TALKS TO ELEMENTARY TEACHERS ON THE STRUCTURE OF MATHEMATICS

A Thesis

Submitted To

The Department of Mathematics and The Graduate Council

of the Kansas State Teachers College of Emporia

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

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August 1960

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ACKNOWLEDGEMENT

To Dr. John M. Burger, Head of the Department of Mathematics of Kansas State Teachers College of Emporia, who so kindly supplied direction and encouragement in the preparation of this thesis, the writer wishes to express her sincere gratitude. The writer also acknowledges her indebtedness to Mr. Lester E. Laird, Professor of Mathematics, for his helpful suggestions.

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CHAPTER I

INTRODUCTION

For the past five years, many individuals and organizations have been actively engaged in improving the mathematics program in the American public schools. Most of their efforts have been directed towards the secondary schools. Teachers at that level have been recipients of various grants to help them acquire more subject matter knowledge. High school curriculums have been examined by authorized committees which have in turn made suggestions for revisions. Sample textbooks have been written and are being tested and improved by some of the best mathematicians and mathematics teachers in the country.

While proposals for improving the program in grades one through six are being studied by such organizations as the School Mathematics Study Group, the National Council of Teachers of Mathematics Elementary School Curriculum Committee, and the University of Illinois Committee on School Arithmetic, very little has been accomplished at the time of writing this thesis.

Yet it is in these grades that the student gets his first formal instruction in mathematics. It is here that the foundation is laid for all future study. Here the student develops likes and dislikes for the various branches of learning. Both his ability and his desire to proceed in the study of mathematics are largely determined during these years in the elementary school.

The Problem

It was the purpose of this study to develop a series of talks to be given to elementary teachers. No attempt has been made to instruct them in methods of teaching arithmetic. Instead, the emphasis has been on giving to the elementary teachers an overview of the number systems, their properties, their limitations, and their relationships.

Importance of the study

Since the secondary teachers have been receiving additional instruction, it seemed that they should be the first to carry the attempts at improvement into the elementary school. These talks have been developed in consideration both of the ability of the secondary teachers to give such and that of the elementary teachers to understand them.

Scope of the study

The talks were completed after they had been given informally to the teachers of the St. John Elementary School, St. John, Kansas. They were written in a style which seemed most comprehensible to the teachers. Both content and form were modified to include answers to some questions which arose at that time.

Organization

There is a total of five talks.

The first one is an introduction concerned with a justification of the talks. It includes some background information on who is criticizing the mathematics program, what those criticisms are, why it is necessary to heed them, what efforts are being made to meet them, and what part the elementary teachers have in these efforts.

The second talk develops the set of natural numbers, their basic properties, and the operations which may be performed with them.

The third talk is concerned with the set of positive rational numbers.

In the fourth talk, consideration is given to the development of the integers. The rationals are then defined in terms of the integers so that the negative rationals are included.

In the final talk, the set of real numbers is developed. Irrationals are introduced as non-repeating infinite decimals. The relationship of all the number systems is illustrated by use of the number line.

The order of the development was determined by the order in which the number systems are introduced in the schools.

Sources of data

The principal sources of material used in the preparation of this thesis have been books and textbooks, essays, and magazine articles. Of particular assistance have been the chapter on "Number and Operation" in the Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, and the book, Understanding Arithmetic, by Robert L. Swain.

CHAPTER II

INTRODUCTORY TALK

The mathematics program in the American public schools is under fire. The battle has been going on for several years.

Business and industry, the armed services, the colleges and universities, research institutes, mathematics teachers and their fellow scientists, governmental agencies, the politicians, and Mr. John Q. Public, himself, are all levelling the charges. Everyone is in the firing squad. Shells fly thick and fast.

The bullets, however, have been cast chiefly in two molds. The first one is that not enough young people are being trained in mathematics. The second one is that their training is not good enough. Concern for both quantity and quality has shaped the molds.

The assailants make no secret of the formula for the powder which they pour into the molds. Criticism of the educational aims provides one of the ingredients. Trying to make "well adjusted" individuals is not enough. These so-called "well adjusted" individuals need to know something. Also, trying to educate all students by the same process is the wrong approach and not really democratic. Individual differences in children should be recognized and accepted by teachers, pupils, and parents alike. This country should, of course, provide equal educational opportunities for all American children. It is flying in the face of nature to insist that those opportunities must be identical. By insisting that all children be put through the same mill, we have not only frustrated and poorly equipped the below-average, provided the average student with a very inadequate background of this-and-that, but we have bored and sadly neglected the above-average student who is the potential mathematician.

A second ingredient is criticism of the curriculum. Mathematics is a live and lusty discipline. It is growing. A large amount of new mathematics has been created in the past one hundred years. Almost none of it is being introduced into the public schools. The mathematics curriculum has remained much as it was a century ago. Consequently, high school graduates are getting no introduction to the new mathematics.

Besides the courses, the content of those courses has been watered down and sugar-coated until the result is superficial. Furthermore, it takes a student too many years to go through the courses now considered the maximum of public school training. Content from courses which, for a long time, have been considered senior high school material could be introduced in the junior high, where the present mathematics courses get repetitious and boring. Many ideas not now considered could well be introduced at the elementary level.

Another ingredient is criticism of the text books. Many of them have been written by people who don't know anything about mathematics. The authors and publishers are more concerned with catching the eye than educating the students. Many of the text books turn out to be mathematically unsound.

Criticism of the teachers is an ingredient. The teachers are not well educated in mathematics. They haven't studied it long enough or thoroughly enough to understand it themselves. Consequently, they teach it as tricks and skills. This is not enough.

As a result, the students have no real understanding either. They can do only what machines can do, and that not as well. It is no longer enough to be able to add, subtract, multiply, divide, period. The need is for people who know why and can tell the machine what to do and when. The need is for people who can solve problems which they have never seen before. What is even more important, they need to be able to recognize what a problem is.

Criticism of the American way of life provides more explosive ingredients. We have failed to recognize that showing a credit in mathematics on paper is not the same as a good, thorough, systematic study and some knowledge of the subject. We childishly cling to the idea that we can eat our cake and have it, too.

We have made heroes of the wrong people. Our children grow up thinking it is more important to be a sexy television "singer," or a daredevil driver of a sportscar, or be ambidextrous with side arms than it is to be an intellectual. We use very unglamourous terms to describe the latter. We hold our noses while we use those terms; brain, egghead, square.

We are also uninterested in anything unless we can see an immediate use for it. That use largely implies physical use. We are so concerned with measuring everything by a material yardstick that we have failed to develop a more important but less tangible method for determining what is important.

These and many other lesser ingredients are all part of the mixture which provides the explosion. The mixture is then poured into the molds, the bullets are cast, and more ammunition is rushed to the firing squad. The barrage continues.

The target, the mathematics program in the American public schools, is literally shot full of holes.

No doubt the battle is justified. Much of what is being said about the program has already been said by the mathematics teachers themselves. Many of them have been genuinely concerned about it for years. However, what is done is over. It is not with the past that our salvation lies. Whether all of the criticisms are justified is not a question with which we are concerned.

In order to avoid arguing the validity of all the criticisms, let's start with the assumption that the mathematics program was adequate for yester years. Does it necessarily follow that it will continue to meet future needs?

Whether we are aware of it or not, we have been undergoing a revolution. It has been described by those who were aware of it as being as great as any revolution man has ever experienced. Those who have been studying it have called it the Scientific Revolution.

Like all upheavals, it has changed conditions. It has created new demands. It has brought new opportunities also. It has resulted in an altered way of life. Standards have changed. The old order is thrown out of gear. Before it is fully understood, scapegoats are sought. Charges are made, blame placed.

When the smoke settles, however, one unalterable fact emerges. In the new order created by the Scientific Revolution, mathematics has become the language for describing our expanding universe and the mathematicians are the authors of such descriptions.

There is little room in this new era for the mathematically illiterate. The untrained laborer will have more and more difficulty finding work. There will be fewer jobs for him in the future.

Rather, the need is developing with the speed of a chain reaction for the trained intellect, for the statesman who has an understanding of science and mathematics, for more mathematics, for more people to understand and create more mathematics, for young people with a background which equips them to solve problems which haven't yet arisen.

A tremendous effort has already been made to meet the demands growing out of the change.

Industry has financed and conducted examinations on its own. It has made evaluations both in view of past achievements and future needs. It has offered suggestions for improvement.

Various government investigating committees and the armed services have uncovered situations. These have led to suggestions and demands. As a result, the government, which is the people of the United States, has spent great sums of money to supplement the training of mathematics teachers already in the field.

The high schools are trying to improve their programs to close the gap.

Outstanding mathematicians and mathematics teachers have met in authorized groups to study needed curricular changes. They have been granted funds to use in writing sample textbooks. Some of those textbooks have already been tested in the classroom. They have been and are being revised.

Efforts are being made, both in the schools and through various other agencies, to seek out those students who are capable of being trained. They are being enticed and persuaded to continue studying mathematics. Scholarships are growing larger and more plentiful.

The whole public school system is being given a good long look. Some of the educational philosophies of bygone days are being re-examined and discarded.

The first of these is that children will learn mathematics incidentally. The schools have been operating under a philosophy which is not even a correct grammatical statement. You've all heard it. "We teach children." 'Children' is not subject matter, hence it is not the direct object of the sentence. A correct statement is, "We teach mathematics to children." Mathematics is what is being taught, 'children' is the indirect object. Both the object and the indirect object are important, and consideration must be given to each. The child is a human being with background, inate abilities, desires, drive, all making their contributions to his individuality. It would be folly to ignore this. But it is equally foolish to think the child is the only problem. When all is said and done, it must be mathematics which is taught to the child if that is what he is expected to learn.

A second philosophy which must go is that mathematics can be taught as a few isolated skills and tricks. The manipulations that a student learns to perform in the course of studying mathematics will have little value if he doesn't understand them. He will neither remember the tricks very long nor know when to use them. Mathematics is integrated, it has purpose and direction, patterns repeat, ideas are related. None of this will ever become evident to the student who gets it in bits.

Another philosophy which is inadequate for present and future needs is that anyone trained to teach can teach mathematics. This is about as foolish as 'teaching children'. I am not saying that a teacher should not be trained to teach. I am saying such training is not all that is needed. In fact, experience is showing that it is a very small factor in the success of a mathematics teacher.

Now, no teacher can possibly know all the mathematics there is. But to know none at all is to fail completely. The fallacy in the philosophy that to teach mathematics all you need is training to teach has been so widely recognized that every effort is being made to expand the mathematical background of the teachers.

Another philosophy, closely related to the ones just mentioned, is equally erroneous. It is, "The teacher needs to know no more mathematics than he is expecting to teach." More than any other philosophy, this one comes closest to exposing the shallowness of our education. One cannot know very much about mathematics until he sees its relationships. The public school student in his day-by-day study won't get this overview. He's too close to details. If the teacher doesn't have it so that he can point out the relationships, the child may never study enough mathematics to discover them.

Also, there is the matter of leading the student. If the teacher is down on the level of day-to-day details, he isn't going to be able to chart any sort of efficient course. He is apt to wander with the student through all the byways and hedges and never arrive any place.

A final error is that mathematics can be taught through applications alone. Playing store or filling out income tax forms or writing formulas to express some social science relationship or natural science law are only some of the things that can be done with mathematics. They are not mathematics itself and will not replace a systematic study of the subject any more than eating food teaches the fundamentals of cooking.

What, then, is mathematics? It can't be defined in a few words--or perhaps many. We can, however, look at some of its characteristics.

It is a science. It is not a natural science in which certain basic laws operate whether man knows what they are or not. Mathematics is an artificial science for it has been created in the minds of men. Men have defined the elements, men have made certain laws, men have outlined the operations. Mathematics is man made. Consequently, it is a science in which both the old and the new must be studied, because both the old and new mathematics is valid.

Mathematics also, "has a triple role as a tool, a language, and a mode of thought."[1] We are most familiar with it as a tool. Its applications are many and varied. Business, industry, and the social sciences apply it in

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[1]Stewart Scott Cairns, "Mathematical Education and the Scientific Revolution," Mathematics Teacher, (February, 1960), pp. 66, 67.

such a diversity of ways that it "is a matter of astonishment to professional mathematicians."[2]

As a language, it describes the world in which we live, both mentally and physically. As a way of thinking, it reaches necessary conclusions in a logical way.

It is to mathematics as a way of thinking that we will give further consideration.

Mathematical ideas present themselves to a child even before he starts to school. His first experiences in kindergarten and the early grades are related to what he has already discovered. Because mathematics is sequential, those early ideas are the foundation for all further study. Their importance cannot be overestimated. His success in those years will have a direct effect on his attitude towards mathematics and his desire and ability to continue his study.

As has been said, the mathematics program in the public schools needs improving. It can't all be done at any one level. It takes twelve years to get a student ready for college mathematics. If we are to improve the program and equip the student in more able fashion, we must begin before his last four years. We must begin at the beginning.

Much is being planned to improve the elementary mathematics program. New text books are being prepared by those

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[2]*Ibid.*

who understand mathematics. The whole curriculum is being scrutinized. Some changes are "just around the corner".

Meanwhile, it seems well to take one of those long looks at what it is we are all trying to teach. We each work in our own grade or department. We are apt to forget that what we are all dealing with is part of one system.

Finally, this is no attempt to tell you how to teach arithmetic. This is an attempt to help you see arithmetic as it is related to the whole structure which is called mathematics.

CHAPTER III

SECOND TALK

The beginning of numbers pre-dates recorded history so that any attempt to reconstruct the origin involves a lot of guessing. Two guesses, however, can be made with confidence. The inception was very early, perhaps even before man learned to speak. Moreover, it was a result. It was the consequence of man's effort to meet certain needs as they arose in his development.

It is thought that primitive man learned early to group together objects which had some quality in common. Undoubtedly he gathered each day those objects which he could eat. Of the edibles, he selected, if only in his own mind, those which he enjoyed eating. The acts which he must perform to insure life and protection for himself and his family were recognized. Friendly animals were members of a special class. All those objects which belonged to him comprised one of the most important sets of all.

This idea of group, collection, class, herd, set was undoubtedly basic in the thinking of primitive man. Some characteristic which certain elements had in common determined whether they could be thought of simultaneously as a unit. If an object possessed that characteristic, then it was a member of the set. If it did not possess that characteristic, then it was not an element of the set. For example, consider all the tribes-women who made up the set of a man's wives. Either a tribes-woman was one of his wives, or she was not. If she was, then she was a member of the set. If she was not, then she was not a member of the set.

A set didn't need to be composed of material elements. The elements could be sensations, ideas, emotions, as well.

Likewise, they need not be grouped in space. The collection of all those people who were friendly to a man could be thought of simultaneously as a unit even though they were not brought together in one meeting place. The days on which the weather was cold could be classed together, although they were scattered throughout the year, or many years.

The formation of a set, then, was a mental process.

If primitive man wanted to describe the collection under consideration, he could use either of two methods. One consisted of naming, whether by gesture, drawing, or sound, the elements. The equipment which he took with him when he left his dwelling in the morning could be listed; his club, his knife, his dog. A second way was to describe the members of the group. This could be done very simply in his mind. All those people who could be counted on to defend the threatened village from ferocious animals or warring tribes constituted a very important set. A second such might be those who would run and hide.

The grouping of elements having some common characteristic into a set is basic in the structure of mathematics.

A second mathematical concept developed equally early from man's need to know 'how many'.

When the shepherd took his small flock of sheep to graze in the early morn, he needed a method of determining if all the sheep returned with him in the evening. He probably used a simple method; a finger, a sheep. In the evening, if a finger was left over, then he knew a sheep had failed to return and must be sought.

As the size of the herd increased, fingers and also toes, were exhausted too soon. The shepherd devised other methods. In the morning for each sheep going out, he put a pebble on the pile; in the evening, for each sheep back, he took a pebble off the pile. Later, for each sheep out he made a notch on a stick or tied a knot in a string.

This setting up of a one-to-one correspondence between the sheep and the pebbles was a powerful idea. Even before primitive man could speak or count, he arrived at a comparison of the size of sets by putting their elements into a one-to-one correspondence. For if the members of the herd could be so related to the pebbles in the pile, the shepherd knew that the size of the flock matched the 'how many' of pebbles in the pile.

If the set of his wives, the set of his spears, the set of earthen pots in his dwelling, and the set of fur garments in his wardrobe could be put into one-to-one correspondence with the fingers on one hand, he came to recognize that the size of the sets matched. They shared a quality, their size or 'manyness' which, by mathematicians, is called their cardinality.

A man's fingers made handy basic sets with which to compare other sets. The next step, then, was to attach a label or tag to all the basic sets. The labels could then be used to name the size of those sets whose elements could be set into one-to-one correspondence with the standard sized sets.

The labels may have been marks traced in the sand; they may have been motions of the fingers or head; they may have been grunts or other odd sounds or words. Whatever form they took, they were the names denoting the manyness or cardinality of the sets and have been called numerals by the mathematicians. Numerals, then, are simply the names or symbols expressing number ideas.

The idea of an empty set, though not given a label until much later, was recognized. If our primitive friend, for example, had the misfortune to have no sons, then the set of all his sons was an empty set.

Other concepts developed.

If a shepherd had only black sheep in his flock, then the set of all his sheep was identical to the set of all his black sheep. The same label could be attached to the size of each herd. If, however, his flock included some white sheep also, then the set of black sheep was a subset of all his sheep. Here the label used to name the size of his whole flock could not be the same as the one he used for the set of black sheep.

It is thought that man early recognized 'more than' and 'less than'. In the instance described above, he could recognize that the subset of black sheep was not only not the same size, but 'less than' the set of all his sheep.

It would have been possible to go on indefinitely matching sets with the basic or standard sized sets and attaching labels there-to. No one knows how long such a system was used. However, as man became aware of unequivalent sets and the need for different tags or numerals, he found it convenient to introduce some order.

He found that he could use the idea of unequal sets, and make an ordered sequence where each set could be put into one-to-one correspondence with the preceding set, with one element left over. This 'and one more' could go on and on. The arrangement of non-matching sets in order of 'one more' was more advantageous than the old random matching of sets with standard sized sets. The ordered sequence of the cardinals is called the ordinals.

From matching equivalent sets to comparing unequivalent sets to arranging unequivalent sets in an ordered sequence, he had reached a great achievement. Man had learned to count.

Each attainment, however, in man's development has been accompanied by new difficulties.

Now he was faced with a different problem. He could perceive the difference in the size of sets, so he realized, by whatever name he called it, 'manyness' or cardinality. He could arrange sets in order so he had a conception of ordinality. Now the labels, the names, the numerals which expressed these ideas demanded attention. Undoubtedly at first the numerals were gestures or marks traced in the sand. Random words were adopted, a different word for each sized set.

When life was simple and possessions few, such a system served very well. Man could invoice his assets without counting very far. Eventually such a system became cumbersome. Introducing a different word or symbol for each number idea taxed the early vocabularies and required memorizing many new numerals.

The system man employed to relieve this new difficulty was very convenient. If he was using his fingers with which to count, then one object called for one finger; two objects, two fingers; ..., five objects, five fingers or one hand; six objects, one hand and one finger; ..., eleven objects, two hands and one finger.

He was using a base of five which seems, along with a base of ten, to have been employed quite early.

Other bases have been and still are being used by various peoples. While the Egyptians and Romans used ten as a base, twenty was used by the Mayas, and sixty by the Babylonians. Two was used by primitive Australian tribes and even yet by the latest high speed computers.

Having learned to employ a base in counting, man now needed fewer words or symbols in numeration. To the idea of base he annexed the idea of adding on. In the Roman system, for example, a V represented the idea of five, and an I the idea of one. The idea of seven which is five and one more and one more could then be written VII. The numeral conveyed the idea of V and I and I.

The notions of a base and adding on simplified the numeration process. Even so, as flocks enlarged, population increased, and wealth accumulated, more and more symbols were needed. It became so difficult to express and handle large number ideas that only a specialist was equal to the task.

Man then discovered that he needed fewer symbols if he was careful as to their placement. Combining the ideas of base, adding on, and placement resulted in a numeration system that was relatively simple, concise, and adequate for the size ideas which he needed to express.

The numeration system which we use makes use of the three concepts. It is called the Hindu-Arabic system. The Hindus, and perhaps others, were responsible for its origin but it was introduced into Europe sometime before 800 A.D. by the Arabs.

The system employs ten symbols, is positional, and additive. One of the ten symbols is zero, the label for the empty set. The whole organization has proved to be so adequate and versatile that it gives every indication of carrying us to infinity--and beyond, with an indebtedness so large that to express even the interest would have been beyond the ability of the early systems of numeration.

Hindu-Arabic numeration made it possible for man to count as far as necessary. Each numeral was followed by a numeral representing one more and preceded by a numeral representing one less, excepting one. The numeral one represented the first of the counting or natural numbers.

Man found that counting numbers were very useful.

Suppose a shepherd had five sheep in one pen and three sheep in another pen. He decided to put them in the same pen. So he moved the five into the pen with the three sheep and counted the result. He had eight sheep.

Another day, however, he decided it would be more simple to move the three sheep into the pen with the five sheep. When he counted the result he had eight sheep.

Five sheep and three sheep is equal to three sheep and five sheep. The order of combining the sets didn't make any difference in the total $(5 + 3) = (3 + 5)$.

With eight sheep now in the pen, he decided to move in four more. When he counted the result, he had twelve sheep. Next time, he grouped them in a different way. He moved the three sheep into a pen with four, and counted the result, seven sheep. Then he moved the seven into the pen with the five. The final count totaled twelve. He thought over the result. Five and three more made eight sheep. Eight sheep and four more, and he had twelve sheep. On the other hand, five sheep and the result of putting the three and four together first still resulted in twelve sheep. $(5 + 3) + 4 = 5 + (3 + 4)$.

Another idea emerged. A set of seven sticks and a set of two sticks, six sticks and three more, or five sticks increased by four, all resulted in the same sized set of nine sticks.

Suppose, instead, that our shepherd was a hunter with a set of seven spears. His least favorite brother-in-law promised to settle an old obligation by paying him the set of all the brother-in-law's spears. He learned to his sorrow that he still had a set of seven spears, since the set of said brother-in-law's spears was an empty set.

Another interesting property of the counting numbers became apparent. No matter how many subsets of varying numbers of elements were joined together, the number of elements in the set created could always be expressed by one of the counting numbers.

Sometimes, however, man suffered reverses.

If he had seven arrows and lost three, how many remained? Or, suppose he had seven arrows and his enemy had only three. How much better armed was he than his enemy? If he left his dwelling in the morning armed with seven arrows and returned in the evening with only three, how many had he lost during the day? Finally, if he had a set of three arrows and needed ten to be adequately armed, how many more arrows did he need?

Counting still provided the answer for these important problems, but the approach was different.

He found also that the order of 'taking away' made a lot of difference. Whereas two arrows united to three was identical to three arrows united to two, five arrows take away three arrows was not the same as three arrows take away five. In fact, the latter operation didn't make any sense. If he had only three arrows to begin with, he certainly couldn't go to the forest and lose five arrows.

This take-away business was tricky. Sometimes, he couldn't perform the operation. When he could, it could be explained like this. If three sheep are in a pen and he put four more sheep in the same pen, the resulting group numbers seven. If from the seven sheep, he took away four, he was back to three sheep again.

Bringing sets together and taking sets apart were operations like building a man's dwelling and tearing it down again, like hunting and being hunted, like daylight and dark.

A short-cut emerged.

It was found that if a set of three stones, another set of three stones, a third set of three stones and yet a fourth set of three stones were united into one set and counted, that the same result could be obtained by another method. Taking the number of elements in one set times the total number of sets was quicker. It couldn't always be used, the sets must each contain the same number of elements.

This new process, besides being more convenient, developed some characteristics on which man could depend.

Four sets of three knives each was equivalent to three sets of four knives each. In either case, the result was twelve knives. (3 x 4) = (4 x 3). Two such sets of twelve knives each gave a set with twenty-four knives. Yet, three sets of two knives each gave a set of six knives. Four of those sets resulted in a set of twenty-four knives. The manner of grouping didn't disturb the answer. (4 x 3) x 2 = 4 x (3 x 2). Besides, it all made sense. The result was always a counting number.

The short-cut operation worked with the empty set, too. Three empty sets resulted in an empty set. Our friend had already discovered that if his miserly neighbor had many choice collections of three knives each, but gave him none of these, he would have no knives. (3 x 0) = 0 = (0 x 3).

One came in for further consideration. Man had long considered it the first of the counting numbers. Now, he discovered an additional property. He found that three sets of one coin each resulted in a new collection of three coins. On the other hand, one set of three coins remained a set of three coins. (3 x 1) = 3 = (1 x 3).

A combination of two operations was the most marvelous of all. Our ancestor learned that three sets of two bricks each joined to three sets of five bricks each resulted in a super set of twenty-one bricks. Yet he arrived at the same result if he joined a set of two bricks to a set of five bricks, making a stack of seven bricks. Three such stacks totaled twenty-one bricks also. $(3 \times 2) + (3 \times 5) = 3(2 + 5)$.

There was the reverse story here, also.

Suppose a man had fourteen very special sheep and he wished to separate them into pairs. To how many of his children could he give a pair? Or perhaps it was the question of giving each of his two children an equal number of sheep.

He could solve each problem by repeated take-away. Also, he found that he could relate it to the 'times' process. If two sets of seven elements gave a set of fourteen elements, then fourteen elements could be separated into two sets of seven elements.

This process, finding how many groups of so many elements each the set could be separated into, had limits. For one, he couldn't reverse the order. Fourteen elements could be separated into two sets of seven elements each, but two couldn't be separated into fourteen sets of seven elements each. $14 \div 2 \neq 2 \div 14$.

A collection of five earthen pots could be made into five different collections of one pot each or remain one collection of five pots. But how many sets of no pots each could he form?

The last operation had other limits. Fifteen live sheep could be apportioned among three people, but how about four people? Then there was the question concerning what a father did if he had two equally deserving sons and only one sheep to give them.

Man had not yet found all the answers.

Great strides he had made, however. He had learned to count. He had a numeration system, and he had certain operations which he could perform with his counting numbers. These achievements were the results over many centuries of meeting his needs as they arose.

The mathematician looked over all the achievements and said, "We aren't concerned in mathematics with whether we're dealing with sheep, arrows, people or what have you. Our concern is the framework. We want to examine the elements to see how we can operate with them and to determine what properties will be present."

"We'll begin with the idea of sets, one-to-one correspondence, and the set of natural numbers. These we will not attempt to define since we must begin someplace. By the elements of a set we mean the members. Zero is the number of elements in the empty set."

"One is the first number in the natural numbers. Each one which follows is one more. There is no stopping place."

"We have two operations which we can always perform with the set of natural numbers."

"The first of these, the bringing together of the elements of two or more subsets into one set and counting the elements, we'll call addition. The result of addition we'll call the sum, and the symbol for addition will be a small cross. Since we can always find the sum in our set of natural numbers, we'll name that property closure. The order of addition doesn't matter; that is, $2 + 3 = 3 + 2$. The property of order we'll call commutativity. Grouping in addition won't affect the result. Two added to three, then added to six is equal to two added to the sum of three and six. That property we'll call associativity. $(2 + 3) + 6 = 2 + (3 + 6)$. We'll call zero the identity element because adding zero to any natural number does not change the identity of that number."

"The inverse of addition we'll call subtraction and define it in terms of addition. One number subtracted from a second equals a third number if, and only if, the sum of the third and first numbers equals the second. The symbol for subtraction will be a small line. The result of subtraction we'll call the difference. $7 - 3 = 4$ if, and only if, $4 + 3 = 7$. There is no closure under subtraction since we cannot always find a solution in our set of natural numbers. $7 - 3 = 4$ but $3 - 7$ has no solution in the natural numbers."

"The second operation we'll define as repeated addition where all the sets to be added are of the same size. This operation we'll call multiplication and the result of the operation we'll call the product. The symbol for multiplication will be a small x. The product of two natural numbers is a natural number which is the property of closure. Order in multiplication doesn't affect the product. That is the property of commutativity and can be illustrated thus; (2 x 3) = (3 x 2). Associativity, which has to do with grouping, is also present. Multiplying two numbers together and their product by the third leads to the same result as multiplying the first by the product of the second and third. Since any natural number multiplied by one equals the natural number, we'll call one the multiplicative identity."

"A combination of multiplication over addition, that is, the product of one number and the sum of two numbers, is equivalent to the sum of the products of the first number and each of the two numbers of the sum. 2 x (3 + 4) = (2 x 3) + (2 x 4). This we'll name distributive property of multiplication over addition."

"The inverse of multiplication we'll call division and we'll define it in terms of multiplication. One number divided by a second equals a third if, and only if, the product of the second and third numbers is equal to the first. The symbol for division is a short line with a dot above and below it. An example of the definition of division can be stated. Six divided by three equals two if, and only if, two times three equals six. The result of division is called the quotient. This inverse operation does not possess closure. Five divided by three is equal to what? Three divided by six is equal to what? Neither quotient can be found in the set of natural numbers."

"Division by zero is never possible because under the definition, the quotient cannot be determined. Six divided by zero is not equal to zero because zero times zero does not equal six. Six divided by zero is not equal to six because zero times six does not equal six. The result of dividing zero by zero is not defined either, since zero times any number is equal to zero. Zero divided by zero equals what? So division by zero is never possible."

"Now, so that we won't be limited to particular natural numbers, we'll generalize by using letters to denote any natural numbers. We'll add a few more symbols to speed the writing. The total result, a mathematical structure, we can then state very concisely."

This is what the mathematician means by mathematics.

Sets--Undefined
Elements--Members of a set
One-to-one correspondence--Undefined

Numeration	Counting (order) Natural numbers 1, 2, 3, 4, 5, 6,...	Number (size) 0 is tag for empty set

Operations

Addition	Basic properties	Basic properties	Subtraction
n(A)+n(B)= n(A + B)	1. Closure; a + b = x If a & b are natural numbers. x also nat. no. 2. 0 additive identity element a + 0 = a 3. Commutative a + b = b + a 4. Associative a + (b + c) = (a + b) + c Distributive a(b + c) = ab + ac	1. No closure Operation limited to a greater than b	a - b = c if & only if c + b = a

Multiplication			Division
(a • b) = b + b + b + ... for 'a' terms if 'a' represents a Natural no.	1. Closure a • b = x If a & b are Nat. nos. then x is also nat. 2. Multiplicative identity 1 a • 1 = a 3. Commutative ab = ba 4. Associative a(bc) = (ab)c	1. No Closure 2. a ÷ 1 = a because a • 1 = a 3. Division by zero. not possible	a ÷ b = c if and only if c • b = a

CHAPTER IV

THIRD TALK

Whatever else may be new under the sun, fractions are not.

One of the oldest mathematical documents in existence today is an Egyptian papyrus called the Rhind. It was written sometime between 1788 and 1580 B. C. by Ahmes. He acknowledged having copied it from an older document dating back to 1849 B. C. There is supposition that the older document may also have been a copy but that has never been verified. Whether that supposition is ever confirmed, the Rhind papyrus gives enough evidence to prove that Egyptian mathematics had reached a considerable degree of usability.

A problem in the papyrus is "the making of loaves 9 for men 10". The actual solution of the problem is not shown but the result is given and proved. The important information to be gained from a study of the problem and others in the papyrus is that the Egyptians by 2000 B. C. were able to work with fractions.

Their methods were not so simple. They could operate only with unit fractions, those with a numerator of one. Excepting 2/3s, every fraction used by the Egyptians had to be expressed as the sum of a series of fractions, having one in the numerator. For example, 3/4 was written $\frac{1}{2}, \frac{1}{4}$ (they did not use a plus sign between). The fraction 2/61 was expressed as 1/40, 1/244, 1/488, 1/610.

Unit fractions continued in use even under Greek mathematicians. Archemedes wrote $\frac{1}{2}$, $\frac{1}{4}$ for 3/4. However, the later Greeks developed a religious attitude towards "unity". It became for them a Diety symbol and so they were disinclined to take parts of a unit. For that reason, the Greeks largely ignored the partitive interpretation of a fraction and treated only ratios of natural numbers.

With the counting numbers, man had already discovered one way to express the comparison of two sets. If one man had a set of two spears while his neighbor had a set of five, his neighbor's set then included three more spears than his own. This was an absolute comparison. It was accomplished by the inverse operation of addition, that of subtraction.

The Greeks emphasized a second way of comparing two sets.

If one Greek master owned six slaves, and a second owned two, the comparison could be expressed as six to two, or 6/2. The first owner had three times as many slaves as the second. But if the first owner had six slaves and and the second one had seven, then the relation between the two sets was expressed as the ratio of six to seven, or 6/7. The set of slaves of the first owner was 6/7s of that of the second.

However, 6/7 was not a natural number. It was a new number, a fraction. The Greeks emphasized the interpretation of the fraction as that of a ratio.

There are two other interpretations of fractions which are important to consider. One of those is the idea of partition. Man was early concerned with 'how much'.

If our citizen was a man of substance and decided to divide his wealth equally into five bags, each bag would contain 1/5 of his wealth. Suppose he decided to give two of these bags to his eldest son. His son received two 1/5ths or 2/5ths of the total wealth.

The 2/5ths could also be regarded as 1/5 of two units. Suppose our rich friend had his wealth in the form of gold coins and jewels, the total value of the coins being equal to that of the jewels. He divided his coins into five bags equally. Each bag then held 1/5 of his coin wealth. The jewels he divided equally into five portions, each portion being 1/5 of his wealth in jewels. Then suppose he gave his son one bag of jewels and one bag of coins. In short the son was being given 1/5 of two different units. The son received the equivalent of 2/5 of his father's wealth in coins, or 2/5 of his father's wealth in jewels.

Two-fifths in the first example named the number property of a set of two elements each of which was 1/5 of a unit. Two fifths in the second case could be regarded as 1/5 of two units.

The third interpretation of a fraction is that of division. A set of six elements could be very nicely divided by a set of two elements under the natural numbers. The result was three sets of two elements each. But six elements divided by five elements or six divided by seven had no solution in that number system. With fractions, six divided by five we can express as 6/5, and six divided by seven as 6/7. Since division by zero was ruled out under the natural numbers, we can never have a denominator of zero.

The elementary student is introduced to all three interpretations: ratios, partition, and division. Each is expressed by the same symbolism. A pair of natural numbers is used. The one written below the line is called the denominator. The other natural number, written above the line, is called the numerator. The order of writing the two numbers is very important for 2/3 is not equal to 3/2.

But what does the student mean by the equality of two fractions? He usually gains an idea by considering some unit such as a freshly baked pie of his favorite flavor.

If the pie is cut into two equal parts, then each piece is one-half of the pie. Suppose that one-half is to be the student's share. He may, of course, eat his share at this time. If, however, he restrains himself, the story grows more interesting. A second equal division of the pie results in four equal portions, each one being $\frac{1}{4}$ of the whole pie. The student's share is two $\frac{1}{4}$ths of the pie. Now if he eats his 2/4ths, he will be devouring the same amount of pie as he would have eaten when his share was one-half of the pie. Evidently one-half is equal to 2/4. Should the student be particularly well disciplined so that he will permit one more cutting of the pie before his feast, and it is divided into eight equal portions, his $\frac{1}{2}$ or 2/4 becomes four 1/8ths or 4/8ths of the pie. By now the pie is mutilated and the student's patience has been tried beyond the breaking point. It is high time that he eats his 4/8ths and considers the results. $\frac{1}{2}$ = 2/4 = 4/8. If he looks at the equality, he can see that 2/4 can be arrived at by simply multiplying the natural number, one, in the numerator by two and natural number, two, in the denominator by two. He thereby arrives at 2/4ths which he knows from his experience with the pie is equal to one-half. What about 2/4 and 4/8? Here again he can arrive at the second fraction by multiplying the numerator and denominator of the first fraction by two.

Why does it work? Well, when he multiplies both numerator and denominator by the same amount, he is really multiplying the whole fraction by one, since 2/2 = 1. He learned with the natural numbers that one was the multiplicative identity because multiplication by it did not change the identity of the number being multiplied. It would seem that the fractions also contain a multiplicative identity element.

If the student examines the equality again but in reverse, 4/8 = 2/4 = 1/2, he can see that he will arrive at the 2/4ths from the 4/8ths by dividing both the numerator and denominator by two.

This idea can be more fully developed when we are ready to consider multiplication of fractions. Meanwhile, we have arrived at a very important rule which sums up the process. If the numerator and the denominator of any fraction are multiplied by (or divided by) the same number, the resulting fraction is equivalent to the original fraction.

Since the student has already seen the equivalence of 1/2, 2/4, 4/8, and the process by which he arrived at equivalent fractions, he knows that 1/2, 2/4, 4/8 are different names for the same number idea. This repeats a concept developed earlier with the natural numbers. Just as (6 + 2), (7 + 1), (5 + 3), and 8 are labels for the same number idea in the natural numbers, so 1/2, 2/4, 3/6, 4/8, 5/10, ... are different labels for the same fraction. This set of fractions constitutes what is called, for very obvious reasons, an equivalence class.

The question also arises about the inequality of fractions. Is one fraction greater than another? If so, how can you tell which is larger?

Consider 2/3 and 3/4. To determine if they are equivalent, let's change them so they will have the same denominators. 2/3 = 4/6 = 6/9 = 8/12. 3/4 = 6/8 = 9/12. Which is greater, eight 1/12ths or nine 1/12ths? There is an order in fractions, a 'greater than' relationship just as there was in the natural numbers.

The pattern begins to be apparent. Having found a new number idea, and having determined when two such numbers are equal or unequal, can we perform the same operations as with the natural numbers?

Fractions can be added.

Since the pie has disappeared and the whole idea of more pie probably doesn't sound too attractive at this time, let's consider a strip of paper.

Suppose we fold the strip into three equal portions and cut the paper on the folded lines. 1/3 of the strip added to 1/3 gives two 1/3rds or 2/3rds of the whole strip. Suppose, however, we want to add 1/3 to 1/2. Let's take a second strip of paper, fold it into two equal parts and cut along the folded line. We can add 1/3 and 1/2 by laying the two side by side. The sum is 1/3 + 1/2 and can be expressed if we are not too impatient. However, such a tag would soon get boring to use. We'd like to express the sum as one fraction and that we can do if we'll use what we know about equivalence classes. 1/3 = 2/6 = 3/9 = 4/12.... 1/2 = 2/4 = 3/6 = 4/8...

Suppose we start with two fresh strips of paper. Let's fold each so it is divided into sixths and cut on the folds. One third is equal to 2/6 so we'll use two 1/6ths of the first strip. One-half is equal to 3/6 so we'll use three 1/6ths of the second. When we place them together and count the result, we find that we have five 1/6ths which is 5/6ths. Five sixths is much more convenient than (1/2 + 1/3).

The addition of two fractions results in a fraction, so we have closure. 1/2 + 1/3 = 1/3 + 1/2. The commutative law regarding order is in operation. The associative law is good; 1/2 + 1/3 + 1/4 will yield the same fraction whether the sum of 1/2 and 1/3, namely 5/6, is added to 1/4; or whether 1/2 is added to the sum of 1/3 and 1/4. The result in each instance is 13/12.

Thirteen twelfths is significant if we think of 13/12 as equal to $\frac{12 + 1}{12}$ which in turn is equal to 12/12 +

1/12. Twelve twelfths are equal to one, so 12/12 + 1/12 = 1 + 1/12 = 1 1/12.

Five and 7/8 added to 7 3/8 = 12 and 10/8 = 12 + 8/8 + 2/8 = 12 + 1 + 2/8 = 13 + 1/4 = $13\frac{1}{4}$. Here the 2/8ths was reduced to 1/4th by using the idea of equivalence classes.

Seven over zero has no meaning, since division by zero is undefined. But 0/7 has meaning. No 1/7ths = zero; or 0 ÷ 7 = 0, because 0 x 7 = 0. Using either interpretation, 5/7 + 0/7 = 5/7. Zero in the numerator over any denominator not zero is the new additive identity since adding it to a fraction does not change the identity of the fraction to which it is added.

Subtraction is again defined in terms of addition. One third subtracted from one half is equal to 1/6 if and only if 1/6 + 1/3 = $\frac{1}{2}$. To perform the operation, equivalence classes again make it easier. 1/2 - 1/3 = 3/6 - 2/6 = 1/6. Just as addition can be thought of as a bringing together, so subtraction can be thought of as a take-away procedure. If from 1/2 of a candy bar, we wish to subtract 1/3, the best method is to divide the candy bar into six equal parts to begin with. One-half is three 1/6ths, 1/3 is two 1/6ths. If 2/6 is taken away from 3/6, 1/6 of the candy bar remains.

Subtraction is still a limited process. 1/2 - 1/3 has meaning, since 1/2 is greater than 1/3. Three sixths is greater than 2/6. But 1/3 - 1/2 is not an operation we can perform with the fractions as we now know them.

What about multiplication?

The approach we used to describe the multiplication of natural numbers as repeated addition doesn't take us very far with fractions. However, it does serve as a starter.

If a board is sawed into five shorter boards of equal length, each board is 1/5th of the original. Three times 1/5 is equal to 1/5 + 1/5 + 1/5 = 3/5. One fifth times three, sometimes spoken of as 1/5 of three, can be regarded as three boards, each divided into five equal parts. One fifth of three boards would be 1/5 of the first plus 1/5 of the second plus 1/5 of the third, or 3/5 of one board.

Patterns can be seen if multiplication of fractions is approached from multiplication of natural numbers. 4 x 4 = 16, 2 x 4 = 8, 1 x 4 = 4, 1/2 x 4 = 2, 1/4 x 4 = 1. Also, 4 x 3/4 = 3, 2 x 3/4 = 6/4 or 3/2, 1 x 3/4 = 3/4, 1/2 x 3/4 = 3/8, 1/4 x 3/4 = 3/16. In each instance, as the number being multiplied was divided by 2, the product was likewise divided by 2. But $\frac{1}{2}$ x 3/4 = 3/8 and $\frac{1}{4}$ x 3/4 = 3/16 gives some idea concerning the multiplication of two fractions. Find the product of the numerators for a new numerator and the product of the denominators for a new denominator.

The product of two fractions is a fraction. These new numbers are closed under multiplication. $\frac{1}{2} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{2}$. The commutative law for multiplication of fractions is present and active. One-half times 1/3 times 1/6 = 1/36 whether the product of 1/2 and 1/3 is multiplied by 1/6, or 1/2 is multiplied by the product of 1/3 and 1/6. The law of associativity is also in operation.

1/2 x (2/3 + 3/4) = 1/3 + 3/8 = 8/24 + 9/24 = 17/24.

1/2 x (2/3 + 3/4) = 1/2 x (8/12 + 9/12) = 1/2 x 17/12 = 17/24.

This is an illustration of the distributive law of multiplication over addition. It works for fractions.

The multiplicative identity, one, is present in this number system. As was pointed out earlier, any number which has the same natural number in the numerator as in the denominator is equivalent to one. Thus 2/2, 4/4, 6/6, all are equivalent to one. 1/2 x 2/2 = (1 x 2)/(2 x 2) = 2/4. Here both numerator and denominator are multiplied by two, yet 2/2 = 1, so 1/2 = 2/4.

A new property appears. One is the multiplicative identity. The product of a fraction and its reciprocal, the fraction inverted, is equal to one. 3/4 x 4/3 = 12/12 = 1. The reciprocal is called the multiplicative inverse of the fraction, since their product is the multiplicative identity.

The idea of division, which is the inverse of multiplication, can be developed intuitively.

If a length of rope is divided into five equal parts, each part is one-fifth of the rope. $1 \div 5 = 1/5$. Yet the same rope, divided by 1/5, equals five pieces or five 1/5ths. $1 \div 1/5 = 5$. Two pieces of rope divided by 1/5 would result in twice as many fifths as in the case of one rope. $2 \div 1/5 = 2\ (1) \div 1/5 = 2\ (5) = 10$ pieces. $3 \div 1/5 = 3\ (5) = 15$ pieces.

On the other hand, 1/2 of a length of rope divided so that each length is 1/5 of the whole rope results in 5/2 or 2 1/2 pieces. In each of these instances:

$$1 \div 5 = 1/5$$
$$1 \div 1/5 = 5$$
$$2 \div 1/5 = 10$$
$$\tfrac{1}{2} \div 1/5 = 5/2$$

the result could be obtained by inverting the divisor and multiplying. The definition of division, $1 \div 1/5 = 5$ if and only if $5 \times 1/5 = 1$ is a check in each of the examples.

The process of division by a fraction is applied when the teacher divides the treat of candy bars into fourths of candy bars so that she will have four times as many treats.

We have looked at the new number system of fractions from the view point of the student. The system has not been completely developed from the mathematical standpoint.

We'll need another number system before we can develop the fractions completely.

Meanwhile, the mathematician looks at this partial development in his own way. Using the undefined and defined terms he developed for the natural numbers, and the laws which were in operation, he builds the fractions.

He defines a fraction as an ordered pair of natural numbers, a/b, where b is not equal to 0.

Two fractions a/b and c/d are equal if and only if a x d = b x c. This takes care of equivalent fractions. 2/3 = 4/6 because 2 x 6 = 3 x 4.

Addition of two fractions $a/b + c/d = \frac{(a \times d) + (b \times c)}{b \times d}$

$$2/3 + 3/4 = \frac{(2 \times 4) + (3 \times 3)}{3 \times 4} = \frac{8 + 9}{12} = \frac{17}{12}$$

The commutative law, concerned with order, and the associative law, concerned with grouping, are both effective. The identity element is 0/a where a represents any natural number.

Subtraction is again defined in terms of addition; $\frac{a}{b} - \frac{c}{d} = \frac{e}{f}$ if and only if $\frac{e}{f} + \frac{c}{d} = \frac{a}{b}$. Subtraction is a limited operation. The problem a/b - c/d does not have a solution in the set of fractions if c/d is greater than a/b.

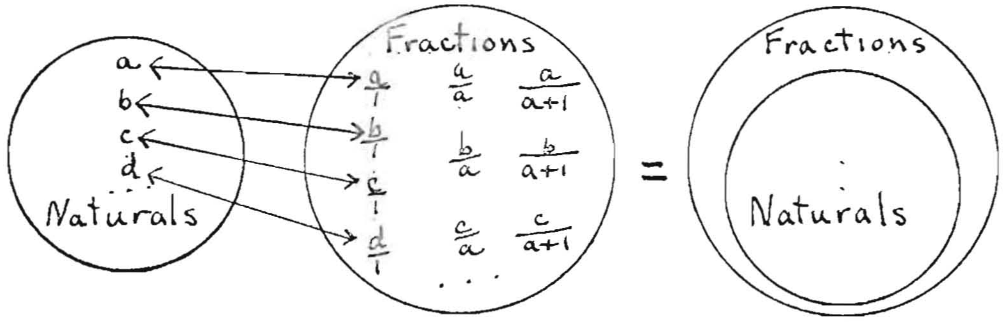
Multiplication of two fractions a/b and c/d is equal to $\frac{a \times c}{b \times d}$. The product of the numerators gives a new numerator, the product of the denominators a new denominator. There is closure since (a x c)/(b x d) is also a fraction.

The multiplicative identity, one, is a/a where a is any natural number. A new property has appeared in multiplication. There is an inverse, such that the product of a fraction and its inverse is one. a/b x b/a = 1. In multiplication, the commutative and associative laws are in operation, plus the distributive law of multiplication over addition.

$$\frac{a}{n} \times \left(\frac{b}{r} + \frac{s}{t}\right) = \frac{a \times b}{n \times r} + \frac{a \times s}{n \times t}$$

Division is defined as the inverse of multiplication. a/b ÷ c/d = N if and only if N x c/d = a/b, so a/b ÷ c/d = a/b . d/c.

The fractions include the natural numbers. If we write the natural numbers a, b, c, ... with denominators of one, a/1, b/1, c/1 we have created ordered pairs of natural numbers, which are fractions. Yet a/1 can be put into one-to-one correspondence with a, b/1 with b, c/1 with c, ... The fractions can thus be said to include the naturals.



Finally, we can now perform three operations; addition, multiplication, and division. Subtraction is still a limited operation. Our next concern, then, will be to explore a number system in which subtraction always has a solution set.

<u>Fraction</u> -- ordered pair of natural numbers, $\frac{a}{b}$ where $b \neq 0$

Equality: $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$

$\frac{a}{b} = \frac{ck}{dk}$; $\frac{a}{b} = \frac{a/k}{d/k}$

Inequality: order

Addition: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

- $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$
- $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$
- 0/a, additive identity
- Closure

Subtraction: $\frac{a}{b} - \frac{c}{d} = \frac{e}{f}$ iff $\frac{e}{f} + \frac{c}{d} = \frac{a}{b}$

Limited operation

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

- $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$
- $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \frac{e}{f}$
- a/a, mult. identity
- Closure
- $\frac{a}{b} \cdot \frac{b}{a} = 1$: mult. inverse

Distributive $\frac{a}{n} + \frac{b}{n} = \frac{1}{n}(a + b)$

Division: $\frac{a}{b} \div \frac{c}{d} = N$ iff

$\frac{c}{d} \cdot N = \frac{a}{b}$ so $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

Closure

CHAPTER V

FOURTH TALK

Man was quite content for many centuries with the two number systems which he had developed. He had the naturals with which he could answer the question 'how many'. He could add and multiply the natural numbers. He had fractions to use when he was concerned with the question 'how much'. Fractions made division always possible. Since even the most imaginative of prophets could not have predicted the United States of America in the twentieth century, no one foresaw the idea of deficit spending. The idea, like the proverbial camel which had gotten its head in the tent, has now become such a part of our economy that we subtract more from less with the greatest facility. The result we express with a negative number.

The history of this new number system, while older than contemporary times, does not extend much farther back than the sixth or seventh century A. D. The Hindus prior to that period had begun to recognize negative numbers and had defined some processes using them. About 700 years later, Johann Widman, in a book published in Leipzig, used plus and minus signs to indicate excess and deficiency. Less than 100 years after that, Michael Stifel, a German, published a book dealing with equations in which the negative roots were discarded. However, at about the same time, 1545, Cardano, who taught mathematics and practiced medicine in Milan, took some notice of the negative roots of an equation. Even so, it was not until the early seventeenth century, after the use of directed numbers on axes was developed, that general recognition was given to negative members and their opposites, the positives.

Likewise, their introduction into the formal study of an American student comes much later than that of the naturals and the fractions. Yet the student has earlier experiences with the idea. He plays games in which it is possible to get a score 'in the hole'. He knows about above and below zero temperatures; probably, also, about above and below sea level. He charges the cost of items at the downtown store when he has no money. He borrows from Mother against his next allowance.

He may even have tried to subtract a larger number from a smaller one and have been told that he couldn't do it. In the system of natural numbers such an operation cannot be performed. There is no closure under subtraction.

Signed or directed numbers, however, provide the answer to any subtraction. These numbers, positive and negative whole numbers, together with zero, constitute the next set with which we are concerned. They are called the integers and they involve the idea both of size and direction. Six take-away three has been understandable for a long time but three minus six now makes sense. $3 - 6 = -3$. If we have six dollars and spend three then we have three dollars left. But if we are modern, we have three dollars and spend six dollars. Subtracting six from three leaves us three dollars in the hole. Both transactions involve a difference of three, called the absolute value, but the signs indicate the opposite states of having and owing.

In fact, money provides an effective intuitive approach to integers and their properties as a number system.

If John has no money and owes none, then his financial situation may be described as zero. Zero is neither positive nor negative. If John has five dollars, he can use +5 to indicate in his bookkeeping that he is solvent. If, instead of having five dollars, he charges five dollars at the store, -5 describes his condition. He is considerably poorer at -5 than at +5, ten dollars poorer, in fact.

Now suppose on the next allowance day, his Dad gives him five dollars and tells him to pay the five dollars he owes at the store. He is again even, he neither owes nor owns. A debt of five dollars and a payment of five dollars balances the account. He is in better financial condition at zero than at -5, but better at +5 than at zero. This suggests the ordering of signed numbers, that +5 is greater than zero is greater than -5, which, by the way, is greater than -10. He's richer if he owes five dollars than if he owes ten dollars. Another property of interest is that -5 and +5 = 0. For each positive integer, there is a corresponding negative integer and their sum is zero.

Let's return to John now and his early struggles to keep 'out of the red'. Suppose he still owes five dollars at the store, -5, the records show. If his Dad gives him seven dollars and tells him to pay off his obligation and keep the change, his assets are greater than his liabilities. He invoices at +2.

Suppose instead, Dad gives John only three dollars, tells him to pay it on his five dollar indebtedness at the store and to ask the merchant to wait for the balance. Then John finds how difficult it is to make ends meet. He pays the three dollars, had no change, and still owes two dollars. But he could be worse off. Suppose when he owed five dollars, he charged three dollars more. -8. He has big problems now. Happy indeed, but rarely so, is John when his financial state is +5 on allowance day and he receives five dollars more. He is then +10 and on his way to being a hoarder.

A quick look at the addition we have just performed shows this. Signs now have two meanings. One sign, the one with the numeral is the sign of the integer; the other is the sign of operation, addition.

$(-5) + (+5) = 0$
$(-5) + (+7) = +2$ -- has more than he owes
$(-5) + (+3) = -2$ -- owes more than he has
$(-5) + (-3) = -8$ -- owes and owes again
$(+5) + (+5) = +10$ -- has and has again

In the last two instances, the sign is the same all the way across. In the second and third instances, the sign depends on whether the absolute value of what he has or what he owes is larger. The final integer is the difference of the sizes. This operation is called adding algebraically.

The operation of the commutative law is evident. If John has five dollars and spends two dollars, his financial state is the same as if he charged two dollars, and then paid it up out of five dollars. In either event, he'll have three dollars left after the transaction is over.

The associative law also applies. If he has five dollars and spends two dollars, he has three dollars left. If, the next day he spends the three dollars, he's as broke as if he had five dollars and spent two dollars and three dollars all in the same place.

If he has five dollars and adds nothing to it, he's +5. Whereas if he owes five dollars and pays nothing on it, he's -5. Zero is the additive identity.

Finally, notice that each time we added two integers, the result was an integer, which illustrates that the integers are closed under the operation of addition.

Subtraction may again be described as the inverse of addition. Since we know how to add signed numbers, we can uncover a pattern which will demonstrate the operation. We shall leave the definition until later for the mathematician to make.

Positive five - (+2) = some number such that the number added to (+2) = (+5). The number is +3, because (+3) added to (+2) = (+5). Similarly, (+5)-(-2) = +7 because (+7) + (-2) = +5. Negative five - (+2) = (-7) because (-7) added to (+2) = (-5). Finally (-5)-(-2) = (-3) because (-3) + (-2) = (-5).

Subtraction of integers can further be illustrated by making thermometer readings on several days throughout the year.

Assume the temperature is read at ten o'clock in the morning and it is twenty degrees above zero, represented by +20. At noon, a second reading is made and it has warmed up to fifty degrees, +50. What is the difference between the reading at mid-morning and that made at noon. The temperature has risen thirty degrees, so the difference between +50 and +20 is +30. On another day, the reading at ten in the morning is ten degrees below zero, -10. At noon it has risen to twenty degrees above, +20. What is the difference between $+20^{\circ}$ and $- 10^{\circ}$? Again thirty degrees, and since the temperature was rising, + to show direction. The difference is +30.

Now, suppose at ten o'clock it is ten degrees above zero, but by noon it is twenty degrees below. The difference between -20 and +10 is -30, because the temperature was dropping. Here we are showing the opposite direction from the rising in the first two examples.

Finally, one cold morning at ten in the morning the thermometer showed ten degrees below zero, -10. Two hours later, the weather was worse and a quick reading showed twenty degrees below, -20. The difference between -20 and -10 is-10, negative because the temperature had fallen.

A second look at the last two examples will indicate a trend.

$$\begin{array}{r} +5 \\ -\ \underline{+2} \\ +3 \end{array} \qquad \begin{array}{r} +5 \\ \underline{-(\ -2)} \\ +7 \end{array} \qquad \begin{array}{r} (\ -5) \\ \underline{-(\ +2)} \\ -7 \end{array} \qquad \begin{array}{r} -5 \\ \underline{-(\ -2)} \\ -3 \end{array}$$

$$\begin{array}{r} +50^{\circ} \\ \underline{-(+20^{\circ})} \\ +30^{\circ} \end{array} \qquad \begin{array}{r} +20^{\circ} \\ \underline{-(-10^{\circ})} \\ +30^{\circ} \end{array} \qquad \begin{array}{r} -20^{\circ} \\ \underline{-(+10^{\circ})} \\ -30^{\circ} \end{array} \qquad \begin{array}{r} -20^{\circ} \\ \underline{-(-10^{\circ})} \\ -10^{\circ} \end{array}$$

To find the difference between two integers, change the sign of the subtrahend (number being subtracted) and add algebraically.

Subtraction deserves a second look because we have finally arrived at a number system in which we have closure under subtraction. The difference between two numbers is no longer limited. We can now take six away from three as easily as we can take three away from six.

The multiplication of the integers may be developed concretely, though it gets a little involved.

Using the idea of multiplication as repeated addition doesn't make the beginning too difficult. If we notice that for all practical purposes there is no difference between the natural numbers and the positive integers, and remember that 3 x 5 = 5 + 5 + 5 = 15, it isn't too difficult to accept the interpretation that (+3) x (+5) can be interpreted as (+5) + (+5) + (+5) = +15. (+3) x (-5) can then be explained as (-5) + (-5) + (-5) = -15. Then, if the commutative law for multiplication is to be valid, (-5) x (+3) can be rewritten as (+3)(-5) = (-5) + (-5) + (-5) = -15. Then the interpretation gets confusing. The product of two positive integers is a positive integer; the product of a positive and a negative, or of a negative and a positive, is a negative integer. The joker is a negative times a negative.

Let's pretend again. We are standing at a railroad station watching the train. It is noon, and the track extends east and west. If we represent direction east by positive, then direction west we'll represent by negative. Three hours later (3 P. M.) we'll call +3; three hours earlier (9 A. M.) we'll call -3. The rate of the train is fifty miles per hour.

If the train is going due east, three hours later it will be 150 miles east of the station. (+50) x (+3) = +150. If the train is going east, three hours earlier (9 A.M.) it was 150 miles west of the station. (+50) x (-3) = -150.

If, instead the train is going west, then three hours later it will be 150 miles west, -150. (-50) x (+3) = (-150). Three hours earlier, it was 150 miles east of the station. (-50) x (-3) = +150.

The rule emerges as this: Multiplying two integers with like signs gives a positive integer for the product. Multiplying together two integers with opposite signs gives a negative.

Both the commutative and associative laws apply. (+50) x (-3) = (-3) x (+50) as we saw with the trains. (+3) x (-4) x (-5) will result in +60, whether we find first the product of (+3) and (-4), which is (-12), and multiply that by (-5); or whether we multiply (+3) times the product of (-4) and (-5). (+3) times (+20) is also (+60).

The distributive law of multiplication over addition is in operation with the integers. (-3) times the sum of (-2) and (+5) is equal to the sum of the two products (+6) and (-15). Either order results in (-9).

There is also a multiplicative identity (+1) since any integer multiplied by (+1) retains its identity. The signed numbers are closed under multiplication; that is, the product of two integers is an integer.

Since division is the inverse of multiplication, the same law of signs will hold as for multiplication. However, until we redefine the fractions in terms of integers, which we shall presently do, we cannot claim that the integers are closed under division. Unlike signs in division result in a negative quotient. (-3) ÷ (+5) = (-3/5), which is not one of the fractions with which we are familiar. Of course, division by zero continues to remain impossible.

The mathematician sees the integers without all the trimmings. He gets them into the organization by defining them in terms of ordered pairs of natural numbers. Since they grew out of the need to find a number system which made subtraction always possible, they are the elements of the set which provide the solution for all problems of the type (a-b). The mathematician uses a sophisticated way of writing the ordered pair. We shall however, cling to the more meaningful way (a-b) knowing that it is not the same as (b-a), and that a and b are any natural numbers.

When a is greater than b, (a-b) is a positive integer. When a is less than b, (a-b) represents a negative integer. When a is equal to b, (a-b) represents zero.

For example, if a is six and b is two, (a-b) = 6 - 2 = 4, a positive integer. On the other hand, if a is three and b is seven, then (a-b) = 3 - 7 = -4, a negative integer. If a = b = 2, then (a-b) = (2 - 2) = 0.

Two integers of the form (a-b) and (c-d) are defined to be equal if a + d = b + c. To illustrate: (2-5) is equal to -3, while (3-6) is equal to -3, also. The definition says that (2-5) is equal to (3-6) if (2 + 6) = (5 + 3).

Equivalence classes emerge with this definition of equality. (3 + 2), (4 + 1), (1 + 4), (6 - 1), and 5 are all members of the same equivalence class in the natural numbers. 1/2, 2/4, 3/6, 4/8, etc. are elements of an equivalence class in the fractions. (5-0), (8-3), (6-1), (7-2), (68-63), etc. are the same type of group in the integers, while (0-5),(3-8), (1-6), (27-32), etc. are also equal by the definition of equality. (1-6) = (27-32) if (1 + 32) = (6 + 27).

Addition of two integers written in the form of ordered pairs (a-b) and (c-d) is defined as (a + c) - (b + d). Two illustrations will make the definition clearer. The addition of (5-2) and (6-4) is equal to (5+6) - (2+4) or 11-6 = 5. (5-2) = 3; (6-4) = 2; 2 + 3 = 5. Also, (2-5) added to (3-6) according to the definition of addition is equal to (2 + 3) - (5 + 6) = (5) - (11) = -6. Going back to the original problem. (2-5) equals (-3) while (3-6) = -3. (-3) added to (-3) is equal to (-6).

Multiplication is defined also in terms of the ordered pairs. The number (a-b) times (c-d) = (ac + bd) - (bc + ad). This looks involved but it isn't so bad when you see how it is applied. (3-6) times (5-2) by the process which we worked out intuitively earlier says (-3) times (+3) = -9. The definition says (15 + 12) - (30 + 6) = (27 - 36) = -9.

The additive identity is (a-a); that's equal to zero, you remember.

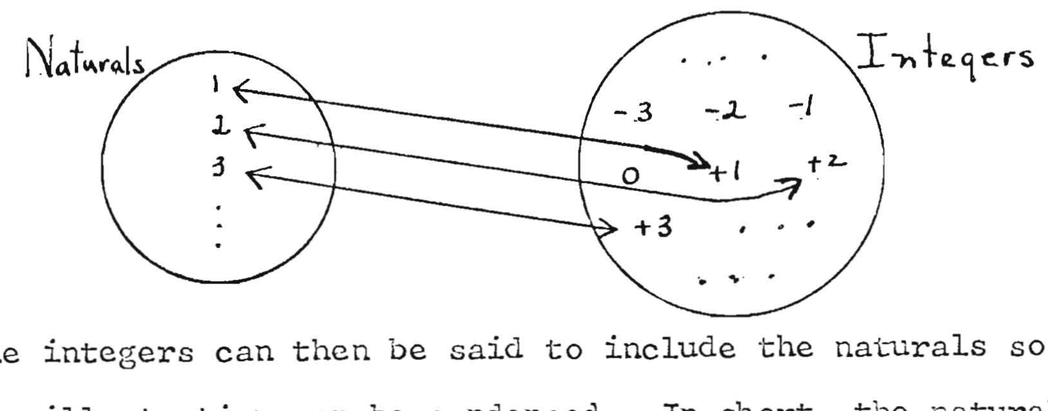
An important property of the integers is that for every positive integer there exists also a negative integer so that their sum equals to zero. (+a) + (-a) = 0.

The multiplicative identity if (1-0), or any element of the equivalence class a-(a-1). Examples are (5-4), (6-5),(27-26), etc.

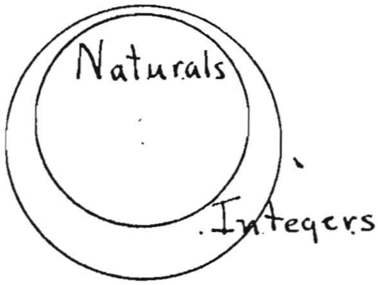
The associative and commutative laws for both addition and multiplication are valid. These were illustrated earlier under the description of the two operations. Of particular interest in the integers is the distributive law. To make it valid under integers it was necessary to define the law of signs in the multiplication of two negative numbers as we determined them to be.

The destributive law says some number 'a' times the sum of two numbers 'b' and 'c' is equal to the product of 'a' and 'b' plus the product of 'a' and 'c'. We saw how it worked in the naturals and the positive fractions. A specific instance will illustrate. We know already that (+5) + (-5) = 0. (-2) times the sum of (+5) and (-5) then must equal zero because (-2) times zero is equal to zero. That is, (-2) x [(+5) - (-5)] = 0. Using the distributive law, [(-2) x (+5)] + [(-2) x (-5)] is equal to (-10) + [(-2) x (-5)]. But we've already established the answer by the other method. The only way (-10) + [(-2) x (-5)] can be equal to zero is for the product of [(-2) and (-5)] to be positive 10. (-10) + (+10) = 0. For that reason, mathematicians defined the product of two negative integers to be a positive integer.

We can set up a one-to-one correspondence between the positive integers and the naturals...

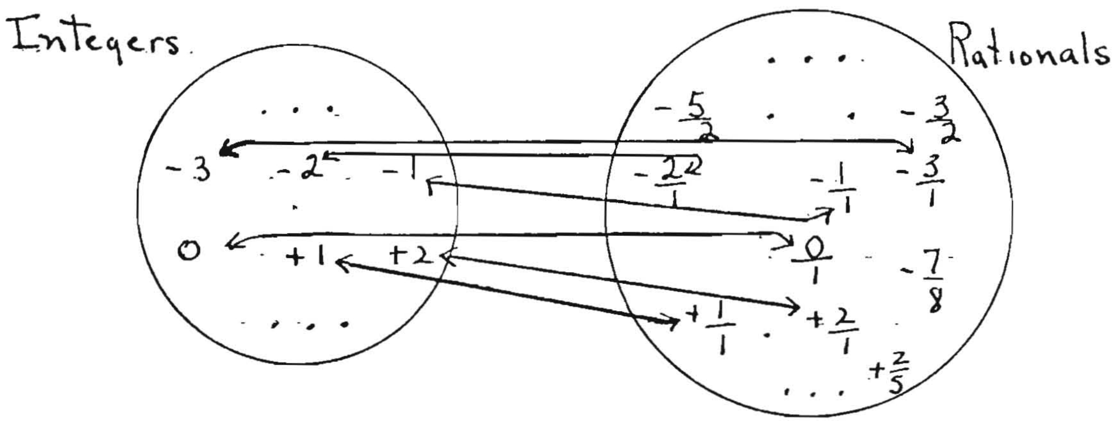


The integers can then be said to include the naturals so the illustration can be condensed. In short, the natural numbers are a subset of the integers.

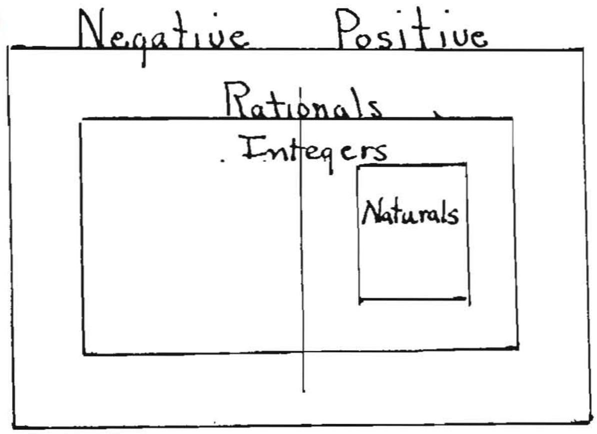


If we go back to the fractions which we defined as the ordered pair of two natural numbers and re-define them as the ordered pair of two integers, then with the law of signs for division for signed numbers, we can include negative fractions as well as positive fractions. This complete system is called the rational number system, where a rational number is defined as an ordered pair of two integers, a/b with b not equal to zero.

Now the diagram can be extended farther. If the integers are written as -3/1..., -2/1, -1/-1, -1/-1... -5/-1... we can show that a one-to-one correspondence can be set up between the integers and some of the rationals.



The integers can be included in the rationals, since they are a subset of the rational number system.



With the development of positive and negative fractions, we have a set of numbers in which addition, multiplication, and their inverse operations, subtraction and division (excepting by zero) are always possible. The five basic laws, commutative for addition and for multiplication, associative for addition and multiplication and the distributive law of multiplication over addition are all valid and lead to no contradictions.

Man has come a long way since the 'one sheep, one finger' days.

CHAPTER VI

FIFTH TALK

These talks have been concerned with showing how a mathematical structure is achieved.

We started with some undefined terms: set, one-to-one correspondence, elements of a set. With these as raw materials, we created some building blocks, the natural numbers. Methods of combining the blocks were evolved. These methods were addition and multiplication. Basic specifications were made and met. They were called the commutative, associative, and distributive laws. We built the foundation.

A problem arose calling for different building blocks. The natural numbers were the raw materials from which we created the new blocks required. The architects called these the integers. They were slightly different in appearance and form. The methods for combining them required slight modifications. Fundamentally the methods were the same, addition and multiplication. The new materials met the basic specifications. The construction continued.

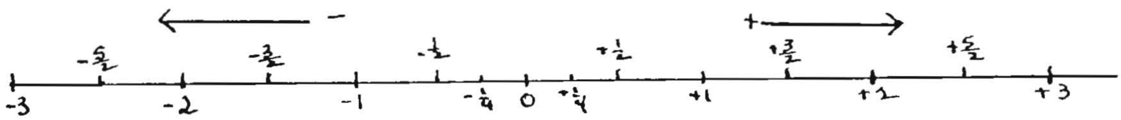
The next phase brought a demand for different building materials. That demand was supplied by using the integers as raw material. The new blocks were called the rationals.

They had a more highly glazed appearance. Their use called for another minor change in methods of combination. The basic specifications remained the same.

No more needs were anticipated. The structure should now be complete.

Actually, what more would seem to be needed in the realm of number? The mathematician had an enlarged system called the rationals, defined as the ratio between two integers, a/b, where b is not equal to zero. With the set of rationals, he could add, multiply, subtract, and divide. The results of these operations were also in the set. Furthermore, these numbers obeyed the laws of commutativity, associativity, and distributivity. What more could the mathematician want?

He could illustrate the rational number system by the use of a number line.



The mid point he labelled zero. Moving to the right he called moving in the positive direction. Moving to the left, which was opposite, he labelled by the negative symbol. He divided the line from zero in each direction into units. At the end of each unit, he located the integers, +1, +2, +3, ... and -1, -2, -3, ... Each integer was a representative of an equivalence class. He then divided the units into smaller parts. These points of division located the rational numbers, +1/2, +3/2, -1/2, -5/8, etc. The +1/2, +3/2, etc. were also representatives of equivalence classes.

We have already shown how the rationals can be set into one-to-one correspondence with the integers, and how the integers could be put in the same relation to the natural numbers. The rationals, then, as has already been pointed out, include the other two number systems.

Every rational number, it was found, could be represented by a point on the number line.

Then, a new crisis arose. It was discovered that while every rational number could be represented by a point on the line, there were points on the line which could not be associated with a rational number.

The story of this new crisis begins with the Egyptians. They discovered that if they needed to construct a right triangle, which they frequently did since they built the Pyramids, they could do so by using a length of rope. They placed two knots in the rope so that the knots divided the length into three, four and five units. One man held the two ends and two other men held the knots, at the same time pulling the rope taut. The figure created was a triangle with a man at each corner. The man standing at the corner between the three and four unit lengths was standing at the right angle.

The Egyptians discovered the method but the Greeks proved it. They stated it in a theorem. The credit for the proof was given to Pythagoras. The Pythagorean theorem says that if a triangle has a right angle, then a square on the side opposite the right angle is equal (in area) to the sum of the areas of the squares on the other two sides. A square on a side means a square using the side of the triangle as a side of the square.

The area of a square is found in the same way as the area of any rectangle, the length times the width. In the case of the square, the length is equal to the width. The area of a square then is a side s times a side s or s times s. The convenient notation is s with a small two written slightly above and to the right of it, s^2

The statement of the Pythagorean theorem was later changed to algebraic form. The square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. We recognize it as a formula $c^2 = a^2 + b^2$, where c represents the length of the hypotenuse, a and b the lengths of theother two sides.

With the theorem, the Greeks knew that what the Egyptians had been doing was right. For $5^2 = 3^2 + 4^2$. $5^2 = 5 \times 5 = 25$; $3^2 = 3 \times 3 = 9$; $4^2 = 4 \times 4 = 16$. Twenty-five did indeed equal nine plus sixteen.

The trouble arose with a square. Let the length of the sides of the square be one unit. Draw in a diagonal which is a line joining two non-consecutive corners. Then the square is divided into two triangles. They are right triangles, because all the angles of a square are right angles. The diagonal is the hypotenuse of the right triangle. Finding the length of the diagonal should be easy. If its length is represented by d, then according to the theorem, $d^2 = 1^2 + 1^2$; $1^2 = 1x1 = 1$. So $d^2 = 2$. If $d^2 = 2$, what did d equal?

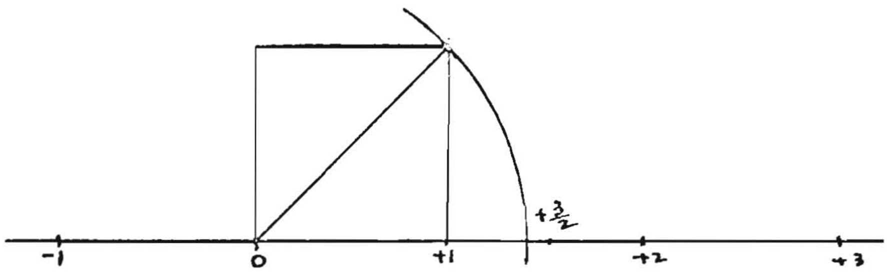
It could have been an easy question. If $d^2 = 4$, then d would equal one of the two equal factors of 4. Since $d^2 = 2^2$, $d = 2$. If $d^2 = 9$, then $d^2 = 3^2$, so $d = 3$. If $d^2 = 25/49$, then $d = 5/7$.

Finding two equal rational numbers the product of which was two was not so simple. In fact, the two equal rational numbers couldn't be found. Some symbolism was devised, written $\sqrt{2}$, and read the square root of two. It represented one of the two equal factors of two. But what was it?

Here was a problem which couldn't be solved by addition, subtraction, multiplication or division.

Yet the square root of two was measurable. It could be located as a point on the number line.

If one constructed a square using one unit on the number line as a side of the square, putting one corner at zero and one at positive one and drew in a diagonal from zero to the opposite corner; the length of that diagonal could be laid off on the number line. The end fell someplace between +1 and +2, a little bit short of +3/2.



It was finally recognized that the $\sqrt{2}$ was not a rational number. A very neat proof was completed, based on the assumption that the $\sqrt{2}$ was equal to the ratio of two integers, which meant it was rational. The proof led to a contradiction. Therefore, the square root was not rational.

Great was the consternation in mathematical circles. The news was so embarrasing that it had top secret priority rating. Gradually, however, the story leaked out to the general public. Legend has it that the one who told lost his life.

Be that as it may, one fact is certain. As a result of the discovery that the $\sqrt{2}$ was not rational, new numbers must be acknowledged. These new numbers were the non-rationals, called the irrationals.

The simplest approach to understanding irrational numbers is to go back and look at the rationals expressed as decimal fractions.

A decimal fraction is one whose denominator is limited to a multiple of ten. Thus, the common fraction, 2/5, becomes a decimal when it is expressed as 4/10. A more convenient notation makes use of position. Then 4/10 is expressed as .4 (read point 4 or 4 tenths), 5/100 as .05 (read point zero five or 5 hundredths), 6/1000 as .006, and so on.

Converting a rational fraction to a decimal was probably first accomplished by thinking, "Two over five is equal to what over ten." (2/5 = 4/10). Here it was necessary to determine by what five was multiplied in order to get ten, then multiplying the two by the same amount. We've met this idea before and called it an equivalence class.

A quicker method evolved. Since two over five indicated division, divide two by five, add a decimal point following the two with the necessary zeros, and divide until there is no remainder.

This method worked as long as the denominator of the common fraction was a multiple of two or five. Fractions like $\frac{1}{2}$, $\frac{1}{4}$, 2/5, 8/25, etc., could all be written as decimal fractions because the division finally 'came out even'. The decimals were called terminating or finite. They ended. One half = .5 = .50 = .500 = .5000. No matter how much farther the division was carried out, the quotient included only more zeros.

Rational numbers like 1/3 or 2/7 or 3/11 were different. One third = .333... The three dots are the mathematician's way of saying "and so on without bounds." Two sevenths = .2857142857142857142... Three elevenths = .27272727... These rationals, and indeed, all rationals whose denominators were not multiples of two or five would never terminate. They were called infinite decimals.

However, in addition to being non-terminating, they did share an interesting second feature. It was found that every rational which did not terminate when it was converted to a decimal repeated after some pattern. Furthermore, every repeating decimal could be converted to a rational number.

It was found, also, that numbers like $\sqrt{2}$ were infinite decimals but they did not repeat. The decimal form of $\sqrt{2}$ begins as if it were going to follow a pattern. We are accustomed to using the approximate value 1.414 for $\sqrt{2}$. But the decimal does not terminate and if carried a bit further, is seen to be non-repeating. Its value is 1.414213... Pi is another non-repeating decimal. Ordinarily we use the approximation 3.14 or 3.1416, rounded off to four decimal places. Not rounded off, it is 3.14159...

William Shanks, an Englishman spent fifteen years of his life working with the decimal which approximates π. He carried it out to 707 places. It was found later that he had made an error in the 528th place. Since then, computers have carried the decimal much farther. Pi, then, is an irrational number. Its decimal is non-repeating and infinite. So, also, are $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$,...

With the recognition and acceptance of the irrationals, a new number system was created. It was called the real number system. It consisted of all the terminating and infinite decimals. The rational numbers, which are terminating and infinite decimals repeating in a pattern comprise a subset of the set of real numbers.

The creation of the real number system made it possible to finish the number line. Every real number could be associated with a point on the line, and what is more, every point on the line could be associated with a real number.

You may well ask, "Now, have all man's number needs been met?"

There are other number systems. The one "just around the corner" is fully developed and provides a solution set for an equation of the type $x^2 + x + 1 = 0$. Third year high school mathematics students are thoroughly introduced to that set of numbers, called complex numbers. There are others, not studied in high school, but fully developed.

However, we shall end our study of mathematical structure with the real numbers.

Remember that each number system was created by man to furnish a solution set for some problem bothering him. The set of natural numbers, 1, 2, 3, 4, ..., enabled him to count, add and multiply. The set of integers, the signed or directed whole numbers and zero, made subtraction always possible. The set of rationals which are the positive and negative fractions, made division always possible. The reals, which are the terminating and infinite decimals, provided a solution set for problems of the type $x^2 = a$.

No one can foretell what the future needs of man may be. Yet, in view of the splendid achievements which have resulted from his efforts to supply necessities in the past, it seems safe to make one prophecy. Whatever crises may arise which call for new systems to provide solution sets, man's intelligence will create the numbers which will supply the answers.

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