = 0$, $m_3 = -a_{23}^2 a_3$, $m_4 = 0$,

$\bar{A}_3, A_4$: $m_1 = -a_{31}^2 a_1$, $m_2 = -a_{32}^2 a_2$, $m_3 = 0$, $m_4 = 0$;

$\bar{A}_1, A_3$: $m_1 = 0$, $m_2 = -a_{12}^2 a_2$, $m_3 = 0$, $m_4 = -a_{14}^2 a_4$,

$\bar{A}_2, A_3$: $m_1 = -a_{21}^2 a_1$, $m_2 = 0$, $m_3 = 0$, $m_4 = -a_{24}^2 a_4$,

$\bar{A}_4, A_3$: $m_1 = -a_{41}^2 a_1$, $m_2 = -a_{42}^2 a_2$, $m_3 = 0$, $m_4 = 0$;

$\bar{A}_1, A_2$: $m_1 = 0$, $m_2 = 0$, $m_3 = -a_{13}^2 a_3$, $m_4 = -a_{14}^2 a_4$,

$\bar{A}_3, A_2$: $m_1 = -a_{31}^2 a_1$, $m_2 = 0$, $m_3 = 0$, $m_4 = -a_{34}^2 a_4$,

$\bar{A}_4, A_2$: $m_1 = -a_{41}^2 a_1$, $m_2 = 0$, $m_3 = -a_{43}^2 a_3$, $m_4 = 0$;

$\bar{A}_2, A_1$: $m_1 = 0$, $m_2 = 0$, $m_3 = -a_{23}^2 a_3$, $m_4 = -a_{24}^2 a_4$,

$\bar{A}_3, A_1$: $m_1 = 0$, $m_2 = -a_{32}^2 a_2$, $m_3 = 0$, $m_4 = -a_{34}^2 a_4$,

$\bar{A}_4, A_1$: $m_1 = 0$, $m_2 = -a_{42}^2 a_2$, $m_3 = -a_{43}^2 a_3$, $m_4 = 0$.

The general equation of a sphere may be written as

$\sum m_i x_i = -\frac{\sum a_{ii}^2 a_i a_i x_i x_i}{\sum a_i x_i} = 0$. Then by using the identity,

$-\frac{\sum a_{ij} a_i a_j x_i x_j}{\sum a_i x_i} \equiv 0$, and substituting the values of m_i as found in the table above, the equations of the spheres associated with the values of m_i are

$\bar{A}_1, A_4$: $\quad a_{12}^2 a_2 x_2 + a_{13}^2 a_3 x_3 = 0,$

$\bar{A}_2, A_4$: $\quad a_{21}^2 a_1 x_1 + a_{23}^2 a_3 x_3 = 0,$

$\bar{A}_3, A_4$: $\quad a_{31}^2 a_1 x_1 + a_{32}^2 a_2 x_2 = 0.$

$\bar{A}_1, A_3$: $\quad a_{12}^2 a_2 x_2 + a_{14}^2 a_4 x_4 = 0,$

$\bar{A}_2, A_3$: $\quad a_{21}^2 a_1 x_1 + a_{24}^2 a_4 x_4 = 0,$

$\bar{A}_4, A_3$: $\quad a_{41}^2 a_1 x_1 + a_{42}^2 a_2 x_2 = 0;$

$\bar{A}_1, A_2$: $\quad a_{13}^2 a_3 x_3 + a_{14}^2 a_4 x_4 = 0,$

$\bar{A}_3, A_2$: $\quad a_{31}^2 a_1 x_1 + a_{34}^2 a_4 x_4 = 0,$

$\bar{A}_4, A_2$: $\quad a_{41}^2 a_1 x_1 + a_{43}^2 a_3 x_3 = 0;$

$\bar{A}_2, A_1$: $\quad + a_{23}^2 a_3 x_3 + a_{24}^2 a_4 x_4 = 0,$

$\bar{A}_3, A_1$: $\quad a_{32}^2 a_2 x_2 + a_{34}^2 a_4 x_4 = 0,$

$\bar{A}_4, A_1$: $\quad a_{42}^2 a_2 x_2 + a_{43}^2 a_3 x_3 = 0.$

By solving simultaneously the equations of the three spheres through A_4, the coordinates of the point of intersection of those spheres are readily found.

Spheres through three vertices and the incenter. The sphere through A_1, A_2, A_3, I passes through the coordinates (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 1). By substituting the coordinates of the four points in the general equation of a sphere, it is found that $m_1 = 0$, $m_2 = 0$, $m_3 = 0$,

$$m_4 = -\frac{(a_{12}^2 a_1 a_2 + a_{13}^2 a_1 a_3 + a_{14}^2 a_1 a_4 + a_{23}^2 a_2 a_3 + a_{24}^2 a_2 a_4 + a_{34}^2 a_3 a_4)}{a_1 + a_2 + a_3 + a_4}.$$

The equation of the sphere $(A_1 A_2 A_3 I)$ may be written

$$(a_1 + a_2 + a_3 + a_4)(a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3$$
$$+ a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4) - x_4(a_{12}^2 a_1 a_2 + a_{13}^2 a_1 a_3 + a_{14}^2 a_1 a_4 + a_{23}^2 a_2 a_3 +$$
$$a_{24}^2 a_2 a_4 + a_{34}^2 a_3 a_4)(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.$$

Similarly the equations of the spheres $(A_1 A_2 A_4 I)$, $(A_1 A_3 A_4 I)$, $(A_2 A_3 A_4 I)$ are

$$\sum a_i \sum a_{ij}^2 a_i a_j x_i x_j - x_3(\sum a_{ij}^2 a_i a_j)(\sum a_i x_i) = 0,$$

$$\sum a_i \sum a_{ij}^2 a_i a_j x_i x_j - x_2(\sum a_{ij}^2 a_i a_j)(\sum a_i x_i) = 0,$$

$$\sum a_i \sum a_{ij}^2 a_i a_j x_i x_j - x_1(\sum a_{ij}^2 a_i a_j)(\sum a_i x_i) = 0.$$

Sphere through the excenters located in the four truncs. The coordinates of the excenters located in the truncs are I_{a_1} (-1, 1, 1, 1), I_{a_2} (1, -1, 1, 1), I_{a_3} (1, 1, -1, 1), I_{a_4} (1, 1, 1, -1). Substituting the coordinates of these in the general equation of a sphere, it is found that

$$-m_1 + m_2 + m_3 + m_4 = -\frac{S_1}{2(S - a_1)},$$

$$m_1 - m_2 + m_3 + m_4 = -\frac{S_2}{2(S - a_2)},$$

$$m_1 + m_2 - m_3 + m_4 = -\frac{S_3}{2(S - a_3)},$$

$$m_1 + m_2 + m_3 - m_4 = -\frac{S_4}{2(S - a_4)},$$

where $S = \dfrac{a_1 + a_2 + a_3 + a_4}{2}$ and

$$S_1 = -a_{12}^2 a_1 a_2 - a_{13}^2 a_1 a_3 - a_{14}^2 a_1 a_4 + a_{23}^2 a_2 a_3 + a_{24}^2 a_2 a_4 + a_{34}^2 a_3 a_4,$$

$$S_2 = -a_{12}^2 a_1 a_2 + a_{13}^2 a_1 a_3 + a_{14}^2 a_1 a_4 - a_{23}^2 a_2 a_3 - a_{24}^2 a_2 a_4 + a_{34}^2 a_3 a_4,$$

$$S_3 = a_{12}^2 a_1 a_2 - a_{13}^2 a_1 a_3 + a_{14}^2 a_1 a_4 - a_{23}^2 a_2 a_3 + a_{24}^2 a_2 a_4 - a_{34}^2 a_3 a_4,$$

$$S_4 = a_{12}^2 a_1 a_2 + a_{13}^2 a_1 a_3 - a_{14}^2 a_1 a_4 + a_{23}^2 a_2 a_3 - a_{24}^2 a_2 a_4 - a_{34}^2 a_3 a_4.$$

$$m_1 = \frac{1}{8}\left[-\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)}\right],$$

$$m_2 = \frac{1}{8}\left[\frac{S_1}{(S - a_1)} - \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)}\right],$$

$$m_3 = \frac{1}{8}\left[\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} - \frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)}\right],$$

$$m_4 = \frac{1}{8}\left[\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} - \frac{S_4}{(S - a_4)}\right].$$

The equation of the sphere through the four excenters may be written as

$$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 + a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 +$$

$$a_{34}^2 a_3 a_4 x_3 x_4 + \frac{1}{8}\Big\{ \Big[-\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)} \Big] x_1 -$$

$$\Big[\frac{S_1}{(S - a_1)} - \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)}\Big] x_2 + \Big[\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} -$$

$$\frac{S_3}{(S - a_3)} + \frac{S_4}{(S - a_4)}\Big] x_3 + \Big[\frac{S_1}{(S - a_1)} + \frac{S_2}{(S - a_2)} + \frac{S_3}{(S - a_3)} - \frac{S_4}{(S - a_4)}\Big] x_4 \Big\}$$

$$(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.$$

CHAPTER VIII

NOTATIONS AND FORMULAS

It is most desirable to have available a list of relations connecting the elements of the fundamental tetrahedron. With the aid of these relations it is often possible to simplify expressions which otherwise would be most awkward and formidable. Thus the determination of concurrency of planes, or collinearity of points, or the equations of a sphere when four of its points are given may be secured.

Following is a short list of formulas. Most of these are formulas actually found in the contents of this thesis.

The notations employed in this list are

1. a_1: The area of the face $A_2A_3A_4$ of the tetrahedron of reference
   a_2: The area of the face $A_1A_3A_4$ of the tetrahedron of reference
   a_3: The area of the face $A_1A_2A_4$ of the tetrahedron of reference
   a_4: The area of the face $A_1A_2A_3$ of the tetrahedron of reference

2. a_{12}: The length of the edge A_1A_2 of the tetrahedron of reference
   a_{13}: The length of the edge A_1A_3 of the tetrahedron of reference
   a_{14}: The length of the edge A_1A_4 of the tetrahedron of reference
   a_{23}: The length of the edge A_2A_3 of the tetrahedron of reference
   a_{24}: The length of the edge A_2A_4 of the tetrahedron of reference
   a_{34}: The length of the edge A_3A_4 of the tetrahedron of reference

3. A_1: Trihedral vertex at intersection of faces a_2, a_3, a_4
   A_2: Trihedral vertex at intersection of faces a_1, a_3, a_4

A_3: Trihedral vertex at intersection of faces a_1, a_2, a_4

A_4: Trihedral vertex at intersection of faces a_1, a_2, a_3

4. A_1': The mid-point of the edge A_2A_3

A_1'': The mid-point of the edge A_1A_4

A_2': The mid-point of the edge A_1A_3

A_2'': The mid-point of the edge A_2A_4

A_3': The mid-point of the edge A_1A_2

A_3'': The mid-point of the edge A_3A_4

5. $\underline{A}_1$: Vertex of the anticomplementary tetrahedron

$\underline{A}_2$: Vertex of the anticomplementary tetrahedron

$\underline{A}_3$: Vertex of the anticomplementary tetrahedron

$\underline{A}_4$: Vertex of the anticomplementary tetrahedron

6. $A_2^{(1)}$: Face angle A_2 in the face $x_1 = 0$

$A_3^{(1)}$: Face angle A_3 in the face $x_1 = 0$

$A_4^{(1)}$: Face angle A_4 in the face $x_1 = 0$

$A_1^{(2)}$: Face angle A_1 in the face $x_2 = 0$

$A_3^{(2)}$: Face angle A_3 in the face $x_2 = 0$

$A_4^{(2)}$: Face angle A_4 in the face $x_2 = 0$

$A_1^{(3)}$: Face angle A_1 in the face $x_3 = 0$

$A_2^{(3)}$: Face angle A_2 in the face $x_3 = 0$

$A_4^{(3)}$: Face angle A_4 in the face $x_3 = 0$

$A_1^{(4)}$: Face angle A_1 in the face $x_4 = 0$

$A_2^{(4)}$: Face angle A_2 in the face $x_4 = 0$

$A_3^{(4)}$: Face angle A_3 in the face $x_4 = 0$

7\. B_1: Intersection of isoclinal plane and edge of tetrahedron

B_2: Intersection of isoclinal plane and edge of tetrahedron

B_3: Intersection of isoclinal plane and edge of tetrahedron

8\. h_1: Altitude of tetrahedron of reference from A_1

h_2: A ltitude of tetrahedron of reference from A_2

h_3: Altitude of tetrahedron of reference from A_3

h_4: Altitude of tetrahedron of reference from A_4

9\. $h_2^{(1)}$: Face altitude from A_2 in face $x_1 = 0$

$h_3^{(1)}$: Face altitude from A_3 in face $x_1 = 0$

$h_4^{(1)}$: Face altitude from A_4 in face $x_1 = 0$

$h_1^{(2)}$: Face altitude from A_1 in face $x_2 = 0$

$h_3^{(2)}$: Face altitude from A_3 in face $x_2 = 0$

$h_4^{(2)}$: Face altitude from A_4 in face $x_2 = 0$

$h_1^{(3)}$: Face altitude from A_1 in face $x_3 = 0$

$h_2^{(3)}$: Face altitude from A_2 in face $x_3 = 0$

$h_4^{(3)}$: Face altitude from A_4 in face $x_3 = 0$

$h_1^{(4)}$: Face altitude from A_1 in face $x_4 = 0$

$h_2^{(4)}$: Face altitude from A_2 in face $x_4 = 0$

$h_3^{(4)}$: Face altitude from A_3 in face $x_4 = 0$

10. θ_1 : Angle between h_1 and an arbitrary line m

θ_2 : Angle between h_2 and an arbitrary line m

θ_3 : Angle between h_3 and an arbitrary line m

θ_4 : Angle between h_4 and an arbitrary line m

11. θ_{12}: Dihedral angle between the faces $x_1 = 0$, $x_2 = 0$

θ_{13}: Dihedral angle between the faces $x_1 = 0$, $x_3 = 0$

θ_{14}: Dihedral angle between the faces $x_1 = 0$, $x_4 = 0$

θ_{23}: Dihedral angle between the faces $x_2 = 0$, $x_3 = 0$

θ_{24}: Dihedral angle between the faces $x_2 = 0$, $x_4 = 0$

θ_{34}: Dihedral angle between the faces $x_3 = 0$, $x_4 = 0$

12. $_{14}$: Trihedral angle between edge a_{14} and face a_4

$_{24}$: Trihedral angle between edge a_{24} and face a_4

$_{34}$: Trihedral angle between edge a_{34} and face a_4

$_{41}$: Trihedral angle between edge a_{41} and face a_1

13. S: One-half the surface area

14. R: The circumradius

15. R_1: Circumradius associated with face a_1

R_2: Circumradius associated with face a_2

R_3: Circumradius associated with face a_3

R_4 : Circumradius associated with face a_4

16. r : The inradius

17. r' : Exradius associated with the roof A_1A_4; A_2A_3

r'' : Exradius associated with the roof A_2A_4; A_1A_3

r''' : Exradius associated with the roof A_3A_4; A_1A_2

r_1 : Exradius associated with the trunc A_1

r_2 : Exradius associated with the trunc A_2

r_3 : Exradius associated with the trunc A_3

r_4 : Exradius associated with the trunc A_4

18. $\triangle$: Volume of the tetrahedron $A_1A_2A_3A_4$

Formulas:

1. $S = (a_1 + a_2 + a_3 + a_4)/2$

$S - a_1 = (-a_1 + a_2 + a_3 + a_4)/2$

$S - a_2 = (a_1 - a_2 + a_3 + a_4)/2$

$S - a_3 = (a_1 + a_2 - a_3 + a_4)/2$

$S - a_4 = (a_1 + a_2 + a_3 - a_4)/2$

2. $S' = (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})/2$

3. $r = \dfrac{3\triangle}{2S} = \dfrac{3\triangle}{(a_1 + a_2 + a_3 + a_4)}$

4. $R = \dfrac{\sqrt{S'(S' - a_{14}a_{23})(S' - a_{13}a_{24})(S' - a_{12}a_{34})}}{6\triangle}$

5. $4R_1a_1 = a_{23}a_{24}a_{34}$

$4R_2a_2 = a_{13}a_{14}a_{34}$

$4R_3a_3 = a_{12}a_{14}a_{24}$

$4R_4a_4 = a_{12}a_{13}a_{23}$

6. $\Delta = 1/3\ a_1h_1 = 1/3\ a_2h_2 = 1/3\ a_3h_3 = 1/3\ a_4h_4$

7. $\Delta = 2/3\ rS = 2/3\ r_1(S - a_1) = 2/3\ r_2(S - a_2) =$

$$2/3\ r_3(S - a_3) = 2/3\ r_4(S - a_4)$$

8. $$144\,\Delta^2 = a_{23}^2 a_{14}^2(-a_{23}^2 - a_{14}^2 + a_{13}^2 + a_{24}^2 + a_{12}^2 + a_{34}^2) +$$

$$a_{13}^2 a_{24}^2(a_{23}^2 + a_{14}^2 - a_{13}^2 - a_{24}^2 + a_{12}^2 + a_{34}^2) +$$

$$a_{12}^2 a_{34}^2(a_{23}^2 + a_{14}^2 + a_{13}^2 + a_{24}^2 - a_{12}^2 - a_{34}^2) -$$

$$a_{23}^2 a_{24}^2 a_{34}^2 - a_{13}^2 a_{14}^2 a_{34}^2 - a_{12}^2 a_{14}^2 a_{24}^2 - a_{12}^2 a_{13}^2 a_{23}^2$$

9. $$\frac{1}{r} - \frac{1}{r_1} = \frac{2a_1}{3\Delta}$$

$$\frac{1}{r} - \frac{1}{r_2} = \frac{2a_2}{3\Delta}$$

$$\frac{1}{r} - \frac{1}{r_3} = \frac{2a_3}{3\Delta}$$

$$\frac{1}{r} - \frac{1}{r_4} = \frac{2a_4}{3\Delta}$$

10. If $a_1 > a_2 > a_3 > a_4$ then $r > r_1 > r_2 > r_3 > r_4$

11. $$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}$$

$$\frac{2}{r_4} = \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}$$

$$\frac{2}{r} - \frac{2}{r_1} = \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}$$

12. $$r' = \frac{3\,\Delta}{2(S - a_1 - a_4)}$$

$$r'' = \frac{3\,\Delta}{2(S - a_2 - a_4)}$$

$$r''' = \frac{3\,\Delta}{2(S - a_3 - a_4)}$$

13. $$\pm\frac{1}{r'} = \frac{1}{r_1} + \frac{1}{r_4} - \frac{1}{r}$$

$$\frac{1}{r''} = \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r}$$

$$\frac{1}{r'''} = \frac{1}{r_3} + \frac{1}{r_4} - \frac{1}{r}$$

14. $$\frac{1}{r} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_4}$$

$$\frac{1}{r_1} = -\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_4}$$

$$\frac{1}{r_2} = \frac{1}{h_1} - \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_4}$$

$$\frac{1}{r_3} = \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{h_4}$$

$$\frac{1}{r_4} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{h_4}$$

15. $$\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{r''^2} + \frac{1}{r'''^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}$$

16. $\frac{2}{h_4} = \frac{1}{r} + \frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r'} = \frac{1}{r_2} - \frac{1}{r''} = \frac{1}{r_3} - \frac{1}{r'''}$

17. $\sin \theta_{12} = \frac{3 \triangle a_{34}}{2\, a_1 a_2} = r3\, \frac{a_{34}}{a_1 a_2}$

$\sin \theta_{13} = \frac{3 \triangle a_{24}}{2\, a_1 a_3} = r3\, \frac{a_{24}}{a_1 a_3}$

$\sin \theta_{14} = \frac{3 \triangle a_{23}}{2 a_1 a_4} = r3 \qquad \frac{a_{23}}{a_1 a_4}$

$\sin \theta_{23} = \frac{3 \triangle a_{14}}{2 a_2 a_3} = r3 \qquad \frac{a_{14}}{a_2 a_3}$

$\sin \theta_{24} = \frac{3 \triangle a_{13}}{2 a_2 a_4} = r3 \qquad \frac{a_{13}}{a_2 a_4}$

$\sin \theta_{34} = \frac{3 \triangle a_{12}}{2 a_3 a_4} = r3 \qquad \frac{a_{12}}{a_3 a_4}$

18. $\cos \theta_{14} = \frac{\cos A_3^{(2)} - \cos A_3^{(1)} \cos A_3^{(4)}}{\sin A_3^{(1)} \sin A_3^{(4)}}$

19. $16\, a_1 a_2 \cos \theta_{12} = a_{14}^2 a_{34}^2 + a_{24}^2 a_{34}^2 + a_{13}^2 a_{34}^2 + a_{23}^2 a_{34}^2 -$

$a_{14}^2 a_{24}^2 - a_{13}^2 a_{23}^2 + a_{14}^2 a_{23}^2 + a_{13}^2 a_{24}^2 +$

$2a_{12}^2 a_{34}^2 - a_{34}^4$

$$16\, a_1 a_3 \cos \theta_{13} = a_{14}^2 a_{24}^2 + a_{24}^2 a_{34}^2 + a_{24}^2 a_{12}^2 + a_{24}^2 a_{23}^2 -$$

$$a_{14}^2 a_{34}^2 - a_{12}^2 a_{23}^2 + a_{12}^2 a_{34}^2 + a_{14}^2 a_{23}^2 -$$

$$2a_{13}^2 a_{24}^2 - a_{24}^4$$

$$16\, a_1 a_4 \cos \theta_{14} = a_{12}^2 a_{23}^2 + a_{23}^2 a_{24}^2 + a_{13}^2 a_{23}^2 + a_{23}^2 a_{34}^2 -$$

$$a_{13}^2 a_{34}^2 - a_{12}^2 a_{24}^2 + a_{12}^2 a_{34}^2 + a_{13}^2 a_{24}^2 -$$

$$2a_{14}^2 a_{23} - a_{24}^4$$

$$16\, a_2 a_3 \cos \theta_{23} = a_{14}^2 a_{12}^2 + a_{14}^2 a_{13}^2 + a_{14}^2 a_{24}^2 + a_{14}^2 a_{34}^2 -$$

$$a_{12}^2 a_{13}^2 - a_{24}^2 a_{34}^2 + a_{13}^2 a_{24}^2 + a_{12}^2 a_{34}^2 -$$

$$2a_{23}^2 a_{14}^2 - a_{14}^4$$

$$16\, a_2 a_4 \cos \theta_{24} = a_{12}^2 a_{13}^2 + a_{13}^2 a_{23}^2 + a_{13}^2 a_{14}^2 + a_{13}^2 a_{34}^2 -$$

$$a_{12}^2 a_{14}^2 - a_{23}^2 a_{34}^2 + a_{14}^2 a_{23}^2 + a_{12}^2 a_{34}^2 -$$

$$2a_{13}^2 a_{24}^2 - a_{13}^4$$

$$16\, a_3 a_4 \cos \theta_{34} = a_{12}^2 a_{13}^2 + a_{12}^2 a_{14}^2 + a_{12}^2 a_{23}^2 + a_{12}^2 a_{24}^2 -$$

$$a_{12}^2 a_{13}^2 - a_{23}^2 a_{24}^2 + a_{13}^2 a_{24}^2 + a_{23}^2 a_{14}^2 -$$

$$2a_{12}^2 a_{34}^2 - a_{12}^4$$

20. $$\sin \varphi_{14} = \frac{3\,\Delta}{a_4 a_{14}}$$

$$\sin \varphi_{24} = \frac{3\,\Delta}{a_4 a_{24}}$$

$$\sin \varphi_{34} = \frac{3\,\Delta}{a_4 a_{34}}$$

$$\sin \varphi_{14} : \sin \varphi_{24} : \sin \varphi_{34} = \frac{1}{a_{14}} : \frac{1}{a_{24}} : \frac{1}{a_{34}}$$

21. $$\cos A_2^{(1)} = \frac{a_{23}^2 + a_{24}^2 - a_{34}^2}{2a_{23}a_{24}}$$

$$\cos A_3^{(1)} = \frac{a_{23}^2 + a_{34}^2 - a_{24}^2}{2a_{23}a_{34}}$$

$$\cos A_4^{(1)} = \frac{a_{24}^2 + a_{34}^2 - a_{23}^2}{2a_{24}a_{34}}$$

$$\cos A_1^{(2)} = \frac{a_{13}^2 + a_{14}^2 - a_{34}^2}{2a_{13}a_{14}}$$

$$\cos A_3^{(2)} = \frac{a_{13}^2 + a_{34}^2 - a_{14}^2}{2a_{13}a_{34}}$$

$$\cos A_4^{(2)} = \frac{a_{14}^2 + a_{34}^2 - a_{13}^2}{2a_{14}a_{34}}$$

$$\cos A_1^{(3)} = \frac{a_{12} + a_{14} - a_{24}}{2a_{12}a_{14}}$$

$$\cos A_2^{(3)} = \frac{a_{12} + a_{24} - a_{14}}{2a_{12}a_{24}}$$

$$\cos A_4^{(3)} = \frac{a_{14} + a_{24} - a_{12}}{2a_{14}a_{24}}$$

$$\cos A_1^{(4)} = \frac{a_{12} + a_{13} - a_{23}}{2a_{12}a_{13}}$$

$$\cos A_2^{(4)} = \frac{a_{12} + a_{23} - a_{13}}{2a_{12}a_{23}}$$

$$\cos A_3^{(4)} = \frac{a_{13} + a_{23} - a_{12}}{2a_{13}a_{23}}$$

22. $$\sin A_2^{(1)} = \frac{a_{34}}{2R_1}$$

$$\sin A_3^{(1)} = \frac{a_{24}}{2R_1}$$

$$\sin A_4^{(1)} = \frac{a_{23}}{2R_1}$$

$$\sin A_1^{(2)} = \frac{a_{34}}{2R_2}$$

$$\sin A_3^{(2)} = \frac{a_{14}}{2R_2}$$

$$\sin A_4^{(2)} = \frac{a_{13}}{2R_2}$$

$$\sin A_1^{(3)} = \frac{a_{24}}{2R_3}$$

$$\sin A_2^{(3)} = \frac{a_{14}}{2R_3}$$

$$\sin A_4^{(3)} = \frac{a_{12}}{2R_3}$$

$$\sin A_1^{(4)} = \frac{a_{23}}{2R_4}$$

$$\sin A_2^{(4)} = \frac{a_{13}}{2R_4}$$

$$\sin A_3^{(4)} = \frac{a_{12}}{2R_4}$$

23. $h_1^{(2)} = \sin A_3^{(2)} a_{13} = \sin A_4^{(2)} a_{14}$

$h_1^{(3)} = \sin A_2^{(3)} a_{12} = \sin A_4^{(3)} a_{14}$

$h_1^{(4)} = \sin A_2^{(4)} a_{12} = \sin A_3^{(4)} a_{13}$

$h_2^{(1)} = \sin A_3^{(1)} a_{23} = \sin A_4^{(1)} a_{24}$

$h_2^{(3)} = \sin A_1^{(3)} a_{12} = \sin A_4^{(3)} a_{24}$

$h_2^{(4)} = \sin A_1^{(4)} a_{12} = \sin A_3^{(4)} a_{23}$

$h_3^{(1)} = \sin A_2^{(1)} a_{23} = \sin A_4^{(1)} a_{34}$

$h_3^{(2)} = \sin A_1^{(2)} a_{13} = \sin A_4^{(2)} a_{34}$

$h_3^{(4)} = \sin A_1^{(4)} a_{13} = \sin A_2^{(4)} a_{23}$

$h_4^{(1)} = \sin A_2^{(1)} a_{24} = \sin A_3^{(1)} a_{34}$

$h_4^{(2)} = \sin A_1^{(2)} a_{14} = \sin A_3^{(2)} a_{34}$

$h_4^{(3)} = \sin A_1^{(3)} a_{14} = \sin A_2^{(3)} a_{34}$

24. $h_1 = \sin \Theta_{14} h_1^{(4)} = \sin \Theta_{12} h_1^{(2)} = \sin \Theta_{13} h_1^{(3)}$

$h_2 = \sin \Theta_{24} h_2^{(4)} = \sin \Theta_{12} h_2^{(1)} = \sin \Theta_{23} h_2^{(3)}$

$$h_3 = \sin \theta_{34} h_3^{(4)} = \sin \theta_{13} h_3^{(1)} = \sin \theta_{23} h_3^{(2)}$$

$$h_4 = \sin \theta_{14} h_4^{(1)} = \sin \theta_{24} h_4^{(2)} = \sin \theta_{34} h_4^{(3)}$$

CHAPTER IX

TABLES OF RESULTS

TABLE I

SUMMARY OF POINTS

No.	Point	Coordinates
1	A_1	$1, 0, 0, 0$
2	A_2	$0, 1, 0, 0$
3	A_3	$0, 0, 1, 0$
4	A_4	$0, 0, 0, 1$
5	A_1'	$0, \frac{1}{a_2}, \frac{1}{a_3}, 0$
6	A_1''	$\frac{1}{a_1}, 0, 0, \frac{1}{a_4}$
7	A_2'	$\frac{1}{a_1}, 0, \frac{1}{a_3}, 0$
8	A_2''	$0, \frac{1}{a_2}, 0, \frac{1}{a_4}$
9	A_3'	$\frac{1}{a_1}, \frac{1}{a_2}, 0, 0$
10	A_3''	$0, 0, \frac{1}{a_3}, \frac{1}{a_4}$
11	A_1	$-\frac{2}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
12	A_2	$\frac{1}{a_1}, -\frac{2}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
13	A_3	$\frac{1}{a_1}, \frac{1}{a_2}, -\frac{2}{a_3}, \frac{1}{a_4}$

Point	Description
A_1	Vertices of fundamental tetrahedron (Figure 1)
A_2	
A_3	
A_4	
A_1'	Mid-point of edge A_2A_3 (Figure 10)
A_1''	Mid-point of edge A_1A_4
A_2'	Mid-point of edge A_1A_3
A_2''	Mid-point of edge A_2A_4
A_3'	Mid-point of edge A_1A_2
A_3''	Mid-point of edge A_3A_4
$\underline{A}_1$	Vertices of anitcomplementary tetrahedron (Figure 16)
$\underline{A}_2$	
$\underline{A}_3$	

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
14	A_4	$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, -\frac{2}{a_4}$
15	B_1	$r, 0, 0, a_{14} - r$
16	B_2	$0, r, 0, a_{24} - r$
17	B_3	$0, 0, r, a_{34} - r$
18	C	$\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_3^2}, \frac{1}{a_4^2}$
19	F	$(1 + \cos \theta_{13})(1 + \cos \theta_{14})(1 + \cos \theta_{24}),$ $(1 + \cos \theta_{12})(1 + \cos \theta_{24})(1 + \cos \theta_{34}),$ $(1 + \cos \theta_{14})(1 + \cos \theta_{23})(1 + \cos \theta_{34}),$ $(1 + \cos \theta_{14})(1 + \cos \theta_{24})(1 + \cos \theta_{34})$
20	I	$1, 1, 1, 1$
21	I'	$1, -1, -1, 1$ or $-1, 1, 1, -1$
22	I''	$-1, 1, -1, 1$ or $1, -1, 1, -1$
23	I'''	$-1, -1, 1, 1$ or $1, 1, -1, -1$
24	I_{a_1}	$-1, 1, 1, 1$
25	I_{a_2}	$1, -1, 1, 1$
26	I_{a_3}	$1, 1, -1, 1$
27	I_{a_4}	$1, 1, 1, -1$
28	I_1 or $I_{\underline{a}_1}$	$0, 1 + \cos \theta_{12}, 1 + \cos \theta_{13}, 1 + \cos \theta_{14}$

Point	Description
A_4	Vertices of anticomplementary tetrahedron
B_1	Intersections of isoclinal plane and the edges of tetrahedron (Figure 11)
B_2	
B_3	
C	Centroid of medial tetrahedron (Figure 16)
F	Fermat point (Figure 19)
I	Incenter
I'	Excenters associated with the roofs
I''	
I'''	
I_{a_1}	Excenters associated with the truncs
I_{a_2}	
I_{a_3}	
I_{a_4}	
I_1	Points of contact of inscribed sphere (Figures 5, 19)

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
29	I_2 or I_{a_2}	$1 + \cos\theta_{12}$, 0, $1 + \cos\theta_{23}$, $1 + \cos\theta_{24}$
30	I_3 or I_{a_3}	$1 + \cos\theta_{13}$, $1 + \cos\theta_{23}$, 0, $1 + \cos\theta_{34}$
31	I_4 or I_{a_4}	$1 + \cos\theta_{14}$, $1 + \cos\theta_{24}$, $1 + \cos\theta_{34}$
32	I_{12}	1, -1, 0, 0
33	I_{13}	1, 0, -1, 0
34	I_{14}	1, 0, 0, -1
35	I_{23}	0, 1, 0, -1
36	I_{24}	0, 1, 0, -1
37	I_{34}	0, 0, 1, -1
38	K	a_1, a_2, a_3, a_4
39	K'	$-a_1$, a_2, a_3, a_4
40	K''	a_1, $-a_2$, a_3, a_4
41	K'''	a_1, a_2, $-a_3$, a_4
42	K''''	a_1, a_2, a_3, $-a_4$
43	K_{12}	0, 0, a_3, a_4
44	K_{13}	0, a_2, 0, a_4
45	K_{14}	0, a_2, a_3, 0
46	K_{23}	a_1, 0, 0, a_4
47	K_{24}	a_1, 0, a_3, 0
48	K_{34}	a_1, a_2, 0, 0
49	K_1'	$-a_1$, a_2, a_3, $-a_4$
50	K_2''	a_1, $-a_2$, a_3, $-a_4$

Point	Description
I_2	
I_3	
I_4	Points of contact of inscribed sphere
I_{12}	Intersection of an external bisecting plane of a dihedral angle and the opposite edge (Figure 14)
I_{13}	
I_{14}	
I_{23}	
I_{24}	
I_{34}	
K	Symmedian point (Figure 8)
K'	Exsymmedian points (Figure 8)
K''	
K'''	
K''''	
K_{12}	Intersection of isogonal conjugates of face medians (Figure 17)
K_{13}	
K_{14}	
K_{23}	
K_{24}	
K_{34}	
K'_1	
K''_2	

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
51	K_3'''	$-a_1, -a_2, a_3, a_4$
52	M	$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
53	M_1	$0, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
54	M_2	$\frac{1}{a_1}, 0, \frac{1}{a_3}, \frac{1}{a_4}$
55	M_3	$\frac{1}{a_1}, \frac{1}{a_2}, 0, \frac{1}{a_4}$
56	M_4	$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, 0$
57	M'	$-\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
58	M''	$\frac{1}{a_1}, -\frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$
59	M'''	$\frac{1}{a_1}, \frac{1}{a_2}, -\frac{1}{a_3}, \frac{1}{a_4}$
60	M''''	$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, -\frac{1}{a_4}$
61	O	$-2a_{23}^2 a_{24}^2 a_{34}^2 + a_{23}^2 A_{11} + a_{24}^2 A_{22} + a_{34}^2 A_{33}$, $-2a_{14}^2 a_{31}^2 a_{34}^2 + a_{14}^2 A_{11} + a_{31}^2 A_{22} + a_{34}^2 A_{33}$,

Point	Description
K_3'''	Vertices of Lemoine tetrahedron (Figure 18)
M	Median point (Figure 7)
M_1	
M_2	Median point
M_3	
M_4	
M'	Exmedian points (Figure 7)
M''	
M'''	
M''''	
Q	

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
[illegible]		$-2a_{14}^2 a_{24}^2 a_{12}^2 + a_{14}^2 A_{11} + a_{24}^2 A_{22} + a_{12}^2 A_{33}$,
		$-2a_{23}^2 a_{31}^2 a_{12}^2 + a_{23}^2 A_{11} + a_{31}^2 A_{22} + a_{12}^2 A_{33}$,
62	P	p_1, p_2, p_3, p_4
63	P_1	$0, p_2, p_3, p_4$
64	P_2	$p_1, 0, p_3, p_4$
65	P_3	$p_1, p_2, 0, p_4$
66	P_4	$p_1, p_2, p_3, 0$
67	P_{12}	$0, 0, p, p$
68	P_{13}	$0, p_2, 0, p_4$
69	P_{14}	$0, p_2, p_3, 0$
70	P_{23}	$p_1, 0, 0, p_4$
71	P_{24}	$p_1, 0, p_3, 0$
72	P_{34}	$p_1, p_2, 0, 0$
73	P'_{12}	$0, 0, -p_3, p_4$
74	P'_{13}	$0, -p_2, 0, p_4$
75	P'_{14}	$0, -p_2, p_3, 0$
76	P'_{23}	$-p_1, 0, 0, p_4$
77	P'_{24}	$-p_1, 0, p_3, 0$
78	P'_{34}	$-p_1, p_2, 0, 0$
79	P'	$-p_1, p_2, p_3, p_4$

Point	Description
P	Given point (Figure 2)
P_1	Pedal points (Figure 2)
P_2	
P_3	
P_4	
P_{12}	Projections of pedal points (Figure 4)
P_{13}	
P_{14}	
P_{23}	
P_{24}	Projections of pedal points (Figure 4)
P_{34}	
P'_{12}	Harmonic conjugate of P_{12} with respect to A_3A_4
P'_{13}	Harmonic conjugate of P_{13} with respect to A_2A_4
P'_{14}	Harmonic conjugate of P_{14} with respect to A_2A_3
P'_{23}	Harmonic conjugate of P_{23} with respect to A_1A_4
P'_{24}	Harmonic conjugate of P_{24} with respect to A_1A_3
P'_{34}	Harmonic conjugate of P_{34} with respect to A_1A_2
P'	Harmonic conjugate of P with respect to A_1P_1

TABLE I (continued)

SUMMARY OF POINTS

No.	Point	Coordinates
80	P''	$p_1, -p_2, p_3, p_4$
81	P'''	$p_1, p_2, -p_3, p_4$
82	P''''	$p_1, p_2, p_3, -p_4$
83	P_1''	$0, -p_2, p_3, p_4$
84	P_1'''	$0, p_2, -p_3, p_4$
85	P_1''''	$0, p_2, p_3, -p_4$
86	P_2'	$-p_1, 0, p_3, p_4$
87	P_2'''	$p_1, 0, -p_3, p_4$
88	P_2''''	$p_1, 0, p_3, -p_4$
89	P_3'	$-p_1, p_2, 0, p_4$
90	P_3''	$p_1, -p_2, 0, p_4$
91	P_3''''	$p_1, p_2, 0, -p_4$
92	P_4'	$-p_1, p_2, p_3, 0$
93	P_4''	$p_1, -p_2, p_3, 0$
94	P_4'''	$p_1, p_2, -p_3, 0$
95	Q	q_1, q_2, q_3, q_4
96	X	x_1, x_2, x_3, x_4
97	Y	$\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}$
98	Z	$\frac{1}{a_1^2 x_1}, \frac{1}{a_2^2 x_2}, \frac{1}{a_3^2 x_3}, \frac{1}{a_4^2 x_4}$

Point	Description
P''	Harmonic conjugate of P with respect to A_2P_2
P'''	Harmonic conjugate of P with respect to A_3P_3
P''''	Harmonic conjugate of P with respect to A_4P_4
P_1''	Harmonic conjugate of P_1 with respect to A_2P_{12}
P_1'''	Harmonic conjugate of P_1 with respect to A_3P_{13}
P_1''''	Harmonic conjugate of P_1 with respect to A_4P_{14}
P_2'	Harmonic conjugate of P_2 with respect to A_1P_{12}
P_2'''	Harmonic conjugate of P_2 with respect to A_3P_{23}
P_2''''	Harmonic conjugate of P_2 with respect to A_4P_{24}
P_3'	Harmonic conjugate of P_3 with respect to A_1P_{31}
P_3''	Harmonic conjugate of P_3 with respect to A_2P_{32}
P_3''''	Harmonic conjugate of P_3 with respect to A_4P_{34}
P_4'	Harmonic conjugate of P_4 with respect to A_1P_{14}
P_4''	Harmonic conjugate of P_4 with respect to A_2P_{24}
P_4'''	Harmonic conjugate of P_4 with respect to A_3P_{34}
Q	Given point (Figure 2)
X	Variable point (Figure 1)
Y	Isogonal conjugate of X
Z	Isotomic conjugate of X

TABLE II

SUMMARY OF LINES

No.	Line	Equation
1	$A_1A_2 = a_{12}$	$x_3 = 0,\ x_4 = 0$
2	$A_1A_3 = a_{13}$	$x_2 = 0,\ x_4 = 0$
3	$A_1A_4 = a_{14}$	$x_2 = 0,\ x_3 = 0$
4	$A_2A_3 = a_{23}$	$x_1 = 0,\ x_4 = 0$
5	$A_2A_4 = a_{24}$	$x_1 = 0,\ x_3 = 0$
6	$A_3A_4 = a_{34}$	$x_1 = 0,\ x_2 = 0$
7	Ideal line	$m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0$
		$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$
8	h_1	$\frac{3\Delta}{a_1}$
9	h_2	$\frac{3\Delta}{a_2}$
10	h_3	$\frac{3\Delta}{a_3}$
11	h_4	$\frac{3\Delta}{a_4}$
12	$h_1^{(2)}$	
13	$h_1^{(3)}$	
14	$h_1^{(4)}$	

Line	Description	Ideal Point
A_1A_2	Edges of tetrahedron	$-a_2, a_1, 0, 0$
A_1A_3		$-a_3, 0, a_1, 0$
A_1A_4		$-a\ , 0, 0, a$
A_2A_3		$0, -a_3, a_2, 0$
A_2A_4		$0, -a_4, 0, a_2$
A_3A_4		$0, 0, -a_4, a_3$
	Ideal line	
h_1	Altitudes of tetrahedron	$-1, \cos\theta_{12}, \cos\theta_{13}, \cos\theta_{14}$
h_2		$\cos\theta_{12}, -1, \cos\theta_{23}, \cos\theta_{24}$
h_3		$\cos\theta_{13}, \cos\theta_{23}, -1, \cos\theta_{34}$
h_4		$\cos\theta_{14}, \cos\theta_{24}, \cos\theta_{34}, -1$
$h_1^{(2)}$	Face perpendiculars	$-\dfrac{a_{34}}{a_1}, 0, \dfrac{a_{14}\cos A_4^{(2)}}{a_3}, \dfrac{a_{13}\cos A_3^{(2)}}{a_4}$
$h_1^{(3)}$		$-\dfrac{a_{24}}{a_1}, \dfrac{a_{14}\cos A_4^{(3)}}{a_2}, 0, \dfrac{a_{12}\cos A_2^{(3)}}{a_4}$
$h_1^{(4)}$		$-\dfrac{a_{23}}{a_1}, \dfrac{a_{13}\cos A_3^{(4)}}{a_2}, \dfrac{a_{12}\cos A_2^{(4)}}{a_3}, 0$

TABLE II (continued)

SUMMARY OF LINES

No.	Line	Equation
15	$h_2^{(1)}$	
16	$h_2^{(3)}$	
17	$h_2^{(4)}$	
18	$h_3^{(1)}$	
19	$h_3^{(2)}$	
20	$h_3^{(4)}$	
21	$h_4^{(1)}$	
22	$h_4^{(2)}$	
23	$h_4^{(3)}$	
24	$h_1^{(j)}$	

Line	Description	Ideal point
$h_2^{(1)}$	Face perpendiculars	$0,\ -\frac{a_{34}}{a_2},\ \frac{a_{24}\cos A_4^{(1)}}{a_3},\ \frac{a_{23}\cos A_3^{(1)}}{a_4}$
$h_2^{(3)}$		$\frac{a_{24}\cos A_4^{(3)}}{a_1},\ -\frac{a_{14}}{a_2},\ 0,\ \frac{a_{21}\cos A_1^{(3)}}{a_4}$
$h_2^{(4)}$		$\frac{a_{23}\cos A_3^{(4)}}{a_1},\ -\frac{a_{13}}{a_2},\ \frac{a_{21}\cos A_1^{(4)}}{a_3},\ 0$
$h_3^{(1)}$		$0,\ \frac{a_{34}\cos A_4^{(1)}}{a_2},\ -\frac{a_{24}}{a_3},\ \frac{a_{32}\cos A_2^{(1)}}{a_4}$
$h_3^{(2)}$		$\frac{a_{34}\cos A_4^{(2)}}{a_1},\ 0,\ -\frac{a_{14}}{a_3},\ \frac{a_{31}\cos A_1^{(2)}}{a_4}$
$h_3^{(4)}$		$\frac{a_{32}\cos A_2^{(4)}}{a_1},\ \frac{a_{31}\cos A_1^{(4)}}{a_2},\ -\frac{a_{12}}{a_3},\ 0$
$h_4^{(1)}$		$0,\ \frac{a_{43}\cos A_3^{(1)}}{},\ \frac{a_{42}\cos A_2^{(1)}}{},\ -\frac{a_{23}}{}$
$h_4^{(2)}$		$\frac{a_{43}\cos A_3^{(2)}}{a_1},\ 0,\ \frac{a_{41}\cos A_1^{(2)}}{a_3},\ -\frac{a_{13}}{a_4}$
$h_4^{(3)}$		$\frac{a_{42}\cos A_2^{(3)}}{a},\ \frac{a_{41}\cos A_1^{(3)}}{a},\ 0,\ -\frac{a_{12}}{a_4}$
$h_i^{(j)}$		$-\frac{a_{kl}}{a_i},\ 0,\ \frac{a_{il}\cos A_l^{(j)}}{a_k},\ \frac{a_{ik}\cos A_k^{(j)}}{a_l}$

TABLE III

SUMMARY OF PLANES

No.	Plane	Equation
1	$A_2A_3A_4$	$x_1 = 0$
2	$A_1A_3A_4$	$x_2 = 0$
3	$A_1A_2A_4$	$x_3 = 0$
4	$A_1A_2A_3$	$x_4 = 0$
5	$A_2A_3A_4$	$a_2x_2 + a_3x_3 + a_4x_4 = 0$
6	$A_1A_3A_4$	$a_1x_1 + a_3x_3 + a_4x_4 = 0$
7	$A_1A_2A_4$	$a_1x_1 + a_2x_2 + a_4x_4 = 0$
8	$A_1A_2A_3$	$a_1x_1 + a_2x_2 + a_3x_3 = 0$
9	$A'_2A'_3A''_1$	$a_1x_1 = a_2x_2 + a_3x_3 + a_4x_4$
10	$A'_1A'_3A''_2$	$a_2x_2 = a_1x_1 + a_3x_3 + a_4x_4$
11	$A'_1A'_2A''_3$	$a_3x_3 = a_1x_1 + a_2x_2 + a_4x_4$
12	$A''_1A''_2A''_3$	$a_4x_4 = a_1x_1 + a_2x_2 + a_3x_3$
13	$A'_1A''_1A'_3A''_3$	$a_1x_1 - a_2x_2 + a_3x_3 - a_4x_4 = 0$
14	$A'_2A''_2A'_3A''_3$	$a_1x_1 + a_2x_2 + a_3x_3 - a_4x_4 = 0$
15	$A''_1A'_2A'_1A''_2$	$a_1x_1 + a_2x_2 - a_3x_3 - a_4x_4 = 0$
16	A_1A_2K	$a_4x_3 - a_3x_4 = 0$
17	A_1A_3K	$a_4x_2 - a_2x_4 = 0$
18	A_1A_4K	$a_2x_3 - a_3x_2 = 0$
19	A_2A_3K	$a_1x_4 - a_4x_1 = 0$
20	A_2A_4K	$a_1x_3 - a_3x_1 = 0$
21	A_3A_4K	$a_1x_2 - a_2x_1 = 0$

Plane	Description
$A_2A_3A_4$	Faces of tetrahedron of reference (Figure 1)
$A_1A_3A_4$	
$A_1A_2A_4$	
$A_1A_2A_3$	
$\underline{A}_2\underline{A}_3\underline{A}_4$	Plane through A_1 parallel to a_1 (Figure 16)
$\underline{A}_1\underline{A}_3\underline{A}_4$	Plane through A_2 parallel to a_2
$\underline{A}_1\underline{A}_2\underline{A}_4$	Plane through A_3 parallel to a_3
$\underline{A}_1\underline{A}_2\underline{A}_3$	Plane through A_4 parallel to a_4
$A'_2A'_3A''_1$	Planes through mid-points of three edges and parallel to faces
$A'_1A'_3A''_2$	
$A'_1A'_2A''_3$	
$A''_1A''_2A''_3$	Bimedian planes (Figure 10)
$A'_1A''_1A'_3A''_3$	
$A'_2A''_2A'_3A''_3$	
$A''_1A'_2A'_1A''_2$	
A_1A_2K	Symmedian planes (Figure 8)
A_1A_3K	
A_1A_4K	
A_2A_3K	
A_2A_4K	
A_3A_4K	

TABLE III (continued)

SUMMARY OF PLANES

No.	Plane	Equation
22		$a_4x_3 + a_3x_4 = 0$
23		$a_4x_2 + a_2x_4 = 0$
24		$a_2x_3 + a_3x_2 = 0$
25		$a_1x_4 + a_4x_1 = 0$
26		$a_1x_3 + a_3x_1 = 0$
27		$a_1x_2 + a_2x_1 = 0$
28	A_1A_2M	$a_3x_3 - a_4x_4 = 0$
29	A_1A_3M	$a_2x_2 - a_4x_4 = 0$
30	A_1A_4M	$a_2x_2 - a_3x_3 = 0$
31	A_2A_3M	$a_1x_1 - a_4x_4 = 0$
32	A_2A_4M	$a_1x_1 - a_3x_3 = 0$
33	A_3A_4M	$a_1x_1 - a_2x_2 = 0$
34	A_1A_2M'''	$a_1x_1 + a_2x_2 = 0$
35	A_1A_3M''	$a_1x_1 + a_3x_3 = 0$
36	A_1A_4M''	$a_1x_1 + a_4x_4 = 0$
37	A_2A_3M'	$a_2x_2 + a_3x_3 = 0$
38	A_2A_4M'	$a_2x_2 + a_4x_4 = 0$
39	A_3A_4M'	$a_3x_3 + a_4x_4 = 0$
40	A_1A_2P	$p_4x_3 - p_3x_4 = 0$
41	A_1A_3P	$p_4x_2 - p_2x_4 = 0$
42	A_1A_4P	$p_3x_2 - p_2x_3 = 0$
43	A_2A_3P	$p_4x_1 - p_1x_4 = 0$

Plane	Description
	Exsymmedian planes (Figure 8)
A_1A_2M	Median planes (Figure 7)
A_1A_3M	
A_1A_4M	
A_2A_3M	
A_2A_4M	
A_3A_4M	
A_1A_2M'''	Exmedian planes (Figure 7)
A_1A_3M''	
A_1A_4M''	
A_2A_3M'	
A_2A_4M'	
A_3A_4M'	
A_1A_2P	Planes through any point P and the edges
A_1A_3P	
A_1A_4P	
A_2A_3P	

TABLE III (continued)

SUMMARY OF PLANES

No.	Plane	Equation
44	A_2A_4P	$p_3x_1 - p_1x_3 = 0$
45	A_3A_4P	$p_2x_1 - p_1x_2 = 0$
46	$B_1B_2B_3$	$a_1a_{14}x_1 + a_2a_{24}x_2 + a_3a_{34}x_3 = r(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$
47	$B_2B_3B_4$	$a_2a_{21}x_2 + a_3a_{23}x_3 + a_4a_{24}x_4 = r(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$
48	$B_1B_2B_4$	$a_1a_{13}x_1 + a_2a_{23}x_2 + a_4a_{43}x_4 = r(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$
49	$B_1B_3B_4$	$a_1a_{12}x_1 + a_3a_{32}x_3 + a_4a_{42}x_4 = r(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$
50	$K'_1K''_2K'''_3$	$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} = 0$
51	$K\,K''_2K'''_3$	$\frac{x_1}{a_1} - \frac{x_2}{a_2} - \frac{x_3}{a_3} + \frac{x_4}{a_4} = 0$
52	$K\,K'_1K'''_3$	$\frac{x_1}{a_1} - \frac{x_2}{a_2} + \frac{x_3}{a_3} - \frac{x_4}{a_4} = 0$
53	$K\,K'_1K''_2$	$\frac{x_1}{a_1} + \frac{x_2}{a_2} - \frac{x_3}{a_3} - \frac{x_4}{a_4} = 0$
54	$M_2M_3M_4$	$-2a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$
55	$M_1M_3M_4$	$a_1x_1 - 2a_2x_2 + a_3x_3 + a_4x_4 = 0$
56	$M_1M_2M_4$	$a_1x_1 + a_2x_2 - 2a_3x_3 + a_4x_4 = 0$
57	$M_1M_2M_3$	$a_1x_1 + a_2x_2 + a_3x_3 - 2a_4x_4 = 0$

Plane	Description
A_2A_3P	
A_2A_4P	
$B_1B_2B_3$	Isoclinal plane cutting edges concurrent at A_4 (Figure 11)
$B_2B_3B_4$	Isoclinal plane cutting edges concurrent at A_1
$B_1B_2B_4$	Isoclinal plane cutting edges concurrent at A_3
$B_1B_3B_4$	Isoclinal plane cutting edges concurrent at A_2
$K'_1K''_2K'''_3$	Lemoine plane (Figure 18)
$K\,K''_2K'''_3$	Faces of Lemoine tetrahedron (Figure 18)
$K\,K'_1K'''_3$	
$K\,K'_1K''_2$	
$M_2M_3M_4$	Planes determined by face centroids (Figure 11)
$M_1M_3M_4$	
$M_1M_2M_4$	
$M_1M_2M_3$	

TABLE III (continued)

SUMMARY OF PLANES

No.	Plane	Equation
58	$P_1P_2P_3$	$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{2x_4}{p} = 0$
59	$P_1P_2P_4$	$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{2x_3}{p_3} + \frac{x_4}{p_4} = 0$
60	$P_1P_3P_4$	$\frac{x_1}{p_1} - \frac{2x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} = 0$
61	$P_2P_3P_4$	$-\frac{2x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} = 0$
62	$P_1'P_2''P_3'''$	$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} = 0$
63	Ideal Plane	$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$
64		$F_1x_1 + F_2x_2 + F_3x_3 + F_4x_4 = 0$
65		$(a_{13}^2a_1a_3p_1 + a_{12}^2a_1a_2p_2 + a_{14}^2a_1a_4p_4)x_1 +$
		$(a_{23}^2a_2a_3p_3 + a_{12}^2a_1a_2p_1 + a_{24}^2a_2a_4p_4)x_2 +$
		$(a_{23}^2a_2a_3p_2 + a_{13}^2a_1a_3p_1 + a_{34}^2a_3a_4p_4)x_3 +$
		$(a_{14}^2a_1a_4p_1 + a_{24}^2a_2a_4p_2 + a_{34}^2a_3a_4p_3)x_4 = 0$
66		$a_{12}^2a_1a_2x_2 + a_{13}^2a_1a_3x_3 + a_{14}^2a_1a_4x_4 = 0$
67		$a_{12}^2a_1a_2x_1 + a_{23}^2a_2a_3x_3 + a_{24}^2a_2a_4x_4 = 0$

Plane	Description
$P_1P_2P_3$	Planes determined by pedal points
$P_1P_2P_4$	
$P_1P_3P_4$	
$P_2P_3P_4$	
$p_1'p_2''p_3'''$	Harmonic or quadriplanar polar plane
	Ideal plane
	Polar with respect to general sphere
	Polar with respect to circumsphere
	Plane through A_1 tangent to circumsphere
	Plane through A_2 tangent to circumsphere

TABLE III (continued)

SUMMARY OF PLANES

No.	Plane	Equation
68		$a_{13}^2 a_1 a_3 x_1 + a_{23}^2 a_2 a_3 x_2 + a_{34}^2 a_3 a_4 x_4 = 0$
69		$a_{14}^2 a_1 a_4 x_1 + a_{24}^2 a_2 a_4 x_2 + a_{34}^2 a_3 a_4 x_4 = 0$
70		$a_1 x_1 + a_2 x_2 + a_3 x_3 = a_4 \sec \theta_{14} x_1$
71		$a_1 x_1 + a_2 x_2 + a_3 x_3 = a_4 \sec \theta_{24} x_2$
72		$a_1 x_1 + a_2 x_2 + a_3 x_3 = a_4 \sec \theta_{34} x_3$
73		$\frac{x_1}{a_1} = \frac{a_2 x_2 + a_3 x_3 + a_4 x_4}{a_2^2 + a_3^2 + a_4^2}$
74		$\frac{x_2}{a_2} = \frac{a_1 x_1 + a_3 x_3 + a_4 x_4}{a_1^2 + a_3^2 + a_4^2}$
75		$\frac{x_3}{a_3} = \frac{a_1 x_1 + a_2 x_2 + a_4 x_4}{a_1^2 + a_2^2 + a_4^2}$
76		$\frac{x_4}{a_4} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1^2 + a_2^2 + a_3^2}$
77		$3a_1 x_1 = a_2 x_2 + a_3 x_3 + a_4 x_4$
78		$3a_2 x_2 = a_1 x_1 + a_3 x_3 + a_4 x_4$
79		$3a_3 x_3 = a_1 x_1 + a_2 x_2 + a_4 x_4$
80		$3a_4 x_4 = a_1 x_1 + a_2 x_2 + a_3 x_3$

Plane	Description
	Plane through A_3 tangent to circumsphere
	Plane through A_4 tangent to circumsphere
	Plane through a vertex perpendicular to the opposite face and parallel to the edges which bound the face (Figure 12)
	Plane through K parallel to a_1 (Figure 13)
	Plane through K parallel to a_2
	Plane through K parallel to a_3
	Plane through K parallel to a_4
	Plane through M parallel to a_1 (Figure 13)
	Plane through M parallel to a_2
	Plane through M parallel to a_3
	Plane through M parallel to a_4

TABLE IV

SUMMARY OF SPHERES

No.	Sphere	Description
1	Any sphere	
2	Circumsphere	
3	Polar sphere	
4	$A_1A_2A_3I$	Spheres through three vertices and the incenter
5	$A_1A_2A_4I$	

Sphere	Equation
Any sphere	$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 +$ $a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 +$ $(m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4)(a_1 x_1 + a_2 x_2 +$ $a_3 x_3 + a_4 x_4) = 0$
Circumsphere	$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 +$ $a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 = 0$
Polar sphere	$a_1^2 a_{12} a_{13} \cos A_1^{(4)} x_1^2 + a_2^2 a_{12} a_{23} \cos A_2^{(4)} x_2^2 +$ $a_3^2 a_{13} a_{23} \cos A_3^{(4)} x_3^2 + a_4^2 a_{24} a_{34} \cos A_4^{(3)} x_4^2 = 0$
$A_1 A_2 A_3 I$	$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 +$ $a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 -$ $x_4 (a_{12}^2 a_1 a_2 + a_{13}^2 a_1 a_3 + a_{14}^2 a_1 a_4 + a_{23}^2 a_2 a_3 +$ $a_{24}^2 a_2 a_4 + a_{34}^2 a_3 a_4)(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0$
$A_1 A_2 A_4 I$	$\sum a_i \sum a_{ij}^2 a_i a_j x_i x_j - x_3 (\sum a_{ij}^2 a_i a_j)(a_i^2 x_i) = 0$

TABLE IV (continued)

SUMMARY OF SPHERES

No.	Sphere	Description
6	$A_1A_3A_4I$	
7	$A_2A_3A_4I$	
8	$A_4B_1B_2B_3$	Isoclinal sphere through vertex A_4
9	$I_{a_1}I_{a_2}I_{a_3}I_{a_4}$	Sphere through the excenters of the four truncs

Sphere	Equation
$A_1A_3A_4I$	$\Sigma a_i \Sigma a_{ij}^2 a_i a_j x_i x_j - x_2(\Sigma a_{ij}^2 a_i a_j)(a_i^2 x_i) = 0$
$A_2A_3A_4I$	$\Sigma a_i \Sigma a_{ij}^2 a_i a_j x_i x_j - x_1(\Sigma a_{ij}^2 a_i a_j)(a_i^2 x_i) = 0$
$A_4B_1B_2B_3$	$\Sigma a_{ij}^2 a_i a_j x_i x_j + [a_{14}(a_{14} - r)x_1 +$
	$a_{24}(a_{24} - r)x_2 + a_{34}(a_{34} - r)x_3]$
	$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 0$
$I_{a_1}I_{a_2}I_{a_3}I_{a_4}$	$a_{12}^2 a_1 a_2 x_1 x_2 + a_{13}^2 a_1 a_3 x_1 x_3 + a_{14}^2 a_1 a_4 x_1 x_4 +$
	$a_{23}^2 a_2 a_3 x_2 x_3 + a_{24}^2 a_2 a_4 x_2 x_4 + a_{34}^2 a_3 a_4 x_3 x_4 +$
	$\frac{1}{s}\Big\{\Big[-\frac{a_1}{(s-a_1)} + \frac{a_2}{(s-a_2)} + \frac{a_3}{(s-a_3)} +$
	$\frac{a_4}{(s-a_4)}\Big] x_1 + \Big[\frac{a_1}{(s-a_1)} - \frac{a_2}{(s-a_2)} + \frac{a_3}{(s-a_3)} +$
	$\frac{a_4}{(s-a_4)}\Big] x_2 + \Big[\frac{a_1}{(s-a_1)} + \frac{a_2}{(s-a_2)} - \frac{a_3}{(s-a_3)} +$
	$\frac{a_4}{(s-a_4)}\Big] x_3 + \Big[\frac{a_1}{(s-a_1)} + \frac{a_2}{(s-a_2)} + \frac{a_3}{(s-a_3)} -$
	$\frac{a_4}{(s-a_4)}\Big] x_4\Big\}$
	$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 0$

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