SOLUTION OF QUARTIC EQUATIONS
BY TABLES

A THESIS

SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE
TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

by

George B. Blodig

August 1959

Approved for the Major Department

John M. Burger

Approved for the Graduate Council

[illegible]

ACKNOWLEDGMENT

To Dr. Oscar J. Peterson, Department of Mathematics of the Kansas State Teachers College of Emporia, who most willingly and cheerfully directed this thesis, the writer is indebted and wishes to express his sincere thanks.

George B. Blodig

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

THE PROBLEM

1.1. Introduction. Many methods are used in the solution of quartic equations in one unknown. Algebraic methods of solution have been devised by several well known mathematicians. There are exact methods which usually express the value of the roots by means of radicals, and approximation methods which require the reapplication of the method if several significant digits are desired. In addition nomographic methods have been applied.

Each of these and similar methods pose both advantages and disadvantages in the solution of quartic equations. Most methods are reasonably accurate but the computations are laborious.

1.2. Statement of the problem. The purpose of this thesis is to develop a tabular method for the solution of a general quartic equation of the form

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

Such a method should assist in the calculation of approximate solutions of the real roots of the general quartic equation listed above. The proposed table will be designed to give the real roots to two and three significant digits and in special cases will yield four and five significant digits. A procedure is to be devised whereby any given quartic equation, with real coefficients, may be solved by tables, after transformations of the equation have been accomplished.

1.3. Limitations of the problem. As stated before, only quartic equations with real coefficients and real roots will be considered. Only the real roots will be given by the tables and the only time all roots will be given is when the given quartic equation has four real roots.

In this thesis, limits have been set for the ranges of the parameters represented by the tables. These limits are sufficiently large for the solution of most quartic equations. However, for cases not within the ranges of the tables an alternate method, similar to tables, is to be devised.

1.4. Importance of the problem. The quartic equation may be solved by several methods, either exactly or approximately. Exact solutions of the quartic equation usually involves radicals, whose evaluation is not always immediately obtainable. Methods of solution by successive approximations, generally require a good approximation of the desired root for the initial application of the method.

Therefore, a rapid method of obtaining all the real roots of the quartic equation with reasonable accuracy by means of tables should be of value. When greater accuracy is desired, the roots given by the tables may be used as a first approximation in one of the methods of solution by successive approximations. A few applications of successive approximations should give several significant digits, when the initial value of the root is taken from the tables.

1.5. Organization of the thesis. Chapter II presents the historical background of the quartic equation. Chapter III presents a discussion of several methods of solving a quartic equation by formulas. Chapter IV presents a discussion of two methods of solving a quartic equation by successive approximations. Chapter V is a development of necessary transformations. Chapter VI is a discussion of how the tables were constructed. Chapter VII discusses how to use the table, with illustrative examples of its use. Chapter VIII is a discussion, with illustrative examples, of how to solve a quartic equation, which is beyond the limits of the tables. In Chapter IX initial values of a desired root are found for the application of two successive approximation methods, Newton's method and the method of series reversion. Chapter X contains a summary of the procedures used in the tabular method of solution, with examples of its application. Chapter XI summarizes the work done in the thesis. Chapter XII comprises all of the tables. Tables I, II, and III, give all the real roots of the transformed equation according to the value of the parameters and in addition aid as guides for the solution of equations not within the limits of these tables. Table IV gives the values of numbers squared and taken to the fourth power.

CHAPTER II

HISTORICAL BACKGROUND OF QUARTIC EQUATIONS

2.1. History of quartic equations. The earliest reference to quartic equations appears in the mathematical records of Babylonia. Discussion of quadratic equations and some discussions of cubic and quartic equations were given in Babylonian algebra, about 2000 B. C. However, there is no record of the general quartic being solved by the Babylonians.[1]

The Arabs had little success in solving quartic equations but did succeed in solving special types. Abul'l-Wefa (c.980) attempted to solve the equation $x^4 + px^3 = q$, but his work has been lost and it is not known what he did in arriving at a solution.[2]

An anonymous Arab or Persian algebraist solved the equation $(100-x^2)(10-x)^2 = 8100$. It was solved by taking the intersection of $(10-x)y = 90$ and $x^2 + y^2 = 100$. However, all evidence indicates that the author was not interested in the algebraic theory.[3]

---

[1] Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, Inc., 1953), p. 32.

[2] David Eugene Smith, History of Mathematics (Boston: Ginn and Co., 1925), Vol. II, p. 466.

[3] Ibid., p. 467.

The Chinese, as early as 1247, solved certain quartic equations by approximation. Evidence indicates that Horner, a European mathematician, never saw these solutions, but developed a similar method about five centuries later.[4]

Mathematics in Europe underwent a great change between the beginning of the thirteenth and the end of the sixteenth century. The main problem of algebra at this time was the solution of equations. During this time in history many mathematicians were trying to find explicit radical solutions for cubic and quartic equations.[5]

One of the most spectacular mathematical achievements of the sixteenth century was the discovery of algebraic solutions of the general cubic and the general quartic equations.[6]

In 1540 the Italian mathematician Zuanne de Tonini da Coi proposed the following problem to Cardan:[7] "Divide 10 into three parts such that they shall be in continued proportion and that the product of the first two shall be 6." It is suggested[8] that the three parts be noted by

---

[4]Florian Cajori, A History of Mathematics (New York: The Macmillan Co., 1924, 2nd ed.), p. 74.

[5]E. T. Bell, The Development of Mathematics (New York: McGraw-Hill Book Co., Inc., 1940), p. 108.

[6]Eves, *op. cit.*, p. 218.

[7]Smith, *op. cit.*, p. 467.

[8]Eves, *op. cit.*, p. 233.

x, y, z, then the relations are $x + y + z = 10$, $xz = y^2$ and $xy = 6$. When x and z are eliminated the result is a quartic equation in y, $y^4 + 6y^2 + 36 = 60y$.

Cardan could not solve this equation and never did find a solution for the general quartic. Cardan's pupil, Lodovico Ferrari, succeeded in solving Coi's problem and while doing so succeeded in finding a solution for the general quartic. Cardan included Ferrari's solution in his Ars Magna, which was published in 1545.[9]

Francois Vieta, a French mathematician of the sixteenth century, is probably best known for his algebraic notations. Many mathematicians claim Vieta took these ideas from other authors. In his algebraic works, Vieta denoted known quantities by consonants and unknown quantities by vowels.[10]

Vieta found a solution of the quartic, but it is essentially the same as Ferrari's.[11]

Joseph Louis Lagrange, a prominent French mathematician of the eighteenth century, also devised a method for the solution of quartic equations. The third edition of

---

[9] W. W. R. Ball, A Short Account of the History of Mathematics (London: The Macmillan Co., Limited, 1935), pp. 223-226.

[10] W. W. R. Ball, A Short Account of the History of Mathematics (London: The Macmillan Co., Limited, 1915), p. 231.

[11] Ibid., p. 233.

Resolution des Equations includes his proof of the theorem:[12]

> A quartic $z^4 + qz^2 + rz + s = 0$ with q, r, s, real, $r \neq 0$, and with the discriminant Δ, has 4 distinct real roots if q and $4s - q^2$ are negative and $\Delta > 0$, no real root if q and $4s - q^2$ are not both negative $\Delta > 0$, 2 distinct real and 2 imaginary roots if $\Delta < 0$, at least 2 equal real roots if $\Delta = 0$.

[12]Leonard Eugene Dickson, Elementary Theory of Equations (New York: John Wiley and Sons, Inc., 1914, 1st ed.), p. 45.

CHAPTER III

SOLUTION OF QUARTICS BY FORMULA

3.1. Ferrari's method of solution. One of the best known methods of solving a quartic equation is Ferrari's solution.

From the general quartic

(3.1) $$x^4 + bx^3 + cx^2 + dx + e = 0$$

form the identity,

(3.2) $$x^4 + bx^3 + cx^2 + dx + e + (px + q)^2 = (x^2 + \frac{b}{2}x + r)^2.$$

Next p, q, and r are determined by equating the coefficients of like powers of x in the first and second members of the identity (3.2). This leads to the relations

(3.3) $$p^2 + c = 2r + b^2/4,$$

(3.4) $$2pq + d = rb,$$

(3.5) $$q^2 + e = r^2.$$

Hence

(3.6) $$(rb - d)^2 = 4p^2q^2 = 4(2r + b^2/4 - c)(r^2 - e)$$

or

(3.7) $$r^3 - \frac{c}{2} r^2 + 1/4(bd - 4e)r + 1/8(4ce - b^2e - d^2) = 0.$$

Now upon solving equation (3.7), r is found. It is not necessary to find all the roots of equation (3.7). This equation is known as the resolvent cubic equation.

If $(px + q)^2$ is added to both members of equation (3.1) it becomes

(3.8) $$(x^2 + \frac{b}{2} x + r)^2 = (px + q)^2.$$

Therefore

(3.9) $$x^2 + \frac{b}{2}x + r = px + q$$

or

(3.10) $$x^2 + \frac{b}{2}x + r = -px - q.$$

The four roots of equation (3.1) may now be found by solving the quadratic equations (3.9) and (3.10) separately.[1]

3.2. Decartes' method of solution. Decartes' method is used only after the quartic equation (3.2) has been deprived of its second term. It is then in the form

(3.11) $$y^4 + qy^2 + ry + s = 0.$$

This is easily done by setting $x = y - \frac{b}{4}$.

It is then assumed

(3.12) $$y^4 + qy^2 + ry + s = (y^2 + ey + f)\ (y^2 - ey + g).$$

The quantities e, f, and g are now found by equating like coefficients of equation (3.12).

The result is the equations

(3.13) $$g + f - e^2 = q,$$

(3.14) $$e(g - f) = r,$$

and

(3.15) $$gf = s,$$

or

(3.16) $$g + f = q + e^2,$$

---

[1]Nelson Bush Conkwright, Introduction to the Theory of Equations (Boston: Ginn and Co., 1941), p. 78.

(3.17) $g - f = r/e,$

and

(3.15) $gf = s.$

Now g and f may be found in terms of e from equations (3.16) and (3.17). The result of substituting these values in equation (3.15) is

(3.18) $(q + e^2 + r/e)\ (q + e^2 - r/e) = 4s.$

Reducing equation (3.18) the result is

(3.19) $e^6 + 2qe^4 + (q^2 - 4s)e^2 - r^2 = 0.$

This may be considered a cubic equation for finding e^2. When e^2 is known e, g, and f become known. Thus the expression $y^4 + qy^2 + ry + s = 0$ is resolved into the product of two real quadratic factors, from which four roots of the proposed quartic equation may be obtained by solving the two quadratic equations

(3.20) $y^2 + ey + f = 0$

and

(3.21) $y^2 - ey + g = 0.$

It should be noted that in one of the quadratic factors ey was introduced and in the other -ey. The reason for this is that there was no term involving y^3 in equation (3.11) which was resolved into quadratic factors.[2]

---

[2] I. Todhunter, An Elementary Treatise on the Theory of Equations (London: The Macmillan Co., 1882, 5th ed.), pp. 112-113.

3.3. Vieta's solution. Vieta's solution of the quartic can be used only after the third degree term has been removed. The general depressed quartic may be written as

(3.23) $$x^4 = q - rx^2 - px$$

adding $x^2y^2 + y^4/4$ to both members of equation (3.22) gives the equation

(3.23) $$(x^2 + y^2/2)^2 = (y^2 - r)\, x^2 - px + (y^4/4 + q).$$

Now a value of y is determined so the right member of equation (3.23) is a perfect square. A condition for this is that

(3.24) $$y^6 - ry^4 + 4qy^2 = 4rq + p^2.$$

Equation (3.24) is a cubic in y^2. One value of y may be found from (3.24), and substituted in (3.23). The solution of the quartic (3.23) is then completed by extracting square roots.[3]

3.4. Lagrange's method of solution. Lagrange's method is dependent on the fact that the four variables acquire only three distinct values when permutated in the twenty-four possible ways. The general quartic may be written as

(3.25) $$x^4 + ax^3 + bx^2 + cx + d = 0$$ with roots x_1, x_2, x_3, x_4.

---

[3]Eves, op. cit., pp. 224-225.

Let

$$p_1 = (x_1 + x_2 - x_3 - x_4)^2, \tag{3.26}$$

$$p_2 = (x_1 + x_3 - x_2 - x_4)^2, \tag{3.27}$$

$$p_3 = (x_1 + x_4 - x_2 - x_3)^2. \tag{3.28}$$

The symmetric combinations are

$$p_1 + p_2 + p_3 = 3a^2 - 8b, \tag{3.29}$$

$$p_1p_2 + p_1p_3 + p_2p_3 = 3a^4 - 16\, a^2b + 16ac + 16b^2 - 64d, \tag{3.30}$$

$$p_1p_2p_3 = (a^3 - 4ab + 8c)^2. \tag{3.31}$$

Thus p_1, p_2, and p_3 may be found from the resolvent cubic equation

$$p^3 - (3a^2 - 8b)\, p^2 + (3a^4 - 16a^2b + 16ac + 16b^2 - 64d)\, p - (a^3 - 4ab + 8c)^2 = 0. \tag{3.32}$$

Now x_1, x_2, x_3, x_4 may be found from the linear equations

$$x_1 + x_2 + x_3 + x_4 = -a, \tag{3.33}$$

$$x_1 + x_2 - x_3 - x_4 = \sqrt{p_1}\,, \tag{3.34}$$

$$x_1 - x_2 + x_3 - x_4 = \sqrt{p_2}\,, \tag{3.35}$$

$$x_1 - x_2 - x_3 + x_4 = \sqrt{p_3}\,, \tag{3.36}$$

hence

$$4x_1 = -a + \sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\,, \tag{3.37}$$

$$4x_2 = -a + \sqrt{p_1} - \sqrt{p_2} - \sqrt{p_3}\,, \tag{3.38}$$

$$4x_3 = -a - \sqrt{p_1} + \sqrt{p_2} - \sqrt{p_3}\,, \tag{3.39}$$

$$4x_4 = -a - \sqrt{p_1} - \sqrt{p_2} + \sqrt{p_3}\,. \tag{3.40}$$

The square roots of p_1, p_2, and p_3 are not independent. There exists the relation

(3.41) $$\sqrt{p_1}\sqrt{p_2}\sqrt{p_3} = -a + 4ab - 8c.$$ [4]

---

[4] J. V. Uspensky, Theory of Equations (New York: McGraw-Hill Book Co., Inc., 1948), pp. 274-275.

CHAPTER IV

APPROXIMATE SOLUTIONS OF QUARTIC EQUATIONS

4.1. Newton's Method of Approximation. Newton's method applies to algebraic equations in the form $f(x) = 0$. An approximate root should be chosen between two numbers a and b. These numbers should be so nearly equal that $f'(x) = 0$ and $f''(x) = 0$ have no roots between a and b.

Since $f''(x)$ has the same sign in the interval from a to b, and $f(x)$ changes in sign, $f''(x)$ and $f(x)$ will have the same sign for one end point whether it be a or b. Here it will be taken that $f''(x)$ and $f(x)$ have the same sign for the end point a. It can be shown that if a root lies in an interval between a and b and closest to a, the root also lies in a smaller interval from c to

$$a - \frac{f(a)}{f'(a)} \tag{4.1}$$

where c is given by the equation.

$$c = \frac{bf(a) - af(b)}{f(a) - f(b)} \tag{4.2}$$

The advantage of having c at each step is that a close limit of the error made in the approximation of the root is known.

Once an approximation of a root is known Newton's method may be applied either using c as described above or not using it for a guide. As before, if a is our approximation of a root, formula

(4.3) $$a_1 = a - \frac{f(a)}{f'(a)}$$

is applied and is essentially the same formula as (4.1).
If more accuracy is desired apply formula

(4.4) $$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$$

If more accuracy is desired the same procedure as in (4.3) and (4.4) is followed.[1]

Newton's method may be applied to transcendental as well as algebraic equations.

4.2. Approximation by Series Reversion. Series reversion is essentially that if given a series representation of the variable y in terms of the independent variable x,

(4.5) $$y = x\,(1 - a_1x - a_2x^2 - a_3x^3 - \ldots),$$

$x \in S$, S being the circle of convergence, y being limited to a region T which contains the orgin, obtain the series representation of the variable x in terms of y,

(4.6) $$x = y\,(1 + b_1y + b_2y^2 + b_3y^3 + \ldots),$$

together with the conditions under which such representation is valid.

In series reversion an approximation of a root must be made and it must be close to the orgin. If z is picked as an approximate root from $f(z) = 0$ and it is not close to the orgin, then it is necessary to make the substitution

---

[1]Leonard Eugene Dickson, First Course in the Theory of Equations (New York: John Wiley & Sons, Inc., 1922), pp. 91-93.

(4.7) $z = x + h$

where h is a value known to be near z. Then the transformed equation.

(4.8) $g(x) = f(x + h) = 0,$

will have a root $x_1 = z_1 - h$, near the orgin.

By expanding equation (4.8) it becomes

(4.9) $0 = g(x) = e_o + e_1x + e_2x^2 + e_nx^n + \ldots .$

This expansion may be done by Taylor's Theorem, synthetic division, binomial theorem, and several others.

Equation (4.9) is then reduced to the form

(4.5) $y = x\ (1 - a_1\ x - a_2x^2 - a_3x^3 - \ldots),$

where

(4.10) $y = e_o/e_1$

and

(4.11) $a_i = -e_{i+1}/e_1.$

Equation (4.5) is then expressed as

(4.6) $x = y(1 + b_1y + b_2y^2 + b_3y^3 + \ldots),$

where

(4.12) $b_1 = a_1,$

(4.13) $b_2 = 2a_1{}^2 + a_2,$

(4.14) $b_3 = 5a_1{}^3\ 5a_1\ a_2\ a_3,$

(4.15) $b_4 = 14a_1{}^4 + 21\ a_1^2\ a_2 + 3a_2^2 + 6a_1a_3 + a^4$

and

(4.16) $bn = \frac{1}{n!}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{f(x)}\right)^n\right]x = 0.$

Now x may be calculated to as many significant digits as desired.[2]

There are several other approximation methods such as Horner's Method, the method of iteration, and by nomograms, but all are similar to Newton's method or else they do not give any more accuracy than the tables in this thesis.

---

[2]George L. Crumley, "Solution of Quartic and Cubic Equations by Series Reversion" (unpublished master's thesis, Kansas State Teachers College, Emporia, Kansas, 1953), pp. 7-46.

CHAPTER V

THE REDUCTION OF THE QUARTIC EQUATION

5.1. Introduction. The general quartic equation

(5.1) $$x^4 + ax^3 + bx^2 + cs + d = 0,$$

involves four parameters a, b, c, and d. The tabulation of roots directly from the values of the four parameters would not be feasible. Therefore, it is necessary to reduce the general quartic equation to a more suitable form for the construction of tables.

5.2. Elimination of the third degree term. The third degree term may be eliminated by the substitution

(5.2) $$x = y - \frac{a}{4}\ .$$

Then equation (5.1) becomes

(5.3) $$y^4 + hy^2 + ky + m = 0,$$

where

(5.4) $$h = b - 3a^2/8,$$

(5.5) $$k = c - ab/2 + a^3/8,$$

(5.6) $$m = d - \frac{ac}{4} + \frac{a^2b}{16} - \frac{3a^4}{256}.$$

5.3. Reduced forms of the quartic equation. Unless h in equation (5.3) is either a negative one, zero, or a positive one and k is positive more transformations must be made.

When h is a positive number, other than one, the substitution

(5.7) $$y = \sqrt{h}\ t$$

transforms equation (5.3) to the form

(5.8) $$t^4 + nt^2 + pt + q = 0,$$

where

(5.9) $$n = \frac{h^2}{h^2} = 1,$$

(5.10) $$p = \frac{k}{h^{3/2}},$$

(5.11) $$q = \frac{m}{h^2}.$$

If h is negative, the substitution

(5.12) $$y = \sqrt{-h}\, t$$

transforms equation (5.3) to the form

(5.13) $$t^4 + n\, t^2 + pt + q = 0$$

where

(5.14) $$n = \frac{-h^2}{h^2} = -1,$$

(5.15) $$p = \frac{k}{(-h)^{3/2}},$$

(5.16) $$q = \frac{m}{h^2}.$$

If p is negative in either equation (5.8) or equation (5.13) the substitution

(5.17) $$t = -z$$

will transform the equations (5.8) and (5.13) and in the same order as they appear above to

(5.18) $$z^4 + nz^2 + pz + q = 0,$$

and n = 1 and p is positive.

(5.19) $z^4 + nz^2 + pz + q = 0,$

and $n = -1$ and p is positive.

The case may arise when h in equation (5.3) is zero. When this happens no further transformations are necessary unless k is negative, then substitute

(5.20) $y = -z,$

then equation (5.3) becomes

(5.21) $z^4 + pz + q = 0,$

and p is positive.

It should be noted here that n is either a negative 1, zero, or a positive one and p is always zero or a positive number. There are no restrictions on q.

Each of the three equations (5.18), (5.19), and (5.21) will be designated the standard reduced quartic equation.

A quartic equation must be reduced to one of the forms expressed by equations (5.18), (5.19), and (5.21) in order to apply the method of this thesis. The tables in Chapter XII are based on these equations. These tables give values for all real roots and give the values of the real roots corresponding to given values of n, p, and q.

5.4. Properties of the standard reduced quartic equation. Equation

(5.18) $z^4 + z^2 + pz + q = 0,$

may have at most two real roots. Equation

(5.19) $z^4 - z^2 + pz + q = 0,$

may have four real roots. Equation

(5.21) $z^4 + py + q = 0,$

may have at most two real roots.[1]

From the above it can be seen that the only equation that need be analyzed is equation (5.19). There is a critical point when the function of z,

$$F(z) = z^4 - z^2 + pz + q,$$

goes from four real roots to two real roots or vice-versa. If F (z) is plotted in cartesian coordinates with z as the independent variable there will be three points of inflection when there are four real roots and only one point of inflection when there are two real roots. The maximum and minimum points may be found by setting F'(z) equal to zero.

$$F'(z) = 4z^3 - 2z + p$$

and $$z^3 - \tfrac{1}{2}z + p/4 = 0.$$

The roots of this equation are,

$$z_1 = \frac{\sqrt{6}}{3} \cos \frac{\phi}{3},$$

$$z_2 = \frac{\sqrt{6}}{3} \cos \left(\frac{\phi}{3} + 120^\circ\right),$$

$$z_3 = \frac{\sqrt{6}}{3} \cos \left(\frac{\phi}{3} + 240^\circ\right),$$

where

$$\cos \phi = p/4\sqrt{54}$$

---

[1]Dickson, 1922, op. cit., p. 81.

on the condition that ϕ is taken in the first or second quadrant according as $p/4$ is negative or positive.[2]

$$F''(z) = 3z^2 - \tfrac{1}{2}.$$

Therefore, the function has a maximum when $F''(z)$ is negative and this happens when $\frac{-\sqrt{6}}{6} > z > \frac{\sqrt{6}}{6}$.

$F(z)$ will have a minimum when $F''(z)$ is positive and this happens when $\frac{-\sqrt{6}}{6} < z < \frac{\sqrt{6}}{6}$.

If the above is applied it can be seen that q merely transfers the function up and down the $F(z)$ axis. When $0 \leq p \leq 2/9\sqrt{6}$ there are four real roots, so when p is outside this interval there are either two real roots or none.

The above information is shown graphically in figures 5.1 and 5.2.

Once the general quartic has been reduced to a standard form it may be helpful to know how many real roots there are, if any. This may be accomplished by applying the following formulas:

$$(5.22) \qquad 27\Delta = 4(12q + n^2)^3 - (27p^2 \quad 2n^3 - 72nq)^2$$

$$(5.23) \qquad L = -9p^2 - 2a^3 + 8aq.$$

If $\Delta < 0$ there are two distinct real roots whether $n = 1$ or $n = -1$.

---

[2]Uspensky, op. cit., p. 92.

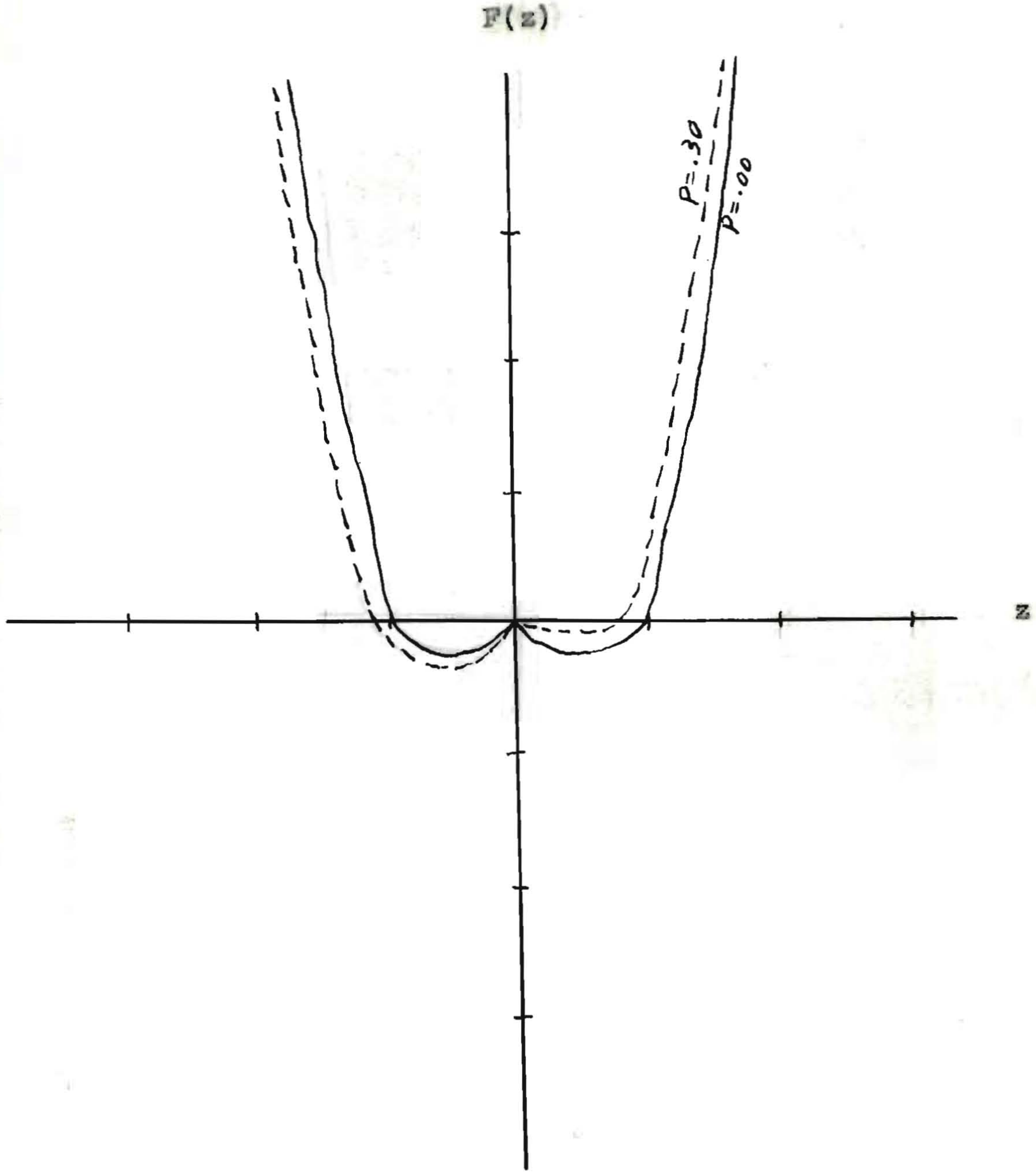

FIGURE 5.1

Graphs of $F(z) = z^4 - z^2 + pz + q$ for selected values of p and q equal to zero.

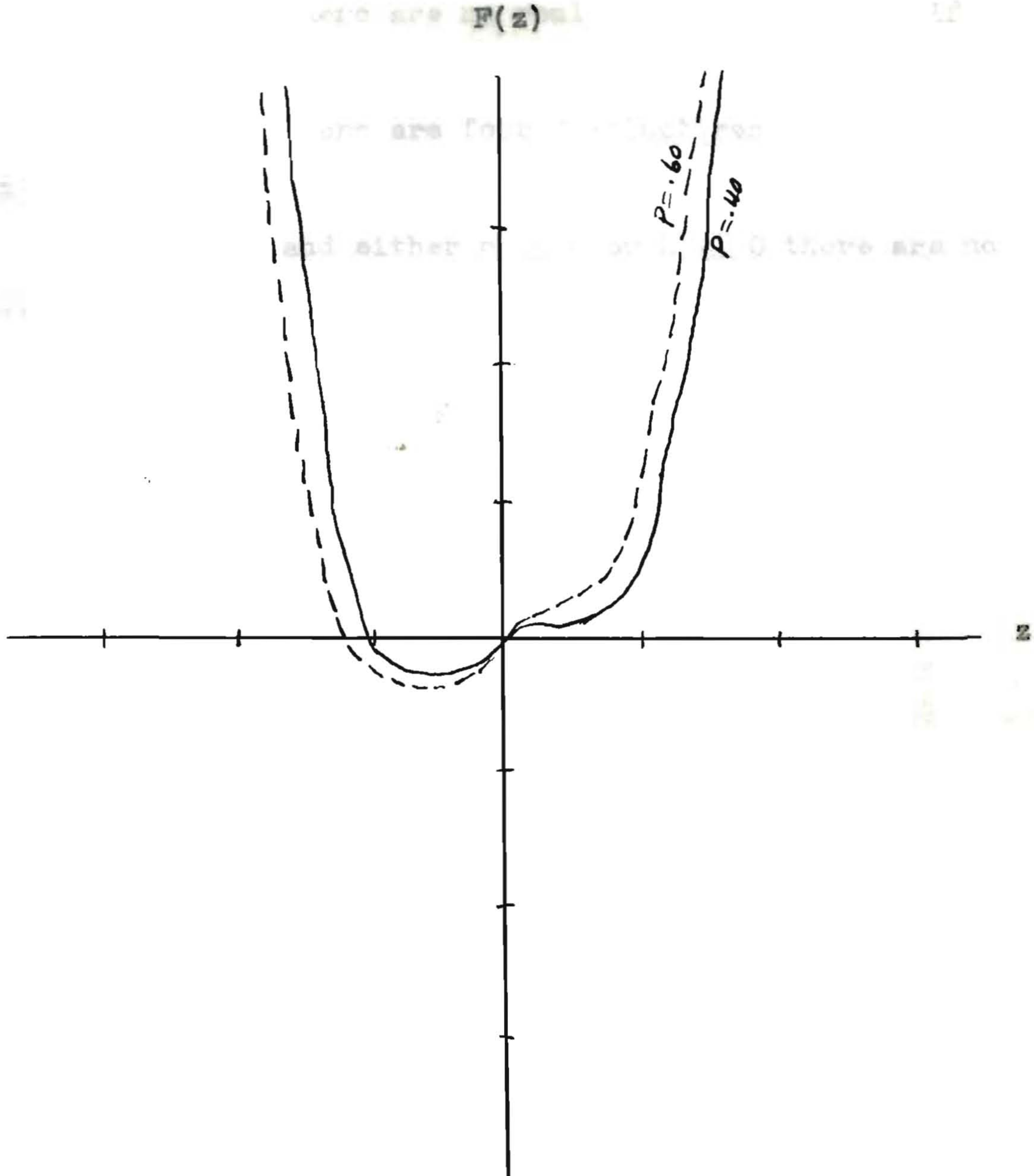

FIGURE 5.2

Graphs of $F(z) = z^4 - z^2 + pz + q$ for selected values of p, and q equal to zero.

If $\Delta > 0$ there are no real roots if $n = 1$ or if $n = -1$ and $L \leq 0$.

If $\Delta > 0$ there are four distinct real roots if $n = -1$ and $L > 0$.

If $\Delta > 0$ and either $n \geq 0$ or $L \leq 0$ there are no real roots.[3]

---

[3]Dickson, 1922, *op. cit.*, pp. 80-81.

CHAPTER VI

CONSTRUCTION OF THE TABLES

6.1. Preliminary considerations. The roots of the standard reduced quartic equations (5.18), (5.19), and (5.21) will vary with the parameters n, p, and q; so, when n, p, and q are given, all of the real roots will be given. The tables in Chapter XII are so constructed that straight line interpolation is necessary when the parameters take on certain values.

6.2. The method of calculating the tables. If all, or any of the roots of the standard reduced quartic equations were to be found for different values of the parameters, the construction of the tables would be very laborious. However, the process may be reversed. The values of q may be found for different values assigned to p and z, and n being equal to either a - 1, o, or 1.

The standard reduced equations (5.18), (5.19), and (5.21) may be written in the following forms and are in the same order as they appear above.

(6.1) $$q = -z^4 - z^2 - pz,$$

(6.2) $$q = -z^4 + z^2 - pz,$$

(6.3) $$q = -z^4 - pz.$$

From these equations for each of the assigned values of n, p, and z, the value of q is determined. These values of q may be calculated to any desired number of significant digits. This method was employed in the construction of the tables in Chapter XII.

When $n = -1$ the range of p is from .00 to 1.00 at intervals of ten-hundredths with $p = .53$, $p = .54$, $p = .55$ added because of the critical point when $p = 2/9\sqrt{6}$. The range of z is from -1.70 to 1.60 at intervals of five-hundredths; except where there is a double root and z is calculated to the nearest one-hundredth close to this double root. The range of q is $-4.000 < q \leq 1.05470$ at no specific intervals.

When $n = 0$ the range of p is from .10 to 1.00 at intervals of ten-hundredths. z is calculated at intervals of five-hundredths; except close to double roots, and here again it is calculated to the nearest one-hundredth. z is calculated from -1.70 to 1.60. The range of q is $-4.000 < q \leq .4725$ at no specific intervals.

When $n = 1$, p is calculated in intervals of ten-hundredths from .00 to 1.00. z is calculated to the nearest one-hundredth close to double roots and five-hundredths elsewhere from -1.35 to 1.20. The range of q is $-4.000 < q \leq .2148$ at no specific intervals.

There are in addition to the tables discussed above other tables corresponding to n equal to -1, o, and 1 for values of p equal to 2, 6, and 10 and a table for z, z^2, and z^4. These tables are designed as guides for equations not solvable by the other tables. They will be discussed in Chapter VIII under special problems.

CHAPTER VII

USE OF THE TABLES

7.1. Introduction. It was pointed out in the preceding chapter that the tables in Chapter XII may require interpolation. However, linear interpolation is sufficient.

It should be noted here that z and p are given in hundredths, and q is given in hundred-thousandths near the double roots and thousands not near the double roots. For illustration limited excerpts from the tables are reproduced here.

7.2. Using the table.

$n = -1$ $\qquad$ $p = .20$

q	z_1	z_2	z_3	z_4
.096				.75
.097	-1.05			
.101			.55	
.109		-.25		
.110			.60	.70
.111				.69
.112			.61	
.1126			.62	.68
.1134			.63	.67
.1138			.64	.66
.1140			.65	.65
.142		-.30		
.177		-.35		
.200	-1.00			
.214		-.40		

Excerpts from tables

Example 7.1. In the equation $z^4 - z^2 + .20z + .1126 = o$ the value of n is -1, p = .20, and q = .1126. From the portion of the table given above two roots may be read directly and they are $z_3 = .62$ and $z_4 = .68$. However, z_1 and z_2 may be found by interpolation.

	q		z_1		
.0156	.097	.103	-1.05	x	.05
	.1126				
	.200		-1.00		

$$\frac{.0156}{.103} = \frac{x}{.05} \quad \text{or} \quad .05\ (0156) = 1030x$$

and $x = .0075$

therefore $z_1 = -1.05 + .008 = -1.04$

	q		z_2		
.0036	.109	.033	-.25	x	.05
	.1126				
	.142		-.30		

$$\frac{.0036}{.033} = \frac{x}{.05} \quad \text{or } 330x = 1.80$$

and $x = .005$

therefore $z_2 = -.25 - .005 = -.26.$

This particular equation has four real roots which are $z_1 = -1.04$, $z_2 = -.26$, $z_3 = .62$, and $z_4 = .68$.

Example 7.2. In the equation $z^4 - z^2 + .20z + .200 = 0$, the value of n is -1, p = .20, and q = .200. Using the portion of the table given above again; it can be seen that this equation has only two real roots, $z_1 = -1.00$ and by interpolation $z_2 = -.38$.

7.3. Special cases in finding z. It should be noted here that the parameters of the standard reduced equations given in the tables will not suffice for the solution of all quartic equations. If the values of n, p, and q lie beyond the range of the tables, another method must be employed.

The above situation will be discussed in Chapter VIII which deals with problems not covered by the tables.

Chapter VIII will also cover interpolation for values of z when the value of p is between two values listed in the table.

CHAPTER VIII

SPECIAL PROBLEMS

8.1. Introduction. As was pointed out in the preceding chapter, all quartics are not solvable by the tables in this thesis; however, they may be solved by the method used to construct the tables. In most cases not covered by the tables, methods may be devised from previous formulas to yield solutions.

8.2. Values of z beyond the tabular limits. Once the general quartic equation is in one of the standard forms (5.18), (5.19), or (5.21); and either p or q are beyond the tabular limits of the tables, new values for p and q must be tabulated by the formulas

(6.1) $$q = -z^4 - z^2 - pz,$$

(6.2) $$q = -z^4 + z^2 - pz,$$

(6.3) $$q = -z^4 - pz,$$

depending upon which formula the general quartic reduced to.

If the value of n is between a negative one and a positive one tables I_a, II_a, and III_a through I_n, II_j, and III_k are used for guides.

If the value of n is not between any of these values then tables I_o, II_k, and III_l through I_q, II_m, and III_n are used for guides.

By inspecting the table that is closest to the standard reduced quartic, values of z are chosen and the values of q are determined by equations.

Once a value of q of the standard reduced quartic is determined then the methods of Chapter VII are applied.

Example 8.1. Given the equation

$$z^4 + z^2 + 3.00\,z - 1.712 = 0,$$

it can be seen that p is not within the limits of the tables. However, p is close to guide table III_1. Upon examination of this table it is seen that z_1 should lie close to -1.40 and z_2 close to .60.

Using table IV and formula (6.1), and setting $z_1 = -1.40$ the corresponding value of q is -1.60, which is close to the value sought for. Now setting $z_1 = -1.45$ the corresponding value of q is -2.173, and the sought after q lies in this interval. Interpolation will yield z_1, and is equal to -1.41. The same procedure may be used for z_2, and if so $z_2 = .44$.

It should be noted here that several substitutions for z may be necessary before the correct values of q are found.

Example 8.2. Given the equation $z^4 + .90\,z - 5.008 = 0$. It may be seen by table II_1 that the desired q is beyond the limits of this table; however, upon examining this table it seems that z_1 should be between -1.50 and -1.60 and z_2 between 1.30 and 1.40. When -1.60 is substituted for z_1 in formula (6.3), q = -5.114 and is larger than the desired q. Therefore, substituting -1.55 for z_1, q = -4.377. Upon interpolating, z_1 is found to equal -1.59. If the same procedure is applied for z_2, it is found that $z_2 = 1.41$.

8.3. <u>Values of p between the limits of the table</u>. When a quartic equation is reduced to a standard form and the p value is between two values in the tables, linear interpolation may be applied to find the value of the desired roots.

Example 8.3. Given the equation $z^4 + .64\,z + .233 = 0$. This type equation lies between tables II_f and II_g. First the roots for p = .60 are found; then the roots for p = .70. Linear interpolation is applied and the desired roots are found.

When p = .60 and q = .233 the roots are read directly from table II_f and are found to be, $z_1 = -.59$ and $z_2 = -.47$.

When p = .70 and q = .233 interpolation should be used for both z_1 and z_2. If this is used, $z_1 = -.72$ and $z_2 = -.36$.

Linear interpolation may now be applied again to find the desired z_1 and z_2.

	p			z_1		
4 [	.60	] 10		-.59	] x	] .13
	.64					
	.70			-.72		

	p			z_2		
4 [	.60	] 10		-.47	] x	] .11
	.64					
	.70			-.36		

When the above interpolations are performed the desired $z_1 = -.64$ and $z_2 = .43$.

8.4. Values of z when p and n are equal to zero. No table was constructed for p and n equal to zero. When this happens formula (6.3) becomes

$$q = -z^4 \tag{8.1}$$

or

$$z^4 = q \tag{8.2}$$

or

$$z^4 = -q \tag{8.3}$$

Equation (8.3) has only imaginary roots and equation (8.2) has two real roots, equal numerically, and two imaginary roots. In the above case table IV may be used to determine the two real roots.

Example 8.4. Given the equation $z^4 - 9.3789 = 0$. Transforming this equation to (8.2) it becomes

$$z^4 = 9.3789$$

or

$$z = \pm\sqrt[4]{9.3789}\,.$$

Now applying table IV it may be seen that $z_1 = 1.75$ and $z_2 = -1.75$.

8.5. Summary. The goal of this chapter was to devise methods for the finding of all the real roots in cases not covered by tables I_a through I_n, II_a through II_j, and III_a through III_k. Four main problems were recognized and solved; the first, that of finding the roots not found in the above tables; second, that of finding the roots when p was given by the above tables, but q and z did not extend far enough; third, that of finding the roots when the desired p was between two values in the tables; and last, the case when n and p were equal to zero.

CHAPTER IX

APPLYING THE TABLES TO METHODS YIELDING ADDITIONAL DIGITS

9.1. Introduction. As was pointed out in Chapter IV, there are several methods that will yield additional digits once an approximation is found for a root. Chapter I emphasizes the fact that this thesis is mainly concerned in evaluating roots of quartic equations to a limited number of significant digits, namely two and three significant digits. If more accuracy is needed, these approximations found in the tables may be used as "starters" for the methods discussed in Chapter IV. Numerical examples will be illustrated here for the application of Newton's method, and the method of series reversion.

9.2. The first approximation for Newton's method. In cases where the roots are desired beyond the accuracy of the tables it is necessary to use some method that will yield the desired number of additional digits.

Example 9.1. Given the equation $z^4 - z^2 + .55z - .757 = 0$ and the negative root is desired to more than three significant digits. Using table I_1, z_1 is read directly and is equal to -1.35.

Using formula

(4.3) $$a_1 = a - \frac{f(a)}{f'(a)}$$

and setting a equal to -1.35,

$$f(a) = (-1.35)^4 - (-1.35)^2 + .55(-1.35) - .757 = -.000494$$

$$f'(a) = 4(-1.35)^3 - 2(-1.35) + .55 = -6.591500$$

or

$$a_1 = -1.35 - \frac{(-.000494)}{(-6.591500)} = -1.350075$$

If more accuracy is desired, formula

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)} \qquad (4.4)$$

may be applied using a equal to the value given above, which is -1.350075.

If z_2 would have been desired the same procedure would be applicable, although linear interpolation should be used to find the first approximation for z_2, which is 1.13.

It should be noted here that table IV may be used in finding the value of the function and the value of the first derivative for the formulas (4.3) and (4.4).

Example 9.2. Given the equation $z^4 - z^2 + .53z - .151 = 0$ and that z_1 and z_2 are desired to more than three significant digits. Using table I_g, z_1 is found to be close to -1.25. This value of z_1 may be used for the value of a in formula (4.3), but additional digits would be obtained more quickly if interpolation were used first and the value of a taken as -1.24.

If z_2 is used in equation (4.3), the value of a, can be read directly from the table, and is equal to .75.

9.3. <u>The first approximation for series reversion</u>. This method is a little more laborious than Newton's method, but it will yield additional significant digits faster once it is applied.

Example 9.3. Given the equation

$$z^4 + z^2 + .20z - 2.842 = 0,$$

and only the root nearest the origin is desired. Using table III_c it may be seen that the above equation has only two real roots, one being positive and the other negative. The desired root is the positive since it lies closest to the origin.

Using z_2 and interpolating, z_2 is found to equal 1.09. This value is substituted in equation

(4.7) $$z = x + h$$

for the value of h. This equation now becomes,

$$z_2 = x + 1.09.$$

Using formula

(4.8) $$g(x) = f(x + h) = 0$$

which becomes formula

(4.9) $$0 = g(x) = e_0 + e_1x + e_2x^2 + e_nx^n + \ldots$$

The above data make formula (4.9) equal to

$$(x + 1.09)^4 + (x + 1.09)^2 + .20(x + 1.09) - 2.842 = 0$$

or

$$x^4 + 4.36x^3 + 8.1286x^2 + 7.560116x - .02431839 = 0$$

Equation (4.9) is now reduced to the form

(4.5) $$y = x(1 - a_1x - a_2x^2 - a_3x^3 - \ldots),$$

where

(4.10) $$y = e_0/e_1$$

and

(4.11) $$a_i = -e_{i+1}/e_1 \ .$$

From the above data, formula (4.5) becomes

$$\frac{.02431839}{7.560116} = x\left(1 + \frac{8.1286}{7.560116}x + \frac{4.36}{7.560116}x^2 + \frac{1}{7.560116}x^3\right)$$

or

$$.003217 = x(1 + 1.075195\,x + .57671\,x^2 + .1323\,x^3) \ .$$

Using formula

(4.6) $$x = y(1 + b_1y + b_2y^2 + b_3y^3 + \ldots)$$

where

(4.12) $$b_1 = a_1$$

(4.13) $$b_2 = 2a_1^2 + a_2 \ ,$$

(4.14) $$b_3 = 5a_1^3 + 5a_1a_2 + a_3,$$

(4.15) $$b_4 = 14a_1^4 + 21a_1^2a_2 + 3a_2^2 + 6a_1a_3 + a_4,$$

and

(4.16) $$b_n = \frac{1}{n!}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{f(x)}\right)^n\right]_{x} = 0,$$

formula (4.6) becomes

$$x = .003217\left[1 + 1.075195\,(.003217) + .21888799\,(.003217)^2 + 9.547334\,(.03217)^3\right]$$

and

$$x = .00322824 \ .$$

Substituting this value in formula (4.7), z_2 is found to equal 1.09322824

If z_1 would have been the sought after root the same procedure could have been used, but no interpolation would have been necessary, since z_1 for $q = -2.842$ is in table III_c.

It may be seen from the above that the computations may be laborious without the use of a calculator and that more accuracy may be obtained by extending the reversion series.

9.4. Summary. In this chapter the goal has been to find the first approximations for Newton's method and the series reversion method.

Since the ease of the above methods depends upon the accuracy of the first approximation of the desired root, this was considered and is the reason interpolation was used instead of taking the first approximation as the closest root appearing in the tables.

CHAPTER X

PROCEDURE

10.1. Introduction. The purpose of this chapter is to present procedures for the application of the methods developed in this thesis for the solution of quartic equations. Numerical problems will be used to illustrate the necessary steps. It should be pointed out again that the methods in this thesis will give all the real roots of quartic equations with real coefficients.

Two main types of situations may arise. Their procedures are presented in this chapter. These are: (1) the solution by tables; and (2) the application of special formulas when the parameters of the reduced quartic equation are beyond the limits of the tables. The above situations have been discussed in preceding chapters and only the procedures will be outlined here. The successive steps of the procedure are identified by the capital letters A, B, C, etc., and the limitations or considerations related to each step are identified by the small letters a, b, c, etc.

6.2. The reduction of a quartic equation. For the tabular solution of a given quartic equation,

(5.1) $$x^4 + ax^3 + bx^2 + cx + d = 0,$$

the following steps are suggested.

A. Eliminate the cube term of the quartic equation.

a. Let $x = y - a/4$. This method was suggested in paragraph 5.2. The resulting equation is $y^4 + hy^2 + ky + m = 0$, where

(1) $h = b - \frac{3a^2}{8}$

(2) $k = c - \frac{ab}{2} + \frac{a^3}{8}$

(3) $m = d - \frac{ac}{4} + \frac{a^2b}{16} - \frac{3a^4}{256}$

B. Transform the quartic equation to a reduced standard form, as suggested in paragraph 5.3.

a. For h equal to a positive number other than one and k not positive the equation becomes $z^4 + nz^2 + pz + q = 0$, by substituting $y = \sqrt{h}\ t$ where

(1) $n = h^2/h^2 = 1$,

(2) $p = k/h^{3/2}$,

(3) $q = m/h^2$,

and substituting $t = -z$.

b. For h equal to a negative number other than one and k not positive, the equation becomes $z^4 + nz^2 + pz + q = 0$, by substituting $y = \sqrt{-h}\ t$ where

(1) $n = -h^2/h^2 = -1$,

(2) $p = k/(-h)^{3/2}$,

(3) $q = m/h^2$,

and substituting $t = -z$.

C. For h equal to zero and k not positive, the equation is $z^4 + pz + q = 0$, by substituting $y = -z$.

6.3. Direct solution by tables. Once the general quartic equation,

(5.1) $$x^4 + ax^3 + bx^2 + cx + d = 0$$

has been reduced to one of the standard forms

(5.18) $$z^4 + z^2 + pz + q = 0,$$

(5.19) $$z^4 - z^2 + pz + q = 0,$$

(5.21) $$z^4 + pz + q = 0,$$

by the steps suggested above, the roots may be determined by tables I_a through I_n, II_a through II_j, and III_a through III_k.

The above procedure is illustrated by the following examples.

Example 1. Find all the real roots of the equation

$$x^4 + 2x^3 - 12x^2 - 10x + 3 = 0.$$

In this equation $a = 2$, $b = -12$, $c = -10$, and $d = 3$.

Using formula

(5.2) $$x = y - a/4$$

or formulas

(5.4) $$h = b - 3a^2/8,$$

(5.5) $$k = c - ab/2 + a^3/8,$$

(5.6) $$m = d - ac/4 \quad a^2b/16 - 3a^4/256,$$

h, k and m are found to be

$h = -13.5,$

$k = 3,$

$m = 4.8125.$

The given equation now becomes

$$y^4 - 13.5\, y^2 + 3y + 4.8125 = 0.$$

Using formula

(5.12) $y = \sqrt{-h}\ t$

and modifying it to $y = \sqrt{-h}\ z$ because the coefficient of y is positive, n, p, and q are found to be

$n = -1$

$p = .060$

$q = .02641$

They may also be found by formulas

(5.14) $n = -h^2/h^2 = -1,$

(5.15) $p = k/(-h)^{3/2},$

(5.16) $q = m/k^2\ .$

The equation now becomes

$$z^4 - z^2 + .06z + .02641 = 0.$$

Since p is between two values in table I, two sets of roots should be found and then linear interpolation used to find the desired roots.

Looking up the z values for $n = -1$, $p = .00$, and $q = .02641$. It is seen by table I_a there are four real roots and linear interpolation should be used. When interpolation is applied $z_1 = -.99$, $z_2 = -.16$, $z_3 = .16$, and $z_4 = .99$.

When $n = -1$, $p = .10$, and $q = .02641$, it is seen by table I_b there are four real roots and applying linear interpolation $z_1 = -1.03$, $z_2 = -.12$, $z_3 = .22$, and $z_4 = .93$. Now upon applying linear interpolation to the p values, the desired $z_1 = -1.01$, $z_2 = -.14$, $z_3 = .20$, and $z_4 = .95$.

Substituting these values in the modified formula (5.12),

$$y_1 = -3.714,$$
$$y_2 = -.515,$$
$$y_3 = .735,$$
$$y_4 = 3.493,$$

and applying formula (5.2)

$$x_1 = -4.2,$$
$$x_2 = -1.0,$$
$$x_3 = .24,$$
$$x_4 = 3.0,$$

which are the roots of the original quartic equation.

Example 2. Find all the real roots of the equation

$$x^4 - 10x^2 - 20x - 16 = 0.$$

This equation has no cube term, therefore formula (5.12) may be applied and

$$n = -1,$$
$$p = -.632,$$
$$q = -.16,$$

and the resulting equation is

$$y^4 - y^2 - .632y - .16 = 0.$$

Now applying formula

(5.17) $t = -z$

in the modified form $y = -z$, the equation becomes

$$z^4 - z^2 + .632z - .16 = 0.$$

By table I_j, and using linear interpolation, $z_1 = -1.26$ and $z_2 = .68$. By table I_k, $z_1 = -1.28$ and $z_2 = .49$. Now using linear interpolation for the p values the desired $z_1 = -1.25$ and $z_2 = .62$. Substituting in formulas (5.17) and (5.12) the original real roots are:

$$x_1 = 4.0,$$

$$x_2 = -2.0.$$

Example 3. Find all the real roots of the equation

$$x^4 - .90x + .4105.$$

Only formula (5.17) need be applied and the resulting equation is

$$z^4 + .90z + .4105 = 0.$$

Using table II_1, the real roots may be read directly and they are $z_1 = -.61$ and $z_2 = -.61$. Substituting in formula (5.17) the original real roots are:

$$x_1 = .61$$

$$x_2 = .61.$$

Example 4. Find all the real roots of the equation

$$x^4 + 4x^3 + 7x^2 + 6.30x + 2.279 = 0.$$

Applying formula (5.2) the equation becomes

$$z^4 + z^2 + .30z - .021 = 0$$

Upon using table III_d the two real roots are $z_1 = -.18$ and $z_2 = -.11$. Substituting these values in formula (5.2), the desired roots are:

$$x_1 = -1.18,$$
$$x_2 = -1.11.$$

6.4. Indirect solution by tables. The case may arise when the standard reduced quartic equation is beyond the limits of the tables. When this happens it is suggested that the number of real roots be determined, unless the number is already known. This may be accomplished by applying formulas

$$(5.22) \qquad 27\Delta = 4(12q + n^2)^3 - (27p^2 + 2n^3 - 72nq)^2$$

and

$$(5.23) \qquad L = -9p^2 - 2n^3 + 8nq.$$

If $\Delta < 0$ there are two distinct real roots whether $n = 1$ or $n = -1$.

If $\Delta > 0$ there are no real roots if $n = 1$ or if $n = -1$ and $L \leq 0$.

If $\Delta > 0$ there are four distinct real roots if $n = -1$ and $L > 0$.

If $\Delta > 0$ and either $n \geq 0$ or $L \leq 0$ there are no real roots.

It should be noted here that the procedure in paragraph 6.2 should be applied before the number of real roots are determined by the above suggested procedure.

Illustrative examples are now presented for the above procedures.

Example 5. Find all the real roots of the equation

$$x^4 + 8x^3 + 25x^2 + 42x + 34.6 = 0.$$

Applying formula (5.2) or formulas (5.4), (5.5), and (5.6) the equation becomes

$$x^4 + z^2 + 6z + 2.6 = 0.$$

Applying formula (5.22), $\Delta < 0$, therefore there are only two real roots. Using table III_m it is seen that $z_1 = -1.40$ and z_2 lies between -.50 and -.40. Calculating q for $z_2 =$ -.45, by formula

(6.1) $\quad q = -z^4 - z^2 - pz$

and table IV, q is found to equal 2.456. Now z_2 lies between -.50 and -.45 and upon interpolation, $z_2 = -.48$. Substituting in formula (5.2) the original roots are:

$$x_1 = -3.40,$$

$$x_2 = -2.48.$$

Example 6. Find the two real roots of the equation,

$$x^4 - x^2 - 9.2x + 8.730 = 0.$$

Applying formula (5.20) the equation becomes

$$z^4 - z^2 + 9.2z + 8.730 = 0.$$

The p value is close to the table I_q; then z_1 should be close to -1.90 and z_2 close to -.90. Using formula

(6.2) $\quad q = -z^4 + z^2 - pz$

and table IV, and p = 9.2, q = 8.058 for $z_1 = -1.85$, therefore $z_1 = -1.85$. Using the same formula, (6.2), and letting $z_2 = -.90$, $q = 8.434$ which is a little small for the desired q. Letting $z_2 = -.95$, $q = 8.828$ which is a little larger than the original q, but using linear interpolation the desired $z_2 = -.94$. Now substituting in formula (5.20), the desired roots are:

$$x_1 = 1.85,$$
$$x_2 = .94.$$

Example 7. Find the two real roots of $x^4 - 2.441 = 0$. This equation is the same as the equation $x = \pm \sqrt[4]{2.441}$, and using table IV it is seen that

$$x_1 = -1.25,$$
$$x_2 = 1.25.$$

Example 8. Find the two real roots of the equation,

$$x^4 + x^2 + .50x - 4.468 = 0.$$

No transformations are necessary but the q value is beyond the limits of table III_f, therefore it is necessary to use formula (6.1) and table IV. Looking at table III_f, it may be seen that x_1 should be close to -1.30, therefore calculating q for $x_1 = -1.35$, $q = -4.469$ which is close enough to the desired q that interpolation is not necessary. Again using table III_f, it may be seen that x_2 is close to 1.20, then calculating $x_2 = 1.25$, by formula (6.1) and table IV, it is seen that $q = -4.629$ which is larger than the desired q, but using linear interpolation the desired $x_2 = 1.23$, therefore the desired roots are

$$x_1 = -1.35,$$
$$x_2 = 1.23.$$

CHAPTER XI

CONCLUSION

11.1. Summary. The general quartic has been solved by several different methods. Each method has its advantages and limitations. The objective of this thesis was not to present a method to displace existing procedures, but to develop an alternate method of solution by tables.

In this thesis formulas were developed which reduced a quartic equation to one of three special forms and tables based on these three forms were computed. These tables which are presented in Chapter XII give approximations of all real roots to two and three significant digits and in special cases four and five significant digits. However, linear interpolation should be used to find certain values of the roots. The steps involved in this solution have been outlined in the preceding chapter, Chapter X.

11.2. Evaluation. This method will yield, with fair ease and speed of operations, roots to two or three significant digits and in special cases four or five significant digits.

A root determined by the methods of this thesis may be used if only an approximation is needed, but if more significant digits are needed they may be used as the first approximation in one of the approximation methods of solution.

11.3. Suggestions for further study. This thesis has dealt only with algebraic equations, of the fourth degree, with real coefficients and real roots. The writer has not attempted to explore the possibilities of applying a tabular method for finding all roots, both real and imaginary, of the quartic equation. It is also suggested that a tabular method might be valuable if applied to quartic equations with imaginary and complex coefficients.

CHAPTER XII

TABLES

TABLE I_a

n=-1 p=.00

q	z_1	z_2	z_3	z_4
-3,994	-1.60	*	*	1.60
-3.370	-1.55	*	*	1.55
-2.812	-1.50	*	*	1.50
-2.318	-1.45	*	*	1.45
-1.882	-1.40	*	*	1.40
-1.499	-1.35	*	*	1.35
-1.166	-1.30	*	*	1.30
- .879	-1.25	*	*	1.25
- .634	-1.20	*	*	1.20
- .426	-1.15	*	*	1.15
- .254	-1.10	*	*	1.10
- .113	-1.05	*	*	1.05
.000	-1.00	.00	.00	1.00
.0001		-.01	.01	
.0004		-.02	.02	
.0009		-.03	.03	
.0016		-.04	.04	
.0025		-.05	.05	
.010		-.10	.10	
.022		-.15	.15	
.038		-.20	.20	
.059		-.25	.25	
.082		-.30	.30	
.088	- .95			.95
.107		-.35	.35	
.134		-.40	.40	
.154	- .90			.90
.161		-.45	.45	
.188		-.50	.50	
.200	- .85			.85
.211		-.55	.55	
.230	- .80	-.60	.60	.80
.244		-.65	.65	
.246	- .75	-.66	.66	.75
.247		-.67	.67	
.248	- .75			.74
.2486		-.68	.68	
.2489	- .73			.73
.2494		-.69	.69	
.2497	- .72			.72
.2499		-.70	.70	
.250	- .71	-.71	.71	.71

*no root

TABLE I_b

n=-1 p=.10

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-4.154		*	*	1.60	.064				.90
-3.834	-1.60	*	*		.072			.35	
-3.525		*	*	1.55	.084		-.25		
-3.215	-1.55	*	*		.094			.40	
-2.962		*	*	1.50	.100	-1.00			
-2.662	-1.50	*	*		.112		-.30		.85
-2.463		*	*	1.45	.115				
-2.173	-1.45	*	*		.116			.45	
-2.022		*	*	1.40	.138			.50	
-1.742	-1.40	*	*		.142		-.35		
-1.634		*	*	1.35	.150				.80
-1.364	-1.35	*	*		.156			.55	
-1.296		*	*	1.30	.164				.77
-1.036	-1.30	*	*		.168				.76
-1.004		*	*	1.25	.170			.60	
- .754	-1.25	*	*	1.20	.171				.75
- .542		*	*	1.15	.174		-.40		.74
- .514	-1.20	*	*		.176			.63	.73
- .364		*	*	1.10	.1777				.72
- .312	-1.15	*	*		.1778			.64	
- .218		*	*	1.05	.1790			.65	.71
- .144	-1.10	*	*		.1799			.66	.70
- .100		*	*	1.00	.1804			.67	.69
- .008	-1.05	*	*		.1806			.68	.68
- .007		*	*	.95	.183	- .95		*	*
- .00251		.05	.05		.206		-.45	*	*
- .00246			.06		.238		-.50	*	*
- .0024		.04			.244	- .90		*	*
- .0021		.03	.07		.266		-.55	*	*
- .0016		.02	.08		.285	- .85		*	*
- .0010		.01	.09		.290		-.60	*	*
- .0001			.10		.309		-.65	*	*
.0000		.00			.310	- .80		*	*
.001			.11		.312		-.66	*	*
.007		-.05			.314	- .79	-.67	*	*
.009			.15		.316	- .78		*	*
.018			.20		.317		-.68	*	*
.020		-.10			.318	- .77	-.69	*	*
.034			.25		.320	- .76	-.70	*	*
.037		-.15			.321	- .75	-.71	*	*
.052			.30		.3217	- .74	-.72	*	*
.058		-.20			.3219	- .73	-.73	*	*

*no root

TABLE I_c

n=-1 p=.20

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-4.314		*	*	1.60	.030		-.10		.85
-3.680		*	*	1.55	.037			.35	
-3.674	-1.60	*	*		.052		-.15		
-3.112		*	*	1.50	.054			.40	
-3.060	-1.55	*	*		.070				.80
-2.608		*	*	1.45	.071			.45	
-2.512	-1.50	*	*		.078		-.20		
-2.162		*	*	1.40	.088			.50	
-2.028	-1.45	*	*		.096				.75
-1.769		*	*	1.35	.097	-1.05			
-1.602	-1.40	*	*		.101			.55	
-1.426		*	*	1.30	.109		-.25		
-1.229	-1.35	*	*		.110			.60	.70
-1.129		*	*	1.25	.111				.69
- .906	-1.30	*	*		.112			.61	
- .874		*	*	1.20	.1126			.62	.68
- .657		*	*	1.15	.1134			.63	.67
- .629	-1.25	*	*		.1138			.64	.66
- .474		*	*	1.10	.1140			.65	.65
- .394	-1.20	*	*		.142		-.30	*	*
- .323		*	*	1.05	.177		-.35	*	*
- .200		*	*	1.00	.200	-1.00		*	*
- .197	-1.15	*	*		.214		-.40	*	*
- .102		*	*	.95	.251		-.45	*	*
- .034	-1.10	*	*		.278	- .95		*	*
- .026		*	*	.90	.288		-.50	*	*
- .01010		.10	.10		.321		-.55	*	*
- .01004			.11		.334	- .90		*	*
- .00996		.09			.350		-.60	*	*
- .0098			.12		.370	- .85		*	*
- .0096		.08			.374		-.65	*	*
- .0094			.13		.384	- .82		*	*
- .0091		.07			.385		-.68	*	*
- .0088			.14		.387		-.69	*	*
- .0084		.06			.388	- .81		*	*
- .008		.05	.15		.390	- .80	-.70	*	*
- .007			.16		.392		-.71	*	*
- .006		.04	.17		.393	- .79		*	*
- .005		.03			.3937		-.72	*	*
- .004		.02			.3942	- .78		*	*
- .002		.01	.20		.3949		-.73	*	*
.000		.00			.3954	- .77		*	*
.009			.25		.3957		-.74	*	*
.012		-.05			.3959	- .76		*	*
.022			.30		.3961	- .75	-.75	*	*

*no root

TABLE I_d

$n=-1$ $p=.30$

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-4.474		*	*	1.60	.021				.75
-3.835		*	*	1.55	.026			.45	
-3.514	-1.60	*	*		.038			.50	
-3.262		*	*	1.50	.040		-.10		.70
-2.905	-1.55	*	*		.0460			.55	
-2.753		*	*	1.45	.0464				.67
-2.362	-1.50	*	*		.0473			.56	
-2.302		*	*	1.40	.0479				.66
-1.904		*	*	1.35	.0483			.57	
-1.883	-1.45	*	*		.0490				.65
-1.556		*	*	1.30	.0492			.58	
-1.462	-1.40	*	*		.0498				.64
-1.254		*	*	1.25	.0499			.59	
-1.094	-1.35	*	*		.0504			.60	.63
- .994		*	*	1.20	.0506			.61	.62
- .776	-1.30	*	*		.067		-.15	*	*
- .772		*	*	1.15	.076	-1.10		*	*
- .584		*	*	1.10	.098		-.20	*	*
- .504	-1.25	*	*		.134		-.25	*	*
- .428		*	*	1.05	.172		-.30	*	*
- .300		*	*	1.00	.202	-1.05		*	*
- .274	-1.20	*	*		.212		-.35	*	*
- .197		*	*	.95	.254		-.40	*	*
- .116		*	*	.90	.296		-.45	*	*
- .082	-1.15	*	*		.300	-1.00		*	*
- .055		*	*	.85	.338		-.50	*	*
- .0231		.16	.16		.373	- .95		*	*
- .0230		.15			.376		-.55	*	*
- .0229			.17		.410		-.60	*	*
- .0228		.14			.424	- .90		*	*
- .0226			.18		.439		-.65	*	*
- .0224		.13			.455	- .85		*	*
- .0222			.19		.4597	- .84		*	*
- .0218		.12			.4599		-.70	*	*
- .0216			.20		.4633	- .83		*	*
- .0210		.11			.4663	- .82		*	*
- .0201		.10			.4686	- .81		*	*
- .016			.25		.4697		-.74	*	*
- .013		.05			.4702	- .80		*	*
- .010				.80	.4711		-.75	*	*
- .008			.30		.4716	- .79		*	*
.000		.00			.4720		-.76	*	*
.002			.35		.4722	- .78		*	*
.014			.40		.4724	- .77	-.77	*	*
.017		-.05							

*no root

TABLE I_e

$n=-1$ $p=.40$

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-4.634		*	*	1.60	-.016				.65
-3.990		*	*	1.55	-.012			.50	
-3.412		*	*	1.50	-.0116			.51	
-3.354	-1.60	*	*		-.0107			.52	
-2.898		*	*	1.45	-.0104				.61
-2.750	-1.55	*	*		-.0100			.53	
-2.442		*	*	1.40	-.0096				.60
-2.212	-1.50	*	*		-.0094			.54	
-2.039		*	*	1.35	-.0091				.59
-1.738	-1.45	*	*		-.0090			.55	
-1.686		*	*	1.30	-.0088				.58
-1.379		*	*	1.25	-.0087			.56	
-1.322	-1.40	*	*		-.00866			.57	.57
-1.114		*	*	1.20	.000		.00	*	*
- .959	-1.35	*	*		.022		-.05	*	*
- .887		*	*	1.15	.035	-1.15		*	*
- .694		*	*	1.10	.050		-.10	*	*
- .646	-1.30	*	*		.082		-.15	*	*
- .533		*	*	1.05	.118		-.20	*	*
- .400		*	*	1.00	.159		-.25	*	*
- .379	-1.25	*	*		.186	-1.10		*	*
- .292		*	*	.95	.202		-.30	*	*
- .206		*	*	.90	.247		-.35	*	*
- .154	-1.20	*	*		.294		-.40	*	*
- .140		*	*	.85	.307	-1.05		*	*
- .090		*	*	.80	.388		-.50	*	*
- .054		*	*	.75	.400	-1.00		*	*
- .04194		.22	.22		.431		-.55	*	*
- .04190			.23		.468	- .95		*	*
- .0418		.21			.470		-.60	*	*
- .0417			.24		.503		-.65	*	*
- .0416		.20	.25		.514	- .90		*	*
- .0414					.5299		-.70	*	*
- .0412		.19			.5377		-.72	*	*
- .0410			.26		.5405	- .85		*	*
- .0406		.18			.5409		-.73	*	*
- .0404			.27		.5437	- .84	-.74	*	*
- .0399		.17			.5461		-.75	*	*
- .0397			.28		.5463	- .83		*	*
- .0391		.16			.5480		-.76	*	*
- .0390			.29		.5483	- .82		*	*
- .0380		.15	.30		.5494		-.77	*	*
- .033			.35		.5496	- .81		*	*
- .030		.10		.70	.5502		-.78	*	*
- .026			.40		.5504	- .80		*	*
- .019			.45		.5516	- .79	-.79	*	*
- .018		.05							

*no root

TABLE I_f

n=-1 p=.50

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-4.794		*	*	1.60	-.0632			.46	
-4.145		*	*	1.55	-.0630				.53
-3.562		*	*	1.50	-.0629			.47	
-3.194	-1.60	*	*		-.0627			.48	.52
-3.043		*	*	1.45	-.0626				.51
-2.595	-1.55	*	*		-.06248			.49	
-2.586		*	*	1.40	-.06250			.50	.50
-2.174		*	*	1.35	-.0616		.20	*	*
-2.062	-1.50	*	*		-.053		.15	*	*
-1.816		*	*	1.30	-.040		.10	*	*
-1.593	-1.45	*	*		-.034	-1.20		*	*
-1.504		*	*	1.25	-.023		.05	*	*
-1.234		*	*	1.20	.000		.00	*	*
-1.182	-1.40	*	*		.027		-.05	*	*
-1.002		*	*	1.15	.060		-.10	*	*
- .824	-1.35	*	*		.097		-.15	*	*
- .804		*	*	1.10	.138		-.20	*	*
- .638		*	*	1.05	.148	-1.15		*	*
- .516	-1.30	*	*		.232		-.30	*	*
- .500		*	*	1.00	.282		-.35	*	*
- .387		*	*	.95	.296	-1.10		*	*
- .296		*	*	.90	.334		-.40	*	*
- .254	-1.25	*	*		.386		-.45	*	*
- .225		*	*	.85	.412	-1.05		*	*
- .170		*	*	.80	.438		-.50	*	*
- .129		*	*	.75	.486		-.55	*	*
- .100		*	*	.70	.500	-1.00		*	*
- .081		*	*	.65	.530		-.60	*	*
- .070		*	*	.60	.563	- .95		*	*
- .06814		.31	.31		.569		-.65	*	*
- .0681		.30	.32		.600		-.70	*	*
- .0680		.29	.33		.604	- .90		*	*
- .0678			.34		.6177		-.74	*	*
- .0677		.28			.6190	- .87		*	*
- .0675			.35		.6211		-.75	*	*
- .0674		.27			.6226	- .86		*	*
- .0670		.26			.6240		-.76	*	*
- .0664		.25			.6255	- .85		*	*
- .0657		.24			.6264		-.77	*	*
- .0656			.40		.6277	- .84		*	*
- .0649		.23			.6282		-.78	*	*
- .0639		.22			.6293	- .83		*	*
- .0628		.21			.6296		-.79	*	*
- .0635			.45	.55	.6303	- .82		*	*
- .0634				.54	.6304		-.80	*	*
					.6306	- .81	-.81	*	*

*no root

TABLE I_g

n=-1 p=.53

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-3.608		*	*	1.50	-.074		.25	*	*
-3.146	-1.60	*	*		-.068		.20	*	*
-3.087		*	*	1.45	-.058		.15	*	*
-2.624		*	*	1.40	-.043		.10	*	*
-2.548	-1.55	*	*		-.024		.05	*	*
-2.215		*	*	1.35	.000		.00	*	*
-1.855		*	*	1.30	.002	-1.20		*	*
-1.550	-1.45	*	*		.029		-.05	*	*
-1.541		*	*	1.25	.065		-.10	*	*
-1.270		*	*	1.20	.101		-.15	*	*
-1.140	-1.40	*	*		.144		-.20	*	*
-1.036		*	*	1.15	.183	-1.15		*	*
- .837		*	*	1.10	.191		-.25	*	*
- .784	-1.35	*	*		.241		-.30	*	*
- .671		*	*	1.05	.293		-.35	*	*
- .530		*	*	1.00	.329	-1.10		*	*
- .477	-1.30	*	*		.346		-.40	*	*
- .416		*	*	.95	.400		-.45	*	*
- .323		*	*	.90	.444	-1.05		*	*
- .250		*	*	.85	.452		-.50	*	*
- .216	-1.25	*	*		.502		-.55	*	*
- .194		*	*	.80	.530	-1.00		*	*
- .151		*	*	.75	.548		-.60	*	*
- .121		*	*	.70	.588		-.65	*	*
- .101		*	*	.65	.592	- .95		*	*
- .088		*	*	.60	.621		-.70	*	*
- .081		*	*	.55	.631	- .90		*	*
- .07801		.35	.35		.6436		-.75	*	*
- .07800		.34	.36		.6468		-.76	*	*
- .07795			.37		.6482	- .86		*	*
- .0779		.33	.38		.6495		-.77	*	*
- .0777		.32	.39		.6510	- .85		*	*
- .0776			.40		.6516		-.78	*	*
- .0775			.41	.50	.6529	- .84		*	*
- .0774		.31			.6533		-.79	*	*
- .0773			.42		.6542	- .83		*	*
- .0772			.43	.49	.6544		-.80	*	*
- .0771		.30	.44	.48	.6549	- .82		*	*
- .0770			.45	.47	.65493	- .81	-.81	*	*
- .07698			.46	.46					

*no root

TABLE I_h

n=-1 p=.54

q	z_1	z_2	z_3	z_4	q	z_1	z_2	z_3	z_4
-3.622		*	*	1.50	-.08148			.44	.44
-3.130	-1.60	*	*		-.0814		.34	*	*
-3.101		*	*	1.45	-.0810		.30	*	*
-2.638	-1.55	*	*	1.40	-.076		.25	*	*
-2.533		*	*		-.070		.20	*	*
-2.228		*	*	1.35	-.059		.15	*	*
-2.002	-1.50	*	*		-.044		.10	*	*
-1.868		*	*	1.30	-.025		.05	*	*
-1.554		*	*	1.25	.000		.00	*	*
-1.535	-1.45	*	*		.014	-1.20		*	*
-1.282		*	*	1.20	.029		-.05	*	*
-1.126	-1.40	*	*		.064		-.10	*	*
-1.048		*	*	1.15	.103		-.15	*	*
- .848		*	*	1.10	.146		-.20	*	*
- .770	-1.35	*	*		.194	-1.15		*	*
- .680		*	*	1.05	.201		-.25	*	*
- .540		*	*	1.00	.244		-.30	*	*
- .464	-1.30	*	*		.296		-.35	*	*
- .425		*	*	.95	.340	-1.10		*	*
- .322		*	*	.90	.350		-.40	*	*
- .259		*	*	.85	.404		-.45	*	*
- .204	-1.25	*	*		.454	-1.05		*	*
- .202		*	*	.80	.458		-.50	*	*
- .159		*	*	.75	.508		-.55	*	*
- .128		*	*	.70	.540	-1.00		*	*
- .107		*	*	.65	.554		-.60	*	*
- .094		*	*	.60	.595		-.65	*	*
- .086		*	*	.55	.601	- .95		*	*
- .083		*	*	.51	.628		-.70	*	*
- .0825		*	*	.50	.640	- .90		*	*
- .0821		*	*	.49	.6511		-.75	*	*
- .0819		*	*	.48	.6595	- .85		*	*
- .0817		*	*	.47	.6544		-.76	*	*
- .08165		.38	.38		.6572		-.77	*	*
- .08164		.37			.6594		-.78	*	*
- .08163			.39		.6612		-.79	*	*
- .08160		.36	.40	.46	.6613	- .84		*	*
- .08156			.41		.6624		-.80	*	*
- .08152			.42		.6625	- .83		*	*
- .08151		.35		.45	.6630		-.81	*	*
- .08149			.43		.6631	- .82	-.82	*	*

*no root

TABLE I_1

n=-1 p=.55

q	z_1	z_2	q	z_1	z_2
-3.638		1.50	-.083		.30
-3.116		1.45	-.079		.25
-3.114	-1.60		-.071		.20
-2.652		1.40	-.065		.15
-2.517	-1.55		-.045		.10
-2.242		1.35	-.025		.05
-1.988	-1.50		.000		.00
-1.881		1.30	.026	-1.20	
-1.566		1.25	.030		-.05
-1.521	-1.45		.065		-.10
-1.294		1.20	.104		-.15
-1.112	-1.40		.148		-.20
-1.059		1.15	.196		-.25
- .859		1.10	.206	-1.15	
- .757	-1.35		.247		-.30
- .691		1.05	.230		-.35
- .550		1.00	.351	-1.10	
- .451	-1.30		.354		-.40
- .435		.95	.409		-.45
- .341		.90	.462		-.50
- .267		.85	.464	-1.05	
- .210		.80	.513		-.55
- .191	-1.25		.550	-1.00	
- .166		.75	.560		-.60
- .135		.70	.601		-.65
- .114		.65	.610	- .95	
- .010		.60	.635		-.70
- .092		.55	.649	- .90	
- .088		.50	.6586		-.75
- .0860		.45	.6620		-.76
- .0859		.44	.6649		-.77
- .0858		.43	.6656	- .86	
- .08572		.42	.6672		-.78
- .08566		.41	.6680	- .85	
- .08560		.40	.6691		-.79
- .08553		.39	.6697	- .84	
- .08555		.38	.6704		-.80
- .0853		.37	.6708	- .83	
- .0852		.36	.6711		-.81
- .0850		.35	.6713	- .82	-.82

TABLE I_j

n=-1 p=.60

q	z_1	z_2	q	z_1	z_2
-3.712		1.50	-.050		.10
-3.188		1.45	-.028		.05
-3.034	-1.60		.000		.00
-2.722		1.40	.032		-.05
-2.440	-1.55		.070		-.10
-2.309		1.35	.086	-1.20	
-1.946		1.30	.112		-.15
-1.912	-1.50		.158		-.20
-1.629		1.25	.209		-.25
-1.448	-1.45		.262		-.30
-1.354		1.20	.263	-1.15	
-1.117		1.15	.317		-.35
-1.042	-1.40		.374		-.40
- .914		1.10	.406	-1.10	
- .743		1.05	.431		-.45
- .689	-1.35		.488		-.50
- .600		1.00	.517	-1.05	
- .482		.95	.541		-.55
- .386	-1.30	.90	.590		-.60
- .310		.85	.600	-1.00	
- .250		.80	.634		-.65
- .204		.75	.658	- .95	
- .170		.70	.670		-.70
- .146		.65	.694	- .90	
- .130		.60	.696		-.75
- .129	-1.25		.699	- .89	
- .119		.55	.700		-.76
- .112		.50	.703	- .88	-.77
- .109		.45	.706	- .87	-.78
- .106		.40	.709	- .86	-.79
- .103		.35	.710	- .85	-.80
- .098		.30	.7116		-.81
- .091		.25	.7117	- .84	
- .082		.20	.71228		-.82
- .068		.15	.71232	- .83	-.83

TABLE I_k

$n=-1$ $p=.70$

q	z_1	z_2	q	z_1	z_2
-4.282	-1.70		-.083		.15
-3.862		1.50	-.060		.10
-3.535	-1.65		-.033		.05
-3.333		1.45	-.004	-1.25	
-2.874	-1.60		.000		.00
-2.862		1.40	.037		-.05
-2.444		1.35	.080		-.10
-2.285	-1.55		.127		-.15
-2.076		1.30	.178		-.20
-1.762	-1.50		.206	-1.20	
-1.754		1.25	.234		-.25
-1.474		1.20	.292		-.30
-1.303	-1.45		.352		-.35
-1.232		1.15	.378	-1.15	
-1.024		1.10	.414		-.40
- .902	-1.40		.476		-.45
- .848		1.05	.516	-1.10	
- .700		1.00	.538		-.50
- .577		.95	.596		-.55
- .554	-1.35		.622	-1.05	
- .476		.90	.650		-.60
- .395		.85	.699		-.65
- .330		.80	.700	-1.00	
- .279		.75	.740		-.70
- .256	-1.30		.753	- .95	
- .211		.65	.771		-.75
- .190		.60	.784	- .90	-.78
- .174		.55	.788	- .89	-.79
- .163		.50	.790		-.80
- .154		.45	.791	- .88	
- .146		.40	.793	- .87	-.81
- .138		.35	.794		-.82
- .128		.30	.7946	- .86	
- .116		.25	.7953		-.83
- .102		.20	.7957	- .84	-.84

TABLE I_1

$n=-1$ $p=.80$

q	z_1	z_2	q	z_1	z_2
-4.102	-1.70		-.070		.10
-4.012		1.50	-.038		.05
-3.478		1.45	.000		.00
-3.370	-1.65		.042		-.05
-3.002		1.40	.090		-.10
-2.714	-1.60		.121	-1.25	
-2.579		1.35	.142		-.15
-2.206		1.30	.198		-.20
-2.130	-1.55		.259		-.25
-1.879		1.25	.322		-.30
-1.612	-1.50		.326	-1.20	
-1.594		1.20	.387		-.35
-1.347		1.15	.454		-.40
-1.158	-1.45		.493	-1.15	
- .953		1.05	.521		-.45
- .800		1.00	.588		-.50
- .762	-1.40		.626	-1.10	
- .566		.90	.651		-.55
- .480		.85	.710		-.60
- .419	-1.35		.727	-1.05	
- .410		.80	.764		-.65
- .354		.75	.800	-1.00	
- .310		.70	.810		-.70
- .276		.65	.846		-.75
- .250		.60	.848	- .95	
- .229		.55	.870		-.80
- .212		.50	.874	- .90	-.81
- .199		.45	.876		-.82
- .186		.40	.877	- .89	
- .173		.35	.878		-.83
- .158		.30	.879	- .88	
- .141		.25	.880	- .87	
- .126	-1.30		.8804		-.84
- .122		.20	.8805		-.85
- .098		.15	.8806	- .86	-.86

TABLE I_m

$n=-1$ $p=.90$

q	z_1	z_2	q	z_1	z_2
-3.932	-1.70		.004	-1.30	
-3.623		1.45	.047		-.05
-3.205	-1.65		.100		-.10
-3.142		1.40	.157		-.15
-2.714		1.35	.218		-.20
-2.554	-1.60		.246	-1.25	
-2.336		1.30	.284		-.25
-2.004		1.25	.352		-.30
-1.975	-1.55		.422		-.35
-1.714		1.20	.446	-1.20	
-1.462	-1.50	1.15	.494		-.40
-1.244		1.10	.566		-.45
-1.058		1.05	.608	-1.15	
-1.013	-1.45		.638		-.50
- .900		1.00	.706		-.55
- .767		.95	.736	-1.10	
- .656		.90	.770		-.60
- .622	-1.40		.829		-.65
- .565		.85	.832	-1.05	
- .490		.80	.880		-.70
- .429		.75	.900	-1.00	
- .380		.70	.921		-.75
- .341		.65	.943	- .95	
- .310		.60	.949	- .94	
- .284	-1.35	.55	.950		-.80
- .262		.50	.954	- .93	
- .244		.45	.955		-.81
- .226		.40	.958	- .92	-.82
- .208		.35	.961	- .91	-.83
- .188		.30	.964	- .90	-.84
- .166		.25	.965		-.85
- .142		.20	.966	- .89	
- .113		.15	.9666		-.86
- .080		.10	.9667	- .88	
- .043		.05	.9670	- .87	-.87
.000		.00			

TABLE I_n

n=-1 p=1.00

q	z_1	z_2	q	z_1	z_2
-3.762	-1.70		-.048		.05
-3.282		1.40	.000		.00
-3.040	-1.65		.052		-.05
-2.849		1.35	.110		-.10
-2.466		1.30	.134	-1.30	
-2.394	-1.60		.172		-.15
-2.129		1.25	.238		-.20
-1.834		1.20	.309		-.25
-1.820	-1.55		.371	-1.25	
-1.577		1.15	.382		-.30
-1.354		1.10	.457		-.35
-1.312	-1.50		.534		-.40
-1.163		1.05	.566	-1.20	
-1.000		1.00	.611		-.45
- .868	-1.45		.688		-.50
- .862		.95	.724	-1.15	
- .746		.90	.761		-.55
- .650		.85	.830		-.60
- .570		.80	.846	-1.10	
- .504		.75	.894		-.65
- .482	-1.40		.937	-1.05	
- .450		.70	.950		-.70
- .406		.65	.996		-.75
- .370		.60	1.00	-1.00	
- .339		.55	1.030		-.80
- .312		.50	1.038	- .95	
- .289		.45	1.043	- .94	
- .266		.40	1.044		-.83
- .242		.35	1.047	- .93	
- .218		.30	1.048		-.84
- .191		.25	1.050	- .92	-.85
- .162		.20	1.052	- .91	
- .149	-1.35		1.053		-.86
- .128		.15	1.054	- .90	-.87
- .090		.10	1.05468	- .89	
			1.05470	- .88	-.88

TABLE I_o

$n = -1$ $p = 2.00$

q	z_1	z_2	q	z_1	z_2
-3.658	-1.80		.188	-1.50	
-3.034		1.20	.210		-.10
-2.454		1.10	.438		-.20
-2.062	-1.70		.682		-.30
-2.000		1.00	.918	-1.40	
-1.646		.90	.934		-.40
-1.370		.80	1.188		-.50
-1.150		.70	1.430		-.60
- .970		.60	1.434	-1.30	
- .812		.50	1.650		-.70
- .794	-1.60		1.766	-1.20	
- .666		.40	1.830		-.80
- .518		.30	1.946	-1.10	
- .362		.20	1.954		-.90
- .190		.10	2.000	-1.00	-1.00
.000		.00			

TABLE I_p

n=-1 p=6.00

q	z_1	z_2	q	z_1	z_2
-8.894	-2.30		1.882		- .30
-6.000		1.00	1.978	-1.90	
-5.246	-2.20		3.188		- .50
-4.570		.90	3.542	-1.80	
-4.570		.80	3.830		- .60
-3.950		.70	4.450		- .70
-3.370		.60	4.740	-1.70	
-2.812		.50	5.030		- .80
-2.438	-2.10		5.554		- .90
-2.266		.40	5.606	-1.60	
-1.718		.30	6.000		-1.00
-1.162		.20	6.188	-1.50	
- .590		.10	6.346		-1.10
.000	-2.00	.00	6.518	-1.40	
.610		-.10	6.566		-1.20
1.238		-.20	6.734	-1.30	-1.30

TABLE I_q

n=-1 p=10.00

q	z_1	z_2	q	z_1	z_2
-12.938	-2.60		4.134		- .40
-10.000		1.00	5.188		- .50
- 8.846		.90	5.962	-2.10	
- 7.812	-2.50		6.230		- .60
- 7.770		.80	7.250		- .70
- 6.750		.70	8.000	-2.00	
- 5.770		.60	8.230		- .80
- 4.812		.50	9.154		- .90
- 3.866		.40	9.578	-1.90	
- 3.418	-2.40		10.000		-1.00
- 2.918		.30	10.742	-1.80	
- 1.962		.20	10.746		-1.10
- .990		.10	11.366		-1.20
.000		.00	11.538	-1.70	
.306	-2.30		11.834		-1.30
1.010		-.10	12.006	-1.60	
2.038		-.20	12.118		-1.40
3.082		-.30	12.188	-1.50	-1.50
3.414	-2.20				

TABLE II_a

n=0 p=.10

q	z_1	z_2	q	z_1	z_2
-3.982		1.40	-.114	-.65	
-3.702	-1.40		-.112		.50
-3.457		1.35	-.086		.45
-3.187	-1.35		-.070	-.60	
-2.986		1.30	-.066		.40
-2.726	-1.30		-.050		.35
-2.566		1.25	-.038		.30
-2.316	-1.25		-.037	-.55	
-2.194		1.20	-.022		.20
-1.954	-1.20		-.016		.15
-1.864		1.15	-.012	-.50	
-1.634	-1.15		-.010		.10
-1.574		1.10	-.005		.05
-1.354	-1.10		.000		.00
-1.321		1.05	.004	-.45	
-1.111	-1.05		.005		-.05
-1.100		1.00	.014	-.40	-.15
- .910		.95	.018		-.20
- .900	-1.00		.019		-.21
- .746		.90	.0197		-.22
- .720	- .95		.0200	-.35	
- .607		.85	.0202		-.23
- .566	- .90		.0206	-.34	
- .490		.80	.0207		-.24
- .437	- .85		.0211	-.33	-.25
- .391		.75	.0214		-.26
- .330	- .80		.0215	-.32	
- .310		.70	.0217		-.27
- .244		.65	.0218	-.31	
- .241	- .75		.02185		-.28
- .190		.60	.02190	-.30	
- .170	- .70		.02193	-.29	-.29
- .147		.55			

TABLE II_b

n=0 p=.20

q	z_1	z_2	q	z_1	z_2
-3.592		1.35	-.162		.50
-3.562	-1.40		-.131		.45
-3.116		1.30	-.106		.40
-3.052	-1.35		-.100	-.70	
-2.691		1.25	-.085		.35
-2.596	-1.30		-.068		.30
-2.314		1.20	-.054		.25
-2.191	-1.25		-.049	-.65	
-1.979		1.15	-.042		.20
-1.834	-1.20		-.010		.15
-1.684		1.10	-.020		.10
-1.519	-1.15		-.010	-.60	.05
-1.426		1.05	.000		.00
-1.244	-1.10		.018	-.55	
-1.200		1.00	.020		-.10
-1.006	-1.05		.029		-.15
-1.005		.95	.038	-.50	-.20
- .900	-1.00		.046		-.25
- .836		.90	.049	-.45	
- .692		.85	.051	-.44	
- .625	- .95		.052	-.43	-.30
- .570		.80	.053	-.42	
- .476	- .90		.0535		-.32
- .466		.75	.0537	-.41	
- .380		.70	.0541		-.33
- .352	- .85		.0544	-.40	
- .309		.65	.0546		-.34
- .250	- .80	.60	.0549	-.39	
- .202		.55	.0550		-.35
- .166	- .75		.05515	-.38	
			.05520		-.36
			.05525	-.37	-.37

TABLE II_c

n=0 p=.30

q	z_1	z_2	q	z_1	z_2
-3.727		1.35	-.120		.35
-3.422	-1.40		-.098		.30
-3.246		1.30	-.091	-.75	
-2.917	-1.35		-.079		.25
-2.816		1.25	-.062		.20
-2.466	-1.30		-.046		.15
-2.434		1.20	-.030	-.70	.10
-2.094		1.15	-.015		.05
-2.066	-1.25		.000		.00
-1.794		1.10	.015		-.05
-1.714	-1.20		.016	-.65	
-1.531		1.05	.030		-.10
-1.404	-1.15		.044		-.15
-1.300		1.00	.050	-.60	
-1.134	-1.10		.058		-.20
-1.100		.95	.071		-.25
- .926		.90	.073	-.55	
- .911	-1.05		.082		-.30
- .777		.85	.088	-.50	
- .700	-1.00		.089	-.49	
- .650		.80	.090		-.35
- .541		.75	.091	-.48	-.36
- .530	- .95		.092	-.47	-.37
- .450		.70	.093	-.46	-.38
- .386	- .90		.0939		-.39
- .310		.60	.0940	-.45	
- .267	- .85		.0944		-.40
- .257		.55	.0945	-.44	
- .222		.50	.0947		-.41
- .176		.45	.0948	-.43	
- .170	- .80		.0949	-.42	-.42
- .146		.40			

TABLE II_d

n=0 p=.40

q	z_1	z_2	q	z_1	z_2
-3.376		1.30	-.128		.30
-3.282	-1.40		-.104		.25
-2.941		1.25	-.090	-.80	
-2.782	-1.35		-.082		.20
-2.554		1.20	-.061		.15
-2.336	-1.30		-.040		.10
-2.209		1.15	-.020		.05
-1.941	-1.25		-.016	-.75	
-1.904		1.10	.000		.00
-1.636		1.05	.020		-.05
-1.594	-1.20		.040	-.70	-.10
-1.400		1.00	.059		-.15
-1.289	-1.15		.078		-.20
-1.195		.95	.081	-.65	
-1.024	-1.10		.096		-.25
-1.016		.90	.110	-.60	
- .862		.85	.112		-.30
- .796	-1.05		.125		-.35
- .730		.80	.128	-.55	
- .616		.75	.134		-.40
- .600	-1.00		.136		-.41
- .520		.70	.137		-.42
- .439		.65	.1375	-.50	
- .435	- .95		.1378		-.43
- .370		.60	.1384	-.49	
- .312		.55	.1385		-.44
- .296	-.90		.1389	-.48	
- .221		.45	.1390		-.45
- .186		.40	.13920	-.47	
- .182	- .85		.13922	-.46	-.46
- .155		.35			

TABLE II_e

n=0 p=.50

q	z_1	z_2	q	z_1	z_2
-3.696	-1.45		-.158		.30
-3.506		1.30	-.129		.25
-3.142	-1.40		-.102		.20
-3.066		1.25	-.097	-.85	
-2.674		1.20	-.076		.15
-2.647	-1.35		-.050		.10
-2.324		1.15	-.025		.05
-2.206	-1.30		-.010	-.80	
-2.014		1.10	.000		.00
-1.816	-1.25		.025		-.05
-1.741		1.05	.050		-.10
-1.500		1.00	.059	-.75	
-1.474	-1.20		.075		-.15
-1.290		.95	.098		-.20
-1.174	-1.15		.110	-.70	
-1.106		.90	.121		-.25
- .947		.85	.142		-.30
- .914	-1.10		.146	-.65	
- .810		.80	.160		-.35
- .691	-1.05	.75	.170	-.60	
- .590		.70	.174		-.40
- .504		.65	.183	-.55	
- .500	-1.00		.184		-.45
- .430		.60	.185	-.54	-.46
- .367		.55	.186	-.53	-.47
- .340	- .95		.1869	-.52	-.48
- .312		.50	.1873	-.51	
- .266		.45	.1874		-.49
- .226		.40	.1875	-.50	-.50
- .206	- .90				
- .190		.35			

TABLE II$_f$

$n=0 \qquad p=.60$

q	z_1	z_2	q	z_1	z_2
-3.636		1.30	-.122		.20
-3.551	-1.45		-.116	-.90	
-3.191		1.25	-.091		.15
-3.002	-1.40		-.060		.10
-2.794		1.20	-.030		.05
-2.512	-1.35		-.012	-.85	
-2.439		1.15	.000		.00
-2.124		1.10	.030		-.05
-2.076	-1.30		.060		-.10
-1.846		1.05	.070	-.80	
-1.691	-1.25		.089		-.15
-1.600		1.00	.118		-.20
-1.385		1.95	.134	-.75	
-1.354	-1.20		.146		-.25
-1.196		.90	.172		-.30
-1.059	-1.15		.180	-.70	
-1.032		.85	.195		-.35
- .890		.80	.211	-.65	
- .804	-1.10		.214		-.40
- .766		.75	.229		-.45
- .660		.70	.230	-.60	
- .586	-1.05		.231		-.46
- .569		.65	.233	-.59	-.47
- .490		.60	.235	-.58	-.48
- .422		.55	.236	-.57	-.49
- .400	-1.00		.2375		-.50
- .362		.50	.2377	-.56	
- .311		.45	.2383		-.51
- .266		.40	.2385	-.55	
- .245	- .95		.2389		-.52
- .225		.35	.2390	-.54	
- .188		.30	.2391	-.53	-.53
- .154					

TABLE II_g

$n=0 \qquad p=.70$

q	z_1	z_2	q	z_1	z_2
-3.766		1.30	-.150	-.95	
-3.406	-1.45		-.142		.20
-3.316		1.25	-.106		.15
-2.914		1.20	-.070		.10
-2.862	-1.40		-.035		.05
-2.554		1.15	-.026	-.90	
-2.377	-1.35		.000		.00
-2.234		1.10	.035		-.05
-1.951		1.05	.070		-.10
-1.946	-1.30		.073	-.85	
-1.700		1.00	.104		-.15
-1.566	-1.25		.138		-.20
-1.480		.95	.150	-.80	
-1.234	-1.20		.171		-.25
-1.196		.90	.202		-.30
-1.117		.85	.209	-.75	
- .970		.80	.230		-.35
- .944	-1.15		.250	-.70	
- .831		.75	.254		-.40
- .730		.70	.274		-.45
- .694	-1.10		.276	-.65	
- .634		.65	.288		-.50
- .550		.60	.289		-.51
- .481	-1.05		.290	-.60	
- .477		.55	.291		-.52
- .412		.50	.292	-.59	-.53
- .356		.45	.293	-.58	-.54
- .306		.40	.2934	-.57	
- .300	-1.00		.2935		-.55
- .260		.35	.2937	-.56	-.56
- .218		.30			
- .179		.25			

TABLE II_h

n=0 p=.80

q	z_1	z_2	q	z_1	z_2
-3.896		1.30	-.121		.15
-3.862	-1.50		-.080		.10
-3.441		1.25	-.055	-.95	
-3.261	-1.45		-.040		.05
-3.044		1.20	.000		.00
-2.722	-1.40		.040		-.05
-2.669		1.15	.064	-.90	
-2.344		1.10	.080		-.10
-2.242	-1.35		.119		-.15
-2.056		1.05	.158	-.85	-.20
-1.816	-1.30		.196		-.25
-1.800		1.00	.230	-.80	
-1.575		.95	.232		-.30
-1.441	-1.25		.265		-.35
-1.376		.90	.284	-.75	
-1.202		.85	.294		-.40
-1.114	-1.20		.319		-.45
-1.050		.80	.320	-.70	
- .916		.75	.338		-.50
- .829	-1.15		.340		-.51
- .800		.70	.341	-.65	
- .699		.65	.343		-.52
- .610		.60	.344	-.64	
- .584	-1.10		.345		-.53
- .532		.55	.346	-.63	
- .462		.50	.347		-.54
- .401		.45	.348	-.62	-.55
- .376	-1.05		.3495	-.61	
- .295		.35	.3497		-.56
- .248		.30	.35040	-.60	
- .204		.25	.35044		-.57
- .200	-1.00		.35083	-.59	
- .162		.20	.35084	-.58	-.58

TABLE II_i

n=0 p=.90

q	z_1	z_2	q	z_1	z_2
-3.712	-1.50		-.136		.15
-3.566		1.25	-.100	-1.00	
-3.154		1.20	-.090		.10
-3.116	-1.45		-.045		.05
-2.784		1.15	.000		.00
-2.582	-1.40		.041	- .95	
-2.454		1.10	.045		-.05
-2.161		1.05	.090		-.10
-2.107	-1.35		.134		-.15
-1.900		1.00	.154	- .90	
-1.670		.95	.178		-.20
-1.686	-1.30		.221		-.25
-1.466		.90	.243	- .85	
-1.316	-1.25		.262		-.30
-1.287		.85	.300		-.35
-1.130		.80	.310	- .80	
- .994	-1.20		.334		-.40
- .991		.75	.359	- .75	
- .870		.70	.364		-.45
- .764		.65	.388		-.50
- .714	-1.15		.390	- .70	
- .670		.60	.401	- .67	
- .587		.55	.403		-.55
- .522		.50	.404	- .66	
- .474	-1.10		.406	- .65	-.56
- .446		.45	.407		-.57
- .386		.40	.408	- .64	
- .330		.35	.409	- .63	-.58
- .278		.30	.4098		-.59
- .271	-1.05		.4102	- .62	
- .229		.25	.4104		-.60
- .182		.20	.4105	- .61	-.61

TABLE II_j

n=0 p=1.00

q	z_1	z_2	q	z_1	z_2
-3.691		1.25	-.151		.15
-3.562	-1.50		-.100		.10
-3.274		1.20	-.050		.05
-2.971	-1.45		.000	-1.00	.00
-2.899		1.15	.050		-.05
-2.564		1.10	.100		-.10
-2.442	-1.40		.135	- .95	
-2.266		1.05	.149		-.15
-2.000		1.00	.198		-.20
-1.972	-1.35		.244	- .90	
-1.765		.95	.246		-.25
-1.556	-1.30	.90	.292		-.30
-1.372		.85	.328	- .85	
-1.210		.80	.335		-.35
-1.191	-1.25		.374		-.40
-1.066		.75	.390	- .80	
- .940		.70	.409		-.45
- .874	-1.20		.434	- .75	
- .829		.65	.438		-.50
- .730		.60	.458		-.55
- .642		.55	.460	- .70	
- .599	-1.15		.462		-.56
- .562		.50	.463	- .69	
- .491		.45	.464		-.57
- .426		.40	.466	- .68	
- .365		.35	.467		-.58
- .364	-1.10		.468	- .67	
- .308		.30	.469		-.59
- .254		.25	.470	- .66	-.60
- .166	-1.05		.4715	- .65	-.61
- .254		.25	.4722	- .64	-.62
- .202		.20	.4725	- .63	-.63

TABLE II_k

n=0 p=2.00

q	z_1	z_2	q	z_1	z_2
-3.664		1.10	- .200		.10
-3.354	-1.60		.000		.00
-3.000		1.00	.200		-.10
-2.456		.90	.326	-1.20	
-2.110		.80	.398		-.20
-2.062	-1.50		.592		-.30
-1.640		.70	.736	-1.10	
-1.330		.60	.774		-.40
-1.062		.50	.938		-.40
-1.042	-1.40		1.000	-1.00	
- .826		.40	1.070		-.60
- .608		.30	1.144	- .90	
- .402		.20	1.160		-.70
- .256	-1.30		1.190	- .80	-.80

TABLE II_m

n=0 p=10.00

q	z_1	z_2	q	z_1	z_2
-14.062	-2.50		2.992		- .30
-11.000		1.00	3.974		- .40
- 9.656		.90	4.000	-2.00	
- 9.178	-2.40		4.938		- .50
- 8.410		.80	5.870		- .60
- 7.250		.70	5.968	-1.90	
- 6.130		.60	6.760		- .70
- 5.062		.50	7.502	-1.80	
- 4.984	-2.30		7.590		- .80
- 4.026		.40	8.344		- .90
- 3.008		.30	8.648	-1.70	
- 2.002		.20	9.000		-1.00
- 1.426	-2.20		9.446	-1.60	
- 1.000		.10	9.536		-1.10
.000		.00	9.926		-1.20
1.000		-.10	9.938	-1.50	
1.552	-2.10		10.144		-1.30
1.998		-.20	10.158	-1.40	-1.40

TABLE III_a

n=1 p=.00

q	z_1	z_2	q	z_1	z_2
-3.514	-1.20	1.20	-.138	-.35	.35
-3.072	-1.15	1.15	-.098	-.30	.30
-2.674	-1.10	1.10	-.066	-.25	.25
-2.000	-1.00	1.00	-.042	-.20	.20
-1.717	- .95	.95	-.023	-.15	.15
-1.466	- .90	.90	-.010	-.10	.10
-1.245	- .85	.85	-.008	-.09	.09
-1.050	- .80	.80	-.006	-.08	.08
- .879	- .75	.75	-.005	-.07	.07
- .730	- .70	.70	-.004	-.06	.06
- .601	- .65	.65	-.003	-.05	.05
- .490	- .60	.60	-.002	-.04	.04
- .394	- .55	.55	-.0009	-.03	.03
- .312	- .50	.50	-.0004	-.02	.02
- .244	- .45	.45	-.0001	-.01	.01
- .187	- .40	.40	.0000	.00	.00

TABLE III_b

n=1 p=.10

q	z_1	z_2	q	z_1	z_2
-3.394	-1.20		-.262	-.50	
-3.634		1.20	-.226		.40
-3.187		1.15	-.199	-.45	
-2.957	-1.15		-.173		.35
-2.894		1.10	-.146	-.40	
-2.454	-1.10		-.128		.30
-2.423		1.05	-.103	-.35	
-2.213	-1.05		-.091		.25
-2.100		1.00	-.068	-.30	
-1.900	-1.00		-.062		.20
-1.812		.95	-.042	-.25	
-1.622	- .95		-.038		.15
-1.556		.90	-.025	-.20	
-1.376	- .90		-.020		.10
-1.330		.85	-.008	-.15	.05
-1.160	- .85		-.006	-.14	.04
-1.130		.80	-.004	-.13	.03
- .970	- .80		-.003	-.12	
- .954		.75	-.002		.02
- .804	- .75		-.001	-.11	.01
- .800		.70	-.0001	-.10	
- .666		.65	.0000		.00
- .660	- .70		.0009		-.01
- .550		.60	.0008	-.09	
- .536	- .65		.0016	-.08	-.02
- .449		.55	.0021	-.07	-.03
- .430	- .60		.0024	-.06	-.04
- .362		.50	.0025	-.05	-.05
- .339	- .55				
- .289		.45			

TABLE III$_c$

n=1 p=.20

q	z_1	z_2	q	z_1	z_2
-3.754		1.20	-.208		.35
-3.302		1.15	-.158		.30
-3.274	-1.20		-.154	-.45	
-2.894		1.10	-.116		.25
-2.842	-1.15		-.106	-.40	
-2.528		1.05	-.082		.20
-2.454	-1.10		-.068	-.35	
-2.200		1.00	-.053		.15
-2.108	-1.05		-.038	-.30	
-1.907		.95	-.030		.10
-1.800	-1.00		-.016	-.25	
-1.646		.90	-.013		.05
-1.527	- .95		-.002	-.20	
-1.415		.85	.000		.00
-1.286	- .90		.0006	-.19	
-1.210		.80	.0019		-.01
-1.075	- .85		.0032	-.18	
-1.029		.75	.0036		-.02
- .890	- .80		.0043	-.17	
- .870		.70	.0051		-.03
- .731		.65	.0059	-.16	
- .729	- .75		.0064		-.04
- .610		.60	.0070	-.15	
- .590	- .70		.0075		-.05
- .504		.55	.0080	-.14	
- .471	- .65		.0084		-.06
- .412		.50	.0088	-.13	
- .370	- .60		.0091		-.07
- .334		.45	.0094	-.12	
- .284	- .55		.0096		-.08
- .266		.40	.0098	-.11	-.09
- .212	- .50		.0099	-.10	-.10

TABLE III_d

$n=1 \qquad p=.30$

q	z_1	z_2	q	z_1	z_2
-3.874		1.20	-.243		.35
-3.629	-1.25		-.162	-.50	
-3.417		1.15	-.141		.25
-3.154	-1.20		-.109	-.45	
-3.004		1.10	-.102		.20
-2.727	-1.15		-.068		.15
-2.633		1.05	-.066	-.40	
-2.344	-1.10		-.040		.10
-2.300		1.00	-.033	-.35	
-2.003	-1.05		-.018		.05
-2.002		.95	-.008	-.30	
-1.736		.90	.000		.00
-1.700	-1.00		.009	-.25	
-1.500		.85	.011	-.24	
-1.432	- .95		.012		-.05
-1.290		.80	.013	-.23	
-1.196	- .90		.014		-.06
-1.104		.75	.015	-.22	
- .990	- .85		.016		-.07
- .940		.70	.017	-.21	
- .810	- .80		.018	-.20	-.08
- .796		.65	.019		-.09
- .670		.60	.020	-.19	-.10
- .654	- .75		.0206	-.18	
- .559		.55	.0208		-.11
- .520	- .70		.0213	-.17	
- .462		.50	.0214		-.12
- .406	- .65		.0217	-.16	
- .379		.45	.0218		-.13
- .310	- .60		.02199	-.15	
- .306		.40	.02202	-.14	-.14
- .229	- .55				

TABLE III$_e$

n=1 p=.40

q	z_1	z_2	q	z_1	z_2
-3.994		1.20	-.278		.35
-3.532		1.15	-.250	-.60	
-3.504	-1.25		-.218		.30
-3.214		1.10	-.174	-.55	
-3.034	-1.20		-.166		.25
-2.738		1.05	-.122		.20
-2.612	-1.15		-.112	-.50	
-2.400		1.00	-.083		.15
-2.234	-1.10		-.064	-.45	
-2.097		.95	-.050		.10
-1.898	-1.05		-.026	-.40	
-1.826		.90	-.023		.05
-1.600	-1.00		.000		.00
-1.585		.85	.002	-.35	
-1.370		.80	.017		-.05
-1.337	- .95		.022	-.30	
-1.179		.75	.030		-.10
-1.106	- .90		.032		-.11
-1.010		.70	.033		-.12
- .905	- .85		.034	-.25	
- .861		.65	.035	-.24	-.13
- .730	- .80	.60	.036	-.23	-.14
- .614		.55	.037	-.22	-.15
- .579	- .75		.0377		-.16
- .512		.50	.0380	-.21	
- .450	- .70		.0383		-.17
- .424		.45	.03840	-.20	
- .346		.40	.03855		-.18
- .341	- .65		.03860	-.19	-.19

TABLE III_f

n=1 p=.50

q	z_1	z_2	q	z_1	z_2
-3.896	-1.30		-.313		.35
-3.647		1.15	-.276	-.65	
-3.379	-1.25		-.248		.30
-3.224		1.10	-.191		.25
-2.914	-1.20		-.190	-.60	
-2.843		1.05	-.142		.20
-2.500		1.00	-.119	-.55	
-2.497	-1.15		-.098		.15
-2.192		.95	-.063	-.50	
-2.124	-1.10		-.060		.10
-1.916		.90	-.028		.05
-1.793	-1.05		-.019	-.45	
-1.670		.85	.000		.00
-1.500	-1.00		.014	-.40	
-1.450		.80	.022		-.05
-1.254		.75	.037	-.35	
-1.242	- .95		.040		-.10
-1.080		.70	.052	-.30	-.15
-1.016	- .90		.054	-.29	-.16
- .926		.65	.055	-.28	-.17
- .820	- .85		.057	-.27	-.18
- .790		.60	.058	-.26	-.19
- .669		.55	.0584		-.20
- .610	- .80		.0586	-.25	
- .562		.50	.0590		-.21
- .504	- .75		.05908	-.24	
- .469		.45	.05926		-.22
- .386		.40	.05930	-.23	-.23
- .380	- .70				

TABLE III_g

n=1 p=.60

q	z_1	z_2	q	z_1	z_2
-3.766	-1.30		-.216		.25
-3.762		1.15	-.211	-.65	
-3.334		1.10	-.162		.20
-3.257	-1.25		-.130	-.60	
-2.948		1.05	-.113		.15
-2.794	-1.20		-.070		.10
-2.600		1.00	-.064	-.55	
-2.382	-1.15		-.033		.05
-2.287		.95	-.013	-.50	
-2.014	-1.10		.000		.00
-2.006		.90	.026	-.45	
-1.755		.85	.027		-.05
-1.688	-1.05		.050		-.10
-1.530		.80	.054	-.40	
-1.400	-1.00		.070		-.15
-1.329		.75	.072	-.35	
-1.150		.70	.075	-.34	
-1.147	- .95		.077	-.33	
- .991		.65	.078		-.20
- .926	- .90		.079	-.32	
- .850		.60	.080		-.21
- .735	- .85		.081	-.31	-.22
- .724		.55	.0819	-.30	
- .612		.50	.0823		-.23
- .570	- .80		.0828	-.29	
- .514		.45	.0831		-.24
- .429	- .75		.0835	-.28	
- .426		.40	.0836		-.25
- .348		.35	.08379	-.27	
- .310	- .70		.08383	-.26	-.26
- .278		.30			

TABLE III_h

n=1 p=.70

q	z_1	z_2	q	z_1	z_2
-3.636	-1.30		-.308		.30
-3.444		1.10	-.241		.25
-3.129	-1.25		-.240	-.70	
-3.053		1.05	-.182		.20
-2.700		1.00	-.146	-.65	
-2.674	-1.20		-.128		.15
-2.382		.95	-.080		.10
-2.267	-1.15		-.070	-.60	
-2.096		.90	-.038		.05
-1.904	-1.10		-.009	-.55	
-1.840		.85	.000		.00
-1.610		.80	.032		-.05
-1.583	-1.05		.038	-.50	
01.404		.75	.060		-.10
-1.300	-1.00		.071	-.45	
-1.220		.70	.082		-.15
-1.056		.65	.094	-.40	
-1.052	- .95		.098		-.20
- .910		.60	.107	-.35	
- .836	- .90		.109	-.34	-.25
- .779		.55	.110	-.33	-.26
- .662		.50	.111	-.32	-.27
- .650	- .85		.1111	-.32	
- .559		.45	.1115		-.28
- .490	- .80		.1117	-.31	
- .466		.40	.1118		-.29
- .383		.35	.1119	-.30	-.30
- .354	- .75				

TABLE III_1

n=1 p=.80

q	z_1	z_2	q	z_1	z_2
-3.606	-1.30		-.266		.25
-3.554		1.10	-.202		.20
-3.158		1.05	-.170	-.70	
-3.004	-1.25		-.143		.15
-2.800		1.00	-.090		.10
-2.554	-1.20		-.081	-.65	
-2.477		.95	-.043		.05
-2.186		.90	-.010	-.60	
-2.142	-1.15		-.004	-.55	
-1.925		.85	.000		.00
-1.794	-1.10		.037		-.05
-1.790		.80	.070		-.10
-1.479		.75	.088	-.50	
-1.478	-1.05		.097		-.15
-1.290		.70	.116	-.45	
-1.200	-1.00		.118		-.20
-1.121		.65	.134	-.40	-.25
- .970		.60	.136		-.26
- .957	- .95		.137	-.39	
- .784		.55	.138		-.27
- .712		.50	.139	-.38	-.28
- .604		.45	.140	-.37	
- .565	- .85		.141		-.29
- .506		.40	.142	-.36	-.30
- .418		.35	.1425	-.35	
- .410	- .80		.1427		-.31
- .338		.30	.1430	-.34	
- .279	- .75		.1432	-.33	-.33

TABLE III_j

$n=1$ $p=.90$

q	z_1	z_2	q	z_1	z_2
-3.664		1.10	-.368		.30
-3.376	-1.30		-.330	-.80	
-3.263		1.05	-.291		.25
-2.900		1.00	-.222		.20
-2.879	-1.25		-.204	-.75	
-2.572		.95	-.158		.15
-2.434	-1.20		-.100	-.70	.10
-2.276		.90	-.048		.05
-2.037	-1.15		-.016	-.65	
-2.010		.85	.000		.00
-1.770		.80	.042		-.05
-1.684	-1.10		.050	-.60	
-1.554		.75	.080		-.10
-1.373	-1.05		.101	-.55	
-1.360		.70	.112		-.15
-1.186		.65	.138	-.50	-.20
-1.100	-1.00		.159		-.25
-1.030		.60	.161	-.45	
- .889		.55	.172		-.30
- .862	- .95		.174	-.40	-.31
- .762		.50	.175		-.32
- .656	- .90		.176	-.39	-.33
- .649		.45	.177	-.38	-.34
- .546		.40	.1774	-.37	
- .480	-.85		.1775		-.35
- .453		.35	.1776	-.36	-.36

TABLE III_k

n=1 p=1.00

q	z_1	z_2	q	z_1	z_2
-3.794	-1.35		-.316		.25
-3.774		1.10	-.250	-.80	
-3.368		1.05	-.241		.20
-3.246	-1.30		-.173		.15
-3.000		1.00	-.129	-.75	
-2.754	-1.25		-.110		.10
-2.667		.95	-.053		.05
-2.366		.90	-.303	-.70	
-2.314	-1.20		.000		.00
-2.095		.85	.047		-.05
-1.922	-1.15		.049	-.65	
-1.850		.80	.090		-.10
-1.629		.75	.110	-.60	
-1.574	-1.10		.127		-.15
-1.430		.70	.156	-.55	
-1.268	-1.05		.158		-.20
-1.251		.65	.184		-.25
-1.090		.60	.188	-.50	
-1.000	-1.00		.202		-.30
- .944		.55	.206	-.45	
- .812		.50	.209	-.44	
- .767	- .95		.211	-.43	
- .694		.45	.212	-.42	-.35
- .586		.40	.214	-.41	-.36
- .566	- .90		.2144	-.40	-.37
- .488		.35	.2147		-.38
- .398		.30	.2148	-.39	-.39
- .395	- .85				

TABLE III_1

n=1 p=2.00

q	z_1	z_2	q	z_1	z_2
-4.312	-1.50		-.442		.20
-4.000		1.00	.000	-1.00	.00
-3.002	-1.40		.190		-.10
-2.650		.80	.334	- .90	
-2.130		.70	.358		-.20
-1.946	-1.30		.502		-.30
-1.690		.60	.550	- .80	
-1.312		.50	.614		-.40
-1.114	-1.20		.670	- .70	
- .986		.40	.688		-.50
- .698		.30	.710	- .60	-.60
- .474	-1.10				

TABLE III_n

$n=1$ $p=10.00$

q	z_1	z_2	q	z_1	z_2
-20.312	-2.50		2.902		- .30
-14.938	-2.40		3.814		- .40
-12.000		1.00	4.262	-1.80	
-10.466		.90	4.688		- .50
-10.274	-2.30		5.510		- .60
- 9.050		.80	5.758	-1.70	
- 7.730		.70	6.270		- .70
- 6.266	-2.20		6.886	-1.60	
- 6.130		.60	6.950		- .80
- 5.312		.50	7.534		- .90
- 4.186		.40	7.688	-1.50	
- 3.098		.30	8.000		-1.00
- 2.858	-2.10		8.198	-1.40	
- 2.042		.20	8.326		-1.10
- 1.010		.10	8.454	-1.30	
.000	-2.00	.00	8.486	-1.20	-1.20
.990		-.10			
1.958		-.20			
2.358	-1.90				

TABLE IV

SQUARES AND FOURTH POWERS

n	n^2	n^4	n	n^2	n^4
1	1	1	46	2 116	4 477 456
2	4	16	47	2 209	4 879 681
3	9	81	48	2 304	5 308 416
4	16	256	49	2 401	5 764 801
5	25	625	50	2 500	6 250 000
6	36	1 296	51	2 601	6 765 201
7	49	2 401	52	2 704	7 311 616
8	64	4 096	53	2 809	7 890 481
9	81	6 561	54	2 916	8 503 056
10	100	10 000	55	3 025	9 150 625
11	121	14 641	56	3 136	9 834 496
12	144	20 736	57	3 249	10 556 001
13	169	28 561	58	3 364	11 316 496
14	196	38 416	59	3 481	12 117 361
15	225	50 625	60	3 600	12 960 000
16	256	65 536	61	3 721	13 845 841
17	289	83 521	62	3 844	14 776 336
18	324	104 976	63	3 969	15 752 961
19	361	130 321	64	4 096	16 777 216
20	400	160 000	65	4 225	17 850 625
21	441	194 481	66	4 356	18 974 736
22	484	234 256	67	4 489	20 151 121
23	529	279 841	68	4 624	21 381 376
24	576	331 776	69	4 761	22 667 121
25	625	390 625	70	4 900	24 010 000
26	676	456 976	71	5 041	25 411 681
27	729	531 441	72	5 184	26 873 856
28	784	614 656	73	5 329	28 398 241
29	841	707 281	74	5 476	29 986 576
30	900	810 000	75	5 625	31 640 625
31	961	923 521	76	5 776	33 362 176
32	1 024	1 048 576	77	5 929	35 153 041
33	1 089	1 185 921	78	6 084	37 015 056
34	1 156	1 336 336	79	6 241	38 950 081
35	1 225	1 500 625	80	6 400	40 960 000
36	1 296	1 679 616	81	6 561	43 046 721
37	1 369	1 874 161	82	6 724	45 212 176
38	1 444	2 085 136	83	6 889	47 458 321
39	1 521	2 313 441	84	7 056	49 787 136
40	1 600	2 560 000	85	7 225	52 200 625
41	1 681	2 825 761	86	7 396	54 700 816
42	1 764	3 111 696	87	7 569	57 289 761
43	1 849	3 418 801	88	7 744	59 969 536
44	1 936	3 748 096	89	7 921	62 742 241
45	2 025	4 100 625	90	8 100	65 610 000

TABLE IV

n	n^2	n^4	n	n^2	n^4
91	8 281	68 574 961	136	18 496	342 102 016
92	8 464	71 639 296	137	18 769	352 275 361
93	8 649	74 805 201	138	19 044	362 673 936
94	8 836	78 074 896	139	19 321	373 301 041
95	9 025	81 450 625	140	19 600	384 160 000
96	9 216	84 934 656	141	19 881	395 254 161
97	9 409	88 529 281	142	20 164	406 586 896
98	9 604	92 236 816	143	20 449	418 161 601
99	9 801	96 059 601	144	20 736	429 981 696
100	10 000	100 000 000	145	21 025	442 050 625
101	10 201	104 060 401	146	21 316	454 371 856
102	10 404	108 243 216	147	21 609	466 948 881
103	10 609	112 550 881	148	21 904	479 785 216
104	10 816	116 985 856	149	22 201	492 884 401
105	11 025	121 550 625	150	22 500	506 250 000
106	11 236	126 247 696	151	22 801	519 885 601
107	11 449	131 079 601	152	23 104	533 794 816
108	11 664	136 048 896	153	23 409	547 981 281
109	11 881	141 158 161	154	23 716	562 448 656
110	12 100	146 410 000	155	24 025	577 200 625
111	12 321	151 807 041	156	24 336	592 240 896
112	12 544	157 351 936	157	24 649	607 573 201
113	12 769	163 047 361	158	24 964	623 201 296
114	12 996	168 896 016	159	25 281	639 128 961
115	13 225	174 900 625	160	25 600	655 360 000
116	13 456	181 063 936	161	25 921	671 898 241
117	13 689	187 388 721	162	26 244	688 747 536
118	13 924	193 877 776	163	26 569	705 911 761
119	14 161	200 533 921	164	26 896	723 394 816
120	14 400	207 360 000	165	27 225	741 200 625
121	14 641	214 358 881	166	27 556	759 333 136
122	14 884	221 533 456	167	27 889	777 796 321
123	15 129	228 886 641	168	28 224	796 594 176
124	15 376	236 421 376	169	28 561	815 730 721
125	15 625	244 140 625	170	28 900	835 210 000
126	15 876	252 047 376	171	29 241	855 036 081
127	16 129	260 144 641	172	29 584	875 213 056
128	16 384	268 435 456	173	29 929	895 745 041
129	16 641	276 922 881	174	30 276	916 636 176
130	16 900	285 610 000	175	30 625	937 890 625
131	17 161	294 499 921	176	30 976	959 512 576
132	17 424	303 595 776	177	31 329	981 506 241
133	17 689	312 900 721	178	31 684	1 003 875 856
134	17 956	322 417 936	179	32 041	1 026 625 681
135	18 225	332 150 625	180	32 400	1 049 760 000

TABLE IV

n	n^2	n^4	n	n^2	n^4
181	32 761	1 073 283 121	226	51 076	2 608 757 776
182	33 124	1 097 199 376	227	51 529	2 655 237 841
183	33 489	1 121 513 121	228	51 984	2 702 336 256
184	33 856	1 146 228 736	229	52 441	2 750 058 481
185	34 225	1 171 350 625	230	52 900	2 798 410 000
186	34 596	1 196 883 216	231	53 361	2 847 396 321
187	34 969	1 222 830 961	232	53 824	2 897 022 976
188	35 344	1 249 198 336	233	54 289	2 947 295 521
189	35 721	1 275 989 841	234	54 756	2 998 219 536
190	36 100	1 303 210 000	235	55 225	3 049 800 625
191	36 481	1 330 863 361	236	55 696	3 102 044 416
192	36 864	1 358 954 496	237	56 169	3 154 956 561
193	37 249	1 387 488 001	238	56 644	3 208 542 736
194	37 636	1 416 468 496	239	57 121	3 262 808 641
195	38 025	1 445 900 625	240	57 600	3 317 760 000
196	38 416	1 475 789 056	241	58 081	3 373 402 561
197	38 809	1 506 138 481	242	58 564	3 429 742 096
198	39 204	1 536 953 616	243	59 049	3 486 784 401
199	39 601	1 568 239 201	244	59 536	3 544 535 296
200	40 000	1 600 000 000	245	60 025	3 603 000 625
201	40 401	1 632 240 801	246	60 516	3 662 186 256
202	40 804	1 664 966 416	247	61 009	3 722 098 081
203	41 209	1 698 818 681	248	61 504	3 782 742 016
204	41 616	1 731 891 456	249	62 001	3 844 124 001
205	42 025	1 766 100 625	250	62 500	3 906 250 000
206	42 436	1 800 814 096	251	63 001	3 969 126 001
207	42 849	1 836 036 801	252	63 504	4 032 758 016
208	43 264	1 871 773 696	253	64 009	4 097 152 081
209	43 681	1 908 029 761	254	64 516	4 162 314 256
210	44 100	1 944 810 000	255	65 025	4 228 250 625
211	44 521	1 982 119 441	256	65 536	4 294 967 296
212	44 944	2 019 963 136	257	66 049	4 362 470 401
213	45 369	2 058 346 161	258	66 564	4 430 766 096
214	45 796	2 097 273 616	259	67 081	4 499 860 561
215	46 225	2 136 750 625	260	67 600	4 569 760 000
216	46 656	2 176 782 336	261	68 121	4 640 470 641
217	47 089	2 217 373 921	262	68 644	4 711 998 736
218	47 524	2 258 530 576	263	69 169	4 784 350 561
219	47 961	2 300 257 521	264	69 696	4 857 532 416
220	48 400	2 342 560 000	265	70 225	4 931 550 625
221	48 841	2 385 443 281	266	70 756	5 006 411 536
222	49 284	2 428 912 656	267	71 289	5 082 121 521
223	49 729	2 472 973 441	268	71 824	5 158 686 976
224	50 176	2 517 630 976	269	72 361	5 236 114 321
225	50 625	2 562 890 625	270	72 900	5 314 410 000

TABLE IV

n	n^2	n^4
271	73 441	5 393 580 481
272	73 984	5 473 632 256
273	74 529	5 554 571 841
274	75 076	5 636 405 776
275	75 625	5 719 140 625
276	76 176	5 802 782 976
277	76 729	5 887 339 441
278	77 284	5 972 816 656
279	77 841	6 059 221 281
280	78 400	6 146 560 000
281	78 961	6 234 839 521
282	79 524	6 324 066 576
283	80 089	6 414 247 921
284	80 656	6 505 390 336
285	81 225	6 597 500 624
286	81 796	6 690 585 616
287	82 369	6 784 652 161
288	82 944	6 879 707 136
289	83 521	6 975 757 411
290	84 100	7 072 810 000
291	84 681	7 170 871 761
292	85 264	7 269 949 696
293	85 849	7 370 050 801
294	86 436	7 471 182 096
295	87 025	7 573 350 625
296	87 616	7 676 563 456
297	88 209	7 780 827 681
298	88 804	7 886 150 416
299	89 401	7 992 538 801
300	90 000	8 100 000 000

BIBLIOGRAPHY

BIBLIOGRAPHY

Ball, W. W. Rouse, A Short Account of the History of Mathematics. London: The Macmillan Company, 1935.

______, A Short Account of the History of Mathematics. London: The Macmillan Company, 1915.

Bell, E. T., The Development of Mathematics. New York: McGraw-Hill Book Company Inc., 1940.

Cajori, Florian, A History of Mathematics. New York: The Macmillan Company, 1924. 2nd ed.

Conkwright, Nelson Busk, Introduction to the Theory of Equations. Boston: Ginn and Company, 1941.

Crumley, George L., "Solution of Quadratic and Cubic Equations by Series Reversion." Unpublished Master's thesis, Kansas State Teachers College of Emporia, Emporia, 1953.

Dickson, Leonard Eugene, Elementary Theory of Equations. New York: John Wiley and Sons, Inc., 1914. 1st ed.

______, First Course in the Theory of Equations. New York: John Wiley and Sons, Inc., 1922.

Eves, Howard, An Introduction to the History of Mathematics. New York: Rinehart and Company, Inc., 1953.

Smith, David Eugene, History of Mathematics. Boston: Ginn and Company, 1925. Vol. II.

Todhunter, I., An Elementary Treatise on the Theory of Equations. London: The Macmillan Company, 1882. 5th ed.

Uspensky, J. V., Theory of Equations. New York: McGraw-Hill Book Company Inc., 1948.