SUPPLEMENTARY MATERIAL FOR THE HIGH SCHOOL MATHEMATICS TEACHER

A Thesis

Submitted to the Department of Mathematics and the Graduate Council of the Kansas State Teachers College of Emporia in Partial Fulfillment of the Requirements for the Degree of Master of Science

by

CLARENCE W. SCHARFF

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Approved for the Major Department

John M. Burger

Approved for the Graduate Council

Laurence C. Boylan

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Clarence W. Scharff

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CHAPTER I

THE PROBLEM

1.1 Introduction. In this day and age our society has become more and more complex due to the technological advances in the fields of science and mathematics. These changes have caused a great deal of attention to be directed to our present system of education and particularly to the mathematics and science that are now being taught in our secondary schools. There is general agreement that our present educational programs need improvement to fit the needs of our changing times, but at present, there is no generally accepted answer to the question of how much the programs of our schools should be altered. Until this problem is adequately solved, there is a need to bring our teaching further into line with the needs of our times by properly supplementing the generally accepted programs of a few years ago.

1.2 Statement of the problem. Many of the teachers of secondary mathematics in our high schools have had little or no introduction to some of the so-called modern aspects of mathematics that are now of such importance. The purpose of this thesis is to provide the high school teacher of mathematics with some information relating to the theory of sets, topology, and non-Euclidean geometry, designed to provide stimulating challenges to the superior student in mathematics.

The following material is not meant to be offered to students in regular class room work. It is intended to supplement the material usually included in current mathematics text books. However, it is expected that some, or all, of the topics represented in this study may well become standard content of future mathematics text books.

1.3 Limitations of the study. During the school year of 1957-58, there were 1,037 teachers of mathematics in the public high schools of Kansas.[1] First year algebra was taught by 692 teachers. Plane geometry was taught by 461 teachers. Second year algebra was taught by 279 teachers. Trigonometry was taught by only 79 teachers and solid geometry by only 71 teachers.[2]

During the 1957-58 school year, sixty-three per cent of all students enrolled in the ninth grade were enrolled in first year algebra. Thirty-five per cent of all tenth

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[1]John M. Burger, "Background and Academic Preparation of the Mathematics Teachers in the Public High Schools of Kansas 1958-1959," The Emporia State Research Studies (Emporia, Kansas: Graduate Division of the Kansas State Teachers College, March 1959), Vol. VII, No. 3, p. 6.

[2]Ibid., p. 17.

grade students were enrolled in plane geometry. Seventeen per cent of all eleventh grade students were enrolled in second year algebra. Fifteen per cent of the twelfth grade students were enrolled in either trigonometry or solid geometry.[3]

On the basis of this information, it was decided to limit the material to that which might be used as supplementing to the high school courses of Algebra I, Plane and Solid Geometry, and Algebra II.

1.4 Background of the problem. In many mathematics classes there are students of superior abilities, who are capable of doing work well beyond that which would be expected of the class as a whole. The teacher of these students must be prepared and able to direct their interests into areas which, at this time, are not generally included in secondary mathematics. This can be done successfully only if the teacher has adequate knowledge of the subject matter involved.

The background of the problem will be discussed in more detail in Chapter II.

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[3]An unpublished report of the mathematics offerings and enrollments in Kansas high schools, obtained from George L. Cleland, Director of Instructional Services, Kansas State Department of Public Instruction, Topeka, Kansas, July 3, 1958.

1.5 Sources of material. The material presented in Chapter II is based upon articles written by leading educators in the field of mathematics. These articles appeared in The Mathematics Teacher and The Kansas Teacher. Much emphasis is also given to the Twenty-third and Twenty-fourth Yearbook of the National Council of Teachers of Mathematics[4] and publications of the Commission on Mathematics of the College Entrance Examination Board.[5]

The material of Chapters III, IV, and V has been based on mathematics textbooks in the general areas of theory of sets, topology, and non-Euclidean geometry. Some of the material has also been condensed from such books of mathematical interest as Riddles in Mathematics by Northrop,[6] Through the Mathescope by Ogilvy,[7] Mathematics, Queen and

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[4]The National Council of Teachers of Mathematics, Insights into Modern Mathematics, Twenty-third Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1957); and The Growth of Mathematical Ideas-Grades K through 12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1959).

[5]Commission on Mathematics of the College Entrance Examination Board, The Commission on Mathematics (New York: The Commission on Mathematics of the College Entrance Examination Board, March, 1958).

[6]Eugene P. Northrop, Riddles in Mathematics (New York: D. Van Nostrand Company, Inc., 1944).

[7]Charles Stanley Ogilvy, Through the Mathescope (New York: Oxford University Press, 1941).

Servant of Science by Bell,[8] and What is Mathematics? by Courant and Robbins.[9] Articles of educational magazines, such as The Mathematics Teacher[10] and Scientific American[11] are noted. The Encyclopedia Britannica[12] and Colliers Encyclopedia[13] have been used as references for technical interpretations.

1.6 Organization of the thesis. Chapter II places the problem in the proper perspective. Modern trends in the teaching of mathematics are discussed, and some of the proposals made by the Commission on Mathematics are presented.

Chapter III presents the fundamental ideas of sets. The more common symbols used in working with sets are introduced, and the meanings of the symbols are explained. Some

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[8]Eric Temple Bell, Mathematics, Queen and Servant of Science (New York: McGraw-Hill Book Company, Inc., 1951).

[9]Richard Courant and Herbert Robbins, What is Mathematics? (New York: Oxford University Press, 1941).

[10]A. J. Meder, "Modern Mathematics and Its Place in the Secondary School," The Mathematics Teacher, October, 1957.

[11]Hans Hahn, "Geometry and Intuition," Scientific American, April, 1954.

[12]Encyclopedia Britannica, "Topology," Chicago, 1956, vol. 22.

[13]Colliers Encyclopedia, "Topology" (New York: P. F. Collier and Son Corporation, 1950), vol. 18.

exercises dealing with sets are suggested to help fix the concept and use of sets in our thinking.

Chapter IV presents some of the basic concepts of topology. Some of the historically famous topological problems and their implications are discussed. Some questions and experiments are proposed to broaden the understanding and stimulate interest in topology.

Chapter V indicates how non-Euclidean geometry has developed as a branch of mathematics. A finite geometry based on twenty-five points is described.

Chapter VI summarizes the material of the preceding chapters and indicates the need for further study of modern mathematics.

CHAPTER II

BACKGROUND OF THE PROBLEM

2.1 Introduction. In order to put the material of this study in its proper perspective, it is helpful to consider the role that modern mathematics is playing in our culture. For many years mathematics has played a major part in the application and development of the fields of physics, engineering, and technology. More recently, the use of mathematical methods has been expanded in the applications to several areas, such as industrial planning, medicine, biochemistry, bio-physics, sociology, philosophy, and linguistics.[1]

2.2 Proposed changes in algebra. The major differences in the algebra of the past and the proposed algebra of the future[2] are not only differences of content but differences in point of view. The emphasis will be shifted from the manipulative processes of algebra to a thorough

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[1]Howard F. Fehr, "Mathematics for the Future," The Kansas Teacher, March, 1958, p. 41.

[2]Commission on Mathematics of the College Entrance Examination Board, Program for College Preparatory Mathematics (New York: Commission on Mathematics of the College Entrance Examination Board, 1959), pp. 20-2.

understanding of the fundamental ideas and basic principles.[3] In order to accomplish this, the concepts and language of the theory of sets will be used extensively in algebra, as also in subsequent work in other areas of high school mathematics.

The set concept, which will be discussed in some detail in Chapter III, is elementary and closely related to experience. It permits a logical approach to a variety of problems that call for creative and original thinking. It is one of the great unifying and generalizing concepts of mathematics.

2.3 Proposed changes in geometry. The Commission on Mathematics of the College Entrance Examination Board has proposed changes in geometry[4] which, if adopted, will result in a geometry course very different from courses as now generally taught. Since logical deduction will be stressed not only in geometry but throughout the study of mathematics, it may not be necessary or profitable to spend as much time as formerly on formal deductive proofs of geometric theorems. This will permit the inclusion of some of the material usually found in the more advanced courses.

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[3]Fehr, op. cit., pp. 43-4.

[4]Commission on Mathematics of the College Entrance Examination Board, op. cit., pp. 22-8.

The length of time now devoted to the study of plane and solid geometry, usually one and one-half years, will be shortened to one school year.

As suggested by the Commission on Mathematics, the unit will start with a look at geometric ideas and a discussion of the nature of deductive reasoning. The formal study will start with the postulation of congruence theorems and proceed rapidly through a chain of six or eight fundamental theorems to the proof of the Pythagorean theorem. It is possible that time will be given to the study of geometries other than the Euclidean geometry which formerly made up the entire course. Algebraic methods may be used in proofs of certain theorems where synthetic methods were used in the past.[5]

2.4 Summary. There will be some major changes in the teaching of high school mathematics if the proposed change in emphasis is adopted. It has been frequently stated that the present programs for the study of mathematics in our secondary schools are inadequate for the needs of our present culture.[6] It is generally accepted that the main body of subject matter which has been studied

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[5]Fehr, op. cit., p. 44.

[6]Ibid., p. 24.

in the past is still useful, and only such topics should be eliminated as now serve no apparently useful purpose.

The transition from teaching the traditional mathematics to the teaching of the new mathematics curriculum can not be brought about without some difficulties. One serious problem is the preparation of teachers for instruction in material which will be part of the revised mathematics program. One recent report by the National Council of Teachers of Mathematics states:

> New processes and new points of view in mathematics have provoked a revolution in many college courses during the past decade which has resulted in emphasis now being given to concepts and techniques that could be found only in graduate courses prior to that time.[7]

Our high school teachers should understand the role of modern mathematics in our scientific culture. The term, "modern mathematics," is used to designate the content and points of view of the proposed new courses. Though this modern mathematics is not universally taught in our secondary schools today, the capable students in our class rooms should be given the opportunity to explore these new fields as far as time and ability will allow.

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[7]Bruce E. Meserve, "Implications for the Mathematics Curriculum," Insights into Modern Mathematics, Twenty-third Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1957), p. 404.

The changes in our mathematics programs have not yet crystalized, and text books to effect these changes are not available at the time of this writing, the summer of 1959. The Commission on Mathematics has given consideration to the following proposals.[8]

(1) That increased emphasis be placed upon the teaching of algebra as the study of mathematical structure in contrast to the development of manipulative skills alone, and that in a limited way the ideas of modern mathematics be introduced into this instruction.

(2) That the ideas of graphing commonly taught be extended into a development of the concepts of elementary analytic geometry, and that analytic as well as synthetic proofs be accepted in geometry.

(3) That increased emphasis be placed upon deductive reasoning in areas of mathematics other than geometry.

(4) That the traditional courses in deductive solid geometry be abandoned, but that spatial concepts be developed in connection with those of the plane.

(5) That increased emphasis be placed upon the trigometric functions and their properties as functions of real numbers, with a consequent lessened emphasis upon such computational trigonometry as solution of triangles by logarithms.

(6) That increased emphasis be placed upon probability and statistical inference as a type of thinking of the greatest importance.

(7) That provision be made for the inclusion of the elementary calculus of polynomials in the high school program, but that a standard course in analytic geometry

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[8]Commission on Mathematics of the College Entrance Examination Board, Commission on Mathematics (New York: Commission on Mathematics of the College Entrance Examination Board, March, 1958).

and calculus be considered as a college-level course, which if taught in high school should be regarded as a college course taught to able students for advanced placement.

(8) That a student who completes a full four year program in secondary school mathematics should be prepared to take analytic geometry and calculus as his freshman college course.

The Commission on Mathematics has recently modified its stand on the seventh proposal. There are differences of opinion as to whether calculus should be taught as a high school subject.

These proposals are most significant to the teachers of secondary schools. If these proposals are adopted, it will be necessary for many teachers to take courses in modern mathematics in order to successfully prepare students in accordance with the eight proposals of the Commission on Mathematics.

The School Mathematics Study Group has taken significant steps in the direction proposed by the Commission on Mathematics. In its session at Yale University, New Haven, Connecticut, from June 23 to July 18, 1958, the group worked out tentative course outlines for grades nine through twelve. This group plans to continue its work in another session from June 21 to August 22, 1959, at Boulder, Colorado.

CHAPTER III

THEORY OF SETS

3.1 Sets. As far back as language has been recorded, man has been thinking in terms of sets. To illustrate this point, one may think of the collective nouns used in the phrases, "herd" of cattle, "flock" of sheep, "swarm" of bees, or "field" of corn. The words, herd, flock, swarm, and field are used to give the mental concept of many things grouped into a single, coherent set. Thus, the basic idea of set is simple and obvious. However, sets did not become a recognized part of mathematics until less than a century ago.[1] Today, mathematicians and scientists regularly use the symbolism and the algebra of sets in their work.

Set is an undefined concept which may be associated with a collection of objects or entities.[2] There may be an element of interest in sets of things in non-mathematical situations, such as football players, bowling pins, class

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[1]E. J. McShane, "Operating with Sets," Insights into Modern Mathematics, Twenty-third Yearbook of the National Council of Teachers of Mathematics, p. 36.

[2]Joseph Breuer, Introduction to the Theory of Sets (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958), p. 4; and A. J. Meder, "Modern Mathematics and Its Place in the Secondary School," The Mathematics Teacher, October, 1957, p. 422.

officers, and so on. In the study of mathematics, however, it is advisable to limit the considerations of sets to collections of mathematical interest, such as sets of numbers or sets of points.

3.2 Set notations. The elements of a set are usually identified by the use of small letters, such as, a, b, c, etc. Capital letters will be used to identify the sets, themselves. The symbol, $\in$, is used to designate the phrase, "is a member of" or "is an element of." To indicate that x is a member of set A, one would write, in symbols, $x \in A$. To indicate that x is not a member of set A, one would write, $x \notin A$.

A finite set, that is, a set with a limited number of members, may be designated by naming all its members. This method of designating a set is called the roster method, or the enumeration method. An example of a finite set is the set, A, of integers between one and four, inclusive, $A = \{1, 2, 3, 4\}$. The set, B, whose elements are the squares of the four elements of A, is $B = \{1, 4, 9, 16\}$. Changing the order of the elements does not change the set. For example, the set A could also be designated by $A = \{4, 2, 1, 3\}$, and the set B might be designated by $B = \{9, 1, 16, 4\}$. In the following material, for easier identification, the sets will be designated with the elements in numerical order.

It may be that the number of elements of a finite set is so large that it is not convenient to identify them all, individually, as in the roster notation. For example, consider the set of the first 100 natural numbers. Here it would be most inconvenient to use the roster notation, indicating each of the one hundred elements. The listing of the elements could be satisfactorily condensed into the convenient and suggestive symbol, $\{1, 2, 3, . . ., 100\}$.

A set may also be named by describing the conditions which identify the members of the set. For example, the set of the first seven positive integers could be designated by the symbol, $\{a: a \in N \text{ and } a \leq 7\}$, where N is the set of natural numbers. The colon, ":," appearing immediately after the element in this set symbol, is to be translated as "such that." The set symbol would then be read as "the set of all elements a, such that a is one of the first seven natural numbers." Similarly, the set of the first one hundred natural numbers could be designated by the symbol $\{a: a \in N \text{ and } a \leq 100\}$. This method of representation of a set is called the description notation or the set-builder notation.

A given set may have an infinite number of elements. For example, the set of all natural numbers has an infinite

Answer. $\{a: a \text{ is a rational number and } 0 < a < 1\}$, or $\{a: a = m/n,\ m < n,\ m,n \text{ relatively prime},\ m,n \in N\}$.

Exercise 6. Designate the set of numbers x which satisfy the equation $x^2 - 3x - 10 = 0$.

Answer. $\{-2, 5\}$, or $\{x: x^2 - 3x - 10 = 0\}$.

3.3 Venn Diagrams. Certain relations of sets will be introduced pictorially in figures. These figures make use of circles to provide graphic representations of sets. These figures are Venn diagrams.[3] The circles in a Venn diagram may represent either finite or infinite sets, and illustrate the relations existing between two or more sets by their relative positions. Figures 3.1, 3.2, 3.3, and 3.4 are Venn diagrams.

3.4 Relations between sets. There are three relations which may exist between two sets, A and B. (1) All elements of A may be the same as some, but not all, of the elements of B. In this case, A is a proper subset of B. Or the elements of A and B are identical, in which case each of the sets, A and B, is said to be a subset of the other. (2) Some of the elements of A may also be elements of B. The set of these common elements is called

[3]John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson, Introduction to Finite Mathematics (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957), pp. 59-60; and McShane, op. cit., pp. 39-40.

the intersection of sets A and B. (3) The elements of A may be entirely different from the elements of B. That is, there is no element of A which is also an element of B.

3.5 Subsets. As has already been indicated, if all the members of one set are also members of another set, then the first set is a subset of the second set. Thus, the set of positive odd integers less than 10 is a proper subset of the positive odd integers less than 12.

As another example of a subset, consider the sets defined by $A = \{x: x^2 - 3x + 2 = 0\}$ and $B = \{x: 0 < x < 5\}$ where x is a real number. The set A may also be represented by the symbol $\{1, 2\}$. It is apparent that the members of A are included in the set B. This relation of the two sets is indicated by the symbolic statement $A \subset B$, read, A is a proper subset of B.

3.6 Intersection of two sets. It may be that, as in the case above, two sets may have some members in common. The set of elements common to the two sets, A and B, is called the intersection of A and B, and is represented by the symbol $A \cap B$. For example, the two finite sets, $A = \{1, 4\}$ and $B = \{4, 7\}$, have the element 4 in common. The set of common elements, in this case, the element 4, is called the intersection of A and B.

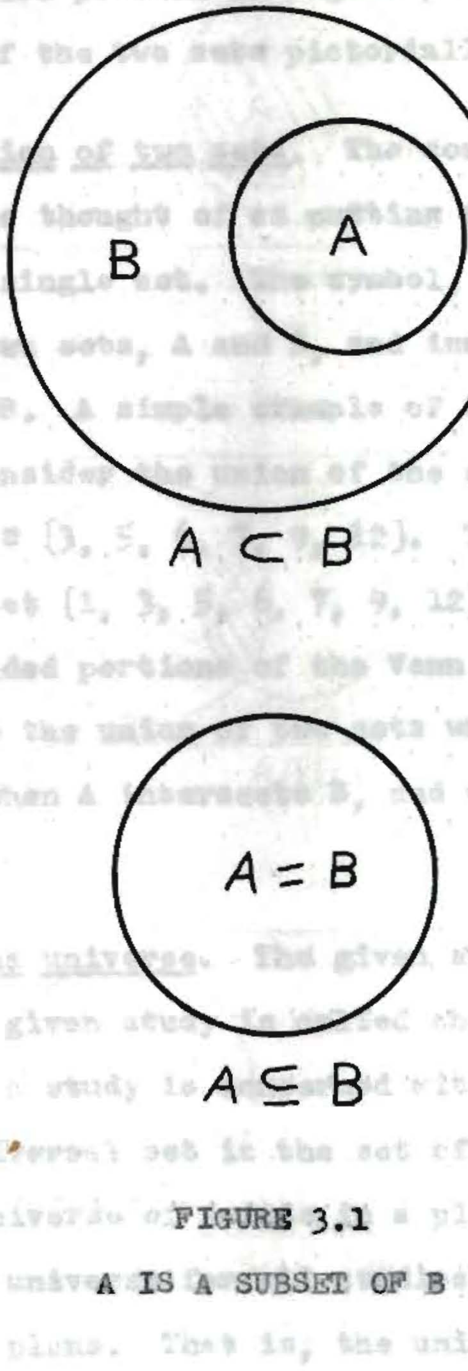


FIGURE 3.1

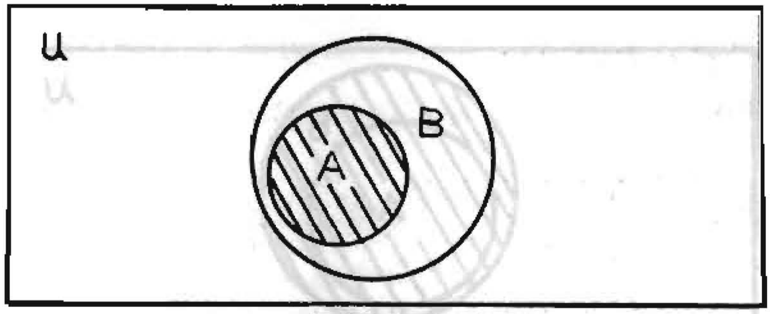
A IS A SUBSET OF B

The shaded portion of Figure 3.2 represents the intersection of the two sets pictorially.

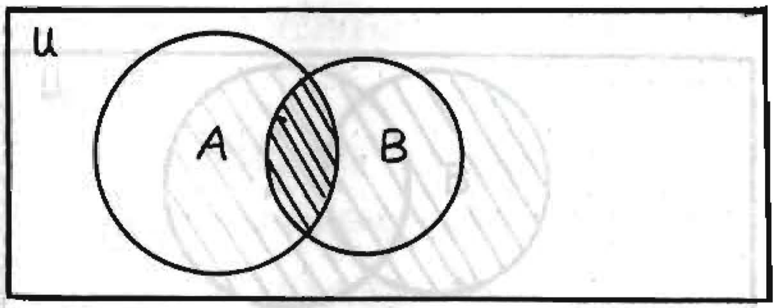
3.7 <u>Union of two sets</u>. The concept of the union of two sets may be thought of as putting the elements of both sets into one single set. The symbol, $A \cup B$, indicates the union of the two sets, A and B, and includes all of the members of A and B. A simple example of this relation is the following. Consider the union of the set $A = \{1, 5, 6, 9\}$ and the set $B = \{3, 5, 6, 7, 9, 12\}$. The set indicated by $A \cup B$ is the set $\{1, 3, 5, 6, 7, 9, 12\}$.

The shaded portions of the Venn diagrams in Figure 3.3 illustrate the union of two sets when A is a proper subset of B, when A intersects B, and when A and B do not intersect.

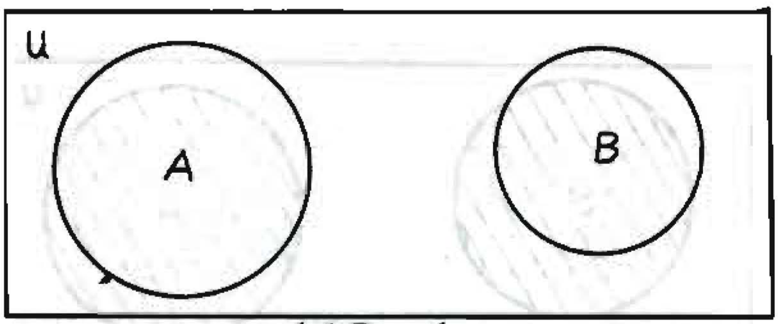
3.8 <u>The universe</u>. The given set which contains all elements of a given study is called the universal set, or universe. If a study is concerned with the points of a plane, the universal set is the set of all points on the plane. The universe of points in a plane would not necessarily be the universe for all studies which might be concerned with a plane. That is, the universal set must be given in order to limit the study to the elements intended for study. The capital letter U is frequently used for the



A ∩ B = A



A ∩ B



A ∩ B = ∅

FIGURE 3.2

THE INTERSECTION OF TWO SETS

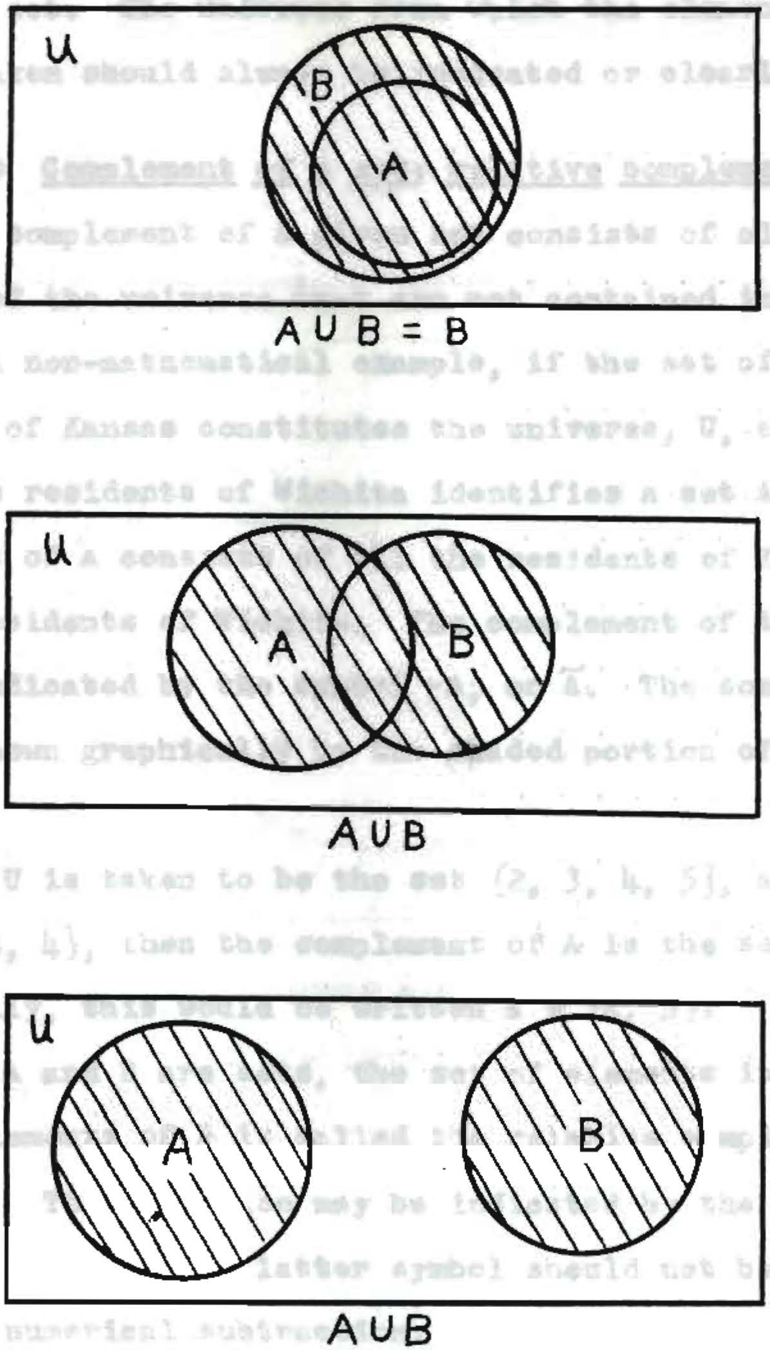


FIGURE 3.3

THE UNION OF TWO SETS

universal set. The universe from which the elements of a set are taken should always be indicated or clearly implied.

3.9 Complement of a set; relative complement of a set. The complement of a given set consists of all of the elements of the universe that are not contained in the given set. As a non-mathematical example, if the set of all the residents of Kansas constitutes the universe, U, and the set of all the residents of Wichita identifies a set A, then the complement of A consists of all the residents of Kansas who are not residents of Wichita. The complement of A is usually indicated by the symbol $\sim A$, or $\widetilde{A}$. The complement of A is shown graphically by the shaded portion of Figure 3.4.

If U is taken to be the set $\{2, 3, 4, 5\}$, and A is the set $\{3, 4\}$, then the complement of A is the set $\{2, 5\}$. Symbolically, this would be written $\widetilde{A} = \{2, 5\}$.

If A and B are sets, the set of elements in A that are not elements of B is called the relative complement of B in A. This relation may be indicated by the symbol $A \cap \widetilde{B}$, or $A - B$. This latter symbol should not be associated with numerical subtraction.

3.10 Ordered pairs. A pair of elements, such as a and b, of which the first element and the second element have separate identifications, is called an ordered pair

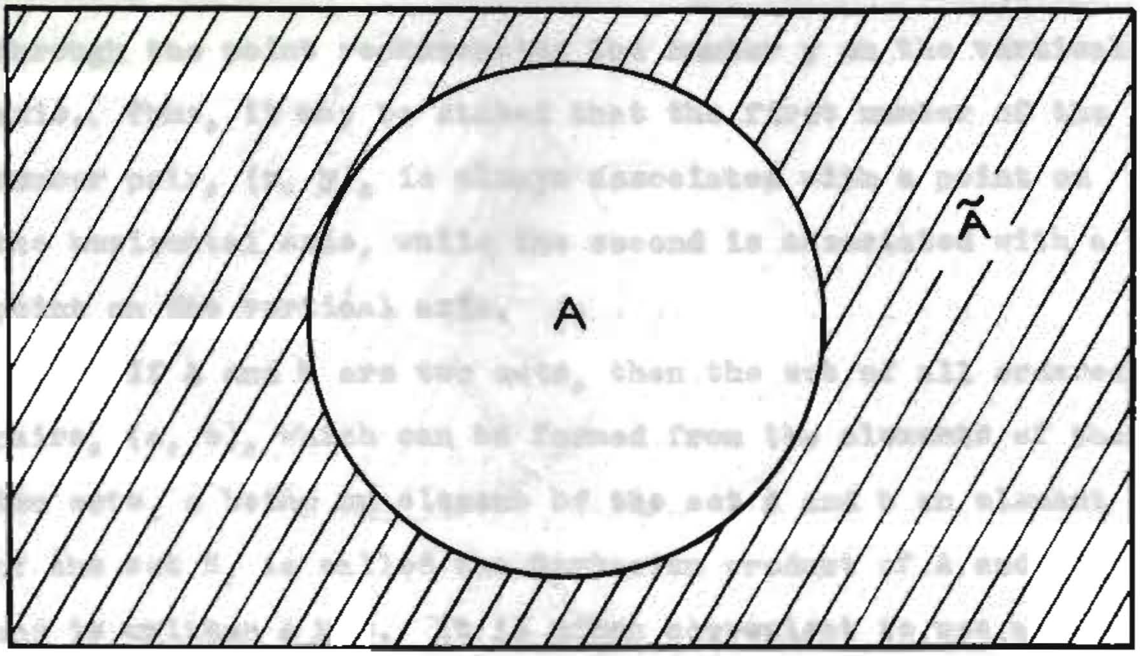


FIGURE 3.4

THE COMPLEMENT OF A

and is represented by the symbol (a, b). Ordered pairs of numbers are used to designate points of a graph in Cartesian coordinates. The ordered pair, (x, y), identifies the point where the vertical line through the point representing the number x on the horizontal axis crosses the horizontal line through the point representing the number y on the vertical axis. Thus, it may be stated that the first number of the number pair, (x, y), is always associated with a point on the horizontal axis, while the second is associated with a point on the vertical axis.

If A and B are two sets, then the set of all ordered pairs, (a, b), which can be formed from the elements of the two sets, a being an element of the set A and b an element of the set B, is called the Cartesian product of A and B and is written A X B. It is often convenient to use a single set A as the source of a Cartesian product, A X A.

3.11 Use of sets in studying equations in two variables. The concept of sets permits a much wider variety of graphs to be introduced into high school work. Since a locus is a set of all points that satisfy a given condition, and only those points, it is quite natural that the set concept be used to study the set of points of a locus.

If x and y are variables replaceable by elements of the set, R, of real numbers, then the solution set of the

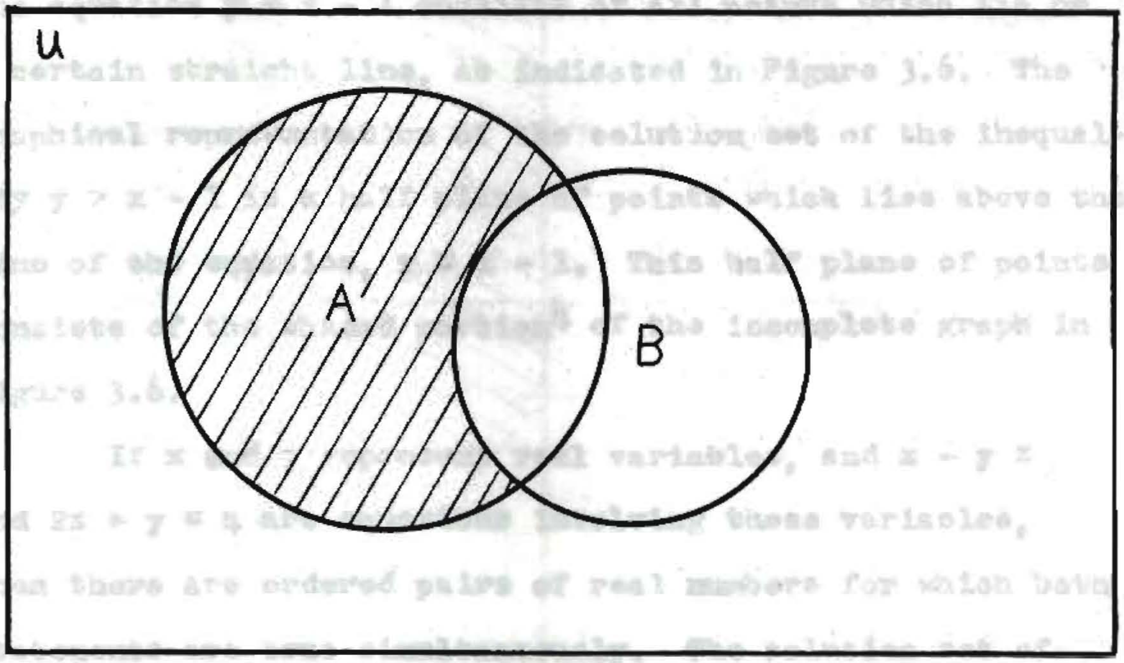


FIGURE 3.5

THE COMPLEMENT OF B IN A

equation, $y = x - 1$, is $\{(x, y): y = x - 1\}$; and the solution set of the inequality, $y > x - 1$, is $\{(x, y): y > x - 1\}$, where $x \in R$ and $y \in R$.

The graphical representation of the solution set of the equation $y = x - 1$ consists of all points which lie on a certain straight line, as indicated in Figure 3.6. The graphical representation of the solution set of the inequality $y > x - 1$ is a half plane of points which lies above the line of the equation, $y = x - 1$. This half plane of points consists of the shaded portion[4] of the incomplete graph in Figure 3.6.

If x and y represent real variables, and $x - y = 2$ and $2x + y = 4$ are equations involving these variables, then there are ordered pairs of real numbers for which both statements are true simultaneously. The solution set of these equations is the set, (2, 0) , consisting of one ordered pair element, (2, 0), which corresponds to the point of intersection of the two lines representing these equations.

Exercise 7. What is the solution set of the pair of equations, $2x + y - 2 = 0$ and $3x + y + 7 = 0$?

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[4]Commission on Mathematics of the College Entrance Examination Board, Sets, Relations, and Functions (New York: Commission on Mathematics of the College Entrance Examination Board, 1958), p. 13; and Introductory Probability and Statistical Inference for Secondary Schools (New York: Commission on Mathematics of the College Entrance Examination Board, 1957), p. 133.

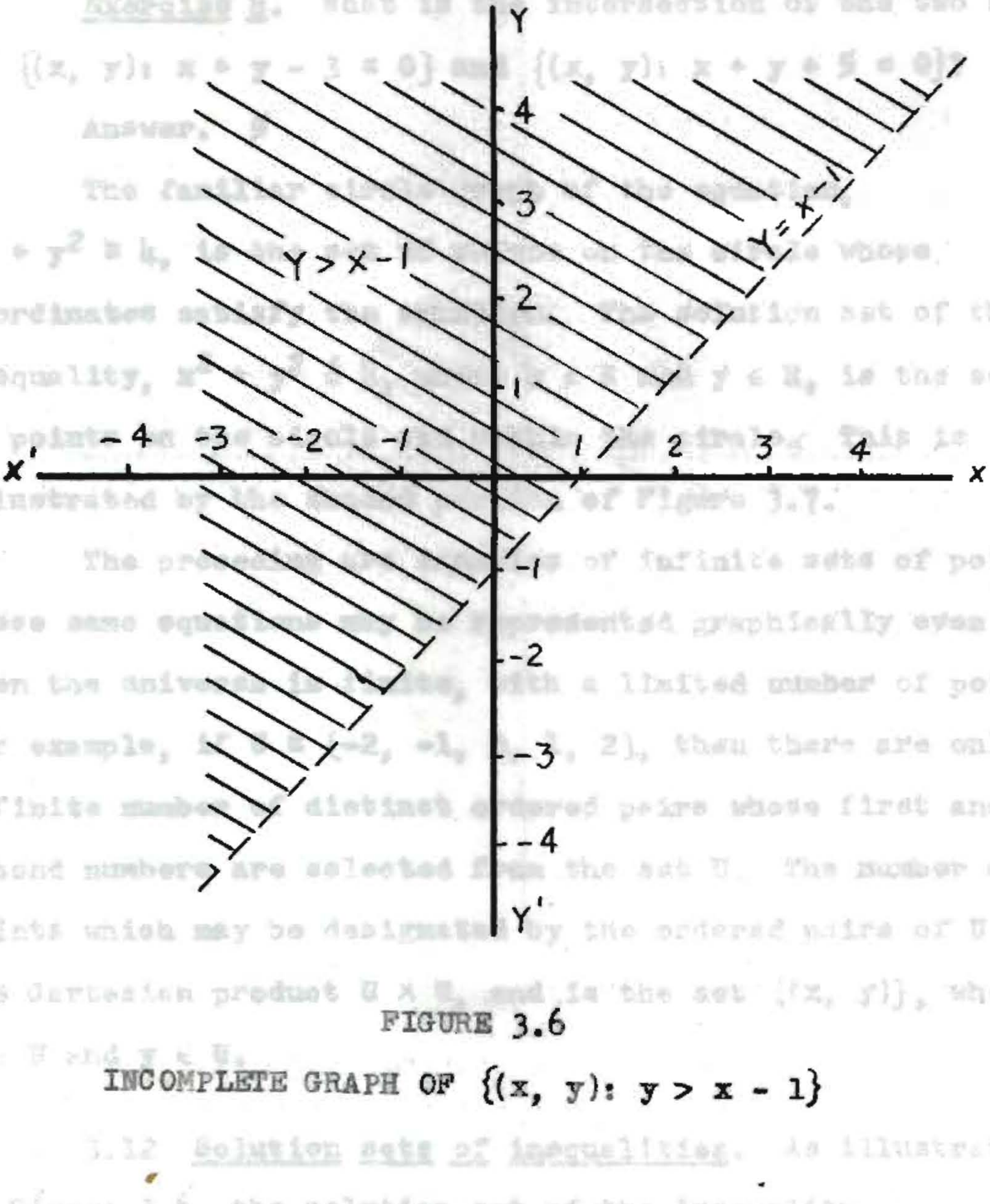

FIGURE 3.6

INCOMPLETE GRAPH OF $\{(x, y): y > x - 1\}$

Answer. $\{(-9, 20)\}$.

Exercise 8. What is the intersection of the two sets of $\{(x, y): x + y - 3 = 0\}$ and $\{(x, y): x + y + 5 = 0\}$?

Answer. $\emptyset$

The familiar circle graph of the equation, $x^2 + y^2 = 4$, is the set of points on the circle whose coordinates satisfy the equation. The solution set of the inequality, $x^2 + y^2 \leq 4$, where $x \in R$ and $y \in R$, is the set of points on the circle and within the circle. This is illustrated by the shaded portion of Figure 3.7.

The preceding are examples of infinite sets of points. These same equations may be represented graphically even when the universe is finite, with a limited number of points. For example, if $U = \{-2, -1, 0, 1, 2\}$, then there are only a finite number of distinct ordered pairs whose first and second numbers are selected from the set U. The number of points which may be designated by the ordered pairs of U is the Cartesian product $U \times U$, and is the set $\{(x, y)\}$, where $x \in U$ and $y \in U$.

3.12 Solution sets of inequalities. As illustrated in Figure 3.6, the solution set of the inequality, $y < x - 1$, corresponds to the half plane of points representing the ordered pairs of $U \times U$. Consider the solution set of the three inequalities, $y > x - 1$, $y > -x + 1$, and

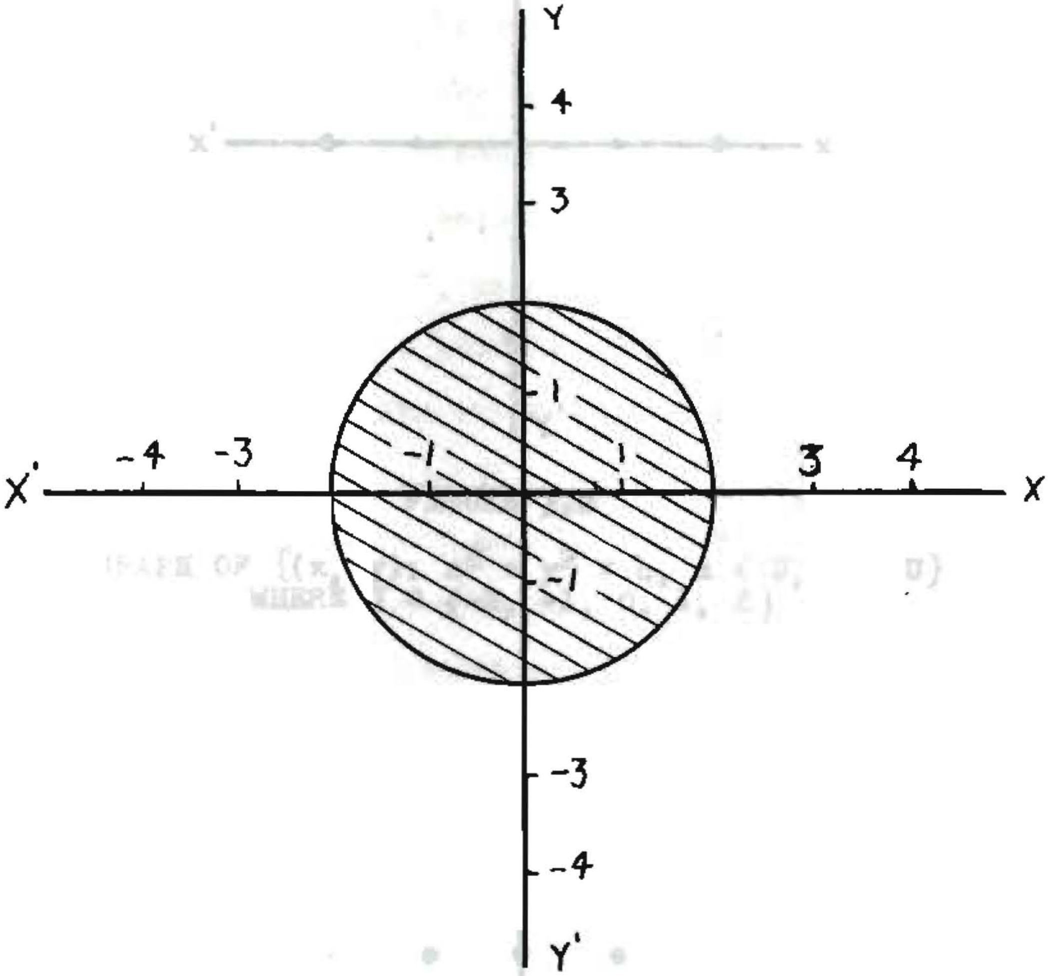

FIGURE 3.7

THE GRAPH OF $\{(x, y): x^2 + y^2 \leq 4\}$

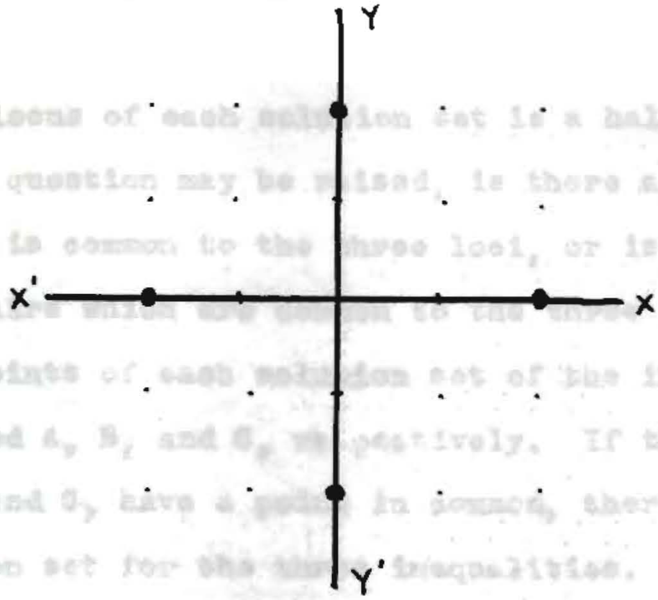

FIGURE 3.8

GRAPH OF $\{(x, y): x^2 + y^2 = 4, x \in U, y \in U\}$
WHERE $U = \{-2, -1, 0, 1, 2\}$

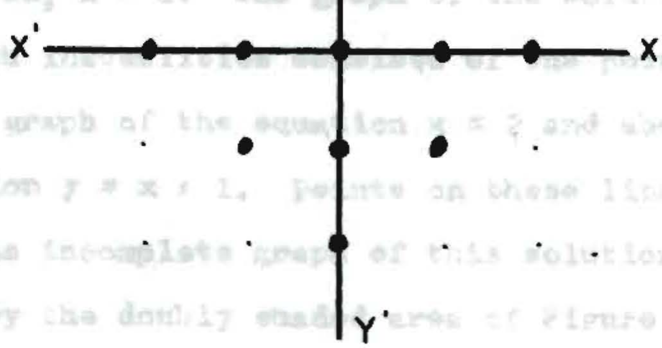

FIGURE 3.9

GRAPH OF $\{(x, y): x^2 + y^2 \leqq 4, x \in U, y \in U\}$
WHERE $U = \{-2, -1, 0, 1, 2\}$

$y < 2$. The locus of each solution set is a half plane of points. The question may be raised, is there a set of points which is common to the three loci, or is there a set of ordered pairs which are common to the three solution sets? The set of points of each solution set of the inequalities will be called A, B, and C, respectively. If the solution sets, A, B, and C, have a point in common, there is a non-empty solution set for the three inequalities. In other words $A \cap B \cap C \neq \emptyset$.

The solution sets of two inequalities may or may not have a set of points common to both sets. Consider the solution sets of the inequalities, $y > x + 1$ and $x > 2$. The solution set of $y > x + 1$ is represented by a half plane of points to the right of the line formed by the graph of the equation, $y = x + 1$. The solution set of $x > 2$ is represented by a half plane of points to the right of the graph of the equation, $x = 2$. The graph of the solution set which satisfies both inequalities consists of the points on the right of the graph of the equation $x = 2$ and above the graph of the equation $y = x + 1$. Points on these lines are not included. The incomplete graph of this solution set is represented by the doubly shaded area of Figure 3.11.

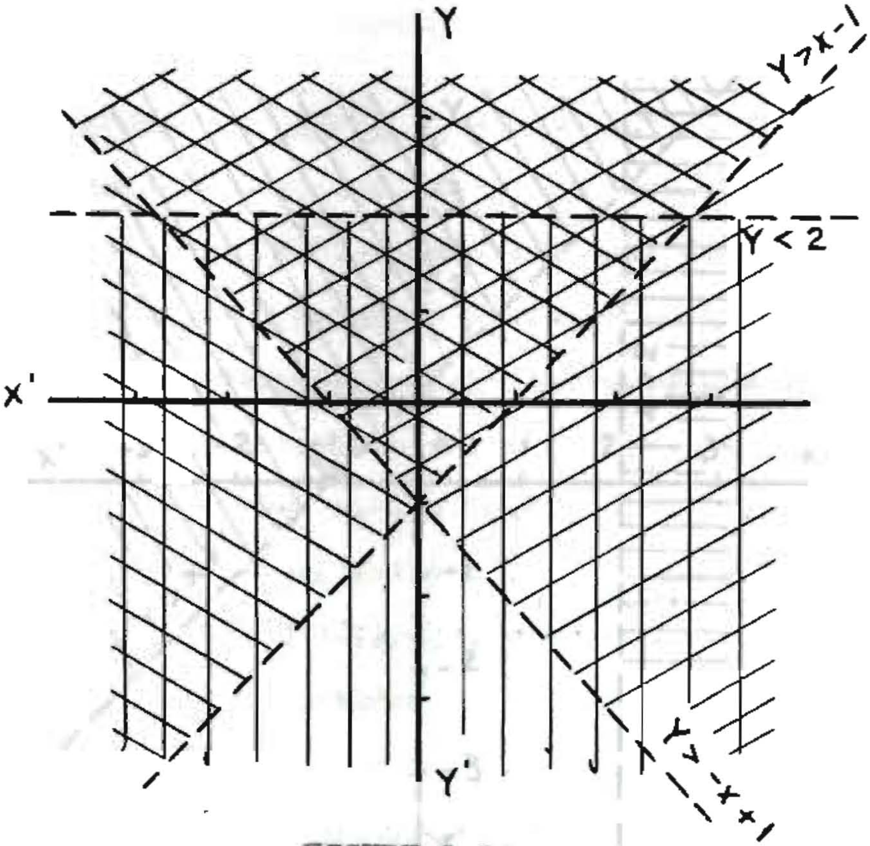

FIGURE 3.10

INCOMPLETE GRAPH OF THE SOLUTION SET OF THREE INEQUALITIES

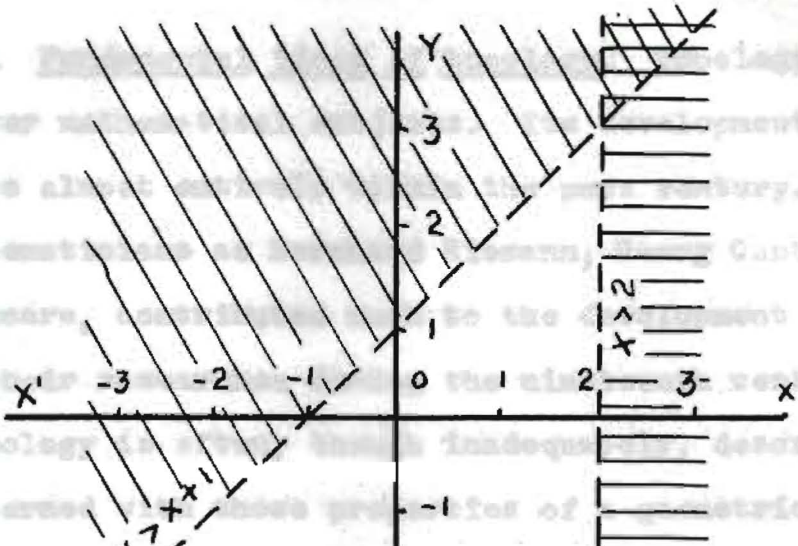

FIGURE 3.11

INCOMPLETE GRAPH OF THE SOLUTION SET
OF TWO INEQUALITIES

CHAPTER IV

TOPOLOGY

4.1 Fundamental ideas of topology. Topology is one of the newer mathematical subjects. Its development has taken place almost entirely within the past century. Such noted mathematicians as Bernhard Riemann, Georg Cantor, and Henri Poincare, contributed much to the development of topology in their researches during the nineteenth century.[1]

Topology is often, though inadequately, described as being concerned with those properties of a geometric figure which are left unchanged after the figure has been subjected to stretching, bending, folding, but without breaking or tearing.[2] For this reason, topology is frequently referred to as a "rubber sheet" geometry.[3] Topological applications of great importance have been made in almost every field of modern mathematics.

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[1]Raymond Louis Wilder, "Topology," Encyclopedia Britannica (Chicago, 1956), XXII, 298.

[2]A. D. Wallace, "Topology," Colliers Encyclopedia, (New York: P. F. Collier and Son Corporation, 1950), XVIII, 603.

[3]Richard Courant and Herbert Robbins, "Topology," The World of Mathematics (New York: Simon and Shuster, 1956), Vol. 1, p. 581; and S. H. Gould, "Origins and Development of Concepts of Geometry," Insights into Modern Mathematics, Twenty-third Yearbook of the National Council of Teachers of Mathematics, Washington, D. C., 1957, p. 304.

4.2 The Jordan Curve Theorem. One of the basic theorems of topology is the Jordan Curve Theorem. The theorem may be stated as follows. Any simple closed curve in a plane divides that plane into exactly two regions.[4] According to this theorem, every simple closed curve has an "inside" and an "outside" which are separated by the set of points of the curve. The entire Euclidean plane, exclusive of the points of the simple closed curve, is divided into two sets of points which have no point in common and neither of which has any point in common with the simple closed curve. One of these sets is called the interior of the simple closed curve, and the other the exterior, the latter being defined as the set which includes points at infinity. The curve is the common boundary of the two regions.

A simple arc in a plane may be thought of as any line which joins two points of the plane and does not cross itself. The notion of "betweenness" of any three distinct points of a simple arc is fundamental in topology. Figure 4.1 illustrates a simple arc.

A simple closed curve in a plane consists of two simple arcs which have their end points in common but have no other points in common. Figure 4.2 is an illustration of a simple closed curve.

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[4]Bruce E. Meserve, Fundamental Concepts of Geometry (Cambridge, Massachusetts: Addison-Wesley Publishing Company, Inc., 1955), p. 294.

FIGURE 4.1

A SIMPLE ARC

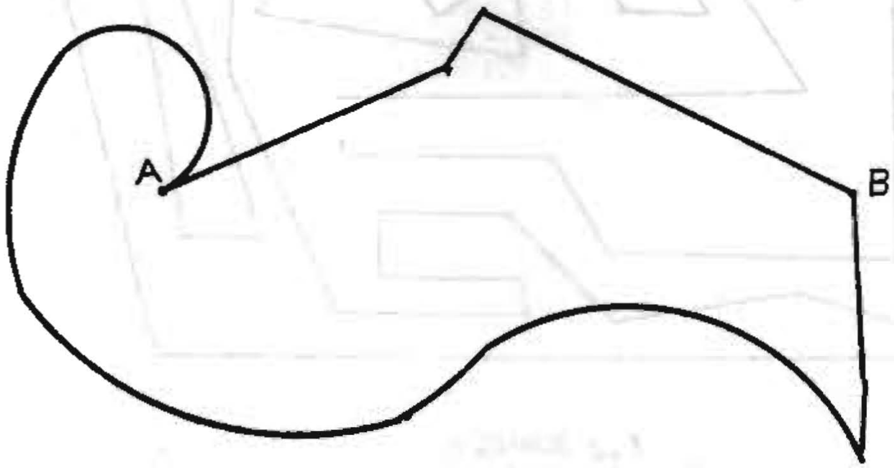

FIGURE 4.2

A SIMPLE CLOSED CURVE

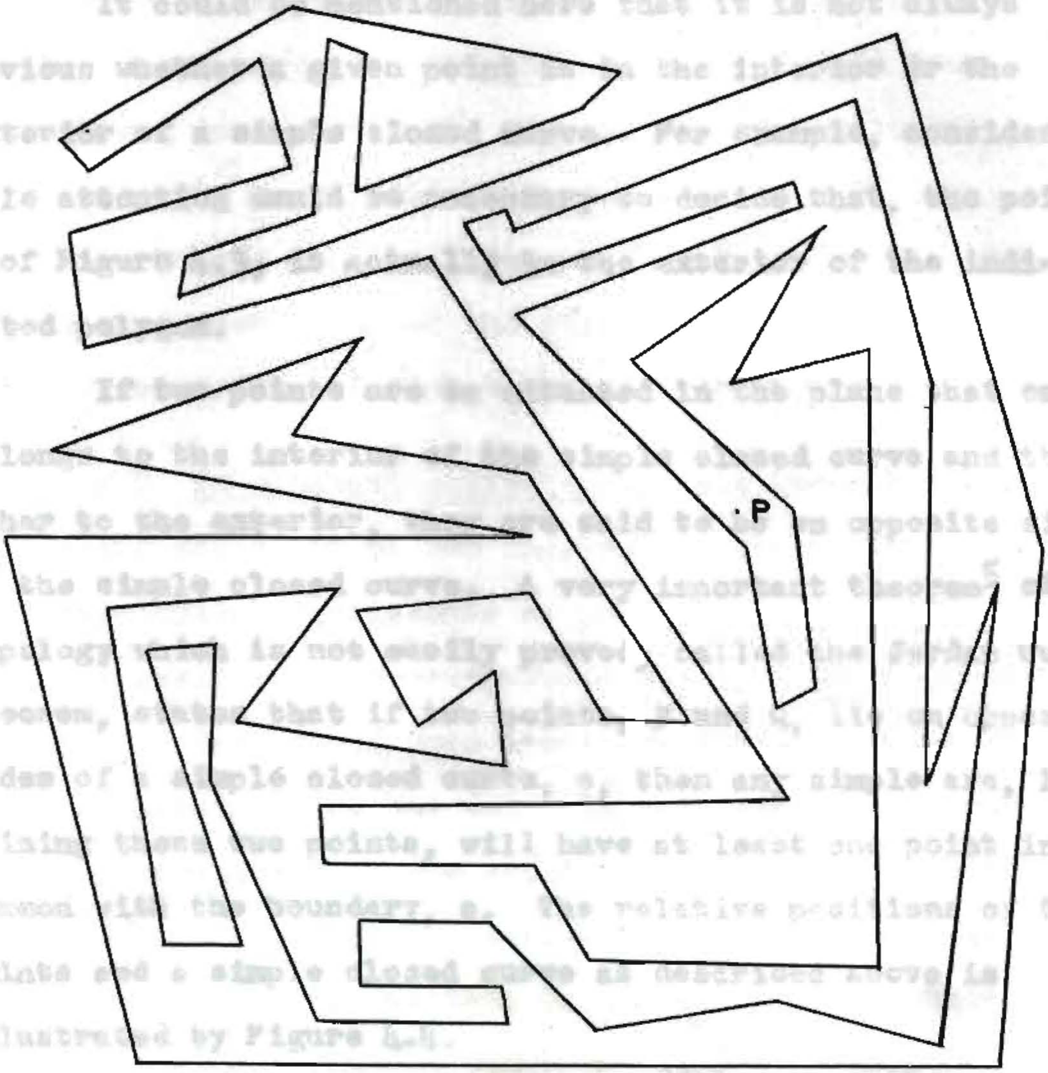

FIGURE 4.3

A POINT OUTSIDE A POLYGON

It could be mentioned here that it is not always obvious whether a given point is in the interior or the exterior of a simple closed curve. For example, considerable attention would be necessary to decide that, the point P of Figure 4.3, is actually in the exterior of the indicated polygon.

If two points are so situated in the plane that one belongs to the interior of the simple closed curve and the other to the exterior, they are said to be on opposite sides of the simple closed curve. A very important theorem[5] of topology which is not easily proved, called the Jordan Curve Theorem, states that if two points, P and Q, lie on opposite sides of a simple closed curve, c, then any simple arc, PQ, joining these two points, will have at least one point in common with the boundary, c. The relative positions of two points and a simple closed curve as described above is illustrated by Figure 4.4.

An elementary exercise in topology might be proposed as follows. Given the circle C, and the point P inside the circle, the point Q outside the circle, and the point R on the circle; if this circle were to be distorted by some continuous transformation, as by stretchings and contractions in several directions, the topological properties of

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[5] *Ibid.*, p. 295. (This is an equivalent statement of Jordan's Curve Theorem to that given on page 36.)

the figure are unchanged. The curve C is distorted into a simple closed curve C', and the points P, Q, and R are transformed into points P', Q', and R', respectively. As P is inside C, so P' will be inside C'. As Q is outside C, so Q' is outside C'. As R is on C, so R' is on C'. The topological distortion of the circle is illustrated by the topological deformation on the circle in Figure 4.5.

4.3 <u>Topological equivalence</u>. Any geometric figure is a point set.[6] Point set topology is generally concerned with topological equivalence of point sets.

A one-to-one correspondence may be thought of as the situation which exists when every element of one set can be made to correspond to one and only one element of a second set, and conversely.[7] Two point sets are topologically equivalent if there is a one-to-one correspondence between them which is continuous both ways. Two sets may be in a one-to-one correspondence though no amount of stretching or bending will take one onto the other. This may be illustrated by two sets of two tangent spheres in which the

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[6] Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Rinehart and Company, Inc., 1958), p. 236.

[7] R. H. Bing, "Point Set Topology," Insights into Modern Mathematics, Twenty-third Yearbook of the National Council of Teachers of Mathematics, Washington, D. C., 1957, p. 309.

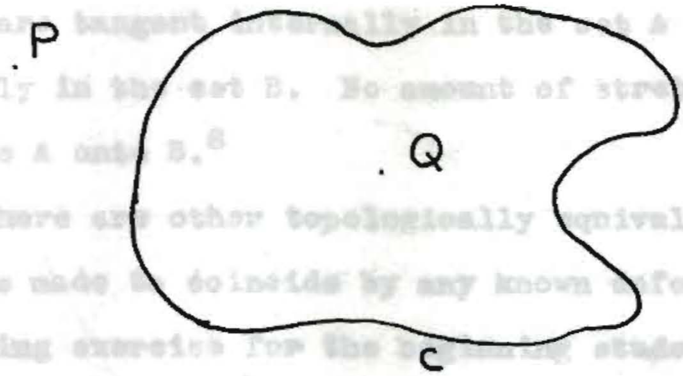

FIGURE 4.4

TWO POINTS SEPARATED BY A SIMPLE CLOSED CURVE

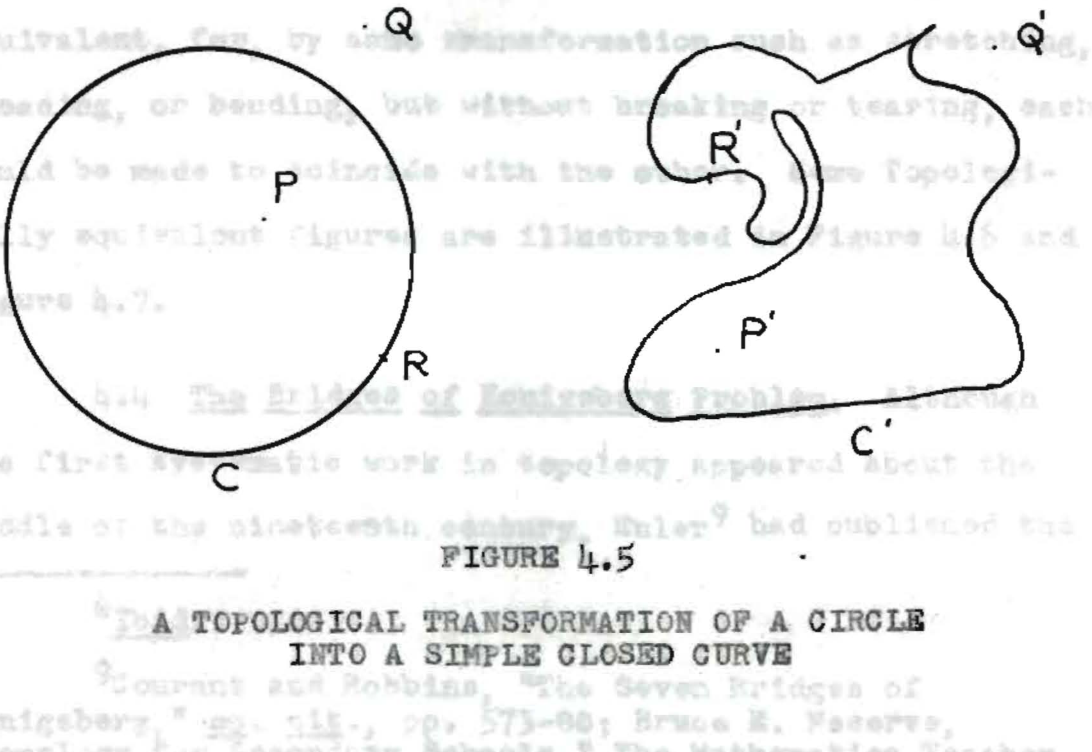

FIGURE 4.5

A TOPOLOGICAL TRANSFORMATION OF A CIRCLE
INTO A SIMPLE CLOSED CURVE

spheres are tangent internally in the set A and are tangent externally in the set B. No amount of stretching or bending will take A onto B.[8]

There are other topologically equivalent curves which cannot be made to coincide by any known deformation. An interesting exercise for the beginning student of topology would be to find a number of such pairs of equivalent figures which cannot be deformed onto each other by any topological transformation.

A simple closed curve is topologically equivalent to any other simple closed curve. Such geometric figures as the circle, square, and triangle are all topologically equivalent, for, by some transformation such as stretching, kneading, or bending, but without breaking or tearing, each could be made to coincide with the other. Some Topologically equivalent figures are illustrated in Figure 4.6 and Figure 4.7.

4.4 The Bridges of Konigsberg Problem. Although the first systematic work in topology appeared about the middle of the nineteenth century, Euler[9] had published the

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[8] Ibid.

[9] Courant and Robbins, "The Seven Bridges of Konigsberg," op. cit., pp. 573-80; Bruce E. Meserve, "Topology for Secondary Schools," The Mathematics Teacher, November, 1953, p. 471; and Eugene P. Northrop, Riddles in Mathematics (New York: D. Van Nostrand Company, 1944), p. 65.

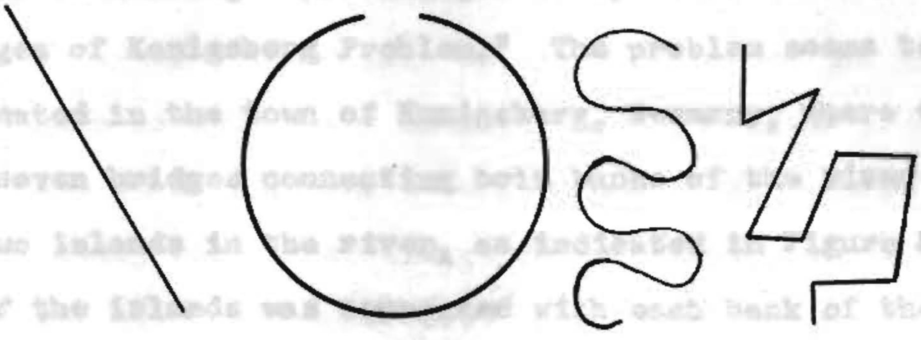

FIGURE 4.6

TOPOLOGICALLY EQUIVALENT LINES

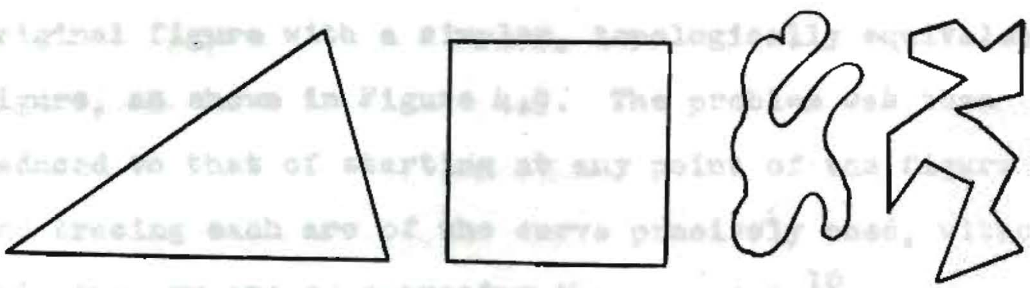

FIGURE 4.7

TOPOLOGICALLY EQUIVALENT SIMPLE CLOSED CURVES first topological study on record nearly one hundred years earlier in solving a problem generally referred to as the "Bridges of Konigsberg Problem." The problem seems to have originated in the town of Konigsberg, Germany, where there were seven bridges connecting both banks of the river Pregel and two islands in the river, as indicated in Figure 4.8. One of the islands was connected with each bank of the river by two bridges, and the other island was connected with each bank of the river by a single bridge. A single bridge connected the islands. The citizens of Konigsberg were fond of taking Sunday walks, and many attempted to cross all seven bridges, in sequence, without crossing the same bridge twice. It finally grew more and more evident that this was no elementary problem, as all attempts met with failure.

In his solution of the problem, Euler replaced the original figure with a simpler, topologically equivalent figure, as shown in Figure 4.9. The problem was then reduced to that of starting at any point of the figure and tracing each arc of the curve precisely once, without skipping any arc or retracing the same arc.[10]

Points A and B represent points on the two sides of the river, and points C and D represent points on the

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[10]Meserve, "Topology for Secondary Schools," op. cit., p. 475.

islands. Euler observed that there were an odd number of lines at each of the vertices of Figure 4.9.

Euler classified each of the vertices of the equivalent figure as either odd or even. Even vertices are those which have an even number of concurrent lines. Since an arc has two ends, there must always be an even number of odd vertices. Under the conditions of the problem, no line may be traced more than once though it is permissible to cross at the vertices any number of times.

Consider the case of a figure with two odd vertices, such as a square and its diagonal. We are to trace the figure without retracing any line, and we may cross a line at a vertex, if we wish. Remember that there are an odd number of lines concurrent at an odd vertex. If the starting point is not at an odd vertex, then each time an odd vertex is reached, there are an even number of lines from this vertex that have not been traced. Continuing from the vertex along one of the untraced lines, there will be left an odd number of untraced lines. The tracing is continued until there is only one untraced line. When this last line is traced, and the final vertex reached, the tracing must stop at this point. Meanwhile, the second odd vertex must have at least one line left untraced. This line of reasoning leads to the following conclusion. If the starting point is an odd vertex, then the tracing cannot end at this

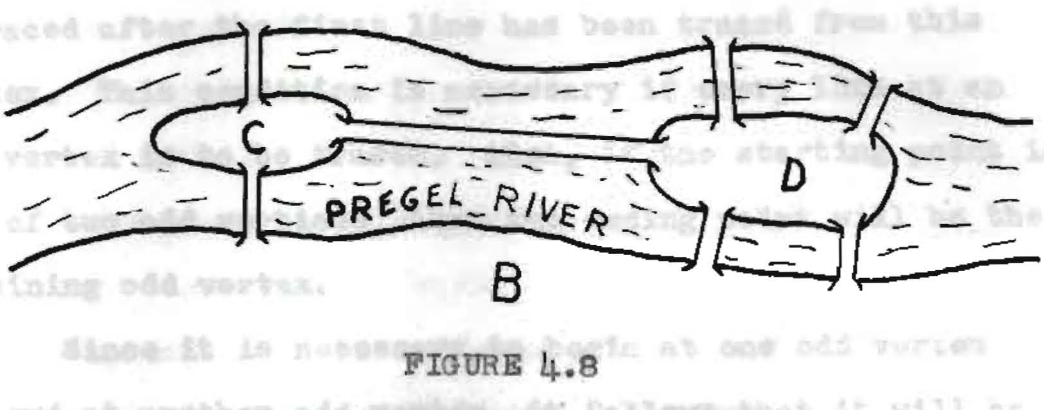

FIGURE 4.8

THE BRIDGES OF KONIGSBERG

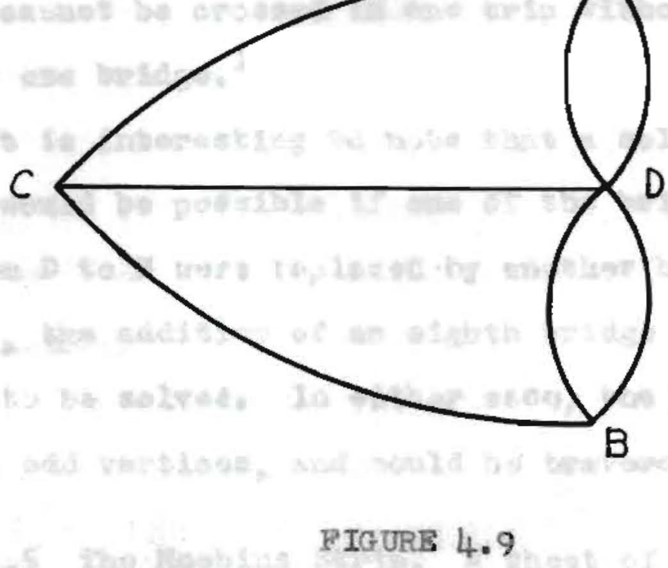

FIGURE 4.9

TOPOLOGICALLY EQUIVALENT FIGURE FOR KONIGSBERG BRIDGES vertex, for there will be an even number of lines left untraced after the first line has been traced from this vertex. This condition is necessary if every line at an odd vertex is to be traced. Also, if the starting point is one of two odd vertices, then the ending point will be the remaining odd vertex.

Since it is necessary to begin at one odd vertex and end at another odd vertex, it follows that it will be necessary to make half as many separate trips as there are odd vertices. The figure for the Konigsberg Bridges has four odd vertices, and must require at least two trips to traverse every line without retracing. Therefore, the bridges cannot be crossed in one trip without recrossing at least one bridge.[11]

It is interesting to note that a solution of the problem would be possible if one of the bridges from A to D or from D to B were replaced by another bridge from A to C. Also, the addition of an eighth bridge would permit the problem to be solved. In either case, the figure would have only two odd vertices, and could be traversed in one trip.

4.5 <u>The Moebius Strip</u>. A sheet of paper has two sides. It is impossible to get from one side to the other without first crossing an edge.

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[11] *Ibid.*

Take an ordinary strip of paper, like adding machine tape, twenty-four inches long and two or more inches wide. Paste both ends together to form a cylinder. With a crayon or colored pencil, color the inside surface of the strip. Also, color one edge of the strip.

Now take a second strip of paper of the same length and width, and give one end a half twist before pasting the two ends together. Take a crayon, as before, and begin coloring any place on the interior surface, and color the edge of the strip as before. How does the surface of the second strip differ from the surface of the first strip? Does the edge of the second strip have any property different from the first strip? How many surfaces does the second strip have? The name for the second type of strip is the Moebius strip, named after its inventor, A. F. Moebius.[12] The strip has been found to have other peculiar properties, as will be found from the following experiments.

<u>Experiment 1</u>. Begin coloring one edge of the Moebius strip, and continue until the starting point has been reached. How many edges does the strip have? How does the length of one edge of a Moebius strip compare with the length of one edge of the two sided surface?

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[12]Northrop, <u>op</u>. <u>cit</u>., p. 71.

Experiment 2. With scissors, start cutting down the center of a Moebius strip, and continue cutting until the starting point has been reached. How many strips will be obtained from this cutting? Color the interior of each strip. Color one edge of each strip. Do the two strips appear to have the same properties?

Experiment 3. Take a strip of paper of convenient length and width as has been suggested before, and give one end of the strip two half twists before pasting the ends together. Color the interior surface of this strip. Color one edge of the strip. What will be the result of cutting this strip down the center? How do the results of this experiment differ from those of Experiment 2?

Variations of these experiments may be carried on with very intriguing results. It is suggested that a number of experiments be performed with strips constructed with twists differing in number from those of the preceding experiments. Intuition will be found to be of little value in predicting the results of such experiments. In these experiments, it is advisable to select strips of paper long enough, in proportion to the width, to permit the desired number of twists without undue entanglement or tearing.

4.6 Topological curves. A simple closed curve, as defined in Paragraph 4.2, divides the plane in which it lies into two parts, one inside the curve, and the other outside the curve. Thus, according to the Jordan Curve Theorem, it is not possible to get from the inside of a simple closed curve to the outside of the curve without crossing the curve at least once. There are other curves, not simple closed curves, where it may be necessary to penetrate the curve more than once in order to reach points outside the given curve. One such curve might be the square with one diagonal. If the regions within the square which are separated by a diagonal are labeled A and B, and if the region outside the square is labeled U, then it is possible to get from any point in A or B to any point in U by crossing the figure at any one point if that point is not on the diagonal. If the diagonal is crossed, a second crossing must still be made before any point of U can be reached. It may be said that relative to the region A, B is neither inside nor outside the curve.[13]

4.7 <u>Topological</u> <u>surfaces</u>. It is possible to start from any point in the inside of a sphere and proceed to any point outside the sphere by a simple arc, penetrating the surface of the sphere only once. This is true for any simple closed surface. Simple closed surfaces can be found

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[13]*Ibid.*, p. 67.

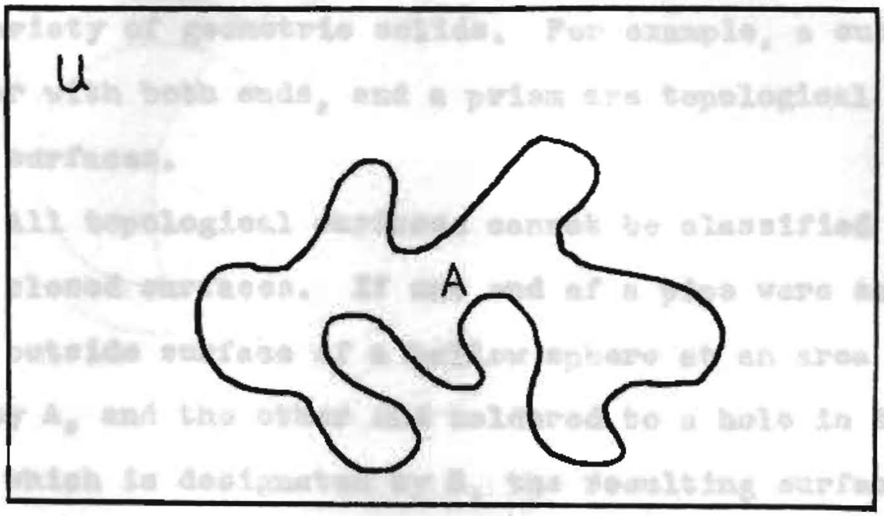

FIGURE 4.10

A SIMPLE CLOSED CURVE

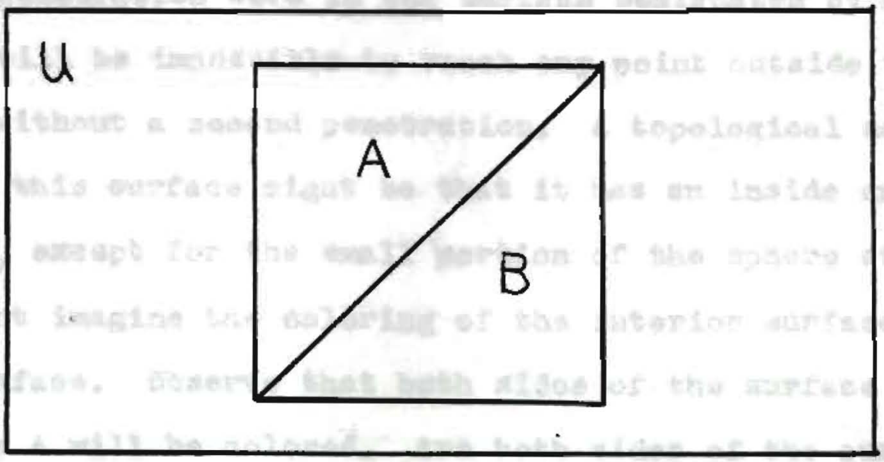

FIGURE 4.11

A CLOSED CURVE WITH TWO INSIDES

in a variety of geometric solids. For example, a cube, a cylinder with both ends, and a prism are topological simple closed surfaces.

All topological surfaces cannot be classified as simple closed surfaces. If one end of a pipe were soldered to the outside surface of a hollow sphere at an area designated by A, and the other end soldered to a hole in the sphere which is designated by B, the resulting surface is no longer a simple closed surface. It is possible to start at any point on the inside of the sphere and proceed to any point outside the sphere by penetrating the surface only once with the exception of that portion of the surface at A. If the penetration were in the surface designated by A, it would still be impossible to reach any point outside the sphere without a second penetration. A topological description of this surface might be that it has an inside and an outside, except for the small portion of the sphere at A.[14] One might imagine the coloring of the interior surface of this surface. Observe that both sides of the surface designated by A will be colored. Are both sides of the surface at A on the inside of the surface?

Another interesting surface, called "Klein's bottle," can be constructed. One end of a hollow tube is bent around

[14]Ibid., p. 68.

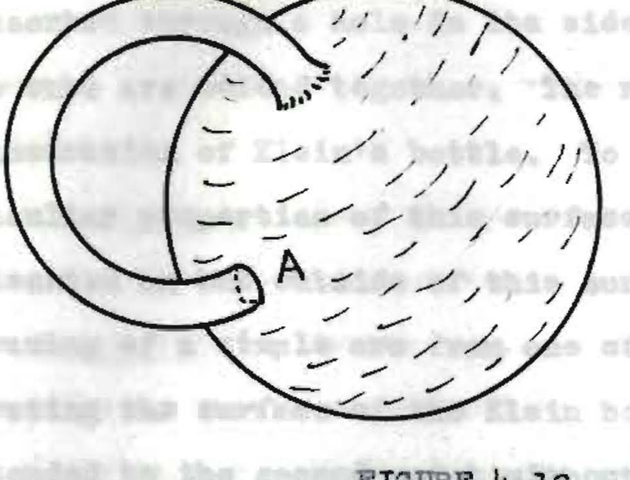

FIGURE 4.12

A CLOSED SURFACE WITH AN INSIDE AND OUTSIDE EXCEPT AT A

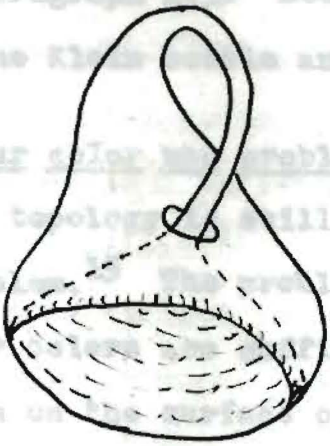

FIGURE 4.13

KLEIN'S BOTTLE

and inserted through a hole in its side, and the two ends of the tube are welded together. The resulting surface is an illustration of Klein's bottle. To illustrate one of the peculiar properties of this surface, imagine two points, each located on the outside of this surface. Then, imagine the drawing of a simple arc from one of the points and penetrating the surface of the Klein bottle. Can this arc be extended to the second point without again penetrating the surface of the Klein bottle?

As another experiment with the Klein bottle, imagine the coloring of the surface of the bottle. How does the result of this coloring compare to the coloring of the Moebius strip in Paragraph 4.5? Does this suggest any relation between the Klein bottle and the Moebius strip?

4.8 The four color map problem. One of the most famous problems of topology is still unsolved. This is the four color map problem.[15] The problem is to determine whether or not four colors are sufficient to color any map that might be drawn on the surface of a plane or a sphere. The requirement of the problem is that no two countries with a common border may have the same color. A single point is not regarded as a common border. The problem is so

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[15]Charles Stanley Ogilvy, Through the Mathescope (New York: Oxford University Press, 1956), p. 125.

simply stated that it may seem surprising that it has not been proved that four colors are either sufficient or are not sufficient.

Experiment 4. Draw a map of several countries that will require no more than two colors.

Experiment 5. Draw a map of several countries that will require no more than three colors.

Experiment 6. Draw a map of several countries that will require at least four different colors.

Oddly enough, corresponding problems have been solved for surfaces more complex than the sphere. It has been established that the torus, or doughnut, requires no more than seven colors.[16]

It has been established that five colors are sufficient for every map on a plane or a sphere, but to this date, no map on a plane or a sphere has ever been produced that requires all five colors.[17] It has also been established that four colors are sufficient for any map containing less than 83 countries, and the number of colors is independent of the sizes and shapes of the countries.[18]

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16 Northrop, op. cit., p. 75.

17 Richard Courant and Herbert Robbins, What is Mathematics? (New York: Oxford University Press, 1941), p. 247.

18 Meserve, "Topology for Secondary Schools," loc. cit.

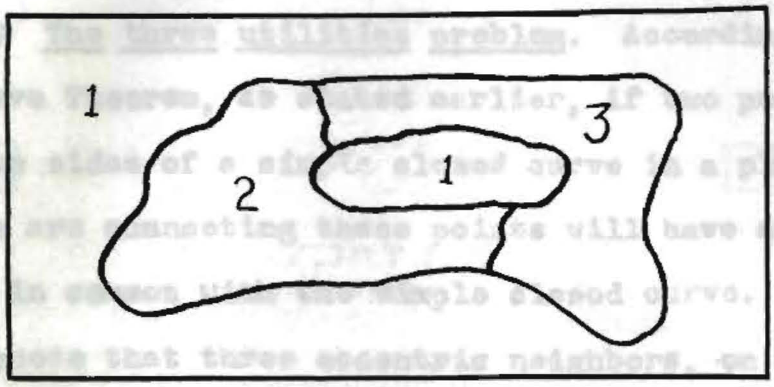

FIGURE 4.14

A MAP REQUIRING THREE COLORS

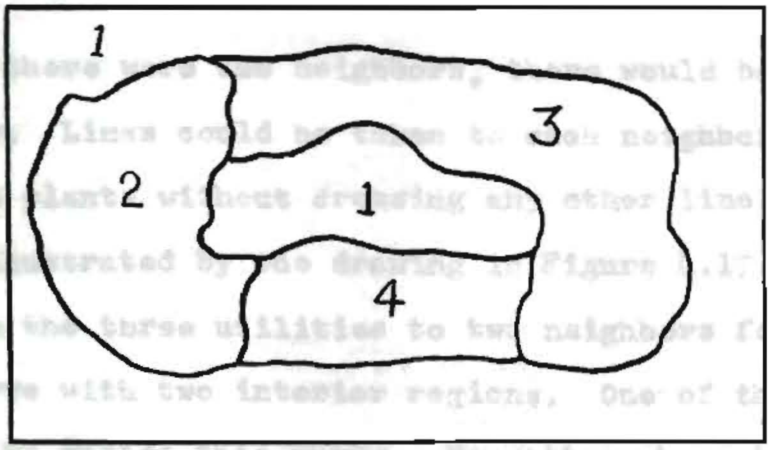

FIGURE 4.15

A MAP REQUIRING FOUR COLORS

4.9 <u>The three utilities problem</u>. According to the Jordan Curve Theorem, as stated earlier, if two points are on opposite sides of a simple closed curve in a plane, then any simple arc connecting these points will have at least one point in common with the simple closed curve.

Suppose that three eccentric neighbors, on the same side of the street, each wanted a gas line, an electric line, and a water line direct to the utility plants. Each neighbor refused to allow his line to be crossed by any other line.

<u>Experiment 7</u>. Make a drawing to show how many of the eccentric neighbors can be satisfied by the three utilities.

If there were two neighbors, there would be no difficulty. Lines could be taken to each neighbor from the respective plants without crossing any other line. This fact is illustrated by the drawing in Figure 4.17. The lines from the three utilities to two neighbors form a closed curve with two interior regions. One of the utilities will be inside this curve. No matter where the third house is located, it will be cut off from one of the three utilities. That is, according to the Jordan Curve Theorem, it will be necessary to cross some line in order to connect the third house with all three utilities.

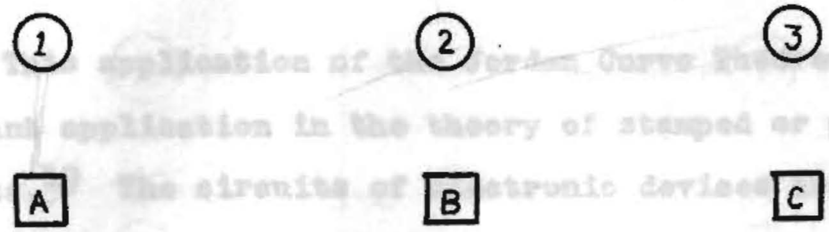

FIGURE 4.16

THE UTILITY COMPANIES PROBLEM

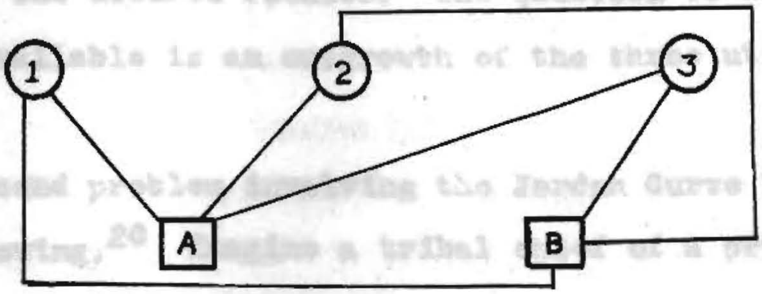

FIGURE 4.17

THREE UTILITIES AND TWO NEIGHBORS

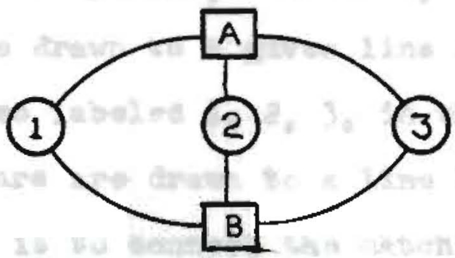

FIGURE 4.18

TOPOLOGICAL EQUIVALENT OF FIGURE 4.16

This application of the Jordan Curve Theorem has important application in the theory of stamped or printed circuits.[19] The circuits of electronic devices are so intricate that it is often simpler to stamp or print the circuits with some conductor of electricity than it is to lay insulated wires. Such circuits cannot cross if they are to give the desired results. The question of what circuits are available is an outgrowth of the three utilities problem.

A second problem involving the Jordan Curve Theorem is the following.[20] Imagine a tribal chief of a primitive tribe who has a beautiful daughter. There are many suitors who wish to marry the daughter. In order to select the best man for his daughter, the chief proposes two problems.

Given a set of points, labeled 1, 2, 3, from which perpendiculars are drawn to a given line a, and a second set of points, also labeled 1, 2, 3, in the same order from which perpendiculars are drawn to a line b, parallel to a. The first problem is to connect the matching numbers with lines which do not intersect or cross any line defined in the above data.

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[19]Ogilvy, op. cit., p. 127.

[20]Meserve, Fundamental Concepts of Geometry, op. cit., p. 469.

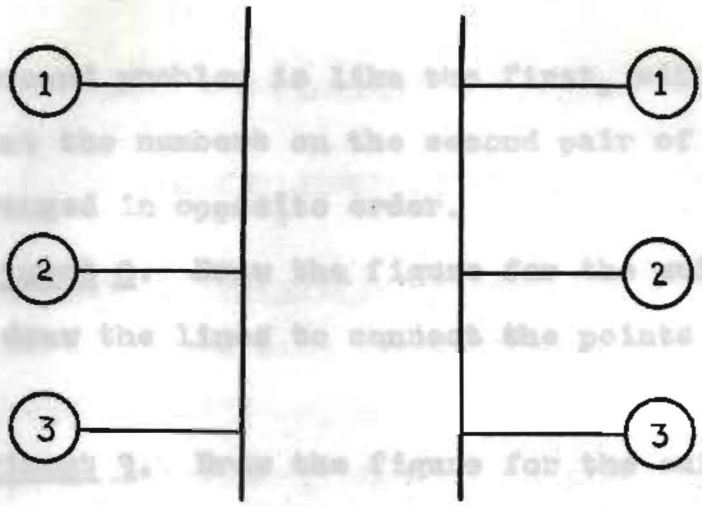

FIGURE 4.19

SUITOR'S FIRST PROBLEM

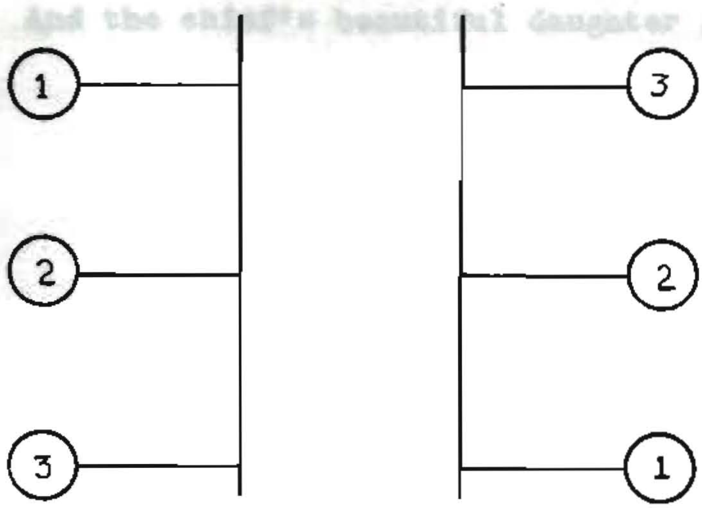

FIGURE 4.20

SUITOR'S SECOND PROBLEM

The second problem is like the first, with the exception that the numbers on the second pair of perpendiculars are arranged in opposite order.

Experiment 8. Draw the figure for the suitor's first problem and draw the lines to connect the points with like numbers.

Experiment 9. Draw the figure for the suitor's second problem. Try to connect the points with like numbers. Is this possible? Why?

The solution of both problems is left to the reader. The second problem can not be solved. The reader will suspect this, when his efforts to solve the problem are not successful. And the chief's beautiful daughter remains unmarried.

CHAPTER V

NON-EUCLIDEAN GEOMETRY

5.1 Meaning of geometry. A geometry is a branch of mathematics which starts with certain space concepts, assumptions, and definitions, and develops into a logical system of spatial relations. The most common type of geometry is the Euclidean geometry which has been traditionally taught in the secondary schools.

When the great geometer, Euclid, compiled his monumental Elements, sometime around 300 B. C., he stated certain postulates, axioms, and definitions. From this list of accepted assumptions, he derived a system of geometry which is logically dependent upon these assumptions. It is now recognized that some of the terms used, such as a point, line, and plane, are properly undefinable. Other concepts or configurations are defined in terms of the fundamental concepts.

Euclid listed ten basic assumptions which he regarded as self-evident truths. These assumptions are to be accepted without proof. The ten assumptions of Euclid are divided into two sets. The first five assumptions, or postulates, deal with geometrical relations so self-evident they are to be accepted without proof. The second five assumptions were called common notions, or statements of generally accepted truths.

The five geometrical postulates of Euclid describe the basic operations which may be accomplished with the prescribed tools of geometry, the unmarked straight edge and compass, and basic geometrical facts. These five geometrical postulates are as follows.[1]

(1) To draw a straight line from any point to any point.

(2) To produce a finite straight line continuously in a straight line.

(3) To describe a circle with any center and distance.

(4) That all right angles are equal to one another.

(5) That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The five common notions of Euclid, mentioned above, are stated as follows.[2]

(1) Things which are equal to the same thing are also equal to one another.

(2) If equals be added to equals, the wholes are equal.

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[1]Thomas L. Heath, The Thirteen Books of Euclid's Elements (New York: Dover Publications, Inc., 1956), pp. 154-5; and Harold E. Wolfe, Introduction to Non-Euclidean Geometry (New York: The Dreyden Press, 1945), p. 4.

[2]Ibid.

(3) If equals be subtracted from equals, the remainders are equal.

(4) Things which coincide with one another are equal to one another.

(5) The whole is greater than the part.

These five geometrical postulates of Euclid were suggested by experience and intuition. No previous knowledge of geometrical relations is assumed other than the basic postulates and definitions. Any additional geometrical truths must be logical derivations of the basic axioms or postulates. Any attempt to prove, logically, any one of the fundamental postulates must lead inevitably to the completion of a vicious circle where one proposition must depend upon another for its proof.[3] However, some critical investigations of Euclid's geometry have revealed a number of defects in the logical structure of the geometry. An example of one such defect might be illustrated as follows. Euclid's fifth postulate gives the condition necessary for two straight lines to intersect at a point, but no postulate is given to indicate the condition necessary for two circles or a circle and a straight line to intersect. The assumption that circles do intersect circles and lines intersect circles is used by Euclid in the proof of certain theorems. This assumption cannot be proved from any of the listed

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[3]Wolfe, op. cit., p. 2.

common notions or postulates and should be listed as an additional postulate.[4]

The structure of all geometries are similar to that of Euclidean geometry. Euclidean geometry is based on a set of assumptions and definitions which are compatible. Nothing else is assumed. Other ideas, or relations, stated as theorems, must be derived logically from what has been assumed or previously established. The following paragraph illustrates how a geometry may be built from a complete and consistent set of definitions and postulates, even if these are contrary to our intuition.

5.2 A geometry of twenty-five points. In developing a geometry, we are free to select any set of basic assumptions we please, provided that these assumptions do not contradict each other directly or by implication, and provided the set of assumptions is sufficiently complete to provide a basis for logical deductions.

As an example of a geometry, very different from ordinary Euclidean geometry, consider the finite geometry

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[4]Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Rinehart and Company, Inc., 1958), pp. 33-4.

of twenty-five points.[5] The twenty-five points are arranged in a square array, with five points in each row and five in each column. Each point is designated by one of the twenty-five letters of the English alphabet from A to Y inclusive, without duplication. These letters are arranged in three blocks of twenty-five letters each. Each block represents a different arrangement of the points so that each pair of points will appear in precisely one row or column of one of the three blocks, no pair appearing in a row or column more than once. No letter representing a point appears twice in the same block.

Figure 5.1 illustrates the positions of the twenty-five points, with three sets of letters representing them. The blocks of letters representing the points will be designated by the colors, red, blue, and yellow, respectively.

As our basic assumptions, let us adopt the following ideas as the foundation for the geometry.[6]

(1) There are exactly twenty-five points in this geometry. Each point is designated by one of the letters of the alphabet from A to Y, inclusive. Each point is

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[5]John C. Brixey and Richard V. Andree, Modern Trigonometry (New York: Henry Holt and Company, 1955), pp. 150-1; and Lillian R. Lieber, The Education of T. C. Mits (New York: W. W. Norton and Company, Inc., 1944), pp. 154-64.

[6]Brixey and Andree, op. cit., p. 150.

Red	Blue	Yellow
A B C D E	A I L T W	A X Q O H
F G H I J	S V E H K	R K I B Y
K L M N O	G O R U D	J C U S L
P Q R S T	Y C F N Q	V T M F D
U V W X Y	M P X B J	N G E W P

FIGURE 5.1

ELEMENTS OF A FINITE GEOMETRY OF TWENTY-FIVE POINTS

represented once in each of the three blocks; however, the points may have different letter designations in each of the three blocks.

(2) A straight line shall mean any row or column in any of the three blocks. Since there are five rows and five columns in each block, there must be exactly thirty distinct lines in this geometry. A line consists of five points, either in a row or in a column.

(3) A line segment is congruent to another line segment when both point pairs occur in rows, or both in columns, and the number of directed steps is the same in each pair. A step is considered to be the distance between any two adjacent points in a row or column. The number of steps between the first and last point of a line segment is always counted to the right in a row line and down in a column. The first letter of a row or column is considered

to follow the last letter of the row or column. For this reason, a line may be thought of as continuous with no end point. Any two point designations determine a distinct line segment in one of the three blocks.

Some examples of congruent line segments are given as follows. In the line ABCDE in the red block and the line GORUD in the blue block, the line segment AD in the red block is congruent to the line segment OD in the blue block. Each segment, AD and OD, contains three steps. The segment LR (blue) is congruent to the segment GK (yellow), but LR (blue) is not congruent to KB (yellow). Each segment, LR and KB, contains the same number of directed steps, but one is in a row and the other is in a column and cannot be considered congruent according to the definition of congruent line segments. Is BF congruent to RX? Is KN congruent to VK? Why?

(4) Two lines are parallel if they have no point in common. The line ABCDE is parallel to line UVWXY, and the line IVOCP is parallel to the line WKDQJ. Can a line have a parallel line in any other block?

The idea of parallels used in the geometry of twenty-five points differs from that of Euclidean geometry in the fact that no mention is made of the straightness of a line. In Euclidean geometry, parallel lines are everywhere equidistant. In the geometry of twenty-five points, parallel lines are everywhere equidistant if one measures this distance in steps, rather than any unit of measurement in the Euclidean sense. Figure 5.2 is used to illustrate the possible arrangement of the points of the red block.

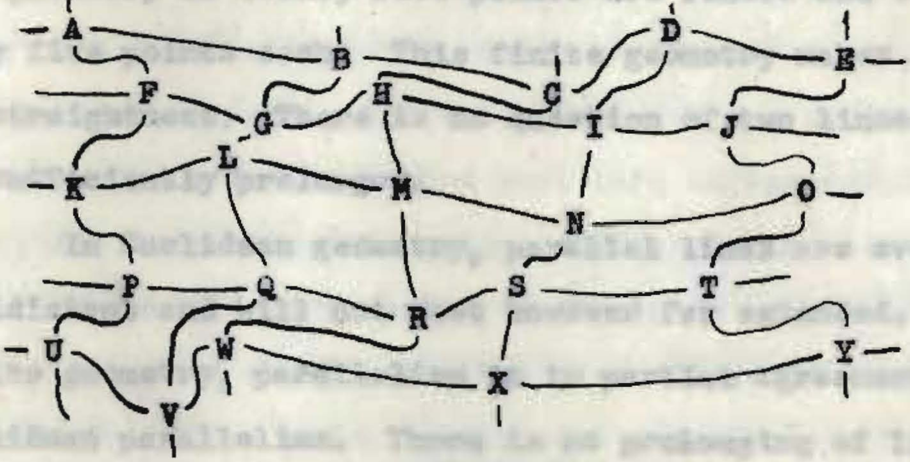

FIGURE 5.2

ONE POSSIBLE ARRANGEMENT OF TWENTY-FIVE POINTS

(5) Two lines are perpendicular if, and only if, there exist two points on one line such that the absolute distance from these two points to any point on the other line are equal. The line ABCDE is perpendicular to the line AFKPU and the points B and E may be taken as the points which are equidistant from each point of the line AFKPU.

It is interesting to note some of the differences between this finite geometry of twenty-five points and the Euclidean geometry. While some of the terms are the same, the interpretations of such terms are quite different. Points are still undefined elements, though they may be represented by letters and are finite in number. In Euclidean geometry, straight lines may be extended indefinitely and contain an infinite number of points. Lines of the geometry of twenty-five points are finite and contain only five points each. This finite geometry makes no mention of straightness. There is no question of two lines meeting if sufficiently prolonged.

In Euclidean geometry, parallel lines are everywhere equidistant and will not meet however far extended. In this finite geometry, parallelism is in partial agreement with Euclidean parallelism. There is no prolonging of lines, for each line is finite, and the term, distance, in the Euclidean sense, has no meaning in this geometry. In finite geometry, the number of steps between parallel lines is the same at all points on the parallel lines, although a step is not defined in any unit of measurement in the Euclidean sense.

A triangle in the geometry of twenty-five points may be defined as any triplet of points, such as H, L, and R. The sides of the triangle are the line segments determined by the points, HL, LR, and RH. Since line segments must be taken from the lines, themselves, and lines are found in rows and columns, the segments must be either in rows or in columns. The line segment HL is found in the yellow block, LR is in the blue block, and HR is in the red block. This triangle is completely dismembered. Each side is either red, blue, or yellow, depending on which block the letters appear in rows or columns. The sides of the triangle may or may not be congruent to each other. It may be convenient to define triangles in which no two sides are congruent as scalene triangles, and triangles with two congruent sides as isosceles triangles. How does this correspond to the Euclidean definition of scalene and isosceles triangles?

In Euclidean geometry, a circle is usually defined as the locus of all points at a given distance from a fixed point. The fixed point is called the center of the circle. In this finite geometry, a circle is defined as the locus of points which are a given number of steps from a fixed point, and such that each of the points, when taken with the center, determines a line segment which is congruent to all other line segments thus formed. The number of steps between a point on the circle and the center of the circle is the radius of the circle.

If we choose the point G as center of a circle, and designate the radius as two steps, then the circle contains only the six points, I, J, R, U, W, and P. Note that GQ, in the red block, cannot be congruent to line segments which are in rows. Therefore, Q is not a point of the given circle.

There are definitions of several terms of Euclidean geometry which might be applicable to the geometry of twenty-five points. A student of this finite geometry could gain much in experience by attempting to give logical definitions of certain Euclidean terms which do not conflict with the previously stated basic assumptions of this geometry of twenty-five points. Give a definition of a quadrilateral which satisfies the geometric assumptions of this finite geometry. Could there be diagonals in this geometry, and if so, how would they differ from the diagonals of Euclidean geometry? Definitions of these and certain other terms common to Euclidean geometry and the twenty-five point geometry are given below.

(1) A point can still be regarded as the intersection of two lines, in much the same sense as in Euclidean geometry. The principal difference is that the number of points in a line is limited in this geometry.

(2) A line is still determined by any two points, with the restriction that the points must lie in the same row or column. Lines cannot be oblique as in Euclidean geometry. It is assumed that any line contains the entire set of five points. Any line containing less than the five points is a line segment.

(3) A quadrilateral is a four sided figure determined by any four point designation. The sides of the

quadrilateral are the line segments determined by any four point designations taken in order. The quadrilateral BJUD has the line segments BJ (blue), JU (yellow), UD (blue), and DB (red) for its sides.

(4) A parallelogram is a quadrilateral whose opposite sides are congruent and parallel.

(5) A rhombus is a quadrilateral whose opposite sides are congruent and parallel and all four sides contain the same number of steps.

(6) Diagonals are the line segments determined by the opposite vertices of a quadrilateral.

(7) The mid-point of a line segment is the point which is the same number of steps from each end of the line segment. Naturally, the point must lie on the line.

(8) A line is tangent to a circle if it contains one and only one point of the circle.

(9) Lines are considered perpendicular if they have one and only one point in common.

(10) Perpendicular bisectors are the lines passing through the mid-points of line segments.

(11) The altitude of a triangle is the line through one vertex of a triangle and perpendicular to the opposite side.

(12) The medians of a triangle are the line segments joining one vertex of a triangle with the midpoint of the opposite side.

It is really quite remarkable how similar many of the Euclidean terms are to the terms used in the geometry of twenty-five points. It must be remembered that certain of the terms, such as point and line, cannot be actually defined in the true sense of the word. At best, the definitions of these terms are merely descriptions which agree with our intuition. With the above list of definitions and assumptions of the twenty-five point geometry, it is actually possible to prove some theorems whose statements are the same as those of the Euclidean system. Some representative theorems of this type are discussed in the following paragraphs of this section.

Theorem 1. If two sides of a quadrilateral are congruent and parallel, so are the other two sides.

Discussion. Two lines must be in the same block to be parallel or they would have a common point. Pick any pair of congruent line segments in any block. For example, in the yellow block, the line segments VEHK and GORU are both congruent and parallel. Now, V, G, U, and K must be the vertices of the determined quadrilateral. There must be a line segment joining the vertices, V and G, and likewise the vertices, K and U. In the red block, column two, we find the line segment VBG determined by V and G, and in the same block, column one, we find a parallel segment KPU determined by the points, K and U. Both VBG and

KPU are parallel because they are segments of lines that have no point in common. The point-pairs, VG and KU, are congruent as they are two step pairs.

Theorem 2. The diagonals of a parallelogram bisect each other.

Discussion. From any block, say block three, take for vertices, the points R, C, U, and K. Thus, R and U are opposite vertices determining one diagonal, and C and K determines the second diagonal. The line UDGOR in the red block, row three, has mid-point G bisecting it. In block yellow, column two, line CTGXK has mid-point G. Since G is the mid-point of both diagonals, the diagonals bisect each other at their common point, G.

The following theorems are stated in the Euclidean form. The corollaries are stated as results pertinent to the geometry of twenty-five points. It is suggested that the reader verify the theorems and corollaries for this new geometry.

Theorem 3. At any point on a circle, there is one and only one tangent.

Corollary 1. There are six and only six tangents to any circle.

Corollary 2. If a circle is contained in rows or in columns, then the tangents must be in columns or rows, respectively.

Theorem 4. The medians of a triangle are concurrent.

Theorem 5. The altitudes of a triangle are concurrent.

Theorem 6. The medians of a triangle are concurrent at a point which divides the medians in the ratio of 1:2.

Other theorems which use the language of Euclidean geometry might be found which are stated exactly the same in the geometry of twenty-five points. It is left for the reader to find and establish other theorems of this nature.

The finite geometry described in the preceding paragraphs does not have a common physical model as entirely acceptable as that of Euclid's geometry, but it has been introduced in order to illustrate more clearly how a geometry is built. The fundamental tools of any geometry are the basic assumptions and definitions of the terms used to build the logical structure. Another example of a simple finite geometry is the six-point geometry briefly described by Adler in the book Modern Geometry.[7]

5.3 The Fifth postulate of Euclidean geometry. It has already been stated in Paragraph 5.1 of this chapter that mathematicians did not readily accept the fifth postulate of Euclid. Even the earliest commentators did not

[7]Claire Fisher Adler, Modern Geometry (New York: McGraw-Hill Book Company, Inc., 1958), pp. 17-9.

believe that the fifth postulate was independent of the others and sufficiently self-evident to be listed with those postulates to be accepted without proof. A great number of noted mathematicians from the time of Euclid to the discovery of non-Euclidean geometry attempted to prove the fifth postulate by deducing it from the other postulates. However, all such attempts failed.[8]

The fifth postulate of Euclidean geometry is not usually stated as in Paragraph 5.1. Geometry text books of today generally replace it with a simpler statement which is equivalent to the original or may be deduced from the original. The usual form of this postulate in present day text books is that of the Playfair axiom, that through a given point not on a given line, one and only one parallel can be drawn to that line.[9]

At the beginning of the nineteenth century there was still no accepted proof for the parallel postulate, and many mathematicians had already admitted the necessity of listing it among the postulates. However, there were still those mathematicians who continued in their efforts to solve the puzzle. Prominent among these mathematicians were the

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[8]Roberto Bonolo, Non-Euclidean Geometry (La Salle, Illinois: The Open Court Publishing Company, 1956), p. 2.

[9]Wolfe, op. cit., p. 16.

three men who are generally considered to be the founders of the study of non-Euclidean geometry, Gauss in Germany, Bolyai in Hungary, and Lobachewsky in Russia. It is remarkable to note that these three men made essentially the same discovery at about the same time, and each worked independently of the other in different countries. Each discovered that a consistent geometry could be built if the fifth postulate of Euclid were replaced by a postulate which assumed more than one parallel to a given line through a given point.[10]

A non-Euclidean geometry is generally regarded as a geometry which is built with a substitute for the fifth postulate of Euclidean geometry. All other axioms and postulates not affected by the fifth postulate are left unchanged. The postulate used in place of the fifth postulate is called the characteristic postulate of the new geometry.

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[10] Ibid., p. 45.

CHAPTER VI

SUMMARY AND CONCLUSION

6.1 Summary. The material contained in this thesis is intended to be of value to the teacher of high school mathematics who has had little or no formal training in modern mathematics. According to one author, the present trends in mathematics are two-fold: "(1) the introduction of modern mathematics into the curriculum, and (2) the introduction of accelerated programs to the gifted students."[1] These trends are accompanied by an emphasis upon the "learning-by-discovery" method of instruction. It is hoped that the material of this study will help some teachers bring their teaching into line with these above stated trends.

The concepts and language of sets enters, in an essential way, much of modern mathematics. It is desirable to introduce the elements of set theory to the high school student as soon as possible in order that he may be better prepared to continue his studies at a more advanced college level of mathematics upon graduation from high school.

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[1]Daymond J. Aiken, "Some Comment on Accelerated Mathematics," The Mathematics Teacher, April, 1958, p. 292.

The introduction of set theory in high school work will not usually go far into the abstract theory of sets. It has been proposed that concepts, vocabulary, and some of the symbolism of sets be used whenever they contribute to the interest and understanding of mathematics.[2]

An important characteristic of both "traditional" and "modern" mathematics is the emphasis upon deduction.[3] The material on topology and non-Euclidean geometry, in Chapter IV and Chapter V, respectively, has been introduced to supplement the teaching of geometry by providing interesting subject matter based on mathematical deduction and to interest students in areas of mathematics not generally included in high school courses.

6.2 Conclusion. The material of this study is not a complete review of all phases of modern mathematics. The teacher of mathematics will benefit himself and his teaching by continuing the study of sets, topology, and non-Euclidean geometry beyond the elementary treatment of those topics in this thesis.

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[2]Commission on Mathematics of the College Entrance Examination Board, Sets, Relations, and Functions, Commission on Mathematics of the College Entrance Examination Board, New York, 1958, p. 4.

[3]Albert E. Meder, Jr., Modern Mathematics and its Place in the Secondary School, Commission on Mathematics of the College Entrance Examination Board, New York, 1957, p. 4.

It is also suggested that other areas of modern mathematics be considered for inclusion in the program of supplementing the teaching of mathematics on the secondary level. Such mathematical topics as symbolic logic, functions, fields and groups, matrices, projective geometry, probability, and statistical analysis are a few of the topics which might provide interesting and stimulating material to supplement material in the present mathematics text books.

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