THE HISTORY AND CALCULATION OF PI

A THESIS

SUBMITTED TO THE DEPARTMENT OF

MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE

TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

by

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August 1959

Approved for the Major Department

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[illegible]

ACKNOWLEDGMENT

To Dr. Oscar J. Peterson, Professor of Mathematics of the Kansas State Teachers College of Emporia, who so willingly directed this thesis, the writer is indebted and wishes to express his sincere gratitude. The writer also wishes to acknowledge his indebtedness to Dr. John Burger, Head of the Department of Mathematics, and Mr. Lester Laird, Professor of Mathematics, for their helpful suggestions on the thesis.

Herman H. Harris Jr.

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CHAPTER I

INTRODUCTION

1.1. Introduction. The number pi occupies a unique place in the history of mathematics. Whether defined as the ratio of the circumerference of a circle to the diameter or as the ratio of the area of the circle to the square on half the diameter, it has been the object of intensive study by peoples of all nations from the earliest times to the present. The number pi has wound itself through the structure of mathematics and woven itself into the fabric of our civilization.

The values of pi used by the ancients were estimates based upon observation rather than calculation. The Hebrews, Egyptians, and Babylonians used 3 as an estimated value of pi. Later, more accurate estimates were adopted. Following Archimedes, about 240 B.C., calculations were used in the determination of pi, and more recently pi has been calculated to 10,000 decimal places.

1.2. Statement of the problem. The purpose of this study is to present, in some detail, the historical background of the number pi, the theoretical aspects for the calculations of pi, and the actual calculations themselves.

1.3. Importance of the study. While teachers, students, and laymen recognize the importance of pi and some of its applications, there is generally vagueness as to its historical background and uncertainty as to the methods of its calculation. Such information is usually obtainable in scattered or inaccessible sources, and it is thought that the present thesis will prove to be of considerable value as a reference and source of information regarding the number pi, its history, and its calculation.

1.4. Sources of information. In getting the material for this thesis the writer has used books, encyclopedias, and magazines as sources of information. Of particular assistance was the article by H. C. Schepler, "The Chronology of Pi," in the Mathematics Magazine.

1.5. Organization. The thesis is divided into eight chapters, each of which is devoted to some aspect of the study which is of major interest. The second chapter presents the history of pi. The third chapter deals with the calculations of pi by geometric methods, whereas the fourth chapter deals with the calculations of pi by infinite products and infinite series. The fifth chapter deals with the determination of pi by experimental methods. The sixth chapter deals with the irrationality and transcendence of the number pi. The seventh chapter presents the place of pi in formulas, and the eighth chapter summarizes the material of the thesis.

CHAPTER II

THE HISTORY OF PI

2.1. Earliest times. The computation of pi is closely connected with one of the three famous problems, the quadrature of the circle.[1] The earliest approximation of pi was 3 being used by the Hebrews, Egyptians, and Babylonians. This assumption that pi was equal to 3 was current for many centuries; it is implied in the Old Testament, 1 Kings 7:23, and in 2 Chronicles 4:2, where the following statement occurs:

> Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.[2]

The earliest traces of pi are to be found in the Rhind Papyrus[3] which is preserved in the British Museum and was translated and explained by Eisenlohr.

> It is there stated that the area of a circle is equal to that of a square whose side is the diameter diminished by one ninth.

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[1] Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, Inc., 1953), p. 90.

[2] E. W. Hobson, Squaring the Circle (New York: Chelsea Publishing Company, 1953), p. 13.

[3] Ibid.

This will give an approximation for pi as 3.1604..., which is greater than 3.1416 by about 0.6%.

2.2. Geometrical method. The geometrical method in computing pi was used by many individuals; some obtained closer approximations than others. A method used in computing pi was inscribing in a circle and circumscribing about it regular polygons. By doubling in succession the number of sides and finding the perimeters or areas of the polygons, the individuals were able to calculate an approximation. If the process was performed a great number of times, a closer approximation was obtained. Some of the individuals that computed pi by inscribing and circumscribing regular polygons were Archimedes, Francois Vieta, Adriaen Van Roomen, Ludolph Van Ceulen, Willebrord Snell, and Grienberger.

It is stated that the first scientific attempt to compute pi, however, was by Archimedes in about 240 B.C. Archimedes found an approximation for pi by circumscribing and inscribing regular polygons to a circle. In finding an approximation for pi, he found an upper and lower limit and came to the conclusion that pi was located between these limits. He started with a regular hexagon circumscribed about the circle. By finding the perimeter of the regular polygon each time, up to 96 sides, he finally came to the conclusion that pi was less than 3 1/7. Next he found a lower limit by inscribing in the circle regular polygons of six, twelve, twenty-four, forty-eight, and ninety-six sides, finding for each successive polygon its perimeter, which is, of course, always less than the circumference. He came to the conclusion that the circumference of the circle exceeds three times its diameter by a part which is less than 1/7 but more than 10/71 of the diameter.[4] Since 3 1/7 is greater than 3.1416 by about 0.04%, and is a simple number for ordinary computations, it is still in common use. Archimedes' approximation for pi is considerably closer than that given in the Bible.[5]

Claudius Ptolemy[6] (c. 150 A.D.), who taught in Athens and Alexandria, gave the first notable value for pi after that of Archimedes.

> His value for pi is given, in sexagesimal notation, as 3°8'30", which is equal to approximately 3.1416. This value was probably derived from the table of chords, which appears in his treatise. This table gives the lengths of the chords of a circle subtended by central angles of each degree and half degree. If the length of the chord of the one degree central angle

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[4] Florian Cajori, A History of Mathematics (New York: The Macmillan Company, 1924), p. 35.

[5] Edward Kasner and James Newman, Mathematics and the Imagination (New York: Simon and Schuster, Inc., 1956), p. 74.

[6] Eves, op. cit., p. 91.

> is multiplied by three hundred sixty, and the result divided by the length of the diameter of the circle, the value of pi is obtained.

The early values[7] of pi used in China were 3 and the $\sqrt{10}$. The most interesting of the Chinese, however, is that of Tsu Ch'ung-chih[8] in the fifth century, who found for the limits of ten pi, 31.415927 and 31.415926, from which he inferred by some reasoning not stated in his works that 22/7 and 355/113 were approximate values.

The early Hindu mathematician Aryabhata (c. 530) gave 62,832/20,000 as an approximate value for pi. This value[9] is equal to 3.1416.

> He showed that, if a is the side of a regular polygon of n sides inscribed in a circle of unit diameter, and if b is the side of a regular inscribed polygon of 2n sides, then $b^2 = \frac{1}{2} - \frac{1}{2}(1 - a^2)^{1/2}$. From the side of an inscribed hexagon, he found successively the sides of polygons of twelve, twenty-four, forty-eight, ninety-six, one hundred ninety-two, and three hundred eighty-four sides. The perimeter of the last is given as equal to $\sqrt{9.8694}$, from which his result was obtained by approximation.

The most prominent of the Hindu mathematicians of the seventh century was Brahmagupta.[10] He (c. 650) gave the $\sqrt{10}$

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[7] Cajori, op. cit., p. 73.

[8] David Eugene Smith, "History and Transcendence of Pi," Monographs on Topics of Modern Mathematics, ed. J. W. A. Young (New York: Longmans, Green, and Company, 1911), p. 394.

[9] W. W. Rouse Ball, Mathematical Recreations and Essays (New York: The Macmillan Company, 1956), p. 341.

[10] Ibid.

as a value of pi, which is equal to 3.1622... . This is approximately 0.66% greater than the value 3.1416.

> He obtained this value by inscribing in a circle of unit diameter regular polygons of twelve, twenty-four, forty-eight, and ninety-six sides, and calculating successively their perimeters, which he found to be $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, and $\sqrt{9.87}$ respectively; and to have [sic] assumed that as the number of sides is increased indefinitely the perimeter would approximate to $\sqrt{10}$.

Bhaskara[11] (c. 1150), a Hindu mathematician, gave 3927/1250 which is equal to 3.14160. He also gave 754/240 which is equal to 3.14166... for pi, but it is uncertain whether this was given only as an approximate value.

Francois Vieta[12] (1579), a French mathematician, found pi correct to nine decimal places.

> He showed that pi was greater than $31415926535/10^{10}$, and less than $31415926537/10^{10}$. This was deduced from the perimeters of the inscribed and circumscribed polygons of 6×2^{16} sides, obtained by repeated use of the formula $2 \sin^2 \frac{1}{2} \theta = 1 - \cos \theta$.

In 1585, Adriaen Anthonisz,[13] a French mathematician, gave the ratio 355/113 which is equal to 3.14159292..., correct to six decimal places. It was apparently a lucky accident since all he showed was that pi was between 377/120

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[11] Ibid., pp. 341-42.

[12] Ibid., p. 343.

[13] Ibid.

and 333/106. He then averaged the numerators and the denominators to obtain the approximate value of pi.[14]

Adriaen Van Roomen,[15] a Dutch mathematician, in 1593 calculated the perimeter of the inscribed regular polygon of 2^{30} sides from which he determined the value of pi correct to fifteen decimal places.

In 1610, Ludolph Van Ceulen,[16] a German mathematician, computed pi to thirty-five decimal places by calculating the perimeter of a polygon having 2^{62} sides.

Willebrord Snell,[17] a Dutch physicist, in 1621 devised a trigonometrical improvement of the classical method for computing pi. From each pair of bounds on pi given by the classical method he was able to obtain considerably closer bounds.

> By his method he was able to get Van Ceulen's thirty-five decimal places by using polygons having only 2^{30} sides. With such polygons the classical method yields only fifteen places. For polygons of ninety-six sides the classical method yields two decimal places whereas Snell's improvement gives seven places.

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[14] Eves, _op. cit._, p. 91.

[15] Ball, _op. cit._, p. 343.

[16] _Ibid._, p. 344.

[17] Eves, _op. cit._, p. 92.

In 1630, Grienberger,[18] using Snell's refinement, carried the approximation to thirty-nine decimal places.

2.3. Analytical method. The geometric method of computing pi has been popular for centuries. With new methods being introduced in mathematics, it became possible to calculate pi in other ways. There was the analytical method of computing pi by the convergent series, products, and continued fractions.

Vieta,[19] about 1593, gave another interesting approximation for pi, using continued products for the purpose. His value may be obtained from the formula

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

In 1650, John Wallis,[20] an English mathematician, proved that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots,$$

and quoted a proposition given a few years earlier by Viscount Brouncker to the effect that

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[18]Ball, op. cit., p. 345.

[19]David Eugene Smith, History of Mathematics (Boston: Ginn and Company, 1925), Vol. II, p. 311.

[20]Ball, op. cit., p. 345.

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdots}}} ,$$

but neither of these theorems was used to any large extent for calculation.

In 1668, James Gregory,[21] a Scotch mathematician, derived a series which was used by others, in connection with other relationships, in calculating a value of pi. The series is

$$\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots .$$

In 1673, Gottfried Wilhelm Leibniz,[22] a German mathematician, by taking Gregory's series and letting $x = 1$, derived the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

This series converges rather slowly for an accurate value of pi.

In 1699, Abraham Sharp,[23] an English mathematician, using Gregory's series and letting $x = \sqrt{\frac{1}{3}}$, derived the series

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[21]Young, *op. cit.*, p. 396.

[22]Kasner and Newman, *op. cit.*, pp. 76-77.

[23]Young, *op. cit.*, p. 397.

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \cdots\right),$$

which is more usable than that for $\frac{\pi}{4}$. He gave an approximation for pi to seventy-one decimal places.

In 1706, John Machin,[24] an English mathematician, by substituting Gregory's infinite series for arc tan $\frac{1}{5}$ and arc tan $\frac{1}{239}$ gave the expression

$$\frac{\pi}{4} = 4 \text{ arc tan } \frac{1}{5} - \text{arc tan } \frac{1}{239}.$$

This convergent series is faster and is more useful in computing pi. He calculated pi correctly to one hundred decimal places.

In 1873, William Shanks,[25] an English mathematician, computed pi to 707 decimal places by using Machin's formula, but in 1946 D. F. Ferguson found an error in the 528th place.

2.4. Pi and probability. The value of pi can be determined experimentally, by applications of probability theory. Probability has been used in many different ways in determining an approximation of pi.

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[24]Cajori, *op. cit.*, p. 206.

[25]Eves, *op. cit.*, p. 94.

In 1760, Comte de Buffon[26] devised his famous needle problem by which pi may be determined by probability.

On a plane a number of equidistant parallel straight lines, distance apart a, are ruled; and a stick of length l, which is less than a, is dropped on to the plane. The probability that it will fall so as to lie across one of the lines is $2 \frac{l}{\pi a}$. If the experiment is repeated many hundreds of times, the ratio of the number of favorable cases to the whole number of experiments will be very nearly equal to this fraction; hence, the value of pi can be found.

2.5. Nature of the number pi. In the middle of the eighteenth century mathematicians began to investigate the nature of the number pi, whether or not it is rational, or whether it is algebraic or transcendental.

The first investigation, of fundamental importance, was that of Johann Heinrich Lambert,[27] a German mathematician, in 1761.

He obtained the two continued fractions:

$$\frac{e^x - 1}{e^x + 1} = \frac{1}{2/x} + \frac{1}{6/x} + \frac{1}{10/x} + \frac{1}{14/x + \ldots},$$

and

$$\tan x = \frac{1}{1/x} - \frac{1}{3/x} - \frac{1}{5/x} - \frac{1}{7/x - \ldots},$$

which are closely related with continued fractions obtained by Euler, but the convergence of which Euler had not established. As the result of an investigation

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[26] Ball, op. cit., pp. 348-49.

[27] Hobson, op. cit., p. 43.

of the properties of these continued fractions, Lambert established the following theorems:

(1) If x is a rational number, different from zero, e^x cannot be a rational number.

(2) If x is a rational number, different from zero, tan x cannot be a rational number.

If $x = \frac{\pi}{4}$, we have tan x = 1, and therefore $\frac{\pi}{4}$ cannot be a rational number, and hence π cannot be a rational number.

When the discovery was made of the distinction between algebraic and transcendental irrationals, the question arose as to which of these categories the number pi belongs. An algebraic irrational is one which is a root of an equation of the form

$$(2.1) \qquad a_o + a_1x + a_2x^2 + \ldots + a_nx^n = 0,$$

where n is a rational number and a_o, a_1, a_2, ..., a_n are rational. A transcendental number is one for which such an equation is not satisfied. In 1882, Ferdinand Lindemann,[28] a German mathematician, proved that equation (2.1) cannot hold, when x = e; n and a_o, a_1, a_2, ..., a_n are algebraic numbers, not necessarily real. Euler had previously shown that $e^{i\pi} + 1 = 0$. Now if π is algebraic then $i\pi$ is algebraic and thus $e^n + 1 = 0$ is satisfied by $n = i\pi$ which

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[28]Young, op. cit., p. 402.

contradicts the theorem of Lindemann, hence π is not algebraic and thus is transcendental.

2.6. *Mnemonics*. There are occasions when it is desired to express pi to more than the well-remembered four decimal places. Then memory can be aided by the use of mnemonics. The number of letters in a word is the key to the appropriate digit. One example, taken from the *School Science and Mathematics*, gives pi correct to twelve digits.[29]

3 1 4 1 5 9
See, I have a rhyme assisting
2 6 5 3 5 9
My feeble brain its tasks resisting.

Another mnemonics, giving pi to thirty-one significant digits, appeared in the *Literary Digest*[30] and has a place in Moritz' *Memorabilia Mathematica*.[31]

3 1 4 1 5 9
Now I, even I, would celebrate
2 6 5 3 5
In rhymes inapt, the great
8 9 7 9
Immortal Syracusan, rivaled nevermore,
3 2 3 8 4
Who in his wondrous lore,

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29 Aaron L. Buchman, "Mnemonics Giving Approximate Values of Pi," *School Science and Mathematics*, LIII, (February, 1953), p. 106.

30 A. C. Orr, "More Mathematical Verse," *Literary Digest*, XXXII, (February, 1906), pp. 83-84.

31 Robert Edouard Moritz, *Memorabilia Mathematica* (New York: The Macmillan Company, 1914), p. 373.

6 2 6
Passed on before,
4 3 3 8 3 2 7 9
Left men his guidance how to circles mensurate.

2.7. <u>The electronic calculator</u>. The value of pi had been known to a great many places of accuracy for centuries. Then, with the advent of electronic calculators, calculation of extraordinary magnitude became possible and practicable, and some of these devices were programmed for the calculation of pi. In 1949, the electronic calculator,[32] the E N I A C, at the Army Ballistic Research Laboratories in Aberdeen, Maryland, in about seventy hours, gave pi to 2035 places.

The latest computation of pi was done by the Paris Data Processing Center in 1958. In only forty seconds, the IBM 704 in the Paris Center computed pi to 707 decimal places, which was the number of places which had been calculated by hand in 1873, by William Shanks. The Paris 704 went on to extend this computation, within a period of an hour and forty minutes, to 10,000 decimal places, a result which makes use of less than half of the 20,500 decimal-place capacity for which the Paris 704 can be programmed.

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[32] Eves, <u>op</u>. <u>cit</u>., p. 95.

In its electronic handling of the computation,[33] the Paris 704 was programmed for the formula

$$\frac{\pi}{4} = 4 \text{ arc tan } \frac{1}{5} - \text{arc tan } \frac{1}{239},$$

which is discussed in Chapter IV.

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[33] Edward Grimm, "The Story of Pi," IBM World Trade News, X, (August, 1958), p. 16.

CHAPTER III

GEOMETRICAL METHODS FOR THE DETERMINATION OF PI

3.1. Introduction. The number which is called pi came to the attention of many persons thousands of years ago. When they tried to measure the area of a circle or its circumference, they found many difficulties. It was discovered, quite early, that there is a connection between the circumference of a circle and its diameter. If the diameter is increased to two or three times its original length, the circumference is increased proportionally.

The history of the determination of the ratio of the circumference to the diameter of a circle could fall into four periods. The periods are divided by fundamentally distinct differences with respect to method, immediate aims, and the advancement of mathematical knowledge.

The first period embraces the time between the first records of the determinations of the ratio of the circumference to the diameter of a circle and the middle of the seventeenth century. This period is here called the geometrical period. For example, the main activity, in this connection, consisted in the approximate determination of pi by calculation of the sides or areas of regular polygons inscribed and circumscribed to the circle. In the earlier part of the period, in spite of unavoidable difficulties, a number of surprisingly good approximations of pi were obtained. Later in the period, geometric methods were devised by which approximations to the value of pi were obtained which required only a fraction of the labor involved in the earlier calculations. At the end of the period,[1] these geometric methods were developed to such a degree of exactness that no further advance could be hoped for along these lines. For further progress more powerful methods were needed.

3.2. A value of pi in the Rhind Papyrus. One of the oldest known mathematical documents, the Rhind Papyrus[2] (c. 1700 B.C.), contains an expression for pi. The Rhind Papyrus is preserved in the British Museum. It is there stated that the area of a circle is equal to that of a square whose side is the diameter diminished by one ninth. No logical reason was given for taking one ninth off the diameter, except that it seemed to lead to a satisfactory value of pi.

Figure 3.1 shows a square presumably equal in area to that of a circle. By taking 1/9 off the diameter AE of the circle and constructing a square upon the remainder AB, the

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[1]E. W. Hobson, Squaring the Circle (New York: Chelsea Publishing Company, 1953), pp. 10-11.

[2]Ibid., p. 13.

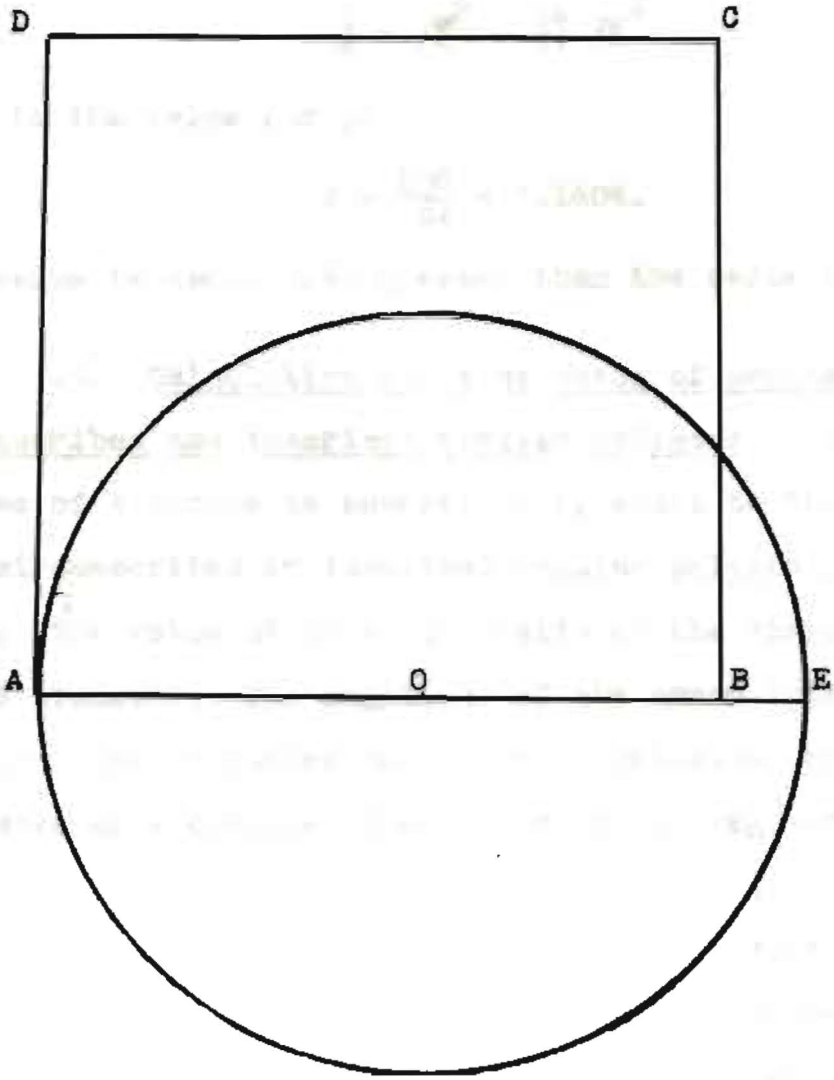

FIGURE 3.1

SQUARE APPROXIMATELY EQUAL IN AREA TO A CIRCLE area S_1 of the square ABCD, equals $64/81\ \overline{AE}^2$; the area S_2 of the circle O equals $\frac{1}{4}\pi\ \overline{AE}^2$.

The relation

$$\frac{1}{4}\pi\ \overline{AE}^2 = \frac{64}{81}\ \overline{AE}^2$$

leads to the value for pi,

$$\pi = \frac{256}{81} = 3.1604.$$

This value is about 0.6% greater than the value 3.1416.

3.3. <u>Calculation of pi by means of perimeters of circumscribed and inscribed regular polygons</u>. The circumference of a circle is approximately equal to the perimeter of a circumscribed or inscribed regular polygon of many sides. The value of pi is the ratio of the circumference to the diameter. The magnitude of the error introduced in replacing the circumference, in this relation, by the perimeter of a circumscribed or inscribed regular polygon depends upon the number of sides of the polygon. In order to obtain the value of pi to several significant digits, the number of sides of the polygon must be progressively increased until the desired accuracy is reached.

If c_n and i_n denote, respectively, sides of regular circumscribed and inscribed n-gons, then the sides of corresponding regular circumscribed and inscribed 2n-gons are given by the formulas,[3]

(3.1) $$c_{2n} = \frac{c_n \cdot i_n}{c_n + i_n} ,$$

and

(3.2) $$i_{2n} = \frac{1}{2} \sqrt{2 \cdot c_{2n} \cdot i_n} .$$

The relation between the sides of regular circumscribed and inscribed polygons is illustrated in Figure 3.2. A_1B_1 and AB are corresponding sides of circumscribed and inscribed regular n-gons, and G_1H_1 and AH are corresponding sides of circumscribed and inscribed 2n-gons.

If C_n and I_n denote, respectively, perimeters of regular circumscribed and inscribed polygons, then the perimeters of circumscribed and inscribed regular n-gons are given by the formulas

(3.3) $$C_n = nc_n ,$$

and

(3.4) $$I_n = ni_n .$$

If C_{2n} denotes the perimeter of a regular circumscribed 2n-gon, then the perimeter is given by the formula,

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[3]Wooster Woodruff Beman and David Eugene Smith, New Plane and Solid Geometry (Boston: Ginn and Company, 1900), pp. 218-19.

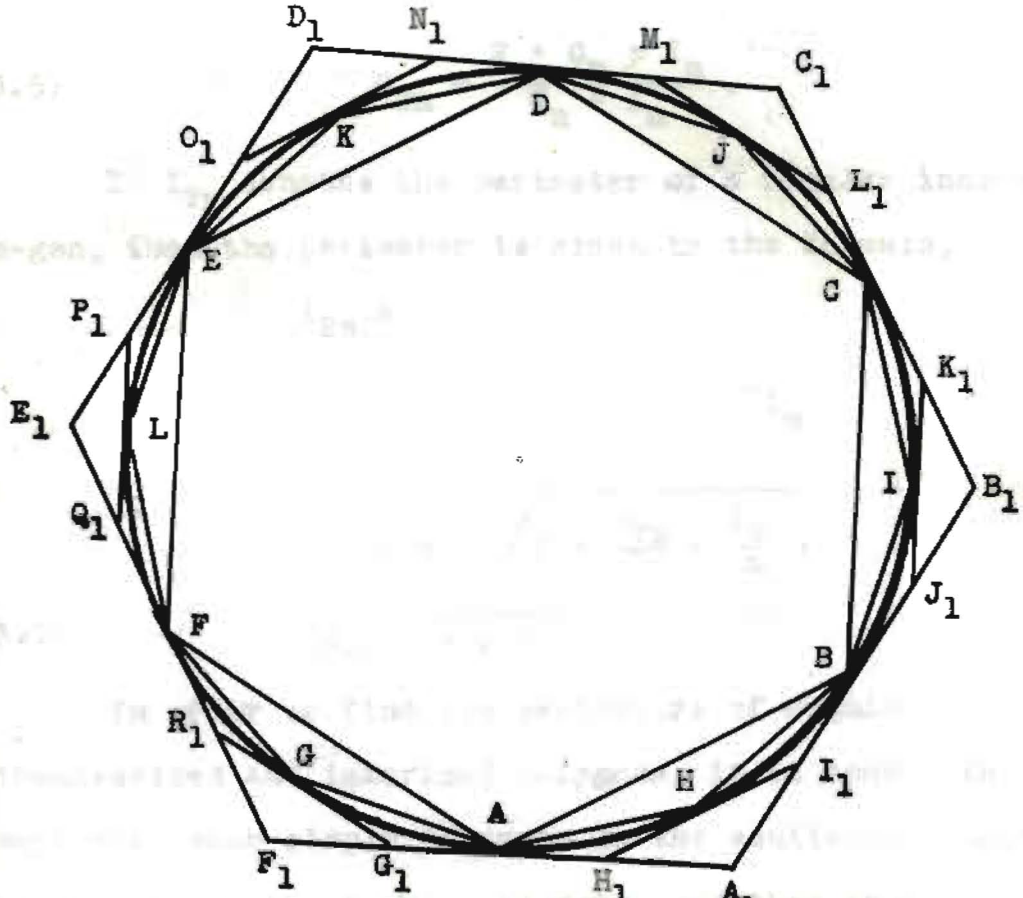

FIGURE 3.2
CIRCUMSCRIBED AND INSCRIBED REGULAR POLYGONS

$$C_{2n} = 2n \cdot c_{2n}$$

$$= 2n \frac{c_n \cdot i_n}{c_n + i_n}$$

$$= 2 \frac{nc_n \cdot ni_n}{nc_n + ni_n},$$

(3.5) $$C_{2n} = \frac{2 \cdot C_n \cdot I_n}{C_n + I_n}.$$

If I_{2n} denotes the perimeter of a regular inscribed 2n-gon, then the perimeter is given by the formula,

$$I_{2n} = 2n \cdot i_{2n}$$

$$= 2n \cdot \tfrac{1}{2} \sqrt{2 \cdot c_{2n} \cdot i_n}$$

$$= n \sqrt{2 \cdot \frac{C_{2n}}{2n} \cdot \frac{I_n}{n}},$$

(3.6) $$I_{2n} = \sqrt{C_{2n} \cdot I_n}.$$

In order to find the perimeters of regular circumscribed and inscribed polygons, it is convenient to start with such simple polygons as the equilateral triangle, the square, or the regular hexagon, and then apply the formulas for the perimeters of polygons with double the number of sides of the given polygons. If the original regular polygon has n sides, then k successive applications of the formulas (3.5) and (3.6) give perimeters of polygons whose number of sides equals $n \cdot 2^k$.

In order to make successively better approximations of pi, we start with the perimeters of circumscribed and inscribed regular hexagons; then, the perimeters of regular circumscribed and inscribed dodecagons; then, in succession, the perimeters of polygons with 24 sides, 48 sides, 96 sides, 192 sides, 384 sides, 768 sides, and 1,536 sides.

In a unit circle the perimeter of the regular circumscribed hexagon C_6 equals $2\sqrt{3}$, and the perimeter of the regular inscribed hexagon I_6 equals 3.

The perimeters of regular circumscribed and inscribed dodecagons are obtained by using formulas (3.5) and (3.6). Thus:

$$\begin{aligned} C_{12} &= \frac{2 \cdot C_6 \cdot I_6}{C_6 + I_6} \\ &= \frac{2(2\sqrt{3})(3)}{2\sqrt{3} + 3} \\ &= 4\sqrt{3}\ (2\sqrt{3} - 3) \\ &= 3.2153903, \text{ approximately.} \end{aligned}$$

Thus:

$$\begin{aligned} I_{12} &= \sqrt{C_{12} \cdot I_6} \\ &= \sqrt{12\sqrt{3}\ (2\sqrt{3} - 3)} \\ &= \sqrt{9.6461709} \\ &= 3.1058285, \text{ approximately.} \end{aligned}$$

The perimeters of regular circumscribed and inscribed polygons of 24 sides are obtained by the relations

$$C_{24} = \frac{2 \cdot C_{12} \cdot I_{12}}{C_{12} + I_{12}} ,$$

and

$$I_{24} = \sqrt{C_{24} \cdot I_{12}} .$$

The above values and those obtained by the continuation of this process are presented in Table I. In this table are given the results of applying formulas (3.5) and (3.6) in eight successive steps following the calculation of the perimeters of the hexagon. The eighth step yields the values of the perimeters of regular circumscribed and inscribed polygons of 1,536 sides. The perimeters are indicated to seven decimal places in the table, and the perimeter of the circumscribed and inscribed regular polygon of 1,536 sides gives an approximate value of pi correct to five decimal places.

If greater accuracy were desired, the process would be continued further, and each calculation would be carried out to an appropriate number of decimal places.

The process of circumscribing and inscribing regular polygons to determine a more accurate value of pi was carried on for a number of years. In 1579, Francois Vieta, a Frenchman, considered polygons of $6 \cdot 2^{16}$ sides—that is,

TABLE I

PERIMETERS OF CIRCUMSCRIBED AND INSCRIBED REGULAR POLYGONS[4]

Number of Sides	Perimeters of Circumscribed Polygons	Perimeters of Inscribed Polygons
6	3.4641016	3.0000000
12	3.2153903	3.1058285
24	3.1596599	3.1326286
48	3.1460862	3.1393502
96	3.1427146	3.1410319
192	3.1418730	3.1414524
384	3.1416627	3.1415576
768	3.1416101	3.1415838
1536	3.1415970	3.1415904

[4]Beman and Smith, op. cit., p. 221.

393,216 sides, and found pi correct to nine places. In 1593, Adriaen Van Roomen, of the Netherlands, found pi correct to 15 decimal places by computing the perimeter of a regular circumscribed polygon of 2^{30} sides—that is, 1,073,741,824 sides. In 1610, a German, Ludolph Van Ceulen,[5] found pi correct to 35 places by finding the perimeter of a polygon of 2^{62} sides—that is, 4,611,686,018,427,387,904 sides. He devoted a considerable part of his life on this task and his achievement was considered so extraordinary that the number was engraved on his tombstone, and to this day is frequently referred to in Germany as "the Ludolphian number."

3.4. Quadratrix of Dinostratus. About the year 425 B.C., Hippias invented a curve known as the Quadratrix, which is often connected with the name of Dinostratus (c. 350 B.C.), who studied the curve carefully, and who showed that the use of the curve gives a construction for pi.

The curve may be described in the following manner. If a circle of unit radius has two perpendicular radii OA and OB, and if two points M and L move with constant

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[5]Herman C. Schepler, "The Chronology of Pi," Mathematics Magazine, XXIII, (March-April, 1950), p. 219.

velocity, one upon the radius OB, the other upon the arc AB (Figure 3.3), starting at the same time at O and A, they arrive simultaneously at B. The point of intersection P(x,y) of OL and the parallel to OA through M describes the quadratrix.[6]

From this definition it follows that y is proportional to θ. Furthermore, since if y = 1,

$$\theta = \frac{\pi}{2} ,$$

therefore

$$\theta = \frac{\pi}{2} y ;$$

and from $\theta = \arctan \frac{y}{x}$ the equation of the curve becomes

$$\frac{y}{x} = \tan \frac{\pi}{2} y .$$

The curve meets the axis of X at the point whose abscissa is

$$x = \lim_{y \to 0} \left(\frac{y}{\tan \frac{\pi}{2} y} \right) ;$$

hence

$$x = \frac{2}{\pi} .$$

According to this formula the radius of the circle is the mean proportional between the abscissa of the intersection

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[6] Felix Klein, Famous Problems of Elementary Geometry, trans. W. W. Beman and D. E. Smith (New York: Dover Publications, Inc., 1956), pp. 57-58.

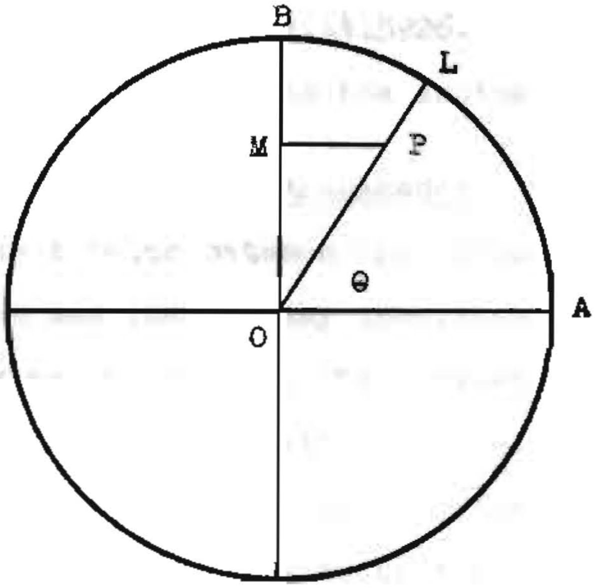

FIGURE 3.3

CONSTRUCTION FOR PI

of the quadratrix with the axis of X and the length of the quadrant. It is necessary that the measurement of the quadrant be known before an approximation of pi can be determined. The formula is

$$\frac{2}{\pi} / 1 = 1 / 1.5707963.$$

Solving for π, we obtain

$$\pi = 3.1415926.$$

This value of pi is correct to the indicated decimal places.

3.5. A discovery by Archimedes. The circumference of a circle has a value between the perimeter of any circumscribed polygon and that of any inscribed polygon. Since it is a simple matter to compute the perimeters of the regular circumscribed and inscribed six-sided polygons, it is easy to obtain bounds for pi. By doubling the sides of the regular polygons, the perimeters of the regular circumscribed and inscribed polygons may be obtained. If this process is continued, this will yield closer bounds for pi. Archimedes[7] (c. 240 B.C.) discovered that pi is less than 3 1/7 and greater than 3 10/71. He established this by circumscribing about a circle and inscribing in it regular polygons of 96 sides. Now, there are formulas, given in

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[7] Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, Inc., 1953), p. 90.

paragraph 3.3, which express the perimeters of given regular circumscribed and inscribed 2n-gons in terms of the perimeters of circumscribed and inscribed n-gons.

In The Works of Archimedes, edited by Heath, T. L. Heath[8] shows the procedure supposedly used by Archimedes in establishing the value of pi. The part dealing with the circumscribing and inscribing regular polygons of 96 sides is given in this thesis.

In Figure 3.4, GH is one side of a regular polygon of 96 sides circumscribed to the given circle. And, since $OA : AG > 4673\frac{1}{2} : 153$, while $AB = 2OA$, $GH = 2AG$, it follows that

$$AB : (\text{perimeter of polygon of 96 sides}) > 4673\tfrac{1}{2} : (153 \cdot 96)$$
$$= 4673\tfrac{1}{2} : 14688.$$

But

$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} = 3\tfrac{1}{7}$$

In Figure 3.5, BG is a side of a regular inscribed polygon of 96 sides. Therefore

$$AB : BG < 2017\tfrac{1}{4} : 66,$$

and

$$BG : AB > 66 : 2017\tfrac{1}{4}.$$

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[8] T. L. Heath, The Works of Archimedes (London: Cambridge University Press, 1897), pp. 95-98.

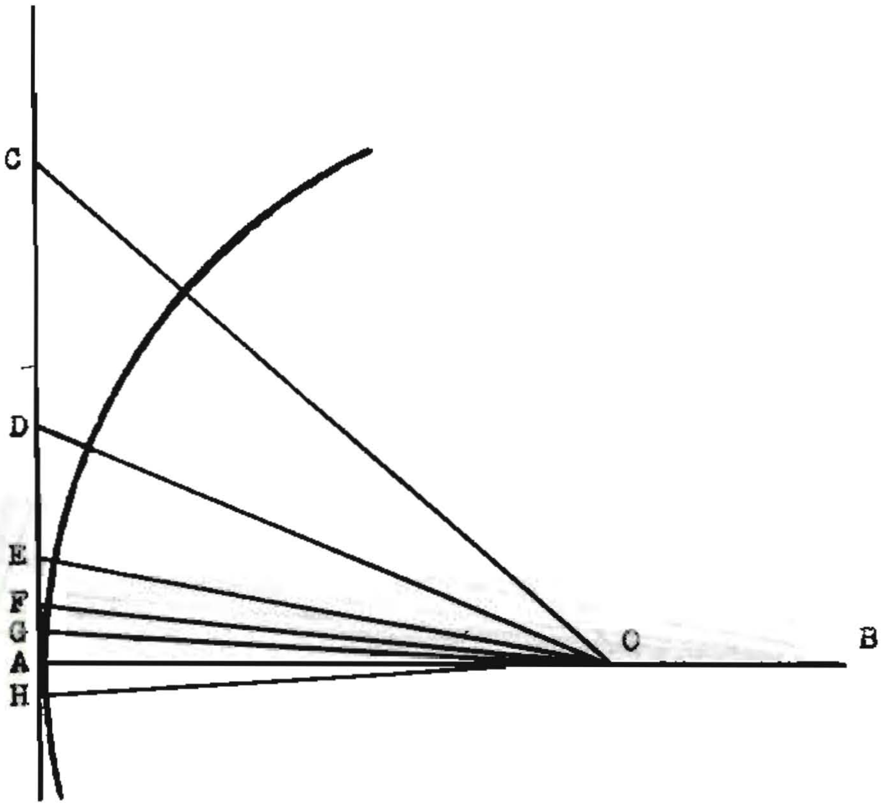

FIGURE 3.4

SIDES OF REGULAR CIRCUMSCRIBED POLYGONS

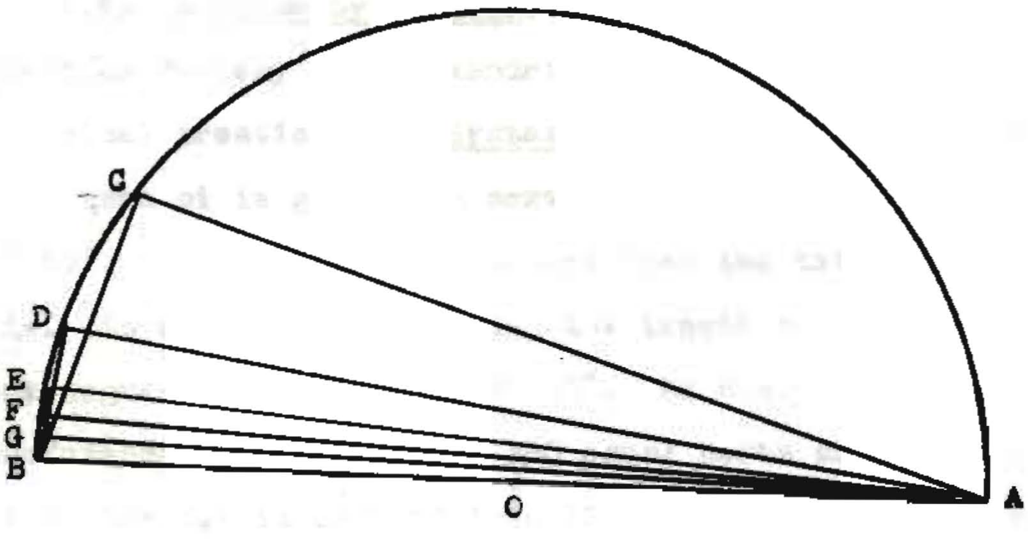

FIGURE 3.5

SIDES OF REGULAR INSCRIBED POLYGONS

It follows that

$$(\text{perimeter of polygon}) : AB > (96 \cdot 66) : 2017\tfrac{1}{4}$$

$$= 6336 : 2017\tfrac{1}{4}$$

But

$$\frac{6336}{2017\frac{1}{4}} = 3\,\frac{10}{71}\,.$$

Thus the ratio of the circumference to the diameter lies between 3 10/71 and 3 1/7.

3.6. A value by Ptolemy. A value of pi was given by Claudius Ptolemy[9] of Alexandria (c. 150 A.D.) in his astronomical treatise, the *Syntaxis* (*Almagest* in Arabic). In this work pi is given, in sexagesimal notation, as 3° 8' 30". It was probably derived from the table of chords. In the table of chords, the length of the chord of one degree is given as 1° 2' 50". He stated that the circumference is divided into 360 equal parts or degrees, and the diameter is divided into 120 equal parts.[10] Since a chord of one degree is found to be 1° 2' 50", the circumference of a circle equals very nearly 360 times (1° 2' 50"); and since the length of the diameter is 120

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[9]Eves, *op. cit.*, p. 91.

[10]E. H. Bunbury, C. R. Bezley, and T. L. Heath, "Ptolemy," *Encyclopaedia Britannica* (1956 edition), XVIII, 734.

equal parts, it follows that pi equals 3 times
$(1° + 2/60 + 50/3600)$, or $3° \; 8' \; 30''$.

In decimal notation the value of pi is found to be

$$3 + \frac{8}{60} + \frac{30}{60^2}$$

$$= \frac{3(3600) + 8(60) + 30}{60^2}$$

$$= \frac{10800 + 480 + 30}{3600} = 3.14166.$$

This value is 0.00007 greater than 3.14159, which is the value of pi correct to the fifth decimal place.

3.7. A value by Tsu Ch'ung-chih. About 480 A.D., the early Chinese worker in mechanics, Tsu Ch'ung-chih, gave the rational approximation 355/113 for pi. His method was probably similar to that outlined in the following paragraph.

In Figure 3.6, AOB is taken equal to unity and is the diameter of the indicated circle.[11] Draw BC equal to 7/8, perpendicular to AB at B. Mark off AD equal to AC on AB produced. Draw DE perpendicular to AD at D and equal to 1/2, and let F be the foot of the perpendicular from D on AE. Draw EG parallel to FB cutting BD in G. Thus

$$\frac{\overline{GB}}{\overline{BA}} = \frac{\overline{EF}}{\overline{FA}} .$$

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[11] Eves, op. cit., p. 105.

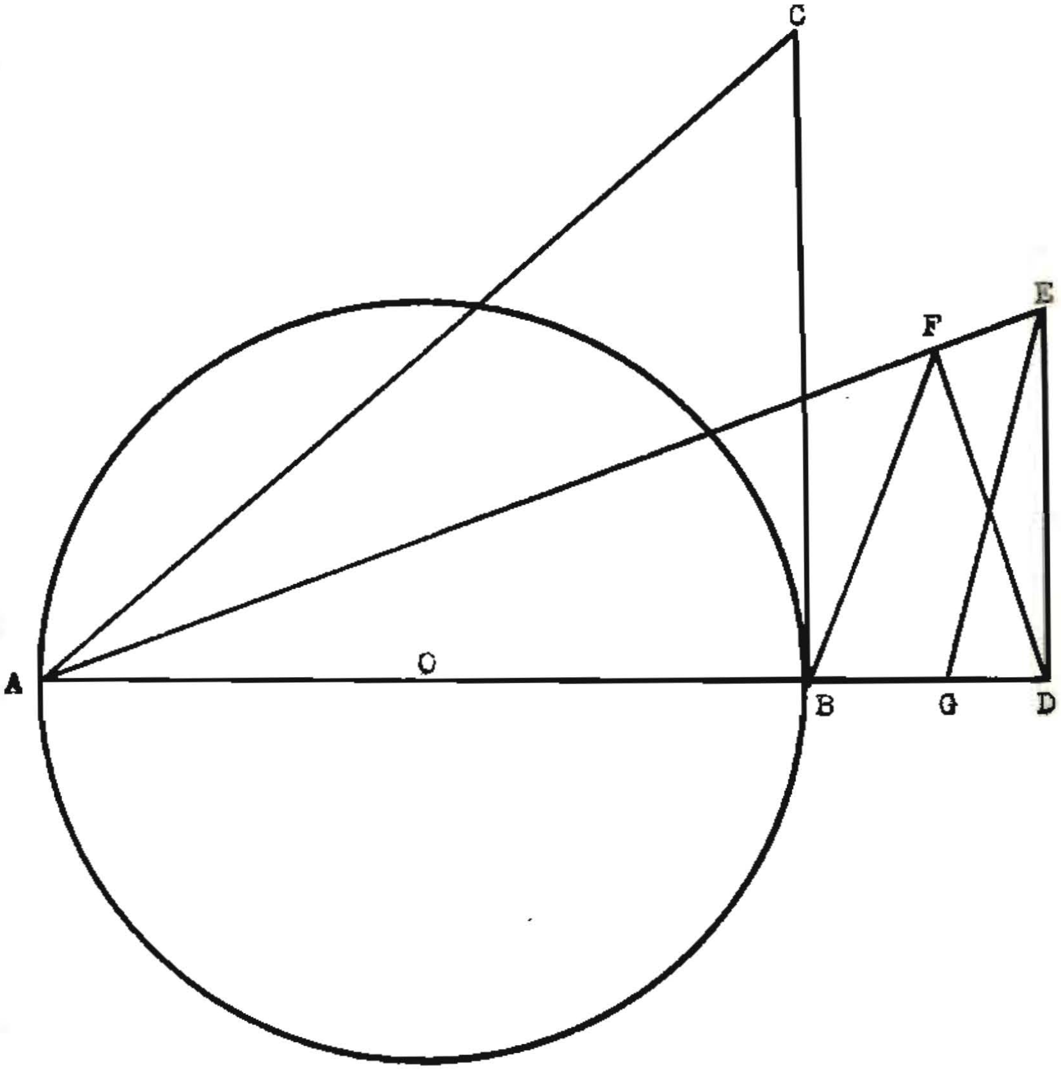

FIGURE 3.6

TSU CH'UNG-CHIH'S APPROXIMATION OF PI

Since, by similar right triangles,

$$\frac{\overline{EF}}{\overline{FD}} = \frac{\overline{DE}}{\overline{AD}} \text{ and } \frac{\overline{FA}}{\overline{FD}} = \frac{\overline{AD}}{\overline{DE}},$$

then

$$\frac{\overline{GB}}{\overline{BA}} = \frac{\overline{DE}^2}{\overline{DA}^2}.$$

Since ABC is a right triangle,

$$\overline{DA}^2 = \overline{BA}^2 + \overline{BC}^2,$$

and

$$\frac{\overline{GB}}{\overline{BA}} = \frac{\overline{DE}^2}{\overline{BA}^2 + \overline{BC}^2}.$$

In solving for GB, we obtain

$$\frac{\overline{GB}}{1} = \frac{(1/2)^2}{(1)^2 + (7/8)^2}$$

$$\overline{GB} = \frac{4^2}{7^2 + 8^2} = \frac{16}{113}.$$

GB is approximately equal to the fractional part of pi.

Thus $3 + \frac{4^2}{7^2 + 8^2} = \frac{355}{113}$, which is very nearly equal to pi.

The decimal equivalent of the rational fraction 355/113 is approximately 3.1415929, which agrees with the value of pi to six decimal places.

3.8. <u>A value by Kochansky</u>. In 1685, Father Kochansky,[12] a librarian of the Polish King John III, gave the following approximate geometrical construction for pi.

In Figure 3.7, AOB is the diameter of the indicated circle. The radius of the circle is equal to unity. Draw AG tangent to circle O at A. Taking point A of the circle, draw a circle that has the same radius as circle O and obtain point D. With point D, trace a second circle with the same radius and obtain point E. The line joining O and E intersects the tangent drawn at A in point F. By measuring off a triple radius upon line AG from point F, point C is obtained. Segment BC is approximately equal to one half of the circumference.

The triangles OAH and FAH are similar, therefore

$$\overline{FA} / 1 = \tfrac{1}{2} / \left(\tfrac{1}{2}\sqrt{3} \right).$$

Solving for $\overline{FA}$, we obtain

$$\overline{FA} = \frac{\sqrt{3}}{3}.$$

The line

$$\overline{AC} = 3 - \frac{\sqrt{3}}{3}$$

$$= \frac{9 - \sqrt{3}}{3}.$$

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[12] H. Steinhaus, <u>Mathematical Snapshots</u> (Warsaw, New York: G. E. Stechert, 1938), p. 40.

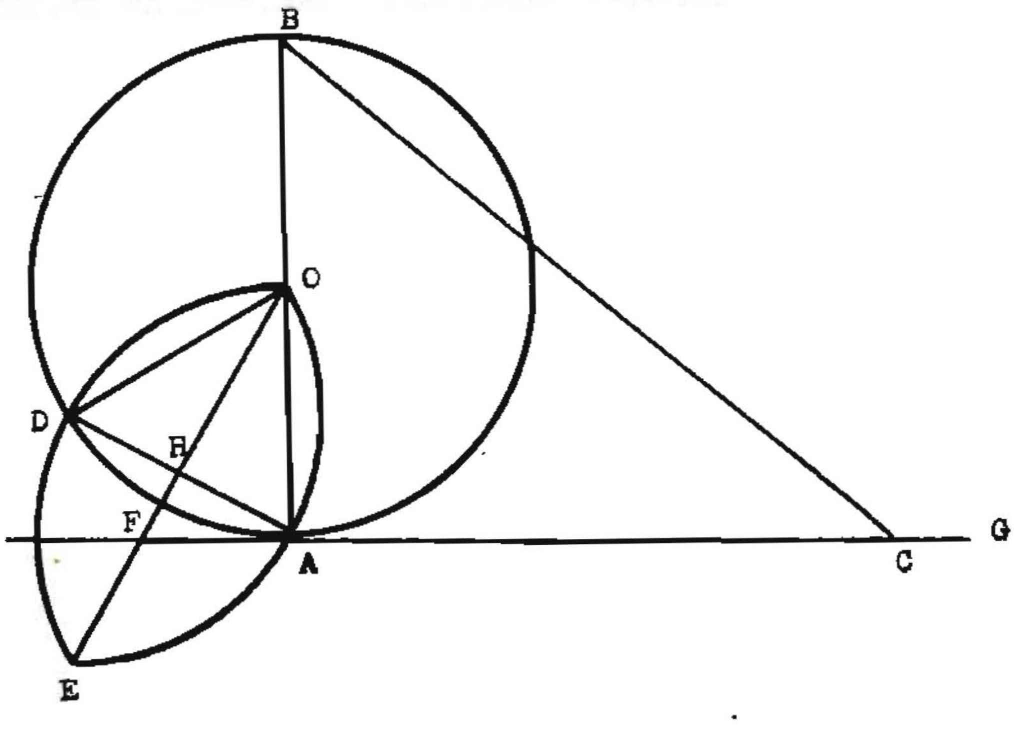

FIGURE 3.7

KOCHANSKY'S APPROXIMATION OF PI

Since ABC is a right triangle,

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 ,$$

and

$$\begin{aligned}\overline{BC}^2 &= 4 + 9 - 2\sqrt{3} + 1/3 \\ &= \frac{40}{3} - 2\sqrt{3} \\ &= 9.869231\ldots , \\ \overline{BC} &= 3.141533, \text{ approximately.}\end{aligned}$$

This value of BC is 0.000059 less than 3.141592, which is the value of pi correct to the sixth decimal place.

CHAPTER IV

CALCULATION OF PI BY INFINITE PRODUCTS AND SERIES

4.1. Introduction. The second period, which commenced in the second half of the seventeenth century, was characterized by the application of powerful analytical methods. With the assistance of new analytical expressions, pi could be expressed by convergent products and convergent series. The older geometrical forms of investigation gave way to analytical processes such as those presented in this chapter. The new methods of systematic representation stimulated fresh activity in the calculation of pi. Key formulas were applied and reapplied to obtain numerical approximations of pi to more, and still more, significant digits.[1]

In this period, which covered about a century, mathematicians were interested in calculating pi by employing convergent products and convergent series. An infinite product contains an unlimited number of factors. An infinite product which has a definite limit for the product of its factors as the number of factors is allowed to increase without limit is called a convergent product. A

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[1] E. W. Hobson, Squaring the Circle (New York: Chelsea Publishing Company, 1953), p. 11.

series is a sum of terms which progress according to some law. If the number of terms is limited, the series is said to be "finite"; if the number of terms is unlimited, the series is "infinite." An infinite series which has a definite limit for the sum of its terms as the number of terms is allowed to increase without limit is called a convergent series.

4.2. A formula by Vieta. About 1593, Francois Vieta,[2] a French mathematician, expressed the value of pi, for the first time, in a regular mathematical pattern. He proved that if two regular polygons are inscribed in a circle, the first having half the number of sides of the second, then the area of the first is to that of the second as the side of the first polygon which is drawn to the extremity of the diameter is to the diameter of the circle. Taking a square, an octagon, then polygons of 16, 32, ... sides, he expressed the side of each polygon which is drawn to the extremity of the diameter, and thus obtained the ratio of the area of each polygon to that of the next. He found that, if the diameter be taken as unity, the area of the circle is

[2] *Ibid.*, p. 26.

$$\frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots},$$

from which was obtained

$$\pi = \frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

The denominator is an infinite product of expressions of square roots with a regular pattern. The accuracy of the approximation of pi obtained by the use of this formula depends upon the number of factors in the denominator of the right member, which are used in the calculation.

4.3. A formula by Wallis. John Wallis,[3] an English mathematician, in 1650 gave an expression for pi as an infinite product.

His expression for $\frac{\pi}{2}$ may be derived in the following manner. Applying the method of integration by parts to the indefinite integral, $\int \sin^n x\, dx$, where $n > 1$,

$$(4.1)\quad \int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

Therefore

$$(4.2)\quad \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx, \quad n > 1.$$

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[3] R. Courant, Differential and Integral Calculus, trans. E. J. McShane (New York: Nordemann Publishing Company, Inc., 1938), Vol. I, pp. 223-24.

Formula (4.2) is a recurrence formula which, in successive applications, yields diminishing positive powers of the sin x factor of the integrand of the right member. Two cases need to be distinguished, according as n is even or odd. If n = 2m,

$$(4.3) \quad \int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} dx.$$

If n = 2m + 1,

$$(4.4) \quad \int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \sin x \, dx.$$

Hence,

$$(4.5) \quad \int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2},$$

and

$$(4.6) \quad \int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{2}{3} \cdot 1 .$$

By division of corresponding members of formulas (4.5) and (4.6),

$$(4.7) \quad \frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots \frac{2m \cdot 2m}{(2m-1) \cdot (2m+1)} \frac{\int_0^{\frac{\pi}{2}} \sin^{2m} x dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x dx}$$

The quotient of the two integrals on the right-hand side converges to 1 as m increases. This may be established by the following considerations. In the interval $0 < x \leqq \frac{\pi}{2}$,

(4.8) $\quad 0 < \sin^{2m+1} x \leqq \sin^{2m} x \leqq \sin^{2m-1} x, \quad m > 0.$

Consequently,

(4.9) $\quad 0 < \int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx \leqq \int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx \leqq \int_0^{\frac{\pi}{2}} \sin^{2m-1} x \, dx.$

If each term of the inequality (4.9) be divided by

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx,$$

(4.10) $$1 \leqq \frac{\int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx} \leqq \frac{\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx}.$$

Since

(4.11) $$\frac{\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx} = \frac{2m+1}{2m} = 1 + \frac{1}{2m},$$

we obtain

(4.12) $$1 \leqq \frac{\int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx} \leqq 1 + \frac{1}{2m},$$

and

$$(4.13)\qquad \lim_{m\to\infty} \frac{\int_0^{\frac{\pi}{2}} \sin^{2m} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, dx} = 1.$$

The relation (4.7) may then be written in the limiting form

$$(4.14)\qquad \frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}.$$

In the numerator we find the even numbers, in the denominator the odd numbers. Both appear in pairs with the exception of the first factor in the denominator.

Wallis showed that the approximation obtained by stopping at any fraction in the expression on the right is in defect or in excess of the value $\frac{\pi}{2}$, according as the fraction is proper or improper.[4] This is illustrated in Figure 4.1. Some of the successive products are:

$$\frac{2}{1} = 2 = 2.0000,$$

$$\frac{2}{1}\cdot\frac{2}{3} = \frac{4}{3} = 1.3333,$$

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3} = \frac{16}{9} = 1.7777,$$

[4] Edward Kasner and James Newman, Mathematics and the Imagination (New York: Simon and Schuster, Inc., 1956), p. 76.

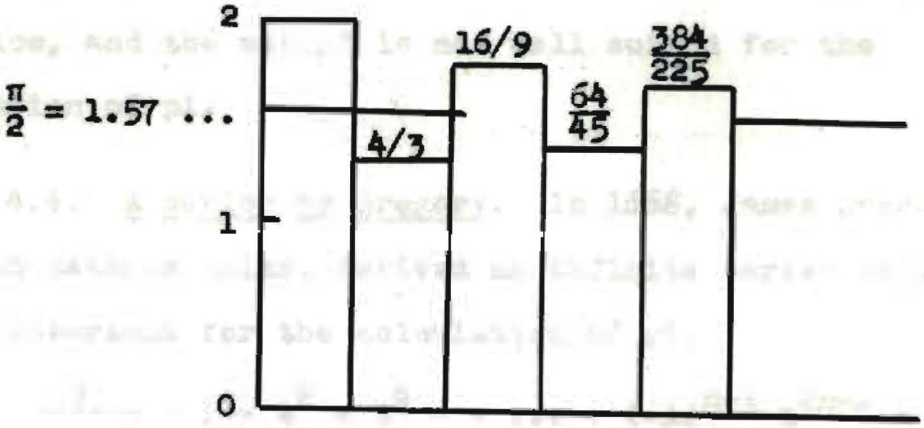

FIGURE 4.1

WALLIS' PRODUCT

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} = \frac{64}{45} = 1.4222,$$

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} = \frac{384}{225} = 1.7066.$$

Thus $$\frac{\pi}{2} = 1.5707.$$

As it is seen, the convergence of the infinite product is very slow, and the method is not well suited for the calculation of pi.

4.4. A series by Gregory. In 1668, James Gregory,[5] a Scotch mathematician, derived an infinite series which proved important for the calculation of pi.

$$(4.15) \quad \frac{1}{1 + t^2} = 1 - t^2 + t^4 - + \ldots + (-1)^{n-1} t^{2n-2} + r_n ,$$

where

$$r_n = (-1)^n \frac{t^{2n}}{1 + t^2}, \text{ where n is positive.}$$

Integration of both members of the equality,

$$(4.16) \quad \int_0^x \frac{dt}{1 + t^2} = \text{arc tan } x,$$

yields

$$(4.17) \quad \text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n - 1} + R_n,$$

where

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[5] Courant, op. cit., pp. 318-19.

$$R_n = (-1)^n \int_0^x \frac{t^{2n}}{1 + t^2}\, dt.$$

In the interval $-1 \leqq x \leqq 1$,

$$\left|R_n\right| \leqq \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n + 1} \leqq \frac{1}{2n + 1}\,. \tag{4.18}$$

Therefore, it is evident that R_n tends to zero as n increases. For $|x| > 1$, the absolute value of the remainder increases beyond all bounds as n increases. His infinite series,

$$\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\,, \tag{4.19}$$

is thus valid for $-1 \leqq x \leqq 1$.

It was with this expression, and with the help of other relationships, that most of the practical methods of calculating pi have been obtained.

4.5. A formula by Leibniz. Gottfried Wilhelm Leibniz,[6] a German mathematician, in 1673 obtained a formula that could be used in calculating pi. He took the series by Gregory, formula (4.19), and let x equal unity. Since arc tan $1 = \frac{\pi}{4}$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,. \tag{4.20}$$

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[6] Kasner and Newman, *op. cit.*, p. 77.

The successive sums of the terms of this series yield a value of pi which may, theoretically, be found as accurate as desired. This process, typical of the powerful methods of approximation used in mathematics, still entails a great deal of calculation.

Figure 4.2 illustrates some of the successive sums of this series.

$$1 = 1 = 1.0000,$$

$$1 - \frac{1}{3} = \frac{2}{3} = 0.6666,$$

$$1 - \frac{1}{3} + \frac{1}{5} = \frac{13}{15} = 0.8666,$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \frac{76}{105} = 0.7238.$$

Thus $$\frac{\pi}{4} = 0.7853.$$

After an approximation for $\frac{\pi}{4}$ has been obtained by taking the first 50 terms of this series, the next 50 will not yield an approximation which is sufficiently more accurate to justify the additional computation, for the series converges very slowly.

4.6. <u>A value by Sharp</u>. Abraham Sharp,[7] an English mathematician, in 1699 calculated a value of pi by making

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[7] Herman C. Schepler, "The Chronology of Pi," <u>Mathematics Magazine</u>, XXIII (March-April, 1950), p. 222.

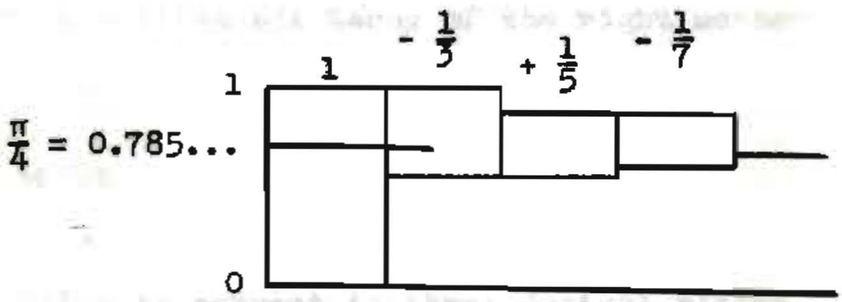

FIGURE 4.2

LEIBNIZ' SERIES

the substitution, $x = \sqrt{\frac{1}{3}}$, in the Gregory series, formula (4.19). Since arc tan $\sqrt{\frac{1}{3}} = \frac{\pi}{6}$,

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \frac{1}{3^5 \cdot 11} + \dots\right) .$$

Replacing each term in the right member by its decimal equivalent,

$$\frac{\pi}{6} = 0.5773502691\ (1.0000000000 - 0.1111111111 + 0.2222222222$$
$$- 0.0052910052 + 0.0013717421 - 0.0003741114 + \dots) .$$

Taking the first six terms of the right member,

$$\frac{\pi}{2} = 0.523551,$$

from which,

$$\pi = 3.141306.$$

This value is correct to three decimal places.

This series converged more rapidly than the series (4.20), and it was used to calculate pi correct to seventy-one decimal places.

4.7. <u>A formula by Machin</u>. In 1706, John Machin,[8] an English mathematician, developed a convenient method for calculating pi. He applied the Gregory series, formula (4.19), to a trigonometric identity and obtained a good value of pi.

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[8] Francis Maseres, <u>A Dissertation on the Use of the Negative Sign in Algebra</u> (London: Samuel Richardson, 1758), pp. 289-93.

If we let

$$A = \text{arc tan } 1/5 \ ,$$

$$B = \text{arc tan } 1/239 \ ,$$

then

$$\tan(4A - B) = \frac{\dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} - \tan B}{1 + \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} \cdot \tan B}$$

$$= \frac{\dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} - 1/239}{1 + \dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} \cdot 1/239}$$

$$= \frac{\frac{28561}{28441}}{\frac{28561}{28441}} = 1.$$

Since $\tan \frac{\pi}{4} = 1$, we have

(4.21) $$\frac{\pi}{4} = 4 \text{ arc tan } \frac{1}{5} - \text{arc tan } \frac{1}{239} \ .$$

By using formulas (4.19) and (4.21), the process of determining a value for pi is established in the following manner. By formula (4.19),

(4.22) $$\text{arc tan } b = b - \frac{b^3}{3} + \frac{b^5}{5} - \frac{b^7}{7} + \dots \ .$$

Since $$A = \text{arc tan } \frac{1}{5} \ ,$$
then

(4.23) $$A = \frac{1}{5} - \frac{1}{3(5)^3} + \frac{1}{5(5)^5} - \frac{1}{7(5)^7} + \dots \ .$$

Since $b = \frac{1}{5}$, the decimal equivalents of the odd powers of b are

b = 0.2,

b^3 = 0.008,

b^5 = 0.000,32,

b^7 = 0. ,012,8,

b^9 = 0. , ,512,

b^{11} = 0. , ,020,48,

b^{13} = 0. , , ,819,2,

b^{15} = 0. , , ,032,768,

b^{17} = 0. , , ,001,310,72,

b^{19} = 0. , , , ,052,428,8,

b^{21} = 0. , , , ,002,097,152,

b^{23} = 0. , , , , ,083,886,08,

b^{25} = 0. , , , , ,003,355,44,

b^{27} = 0. , , , , , ,134,21,

b^{29} = 0. , , , , , ,005,36,

b^{31} = 0. , , , , , , ,21,

... .

Hence the positive terms of the series of formula (4.22) are

b = 0.2,

$\frac{b^5}{5}$ = 0.000,064,

$\frac{b^9}{9}$ = 0. , ,056,888,888,888,888,8,

$$\frac{b^{13}}{13} = 0.\quad ,\quad ,\quad ,063,015,384,615,3,$$

$$\frac{b^{17}}{17} = 0.\quad ,\quad ,\quad ,\quad ,077,101,176,4,$$

$$\frac{b^{21}}{21} = 0.\quad ,\quad ,\quad ,\quad ,\quad ,099,864,3,$$

$$\frac{b^{25}}{25} = 0.\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,134,2,$$

$$\frac{b^{29}}{29} = 0.\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,1,$$

$$\ldots\ .$$

The sum of the positive terms of the series of formula (4.23) is

(4.24) $$0.200,064,056,951,981,474,679,1.$$

The negative terms of the series of formula (4.22) are

$$-\frac{b^{3}}{3} = -\ 0.002,666,666,666,666,666,666,6,$$

$$-\frac{b^{7}}{7} = -\ 0.000,001,828,571,428,571,428,5,$$

$$-\frac{b^{11}}{11} = -\ 0.\quad ,\quad ,001,861,818,181,818,1,$$

$$-\frac{b^{15}}{15} = -\ 0.\quad ,\quad ,\quad ,002,184,533,333,3,$$

$$-\frac{b^{19}}{19} = -\ 0.\quad ,\quad ,\quad ,\quad ,002,759,410,5,$$

$$-\frac{b^{23}}{23} = -\ 0.\quad ,\quad ,\quad ,\quad ,\quad ,003,647,2,$$

$$-\frac{b^{27}}{27} = -\ 0.\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,004,9,$$

$$-\frac{b^{31}}{31} = -\ 0.\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,\quad ,0,$$

$$\ldots\ .$$

The sum of the negative terms of the series of formula (4.23) is

$$(4.25) \qquad - 0.002,668,497,102,100,716,309,1.$$

Adding the numbers (4.24) and (4.25), the value of arc tan $\frac{1}{5}$ is found to be 0.197,395,559,849,880,758,370,0; and

$$\frac{\pi}{4} = 4(0.197,395,559,849,880,758,370,0) - \text{arc tan } \frac{1}{239} .$$

By formula (4.19),

$$(4.26) \quad \text{arc tan } e = e - \frac{e^3}{3} + \frac{e^5}{5} - \frac{e^7}{7} + \ldots .$$

Since $\qquad B = \text{arc tan } 1/239,$

then

$$(4.27) \qquad B = \frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \ldots .$$

Since $e \doteq 1/239$, the decimal equivalents of the odd powers of e are

$$e = 0.004,184,100,418,410,041,841,0,$$
$$e^3 = 0.000,000,073,249,775,361,251,4,$$
$$e^5 = 0. \quad , \quad , \quad ,001,282,361,572,1,$$
$$e^7 = 0. \quad , \quad , \quad , \quad , \quad ,022,449,9,$$
$$e^9 = 0. \quad , \quad , \quad , \quad , \quad , \quad , \quad ,3,$$

. . . .

Hence the positive terms of the series of formula (4.26) are

$$e = 0.004,184,100,418,410,041,841,0,$$
$$\frac{e^5}{5} = 0.000,000,000,000,256,472,314,4,$$

$$\frac{e^9}{9} = 0. \quad , \quad , \quad , \quad , \quad , \quad , \quad ,0,$$

$$\ldots .$$

The sum of the positive terms of the series of formula (4.27) is

(4.28) $\qquad 0.004,184,100,418,666,514,155,4.$

The negative terms of the series of formula (4.26) are

$$-\frac{e^3}{3} = -\ 0.000,000,024,416,591,787,083,8,$$

$$-\frac{e^7}{7} = -\ 0. \quad , \quad , \quad , \quad , \quad ,003,207,1,$$

$$\ldots .$$

The sum of the negative terms of the series of formula (4.27) is

(4.29) $\qquad -\ 0.000,000,024,416,591,790,290,9.$

Adding the numbers (4.28) and (4.29), the value of arc tan 1/239 is found to be 0.004,184,076,002,074,723,864,5.

Therefore,

$$\frac{\pi}{4} = 4(0.197,395,559,849,880,758,370,0)$$
$$-\ 0.004,184,076,002,074,723,864,5;$$

or

$$\frac{\pi}{4} = 0.785,398,163,397,448,309,615,5.$$

Multiplying each side by 4, we obtain

$$\pi = 3.141,592,653,589,793,238,462,0.$$

This is correct to the 21st decimal place, the error being in the twenty-second place, which in the correct value would be a 6 instead of a 0.

4.8. A formula by Euler. Leonhard Euler,[9] a Swiss mathematician, in 1779 applied the series (4.19) to a simple trigonometric identity and obtained a fairly good value of pi.

If we let

$$A = \text{arc tan } 1/2 \text{ ,}$$
$$B = \text{arc tan } 1/3 \text{ ,}$$

then

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$
$$\frac{1/2 + 1/3}{1 - (1/2)(1/3)} = \frac{5/6}{5/6} = 1.$$

Since $\tan \frac{\pi}{4} = 1$, we have

$$\frac{\pi}{4} = \text{arc tan } 1/2 + \text{arc tan } 1/3 \text{ .}$$

Applying the series (4.19) to each term of the right member,

$$\begin{aligned} \frac{\pi}{4} = \quad & \frac{1}{2} & + \quad & \frac{1}{3} \\ - \quad & \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3 & - \quad & \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^3 \\ + \quad & \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5 & + \quad & \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^5 \end{aligned}$$

---

[9] S. L. Loney, Plane Trigonometry (London: Cambridge University Press, 1933), p. 110.

$$- \left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^7 \qquad - \left(\frac{1}{7}\right)\left(\frac{1}{3}\right)^7$$

$$+ \left(\frac{1}{9}\right)\left(\frac{1}{2}\right)^9 \qquad + \left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^9$$

$$- \left(\frac{1}{11}\right)\left(\frac{1}{2}\right)^{11} \qquad - \left(\frac{1}{11}\right)\left(\frac{1}{3}\right)^{11}$$

$$+ \cdots \qquad + \cdots$$

Replacing each term in the right member by its decimal equivalent,

$$\begin{aligned} \frac{\pi}{4} = &+ 0.500,000,000,0 + 0.333,333,333,3 \\ &- 0.041,666,666,6 - 0.012,345,679,0 \\ &+ 0.006,250,000,0 + 0.000,823,045,2 \\ &- 0.001,116,071,4 - 0.000,065,321,0 \\ &+ 0.000,217,013,8 + 0.000,005,645,0 \\ &- 0.000,044,389,2 - 0.000,000,513,1 \\ &+ \cdots . \end{aligned}$$

Simplifying the right member,

$$\frac{\pi}{4} = 0.785,390,397,0.$$

Multiplying each side by 4, we obtain

$$\pi = 3.141561.$$

This value is correct to four decimal places. It would take more than twenty-two terms of the combined series to give pi correct to 7 decimal places.

4.9. <u>A formula by Dase</u>. In 1844, Zacharias Dase,[10] a German mathematician, replaced the trigonometric identity by one somewhat more involved and obtained a value of pi correct to 200 decimal places, a feat which he completed in two months.

Let

$$A = \text{arc tan } 1/2 ,$$
$$B = \text{arc tan } 1/5 ,$$
$$C = \text{arc tan } 1/8 .$$

Then

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$
$$= \frac{1/2 + 1/5}{1 - (1/2)(1/5)} = \frac{7/10}{9/10} = \frac{7}{9} .$$

Since tan (A + B) equals 7/9, we have

$$\tan(\overline{A + B} + C) = \frac{\tan(A + B) + \tan C}{1 - \tan(A + B) \cdot \tan C}$$
$$= \frac{7/9 + 1/8}{1 - (7/9)(1/8)} = \frac{65/72}{65/72} = 1.$$

Since $\tan \frac{\pi}{4} = 1$, we have

$$\frac{\pi}{4} = \text{arc tan } \frac{1}{2} + \text{arc tan } \frac{1}{5} + \text{arc tan } \frac{1}{8} .$$

With the aid of the formula (4.19),

$$\frac{\pi}{4} = \quad \frac{1}{2} \quad + \quad \frac{1}{5} \quad + \quad \frac{1}{8}$$
$$- \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3 \quad - \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)^3 \quad - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right)^3$$

---

[10] Hobson, <u>op. cit.</u>, p. 39.

$$
\begin{array}{lll}
+ \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5 & + \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^5 & + \left(\frac{1}{5}\right)\left(\frac{1}{8}\right)^5 \\
- \left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^7 & - \left(\frac{1}{7}\right)\left(\frac{1}{5}\right)^7 & - \left(\frac{1}{7}\right)\left(\frac{1}{8}\right)^7 \\
+ \left(\frac{1}{9}\right)\left(\frac{1}{2}\right)^9 & + \left(\frac{1}{9}\right)\left(\frac{1}{5}\right)^9 & + \left(\frac{1}{9}\right)\left(\frac{1}{8}\right)^9 \\
- \left(\frac{1}{11}\right)\left(\frac{1}{2}\right)^{11} & - \left(\frac{1}{11}\right)\left(\frac{1}{5}\right)^{11} & - \cdots . \\
+ \left(\frac{1}{13}\right)\left(\frac{1}{2}\right)^{13} & + \cdots & \\
- \left(\frac{1}{15}\right)\left(\frac{1}{2}\right)^{15} & & \\
+ \left(\frac{1}{17}\right)\left(\frac{1}{2}\right)^{17} & & \\
- \left(\frac{1}{19}\right)\left(\frac{1}{2}\right)^{19} & & \\
+ \left(\frac{1}{21}\right)\left(\frac{1}{2}\right)^{21} & & \\
- \left(\frac{1}{23}\right)\left(\frac{1}{2}\right)^{23} & & \\
+ \left(\frac{1}{25}\right)\left(\frac{1}{2}\right)^{25} & & \\
- \left(\frac{1}{27}\right)\left(\frac{1}{2}\right)^{27} & & \\
+ \cdots & &
\end{array}
$$

Replacing each term in the right member by its decimal equivalent,

$$
\begin{aligned}
\frac{\pi}{4} = &+ 0.500,000,000,0 + 0.200,000,000,0 + 0.125,000,000,0 \\
&- 0.041,666,666,6 - 0.002,666,666,6 - 0.000,651,041,6 \\
&+ 0.006,250,000,0 + 0.000,064,000,0 + 0.000,006,103,5
\end{aligned}
$$

- 0.001,116,071,4 - 0.000,001,828,5 - 0.000,000,068,1
+ 0.000,217,013,8 + 0.000,000,056,8 + 0.000,000,000,8
- 0.000,044,389,2 - 0.000,000,001,8
+ 0.000,009,390,0
- 0.000,002,034,5
+ 0.000,000,448,7
- 0.000,000,100,3
+ 0.000,000,022,7
- 0.000,000,005,1
+ 0.000,000,001,1
- 0.000,000,000,2
+ . . . .

Simplifying the right member,

$$\frac{\pi}{4} = 0.785,398,163,5.$$

Multiplying each side by 4, we obtain

$$\pi = 3.141,592,654,0.$$

This value is correct to nine decimal places.

4.10. A formula by Rutherford. In 1841, William Rutherford, an English mathematician, applied the series (4.19) to a more involved trigonometric identity and obtained a value of pi correct to 152 decimal places.[11]

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[11] Schepler, op. cit., pp. 226-27.

He returned to the problem in 1853 and obtained a value correct to 440 decimal places.

Let

$$A = \text{arc tan } 1/5\ ,$$

$$B = \text{arc tan } 1/70\ ,$$

$$C = \text{arc tan } 1/99\ .$$

Then

$$\tan(4A - B) = \frac{\dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} - \tan B}{1 + \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} \cdot \tan B}$$

$$= \frac{\dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} - 1/70}{1 + \dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} \cdot 1/70}$$

$$= \frac{\dfrac{8281}{8330}}{\dfrac{8450}{8330}} = \frac{8281}{8450}\ .$$

Since tan (4A - B) equals 8281/8450 ,

$$\tan(4A - B + C) = \frac{\tan(4A - B) + \tan C}{1 - \tan(4A - B) \cdot \tan C}$$

$$= \frac{8281/8450 + 1/99}{1 - (8281/8450)(1/99)}$$

$$= \frac{\dfrac{828269}{836550}}{\dfrac{828269}{836550}} = 1.$$

Since $\tan \frac{\pi}{4} = 1$, we have

$$\frac{\pi}{4} = 4 \text{ arc tan } 1/5 - \text{arc tan } 1/70 + \text{arc tan } 1/99\ .$$

707 decimal places by using the formula (4.21). In 1946, D. F. Ferguson of England discovered errors, starting with the 528th place. The value of pi correct to 527 decimal places is

3.141,592,653,589,793,238,462,643,383,279,502,884,197,169,
399,375,105,820,974,944,592,307,816,406,286,208,998,628,
034,825,342,117,067,982,148,086,513,282,306,647,093,844,
609,550,582,231,725,359,408,128,481,117,450,284,102,701,
938,521,105,559,644,622,948,954,930,381,964,428,810,975,
665,933,446,128,475,648,233,786,783,165,271,201,909,145,
648,566,923,460,348,610,454,326,648,213,393,607,260,249,
141,273,724,587,006,606,315,588,174,881,530,920,962,829,
254,091,715,364,367,892,590,360,011,330,530,548,820,466,
521,384,146,951,941,511,609,433,057,270,365,759,591,953,
092,186,117,381,932,611,793,105,118,548,074,462,379,962,
749,567,351,885,752,724,891,227,938,183,011,949,129,833,
673,362,440,656,643,086,021,39.

CHAPTER V

PI AND PROBABILITY

5.1. Introduction. It is perhaps correct to say that the mathematical treatment of probability came in the latter part of the fifteenth century and the early part of the sixteenth century. Some of the Italian writers,[1] notably Pacioli (1494), Tartaglia (1556), and Cardan (1545), had discussed the problem of the division of a stake between two players.

It is generally accepted that the one problem to which can be credited the origin of the science of probability is the so-called "problem of the points." This problem requires the determination of the division of the stakes of an interrupted game of chance between two supposedly equally skilled players, knowing the scores of the players at the time of interruption and the number of points needed to win the game. This problem was proposed to Blaise Pascal and Pierre de Fermat probably in 1654, by the Chevalier de Méré, an able and experienced gambler. There followed a remarkable correspondence between Pascal and Fermat, in which the problem was correctly but differently solved by each. It

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[1] David Eugene Smith, A Source Book in Mathematics, (New York: McGraw-Hill Book Company, 1929), pp. 546-65.

was in their correspondence that Pascal and Fermat laid the foundations of the science of probability.

There has been a number of experiments in connection with pi and probability, such as: tossing a needle onto a ruled surface, tossing a coin onto a cross-ruled board, and writing two numbers at random.

5.2. Buffon's experimental method. Comte de Buffon in 1760 presented his famous Needle Problem in which an approximate value of pi is found by an experiment involving the random tossing of a needle on a ruled surface. On a plane, he ruled off parallel lines equally spaced and 2a units apart. A needle is taken whose length is 2b, which is less than 2a. Let the needle be tossed N times at random upon the ruled surface, and let the number of intersections be K. Then the probability that the needle will intersect the line in a single random toss of the needle is K/N.

This same probability may be deduced also in the following manner.[2] In Figure 5.1, let M denote the midpoint of the needle PQ in a random position, and let O be the midpoint of that segment, AB, perpendicular to the parallel lines which passes through M. In the segment AB, A is the endpoint nearer M, being the intersection of the

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[2] William Elwood Byerly, Elements of the Integral Calculus (Boston: Ginn and Company, 1888), pp. 209-10.

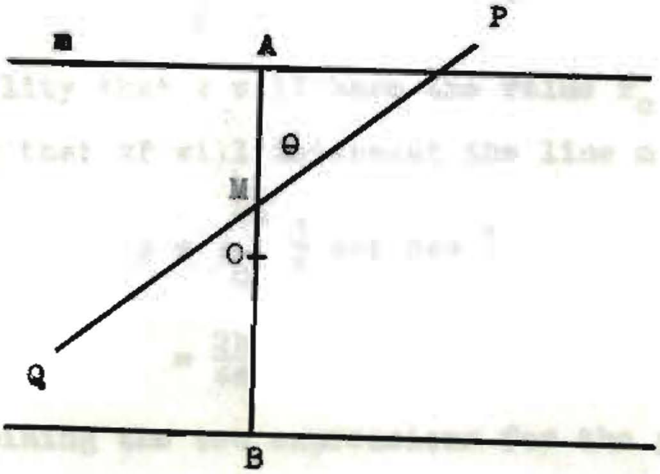

FIGURE 5.1

BUFFON'S NEEDLE PROBLEM

perpendicular with the parallel m. Denote MA by x, and denote the angle AMP by θ. Then, for a given value x_0 of x, the probability that MP will intersect line m when $x = x_0$ equals

$$\frac{\theta_0}{\frac{\pi}{2}} = \frac{2}{\pi} \text{ arc cos } \frac{x_0}{b} \text{ .}$$

The probability that x will have the value x_0 is $\frac{dx}{a}$. The probability that MP will intersect the line m is

$$p = \int_0^a \frac{2}{\pi} \text{ arc cos } \frac{x}{b} \cdot \frac{dx}{a}$$

$$= \frac{2b}{\pi a} \text{ .}$$

Combining the two expressions for the probability, p, that the needle will intersect one of the parallel lines,

$$\frac{K}{N} = \frac{2b}{\pi a} \text{ ,}$$

or

$$\pi = \frac{2bN}{aK} \text{ .}$$

5.3. <u>Experiments for the determination of pi by random tosses</u>. By using the experimental method presented by Comte de Buffon, which is explained in paragraph 5.2, the writer obtained a reasonably close approximation of pi. The distance between the parallel lines was 1 1/8 inches, and the length of the needle was 9/16 of an inch. In 788 random tosses, there were 251 intersections. Dividing the number of tosses by the number of intersections, the value obtained was 3.13944. This value is 0.00215 less than 3.14159, which is the value of pi correct to the fifth decimal place.

In 1855, Mr. A. Smith of Aberdeen, Scotland, made 3204 random tosses and obtained an approximation of 3.1553. This value[3] is approximately 0.0138 greater than the true value of pi. A pupil of Professor DeMorgan, from 600 trials, obtained an approximation of 3.137, which is less than the true value of pi by 0.004. Of the many experiments that have been performed, perhaps the most accurate approximation of pi was made by Lazzerini,[4] an Italian mathematician, in 1901. He made 3,408 tosses and obtained an approximation of 3.1415929, which was in error by only 0.000,000,3.

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[3] W. W. Rouse Ball, Mathematical Recreations and Essays (New York: The Macmillan Company, 1956), pp. 348-49.

[4] Herman C. Schepler, "The Chronology of Pi," Mathematics Magazine, XXIII (March-April, 1950), p. 224.

CHAPTER VI

IRRATIONALITY AND TRANSCENDENCE OF PI

6.1. Introduction. The first period in the history of pi, from 3000 B.C. to the middle of the seventeenth century, was the geometrical period, in which the main interest was the approximate determination of pi by calculation of the sides of regular polygons inscribed and circumscribed to a circle.[1]

The second period, which commenced in the middle of the seventeenth century and lasted for about a century, was characterized by the application of the powerful analytical methods then available. The value of pi was determined by convergent series, products, and continued fractions.

In the third period, which lasted from the middle of the eighteenth century until late in the nineteenth century, attention was turned to critical investigations of the true nature of the number pi. The number was first studied to determine its rationality or irrationality, and it was proved to be irrational. When discovery was made of the fundamental distinction between algebraic and transcendental numbers, that is, between those numbers which can be, and those numbers which cannot be, roots of an algebraical

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[1] E. W. Hobson, Squaring the Circle (New York: Chelsea Publishing Company, 1953), pp. 10-12.

equation with rational coefficients. The question arose as to which of these categories the number pi belongs. It was finally established that the number pi is transcendental.

6.2. Irrationality of pi. Johann Heinrich Lambert,[2] a German mathematician, in 1761 proved that pi was irrational, that is, it cannot be expressed as an integer or as the quotient of two integers.

Proof of the irrationality of pi depends on the continued fraction

$$(6.1) \qquad \tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots - \cfrac{x^2}{2n + 1 - \cdots}}}}} .$$

The derivation of this continued fraction is given by E. W. Hobson in A Treatise on Plane and Advanced Trigonometry.

For, if π were rational, $\frac{\pi}{4}$ would be rational. Put $x = \frac{\pi}{4}$, and if possible let $\frac{\pi}{4} = p/q$. We have then, by (6.1),

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[2] E. W. Hobson, A Treatise on Plane and Advanced Trigonometry (New York: Dover Publications, Inc., 1957), p. 374.

$$1 = \cfrac{p/q}{1 - \cfrac{p^2/q^2}{3 - \cfrac{p^2/q^2}{5 - \cfrac{p^2/q^2}{7 - \ddots - \cfrac{p^2/q^2}{2n + 1 - \cdots}}}}},$$

$$(6.2) \quad 1 = \cfrac{p}{q - \cfrac{p^2}{3q - \cfrac{p^2}{5q - \cfrac{p^2}{7q - \ddots - \cfrac{p^2}{(2n + 1)q \cdots}}}}}.$$

Now, since p and q are fixed finite integers, if we take n large enough we shall have $(2n + 1)q > p^2 + 1$.

If $a_2, a_3, \ldots, a_n, b_2, b_3, \ldots, b_n$ be all positive integers, then the infinite continued fraction

$$\cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \ddots - \cfrac{b_n}{a_n - \cdots}}}$$

converges to an irrational limit provided that, after some finite value of n, the condition $a_n \geqq b_n + 1$ be always satisfied, where the sign $>$ need not always occur but must occur infinitely often.

If π is rational, then the continued fraction in the right member of (6.2) converges to an irrational limit.[3] But the right member actually converges to the rational value 1. Hence $\frac{\pi}{4}$ cannot be equal to a fraction p/q in which p and q are integers, and therefore π is irrational.

6.3. Transcendence of pi. In 1882, Ferdinand Lindemann, a German mathematician, proved that the number pi is transcendental.

In 1873, Charles Hermite,[4] a French mathematician, proved that the number e is transcendental, that is, that no equation of the form

$$ae^m + be^n + ce^r + \dots + ke^y = 0$$

can exist, if m, n, r, ..., y, a, b, c, ..., k are whole numbers. In 1882, Lindemann proved that such an equation cannot hold, when m, n, r, ..., y, a, b, c, ..., k are algebraic numbers, not necessarily real. In the particular case, $e^{ix} + 1 = 0$, x cannot be an algebraic number. But $e^{i\pi} + 1 = 0$ is known to be true, and therefore pi is not algebraic but transcendental.

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[3] G. Chrystal, Algebra (London: Adam and Charles Black, 1906), Vol. II, pp. 512-23.

[4] Hobson, op. cit., p. 51.

CHAPTER VII

FORMULAS INVOLVING PI

7.1. Introduction. Pi represents the ratio of the two most significant measurements associated with the circle, the distance around it to the distance across it. This ratio, c/d or c/2r, is only one of several ratios equivalent to pi. For instance, pi is also the ratio of the area of a circle to the area of the square erected on its radius.

It further appears in many other formulas expressing geometric relations. Pi is the ratio of the volume of a circular cylinder to the radius squared of the base times the altitude; also, it is the ratio of three times the volume of a circular cone to the altitude times the square of the radius of the base. It appears in the formulas for finding the surface area of a sphere and the volume of a sphere. The influence of pi also extends beyond circular structures. It can be found in the area of an ellipse and the volume of an ellipsoid.

In this chapter the formulas dealing with the circle, sphere, cylinder, and cone will be used in finding an approximation of pi. It is necessary that the measurement of the radius, diameter, circumference, area, altitude, slant height, and volume in the formulas be known before an approximation of pi can be determined. The measurements in the following problems are approximate.

7.2. <u>Circle</u>. The circumference of a circle is 62.5 inches, and the radius is 10 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{C}{2r} .$$

Substituting the known values in the formula, we have

$$\pi = \frac{62.5}{2(10)} = 3.12500.$$

This approximation of pi is 0.01659 less than the value 3.14159.

The circumference of a circle is 154 inches, and the diameter is 49 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{C}{d} .$$

Substituting the known values in the formula, we have

$$\pi = \frac{154}{49} = 3.14285.$$

This approximation of pi is 0.00126 greater than the value 3.14159.

The area of the circular base of a cylinder is 38.5 square inches, and the radius is 3.5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{A}{r^2}\,.$$

Substituting the known values in the formula, we have

$$\pi = \frac{38.5}{(3.5)^2} = 3.14285.$$

This approximation of pi is 0.00126 greater than the value 3.14159.

7.3. Sphere. The surface area of a sphere is 452 square inches, and the radius is 6 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{S}{4r^2}\,.$$

Substituting the known values in the formula, we have

$$\pi = \frac{452}{4(6)^2} = 3.13888.$$

This approximation of pi is 0.00271 less than the value 3.14159.

The volume of a sphere is 904 cubic inches, and the radius is 6 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{3V}{4r^3}\,.$$

Substituting the known values in the formula, we have

$$\pi = \frac{3(904)}{4(6)^3} = 3.13888.$$

This approximation of pi is 0.00271 less than the value 3.14159.

7.4. Cylinder. The total surface area of a right circular cylinder is 376 square inches. The altitude of the cylinder is 7 inches, and the radius of the base is 5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{T}{2r(r + h)} \cdot$$

Substituting the known values in the formula, we have

$$\pi = \frac{376}{2(5)(5 + 7)} = 3.13333.$$

This approximation of pi is 0.00826 less than the value 3.14159.

The lateral surface area of a cylinder is 377 square inches. The cylinder is 12 inches deep, and the radius of the base is 5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{S}{2rh} \cdot$$

Substituting the known values in the formula, we have

$$\pi = \frac{377}{2(5)(12)} = 3.14166.$$

This approximation of pi is 0.00007 greater than the value 3.14159.

The volume of a cylinder is 942 cubic inches. The cylinder is 12 inches deep, and the radius of its base is 5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{V}{r^2 h}.$$

Substituting the known values in the formula, we have

$$\pi = \frac{942}{(5^2)(12)} = 3.14000.$$

This approximation of pi is 0.00159 less than the value 3.14159.

7.5. Cone. The total area of a right circular cone is 283 square inches. The slant height of the cone is 13 inches, and the radius of the base is 5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{T}{r(r + s)}.$$

Substituting the known values in the formula, we have

$$\pi = \frac{283}{5(5 + 13)} = 3.14444.$$

This approximation of pi is 0.00285 greater than the value 3.14159.

The lateral area of a right circular cone is 204 square inches. The slant height of the cone is 13 inches, and the radius of the base is 5 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{s}{rs} .$$

Substituting the known values in the formula, we have

$$\pi = \frac{204}{5(13)} = 3.13846.$$

This approximation of pi is 0.00313 less than the value 3.14159.

The volume of a right circular cone is 3284 cubic inches. The radius of the base of the cone is 14 inches, and the altitude is 16 inches. Find the approximation of pi.

The formula is

$$\pi = \frac{3V}{r^2h} .$$

Substituting the known values in the formula, we have

$$\pi = \frac{3(3284)}{(14)^2(16)} = 3.14158.$$

This approximation of pi is 0.00001 less than the value 3.14159.

In this chapter it has been shown that the accuracy of pi as calculated here depends upon the accuracy of the values in the formula.

CHAPTER VIII

SUMMARY

8.1. Introduction. One purpose of this thesis has been to present in compact form, as much as feasible of the history and development of the number pi. An attempt has been made to include significant developments. It appeared advisable to omit certain lengthy mathematical proofs, such as proof of the transcendence of pi, but appropriate references to such proofs have been supplied.

8.2. History of pi. It is of interest to review a few high lights of the history of pi. The determination of the value of pi and the calculation of the area of a circle of given diameter were recognized early as closely related problems. The mathematicians of antiquity had learned how to calculate precisely the areas of plane figures bounded by straight lines, such as rectangles. The circle, however, confounded their best efforts, although many approximations for its area were developed.

The early Babylonians, Chinese, and Hebrews used 3 as the value of pi. In Kings 7:23, of the Old Testament, is found the passage "And he made a molten sea ten cubits from one brim to the other; it was round all about, and a line of 30 cubits did encompass it." In the Rhind Papyrus of ancient Egypt (c. 1700 B.C.) the area of a circle is found by subtracting from the diameter 1/9 of its length and squaring the remainder. This is equivalent to taking pi equal to 3.1605.

In Greece the problem took the form of requiring the construction by straight-edge and compass alone of a square whose area equals that of a given circle. Archimedes (c. 240 B.C.) stated that the area of a circle equals the area of a right triangle, one of whose sides is the radius, the other the circumference. He also made use of the method of exhaustion to determine the value of pi. This method makes use of the fact that the areas of the regular inscribed and the regular circumscribed polygons of a circle approach that of the circle as the number of sides is increased. Using polygons of 96 sides, he found that the ratio of the circumference of a circle to its diameter lies between 3 1/7 and 3 10/71.

After the time of Archimedes, 3 1/7 and $\sqrt{10}$ were frequently taken as satisfactory approximations of pi. The Indian mathematician Aryabhata (c. 530) gave the value 3.1416.

After the invention of decimals the calculation of pi was carried out to fifteen decimal places by Adriaen Van Roomen in 1593, and to thirty-five places by Ludolph

Van Ceulen in 1610. In Germany pi is sometimes called the Ludolphian number, in honor of the latter mathematician.

Toward the end of the seventeenth century new methods of analysis were developed which enabled pi to be expressed as the sum or product of an infinite number of terms. One of the earliest of such expressions, due to Francois Vieta (c. 1593) was

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} \quad \frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2} \quad \frac{1}{2}\sqrt{\frac{1}{2} \quad \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots .$$

In 1650, John Wallis proved that

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots .$$

Gottfriend Wilhelm Leibniz in 1673 published the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

By the use of these and other series, the value of pi may be computed to any desired number of decimal places. Zacharias Dase in 1844 carried out the computation to 200 places. William Shanks in 1873 carried out the computation to 707 places, but in 1946 D. F. Ferguson found an error in the 528th place. The value of pi to ten decimal digits is 3.1415926535. In a circle equal to the mean radius of the earth, if the circumference is calculated from the radius by using this value of pi, then the error in the calculated circumference will be less than one inch.

In 1761, Johann Lambert proved that pi is an irrational number—that is, it cannot be expressed as the ratio of two integers.

In 1882, Ferdinand Lindemann proved that pi is a transcendental number—that is, it cannot be the root of an algebraic equation. This also established the impossibility of squaring the circle.

In 1949, the electronic calculator, the E N I A C, at the Army Ballistic Research Laboratories in Aberdeen, Maryland, in about seventy hours, gave pi to 2035 places.

The IBM 704 in Paris, in 1958, within an hour and forty minutes, computed the value of pi to 10,000 decimal places.

8.3. Suggestions for further study. While the number pi was originally, and for a long time, directly associated with measurements of circles, it is now regarded as an important and fundamental constant appearing in a wide variety of mathematical and physical situations having no evident involvement with circles.

A suggestion for further study would be to present the many, varied applications of pi in diverse fields, or the many situations, mathematical or physical, in which pi plays a critical role.

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