# COMPUTATION OF SQUARE ROOT
# BY THE ERROR FORMULA

A THESIS

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# CHAPTER I

## THE PROBLEM

1.1. Introduction. The computation of square roots is encountered in nearly every course in mathematics from arithmetic through calculus. Usually a set of rules is given to be followed in order to get an answer, and the understanding of the process is disregarded. Most of the processes at least require several calculations to obtain the roots to the desired number of decimal places. A method is needed in mathematics which can be easily mastered and understood. At the same time, it should require a minimum amount of calculation.

1.2. Statement of the problem. This study will investigate the use of the error formula to compute square roots. It will compare this method with other methods, investigate the rapidity of convergence using the correction, determine the accuracy achieved by a one-digit estimate with and without the correction term, extend the correction formulas, and develop a procedure for the application of the method. The devised method will shorten the amount of calculation need to extract square roots.

1.3. Importance of the study. Tables of square root are commonly found for the integers up to one hundred. Sometimes these tables will be extended to five hundred. The use of tables will usually save time and reduce the chances of mechanical error. But they are limited to only a few values. For this reason, a knowledge of some method to compute square roots is necessary.

Various methods may be used to find square roots. Some of the more common methods will be discussed in Chapter II.

This study will seek to develop an improved method for computing square root. It will use as its foundation the iteration method and apply a correction based on the error of the approximations of this iteration. Hence it will be a computation rather than an iteration. It will have all the advantages of the iteration method and yet shorten the amount of computation necessary to find the roots to the desired number of decimal places.

Square roots are of great importance in many fields of physical application. It is thought that this method of calculation will reduce the amount of calculations needed in finding square roots.

1.4. Limitations of the study. The scope of this study is limited only to the computation of positive square roots of real numbers. Its purpose will be to establish a method and to test this method. A procedure will be developed based on the understanding of the method.

1.5. Organization of the thesis. Chapter II is devoted to a review of the literature about computation of square roots. Chapter III is the development of the iteration method and the error formulas. Chapter IV shows the accuracy of the method with a one-digit first estimate. Chapter V extends the correction formulas and a different extension is developed in Chapter VI. Chapter VII develops a procedure for using this method with and without computing machines. Chapter VIII, the final chapter, is a summary of the findings of the study, together with suggestions for future related study.

## CHAPTER II

## REVIEW OF LITERATURE

2.1. Introduction. This study required two phases in the review of the literature. First was a historical overview of square roots. The second involved literature more specifically dealing with the iteration method and error formulas. Books of mathematical history provided the first, and articles in periodicals provided most of the last.

2.2. History of square roots. Square roots have been in use for a long time. Hofmann mentions that they were computed and used by the Babylonians (2000-200 B.C.).[1] Although the symbolism was different, they approximated the square roots by the simple and repeated application of the arithmetical-geometrical mean,

$$\sqrt{a^2 + B} \approx a + \frac{B}{2a}$$

which they probably derived from geometric considerations.

Hooper traces the development of the symbols used in connection with square root.[2] The word "root" came from the geometric concept that numbers grow somewhat like a

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[1]Joseph Ehrenfried Hofmann, The History of Mathematics, (New York: The Philosophical Library, Inc., 1957), p. 6.

[2]Alfred Hooper, Makers of Mathematics, (New York: Random House, 1948), pp. 74-75.

plant. The square root sign √ can be traced to the Latin word _radix_. When the works of al-Khowarizmi and others were translated from Arabic to Latin, the Arab concept of root was put in this Latin word. This was shortened to the symbol R_x during the later part of the middle ages and was used to indicate the root of a number. Eventually this was shortened to a small r and the square root sign is merely this letter slightly changed in shape. The present concept of a radical comes from this word also.

The Greek civilization gave quite a bequest of mathematical knowledge to the world. In the works of Pythagorus, Euclid, Plato, and others, there are many instances of the use of square roots. The irrational numbers were developed by the Greek culture. This broadened the concept of square root to mean the rational approximation of a factor of a number which, when the factor is multiplied by itself, will give the number.

2.3. _Methods of computing square root_. Boyer says, "An algorithm in mathematics, is a procedure which yields the desired result without much attention to the reasoning back of the procedure."[3] An algorithm is one of the most used methods for computing square roots. There are many

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[3]Lee Emerson Boyer, _Mathematics A Historical Development_ (New York: Henry Holt and Company, 1945), p. 50.

sources which give the rules for an algorithm for finding square roots. Most of them will agree with the set of rules found in Boyer:[4]

> 1. Separate the number into periods of two figures each, beginning at the decimal point. (Sometimes the farthest to the left contains only one figure.)
>
> 2. Find the largest number whose square is not more than the left hand period; write it as the first figure of the root; subtract its square from the number; and bring down the next period for a new dividend.
>
> 3. Divide the new dividend, omitting the last figure, by twice the root already found, and annex the quotient to the root and also to the divisor.
>
> 4. Multiply the complete divisor by the second term of the root and subtract the product from the dividend.
>
> 5. Continue in this fashion until all the periods are used.

This algorithm can be traced to the expansion of the binomial theorm to the second power. Using t for the tens digit of a number and u for the units digit, then $(10t+u)^2 = 100t^2+20tu+u^2$. By partial factoring this becomes $100t^2+u\ (20t+u)$. The method can then be followed through with the steps listed above.

Mallory attributes the development of the binomial theorm to Euclid (300 B.C.) and the development of the algorithm to Theon of Alexandria (400 A.D.)[5]

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[4] Ibid.

[5] Virgil S. Mallory, First Algebra (Chicago: Benj. H. Sanborn & Co., 1950), p. 376.

The principal criticisms of the algorithm are that it is difficult to remember, requires a good understanding of the binomial theorem to construct, requires quite a bit of calculation, and is inefficient. It demands a separate computation for each digit of accuracy.

Sample makes the following statement before listing the steps of the algorithm:

> As a preliminary . . . in the recommended procedure which is outlined below, it is presumed, therefore, that the relatively simple expansion of a binomial has been mastered . . . .[6]

Portz presents a different approach to the algorithm method and suggests that it be used only after the student has been exposed to the iteration method and then as a method of application of the binomial theorem.[7]

A common method suggested and used in many textbooks is the use of a table. The view expressed by the School Mathematics Study Group probably sums up the feeling of many authors on the topic. They say, "You should extract square root approximations from tables if tables are available."[8] Tables cannot replace a method of calculation, however; they

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[6] P. A. Sample, "A Device for Teaching the Extraction of Roots," The Mathematics Teacher, 40:340-44, November, 1947.

[7] Bernard J. Portz, "Square Root: An Algebraic Approach," School Science and Mathematics, 55:312-14, April, 1955.

[8] School Mathematics Study Group, Mathematics for High School, Algebra, Part 2 (New Haven, Conn.: Yale University).

are only a supplemental device that should be used when possible.

A method of iteration is sometimes regarded as the best for computing square roots. A discussion of the iteration method is saved for Chapter III. The usual iterative method is essentially the application of the arithmetic mean. The formula

$$x = \frac{1}{2}(x_o + \frac{N}{x_o})$$

is repeated over and over until $x = \sqrt{N}$. Certain refinements of this statement are made in Chapter III.

The earliest recorded use of this formula according to Eves was in Book II of Metrica by Heron of Alexandria.[9] The dates for Heron's life are not definitely established but it is believed he lived sometime between 300 B.C. and 150 A.D.

The School Mathematics Study Group lists the following ten reasons why they support the iteration process in teaching the extractions of square roots.[10] They are applicable to this discussion also.

> 1. The iteration method can be made meaningful. It is based on the definition of a square root.

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[9]Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart, 1953), p. 155.

[10]School Mathematics Study Group, op. cit., p. 363.

2. The student is more likely to realize that he is finding an approximation to $\sqrt{N}$, than when he uses the algorithm. In fact, he can be taught to estimate the size of the error.

3. The student is estimating his results; so he is not likely to make a bad error without realizing it.

4. The second approximation can very often be done mentally; always with very little arithmetic. In many cases it is all that is needed.

5. An easy division with a two-digit divisor yields a result in which the error is in the fourth digit. This is sufficient for most purposes.

6. The method can be completely justified algebraically.

7. A formula for the error of any approximation can be derived.

8. The method is ideal for machine calculation.

9. If the first approximation is obtained from the slide rule, the second approximation is likely to be correct to 7 or 8 digits.

10. The method is self-correcting. That is, if an error is made, it will still give the correct figures providing the error is not made on the last approximation.

Statement number seven of this extract is the basic premise of this thesis. The School Mathematics Study Group includes the ideas of the error formulas as an optional part of the ninth grade course in mathematics.[11] This is one of the few articles using the error formula to compute square roots.

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[11] Ibid., p. 369.

Ward develops an error formula but uses it to only show how the iteration process works.[12] Bouton uses an error formula in a similar manner as early as 1908.[13] He infers that his knowledge is gained from earlier sources but gives no reference.

Chittenden used the error formula for computing square roots in a manner very similar to the one developed in this thesis.[14] He wrote his article in response to an article in the Mathematics Teacher by Tobey a year earlier.[15] Tobey defends the iteration process and Chittenden merely shows that the method can be improved with the use of error formulas.

Brown states the iteration method in a simpler form.[16] This simple statement has only three rules.

1. Divide the number by anything.

2. Divide the number by something between the previous divisor and its quotient.

3. Repeat step two until the divisor equals the quotient to the required number of decimal places or differs by only one in the last decimal place.

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[12] Lewis E. Ward, "On the Computation of Square Roots," The Mathematics Teacher, 24:86-88, March, 1931.

[13] ______ Bouton, "Square Root", Annals of Mathematics, Series 2, 10:167-172, 1909.

[14] E. W. Chittenden, "On Square Root by Division", Mathematics Teacher, 39:75-76, March, 1946.

[15] William S. Tobey, "Square Root by Approximation and Division", Mathematics Teacher, 38:131-2, March, 1945.

[16] Elizabeth F. Brown, "Roots and Logrithms", Mathematics Teacher, 49:544-7, November, 1956.

Shuster adapts the idea of an error in any approximation of a square root to the use of a slide rule although the formula for the error is developed from different considerations.[17]

Arguments may be found to support the method of iteration, the algoritham, the use of tables, the use of a slide rule, the use of logrithms, the use of a binomial series expansion, the use of continued fractions, and others. Each will have advantages and limitations.

Luke maintains that the Newton-Raphson method is very efficient when applied to an equation of the form $x^2 - N = 0$.[18] He develops the formula

$$x = \frac{N + x_o^2}{2x_o}$$

from the Newton-Raphson method. When this is simplified it is exactly the iteration formula. Therefore the Newton-Raphson method applied to finding square roots is the same as the iteration process.

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[17] Carl N. Shuster, "Approximate Square Roots", Mathematics Teacher, 45:17-18, January, 1952.

[18] Yudell L. Luke, "Numerical Analysis and High School Mathematics", Mathematics Teacher, 50:507-12, November, 1957.

## CHAPTER III

## THE ERROR FORMULA AND THE ITERATION PROCESS

3.1. Introduction. The iteration process for extracting square root was mentioned in Chapter II. This process will be used as a starting point from which the error formula will be developed. There is nothing original or unique in the use of iteration. It has been used in some way by mathematicians since the time of Heron. By combining the concepts of the iteration process and the fact that each approximation has some error, the result is an improved method for extracting a square root.

3.2. Definition of square root. The square root of a number may be defined as one of two equal factors of the number. Two observations need to be made about this definition for the purposes of this discussion.

First, for any number there exists two square roots equal in numerical value and opposite in sign. Since one is merely the negative of the other, only the positive value is considered here.

The second observation is that although all square roots exist by the definition, some will be irrational numbers. An irrational number cannot be expressed as an exact rational quantity. Where the square root is irrational, it is desirable to find a rational approximation of the root.

Therefore the term "square root" throughout this discussion is used to mean the "rational approximation of the positive square root of a number."

3.3 <u>The principle of iteration</u>. The definition of the square root of a number is one of two equal factors of the number. If $N$ is the number whose square root is desired, then $x$ is a square root of $N$ if and only if $x = \frac{N}{x}$ .

If $x_1$ is taken as an approximation of $\sqrt{N}$, a second approximation, better than $x_1$, can be found by the equation

$$x_2 = \tfrac{1}{2}(x_1 + \frac{N}{x_1}).$$

Form $x_3$ by

$$x_3 = \tfrac{1}{2}(x_2 + \frac{N}{x_2})$$

and $x_n$ by

$$x_n = \tfrac{1}{2}(x_{n-1} + \frac{N}{x_{n-1}})$$

Hence, the sequence of numbers $x_1, x_2, x_3, \ldots, x_n, \ldots$ tends to a limit which is $\sqrt{N}$, $x_1 \neq 0$.

To prove this it is necessary to show that $x_n$ and $\frac{N}{x_n}$ form an interval around $\sqrt{N}$ and that $x_n > \tfrac{1}{2}(x_{n-1} + \frac{N}{x_{n-1}})$. If $x_n$ and $\frac{N}{x_n}$ form an interval around $N$, then either of the two cases

$$\frac{N}{x_n} < \sqrt{N} < x_n \qquad \text{or} \qquad x_n < \sqrt{N} < \frac{N}{x_n}$$

may exist. If $x_n > \sqrt{N}$, then multiplying both sides of the inequality by $\sqrt{N}$ gives $x_n\sqrt{N} > N$ and multiplying both sides of this inequality by $\frac{1}{x_n}$ gives $\sqrt{N} > \frac{N}{x_n}$ and thus $x_n > N > \frac{N}{x_n}$. Taking $x_n < \sqrt{N}$ and multiplying both sides of the equality by $\sqrt{N}$ makes $x_n\sqrt{N} < N$. And multiplying both sides of this inequality by $\frac{1}{x_n}$ gives $\sqrt{N} < \frac{N}{x_n}$. The conclusion can be drawn then that $x_n$ and $\frac{N}{x_n}$ will always be on opposite sides of $\sqrt{N}$ or in other words, they will always form an interval around $\sqrt{N}$.

From an intuituve standpoint, this is a logical condition. For instance, if a number $r$ is taken as a first estimate of $\sqrt{N}$, then $r \cdot s = N$. If $r$ is too large, then $s$ has to be too small. It follows that the $\sqrt{N}$ lies between them and a likely second approximation would be some number that is half way between $r$ and $s$.

Taking the case where $x_1 > \sqrt{N}$, and squaring both sides of the inequality, then

$$x_1^2 > N ,$$

multiplying by $\frac{1}{x_1}$ on both sides gives

$$x_1 > \frac{N}{x_1} ,$$

adding $x_1$ to each

$$2x_1 > x_1 + \frac{N}{x_1} ,$$

and multiplying both sides by ½

$$x_1 > \tfrac{1}{2}(x_1 + \frac{N}{x_1}).$$

Taking this procedure and repeating it for $x_2$, $x_3$, ...,
it would eventually yield the condition

$$x_n > \tfrac{1}{2}(x_{n-1} + \frac{N}{x_{n-1}}).$$

If the difference

$$\tfrac{1}{2}(x_1 + \frac{N}{x_1}) - \sqrt{N}$$

is taken, it becomes

$$= \frac{x_1^2 - 2\sqrt{N}x_1 + (\sqrt{N})^2}{2x_1}$$

$$= \frac{(x_1 - \sqrt{N})^2}{2x_1}$$

Since the numerator of this term is squared, its value will always be positive. From this the conclusion can be drawn that

$$x_1 > x_2 > x_3 > x_n > \sqrt{N}$$

Another observation to be gleaned from this is that regardless of whether $x_1$ is greater than or less than $\sqrt{N}$, $x_2$ will always be larger than $\sqrt{N}$. For this reason in the following section, $x_1$ will be used as the larger of the first estimate or the number divided by the first estimate.

3.4. <u>The error formula</u>. Any approximation of a square root will have an error involved. Let $C_1$ be the true error of the approximation of $\sqrt{N}$, $x_1$. Or

$$C_1 = x_1 - \sqrt{N}.$$

So then

$$C_2 = x_2 - \sqrt{N} ,$$

$$C_3 = x_3 - \sqrt{N} ,$$

$$C_n = x_n - \sqrt{N} .$$

Taking the iteration process, it is noted that

$$x_2 = \tfrac{1}{2}(x_1 + \frac{N}{x_1}).$$

Then

$$C_2 = \tfrac{1}{2}(x_1 + \frac{N}{x_1}) - \sqrt{N}$$

and clearing parentheses and combining terms

$$C_2 = \frac{x_1{}^2 - 2\sqrt{N}\, x_1 + (\sqrt{N})^2}{2x_1} .$$

The numerator is a perfect square, so

$$C_2 = \frac{(x_1 - \sqrt{N})^2}{2x_1}$$

and substituting $C_1$ for $( x_1 - \sqrt{N} )$

$$C_2 = \frac{C_1{}^2}{2x_1} \quad .$$

Thus there is a formula for the error of a term expressed in relation to the error of the preceding term. The general case would give

$$C_n = \frac{C_{n-1}{}^2}{2x_{n-1}}$$

It would be very convenient to merely compute this error and subtract it from the estimate. However it is necessary to know the square root in order to compute the error and so a different view of the situation is needed.

Borrowing an idea from the Newton-Raphson method, the possibility becomes evident that it might be desirable to estimate the error and subtract this estimate of the error from the approximation. This is essentially the idea behind the Newton-Raphson method but the estimate of the error is found in a different manner.

Now to develop a method for estimating the error, take $x_1$ as an approximation of $\sqrt{N}$ and find $x_2$ by the iteration process. Before continuing the iteration process any farther it would be desirable to subtract the estimate of the error of $x_2$. The error of $x_1$ is $x_1 - \sqrt{N}$. But $\sqrt{N}$ is not known and the best approximation of $\sqrt{N}$ is going to be $\frac{1}{2}(x_1 + \frac{N}{x_1})$

or that is, $x_2$. Replacing $\sqrt{N}$ by this best approximation gives the error of $x_1$ to be estimated, letting e be the estimate,

$$e_1 = x_1 - x_2 .$$

Using $e_1$ as an estimate of $C_1$, then the estimate $e_2$ $C_2$

$$e_2 = \frac{e_1^{\,2}}{2x_1} ,$$

or

$$e_2 = \frac{(x_1 - x_2)^2}{2x_1} .$$

Going to the general case,

$$C_n \approx e_n = \frac{e_{n-1}^{\,2}}{2x_{n-1}}$$

$$= \frac{(x_{n-1} - x_n)^2}{2x_{n-1}} .$$

Since $x_2 > x_3 > \sqrt{N}$,

$$(x_2 - x_3)^2 < (x_2 - \sqrt{N})^2$$

and

$$\frac{(x_2 - x_3)^2}{2x_2} < \frac{(x_2 - \sqrt{N})^2}{2x_2}$$

So

$$e_3 < C_3$$

Replacing $x_2$ and $x_3$ by $x_n$ and $x_{n-1}$ would lead to the general case that

$$e_n < C_n$$

Therefore estimating the error at any step in the iteration method still keeps the process valid. It will make the amount of computation less to extract a root and the iteration actually becomes a calculation.

The method then becomes one where

$$x \approx x_n - \frac{(e_{n-1})^2}{2x_{n-1}} .$$

This improved approximation is then used in the iteration procedure to obtain a new approximation from which a better approximation is calculated by subtracting the error.

3.5. <u>Convergence</u>. Since each successive approximation is closer to $\sqrt{N}$, it is easily seen that the process will converge to $\sqrt{N}$. The use of the error formula will cause the process to converge faster.

Take a number line on which $\sqrt{N}$ is plotted. The first approximation $x_1 > \sqrt{N}$ is plotted to the right of $\sqrt{N}$. Then $\frac{N}{x_1}$ is plotted to the left. Using the iteration process,


$x_2$ is found between $x_1$ and $\sqrt{N}$. When the estimate of the error in $x_2$ is subtracted from $x_2$ a closer approximation is obtained. Using this value to find $x_3$ will cause $x_3$ to be closer to $\sqrt{N}$ than using $x_2$ to find $x_3$.

# CHAPTER IV

## ACCURACY OF THE SECOND APPROXIMATION

4.1. Introduction. The process developed in the proceeding chapter now must be tested to see how good it really might be. In this chapter a table will be constructed to show and compare the value of $x_2$ before and after the correction term is applied.

4.2. Procedure. The table constructed has values of N ranging from 1 to 100. Any other number could be expressed as a multiple of an even power of 10 and an N between 1 and 100. That is, if N is a number between 1 and 100, then any number not between 1 and 100 could be expressed in the form

$$N \cdot 10^{2k} \quad , (k \text{ any integer}),$$

and the square root of this number would be

$$\sqrt{N} \cdot 10^{k} \quad .$$

Hence only values in this range need be considered to compute square roots. This aids greatly in locating a good first estimate.

Take the first estimate as the nearest integer to $\sqrt{N}$. Compute $\frac{N}{x}$ and use the greater of these as the $x_1$ with which $x_2$ is computed by the iteration formula. Then $e_1$ is computed by subtracting $x_2$ from $x_1$ and $e_2$ is computed by the error formula. The final step is to subtract $e_2$ from $x_2$. All of the values are shown in the table to give a comparison and provide information to evaluate the usefulness of the process.

Picking the first estimate at the nearest integer will give the best one-digit first estimate possible for the desired square root. This is not a necessary condition. Any number would do for the first estimate. It would just require more steps to get an accurate answer.

The reason that the larger of $x_1$ and $\frac{N}{x_1}$ is used as the $x_1$ in the error formula is merely that it is more convenient to have $e_1$ positive. Again, it is not a necessary condition.

4.3. Interpretation of the table. The accuracy of the process varies. The greatest error in the first estimate is 0.5. This is a result of using the nearest integer for the first approximation. As N becomes closer to a perfect square, the accuracy improves. It is only logical that $e_2$ would become smaller as $e_1$ becomes smaller.

The first digit of error may occur in the decimal place of the first significant digit of $e_2$. The first digit of error will not always occur at that point however. In many cases the corrected value of the second approximation is accurate to more digits than the error indicates. A method is used in Chapter V to locate these cases where more accuracy exists.

TABLE I

APPROXIMATIONS OF SQUARE ROOTS FROM 1 TO 100
USING A ONE-DIGIT FIRST ESTIMATE

| N | $x_1$ | $\frac{N}{x_1}$ | $x_2$ | $e_1$ | $e_2$ | $x_2$ corrected |
|---|---|---|---|---|---|---|
| 1 | 1 | 1.0000 | 1.0000 | --- | --- | --- |
| 2 | 2 | 1.0000 | 1.5000 | .5 | .0625 | 1.4375* |
| 3 | 2 | 1.5000 | 1.7500 | .25 | .015 | 1.734 |
| 4 | 2 | 2.0000 | 2.0000 | --- | --- | --- |
| 5 | 2 | 2.5000 | 2.2500 | .25 | .0125 | 2.237 |
| 6 | 3 | 2.0000 | 2.5000 | .5 | .041 | 2.459 |
| 7 | 3 | 2.3333 | 2.6666 | .33 | .013 | 2.654 |
| 8 | 3 | 2.6666 | 2.8333 | .17 | .0048 | 2.8285 |
| 9 | 3 | 3.0000 | 3.0000 | --- | --- | --- |
| 10 | 3 | 3.3333 | 3.16666 | .17 | .004 | 3.1626 |
| 11 | 3 | 3.6666 | 3.3333 | .33 | .014 | 3.319 |
| 12 | 4 | 3.0000 | 3.5000 | .5 | .031 | 3.469 |
| 13 | 4 | 3.2500 | 3.6250 | .375 | .017 | 3.608 |
| 14 | 4 | 3.5000 | 3.7500 | .25 | .0078 | 3.7422 |
| 15 | 4 | 3.7500 | 3.8750 | .125 | .0019 | 3.87381 |
| 16 | 4 | 4.0000 | 4.0000 | --- | --- | --- |
| 17 | 4 | 4.2500 | 4.1250 | .125 | .0018 | 4.12382 |
| 18 | 4 | 4.5000 | 4.2500 | .25 | .0069 | 4.2471 |
| 19 | 4 | 4.7500 | 4.3750 | .375 | .015 | 4.360 |
| 20 | 5 | 4.0000 | 4.5000 | .5 | .025 | 4.475 |
| 21 | 5 | 4.2000 | 4.6000 | .4 | .016 | 4.584 |
| 22 | 5 | 4.4000 | 4.7000 | .3 | .009 | 4.691 |
| 23 | 5 | 4.6000 | 4.8000 | .2 | .004 | 4.796 |
| 24 | 5 | 4.8000 | 4.9000 | .1 | .001 | 4.899 |
| 25 | 5 | 5.0000 | 5.0000 | --- | --- | --- |
| 26 | 5 | 5.2000 | 5.1000 | .1 | .00097 | 5.09903 |
| 27 | 5 | 5.4000 | 5.2000 | .2 | .0037 | 5.1963 |
| 28 | 5 | 5.6000 | 5.3000 | .3 | .008 | 5.292 |
| 29 | 5 | 5.8000 | 5.4000 | .4 | .014 | 5.386 |
| 30 | 6 | 5.0000 | 5.5000 | .5 | .02 | 5.48 |

*Note: The underscored digit is the digit in which the first error may occur.

TABLE I (continued)

| N | $x_1$ | $\frac{N}{x_1}$ | $x_2$ | $e_1$ | $e_2$ | $x_2$ corrected |
|---|---|---|---|---|---|---|
| 31 | 6 | 5.1666 | 5.5833 | .42 | .0147 | 5.5686 |
| 32 | 6 | 5.3333 | 5.6666 | .33 | .0091 | 5.6575 |
| 33 | 6 | 5.5000 | 5.7500 | .25 | .0052 | 5.7448 |
| 34 | 6 | 5.6666 | 5.8333 | .17 | .0024 | 5.83093 |
| 35 | 6 | 5.8333 | 5.9166 | .08 | .00026 | 5.91640 |
| 36 | 6 | 6.0000 | 6.0000 | --- | --- | --- |
| 37 | 6 | 6.1666 | 6.0833 | .08 | .0005 | 6.08283 |
| 38 | 6 | 6.3333 | 6.1666 | .17 | .0023 | 6.16443 |
| 39 | 6 | 6.5000 | 6.2500 | .25 | .0048 | 6.2452 |
| 40 | 6 | 6.6666 | 6.3333 | .33 | .0081 | 6.3251 |
| 41 | 6 | 6.8333 | 6.41666 | .42 | .013 | 6.4036 |
| 42 | 7 | 6.0000 | 6.5000 | .5 | .018 | 6.482 |
| 43 | 7 | 6.1430 | 6.5715 | .43 | .0132 | 6.5583 |
| 44 | 7 | 6.2860 | 6.6430 | .36 | .0092 | 6.6338 |
| 45 | 7 | 6.4290 | 6.7145 | .29 | .006 | 6.7085 |
| 46 | 7 | 6.5710 | 6.7855 | .22 | .0034 | 6.7821 |
| 47 | 7 | 6.7143 | 6.8571 | .14 | .0014 | 6.8557 |
| 48 | 7 | 6.8571 | 6.92855 | .07 | .00035 | 6.92820 |
| 49 | 7 | 7.0000 | 7.0000 | --- | --- | --- |
| 50 | 7 | 7.142857 | 7.0714285 | .07 | .000357 | 7.07107 |
| 51 | 7 | 7.2857143 | 7.142857 | .14 | .0013 | 7.1415 |
| 52 | 7 | 7.4290 | 7.2145 | .21 | .0029 | 7.2116 |
| 53 | 7 | 7.5710 | 7.2855 | .28 | .0051 | 7.2804 |
| 54 | 7 | 7.7140 | 7.357 | .35 | .008 | 7.349 |
| 55 | 7 | 7.8570 | 7.4285 | .42 | .012 | 7.4175 |
| 56 | 8 | 7.0000 | 7.5000 | .5 | .0156 | 7.4844 |
| 57 | 8 | 7.1250 | 7.5625 | .44 | .0121 | 7.5504 |
| 58 | 8 | 7.2500 | 7.6250 | .37 | .0085 | 7.6165 |
| 59 | 8 | 7.3570 | 7.6875 | .31 | .006 | 7.6815 |
| 60 | 8 | 7.5000 | 7.7500 | .25 | .0039 | 7.7461 |
| 61 | 8 | 7.6250 | 7.8125 | .19 | .0022 | 7.8103 |
| 62 | 8 | 7.7500 | 7.8750 | .12 | .0009 | 7.8741 |
| 63 | 8 | 7.8750 | 7.9357 | .06 | .000125 | 7.937375 |
| 64 | 8 | 8.0000 | 8.0000 | --- | --- | --- |
| 65 | 8 | 8.1250 | 8.0625 | .06 | .00022 | 8.06228 |

TABLE I (continued)

| N | $x_1$ | $\frac{N}{x_1}$ | $x_2$ | $e_1$ | $e_2$ | $x_2$ corrected |
|---|---|---|---|---|---|---|
| 66 | 8 | 8.2500 | 8.1250 | .12 | .00088 | 8.12412 |
| 67 | 8 | 8.3750 | 8.1875 | .19 | .0021 | 8.1854 |
| 68 | 8 | 8.5000 | 8.2500 | .25 | .0036 | 8.2464 |
| 69 | 8 | 8.6250 | 8.3125 | .31 | .0055 | 8.3070 |
| 70 | 8 | 8.7500 | 8.3750 | .37 | .0081 | 8.3669 |
| 71 | 8 | 8.8750 | 8.4375 | .44 | .011 | 8.4265 |
| 72 | 9 | 8.0000 | 8.5000 | .5 | .0139 | 8.4861 |
| 73 | 9 | 8.1111 | 8.5555 | .44 | .0107 | 8.5448 |
| 74 | 9 | 8.2222 | 8.61111 | .39 | .0084 | 8.6027 |
| 75 | 9 | 8.3333 | 8.66666 | .33 | .006 | 8.6606 |
| 76 | 9 | 8.4444 | 8.72222 | .28 | .0043 | 8.7179 |
| 77 | 9 | 8.5555 | 8.77777 | .22 | .0027 | 8.77507 |
| 78 | 9 | 8.6666 | 8.83333 | .17 | .0015 | 8.8318 |
| 79 | 9 | 8.7777 | 8.88888 | .11 | .00067 | 8.88821 |
| 80 | 9 | 8.8888 | 8.94444 | .05 | .00017 | 8.94427 |
| 81 | 9 | 9.0000 | 9.0000 | --- | --- | --- |
| 82 | 9 | 9.1111 | 9.0555 | .05 | .00013 | 9.05542 |
| 83 | 9 | 9.2222 | 9.1111 | .11 | .00065 | 9.11046 |
| 84 | 9 | 9.3333 | 9.1666 | .17 | .0012 | 9.1654 |
| 85 | 9 | 9.4444 | 9.2222 | .22 | .0025 | 9.2197 |
| 86 | 9 | 9.5555 | 9.2777 | .28 | .0041 | 9.27367 |
| 87 | 9 | 9.6666 | 9.3333 | .33 | .0056 | 9.3277 |
| 88 | 9 | 9.7777 | 9.3888 | .39 | .00804 | 9.38084 |
| 89 | 9 | 9.8888 | 9.4444 | .44 | .0093 | 9.4351 |
| 90 | 10 | 9.0000 | 9.5000 | .5 | .0125 | 9.4875 |
| 91 | 10 | 9.1000 | 9.5500 | .45 | .01 | 9.54 |
| 92 | 10 | 9.2000 | 9.6000 | .4 | .008 | 9.592 |
| 93 | 10 | 9.3000 | 9.6500 | .35 | .0061 | 9.6439 |
| 94 | 10 | 9.4000 | 9.7000 | .3 | .0045 | 9.6955 |
| 95 | 10 | 9.5000 | 9.7500 | .25 | .0031 | 9.7469 |
| 96 | 10 | 9.6000 | 9.8000 | .2 | .002 | 9.798 |
| 97 | 10 | 9.7000 | 9.8500 | .15 | .0011 | 9.8489 |
| 98 | 10 | 9.8000 | 9.9000 | .1 | .0005 | 9.8995 |
| 99 | 10 | 9.9000 | 9.9500 | .05 | .00012 | 9.94988 |
| 100 | 10 | | | | | |

In every case the accuracy is increased and in every case the corrected value of the second approximation is still greater than the square root of N. This would be modified in some cases if the digits were rounded off.

A general statement could be made that this process is good for three-digit accuracy since most of the cases show this. The exceptions come in two types of situations. First, when $x_1$ and $\frac{N}{x_1}$ are both integers one unit apart or in cases approaching this situation the error is the greatest and the accuracy is less. Second, when $x_1$ is relatively small. Actually these are merely the conditions when $e_1^2/2x_1$ would be the greatest; $e_1$ larger and/or $x_1$ smaller.

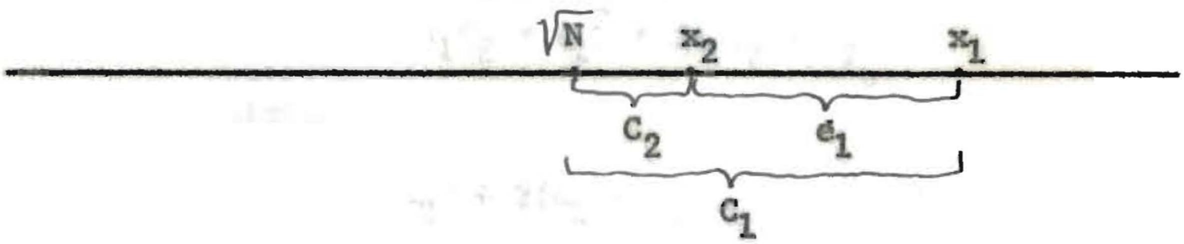
In many cases, this would be as much accuracy as is desired. If not, using this three-digit estimate to find $x_3$ would be accurate to about six significant digits and application of the error formula would possibly extend this to seven or eight significant digits or more.

## CHAPTER V

## EXTENDED CORRECTION FORMULAS

5.1. Introduction. Many of the values of the corrected $x_2$ terms in Table I were accurate to more digits than the error formula indicated. In this chapter, a different approach is used to get the approximate value of $C_2$. A second correction factor formula is incorporated into the method to give even greater accuracy. A short table is used to show the improvement over Table I. This improved method is a desirable method for accuracy of about four or five significant digits in most cases on the second approximation.

5.2. The improved correction formulas. The definitions of Chapter III for $C_1$, $C_2$, $e_1$, and $e_2$ are assumed for this chapter and the derivation of the improved formulas. When these values are plotted on a number line some relationships not noted before may be brought out. One of these




relationships useful for this discussion is the obvious fact that $e_1 + C_2 = C_1$. This can be derived from the definitions in an algebraic manner. Take the definitions of $C_1$, $C_2$, and $e_1$ from Chapter III:

$$C_1 = x_1 - \sqrt{N}, \qquad C_2 = x_2 - \sqrt{N}, \qquad e_1 = x_1 - x_2.$$

By adding

$$C_2 + e_1 = x_2 - \sqrt{N} + x_1 - x_2 = x_1 - \sqrt{N} = C_1.$$

But, a formula was developed in Chapter III which gave the value of $C_2$ in terms of $C_1$. That is,

$$C_2 = \frac{C_1^2}{2x_1}$$

Replacing $C_1$ by $e_1 + C_2$ gives

$$C_2 = \frac{(e_1 + C_2)^2}{2x_1}\ .$$ [1]

This is an interesting formula as the whole left member of the equation is contained in the right member as one of its parts and it is only logical that it should be solved for $C_2$ if possible. Removing fractions and expanding the numerator of the left member gives

$$2x_1C_2 = e_1^2 + 2e_1C_2 + C_2^2$$

Collecting terms

$$C_2^2 + 2(e_1 - x_1)C_2 + e_1^2 = 0$$

---

[1] Statement by Dr. Oscar J. Peterson, personal interview. Permission to use granted.

Solve for $C_2$ by completing the squares.

$$C_2^2 + 2(e_1 - x_1)C_2 + (e_1 - x_1)^2 = (e_1 - x_1)^2 - e_1^2$$

$$[C_2 + (e_1 - x_1)]^2 = e_1^2 - 2e_1x_1 + x_1^2 - e_1^2$$

$$C_2 + e_1 - x_1 = \pm(x_1^2 - 2e_1x_1)^{\frac{1}{2}}$$

$$C_2 = x_1 - e_1 \pm x_1\left(1 - \frac{2e_1}{x_1}\right)^{\frac{1}{2}}.$$

The last term of the right member of the equation is a binomial to a power less than one. Hence it will expand into an infinite series of terms. This means that the error in the second approximation of the square root will actually be an infinite series of terms. Since the series has a sum, $C_2$, it must be convergent. Any approximation of $C_2$, say $e_2$, could then be the sum of the first n terms of the series.

But there is a negative series and a positive series. Since $C_2$ is positive, the value of the series valid would be positive. When the series is expanded, all the terms except the first would be negative, and taking the negative part of the equation will give a positive value for the series and $C_2$.

The equation with the series expanded becomes

$$C_2 = x_1 - e_1 - \left[x_1 - e_1 - \frac{e_1^2}{2x_1} - \frac{e_1^3}{2x_1^2} - \frac{5e_1^4}{8x_1^3} - \frac{7e_1^5}{8x_1^4} - \cdots - \frac{1\cdot3\cdot5\cdots(2n-3)\,e^n}{n!\;x_1^{n-1}}\right].$$

When the parentheses are removed, the first two terms of the series combine with the terms of the equation preceding them and the rest have positive signs. Hence,

$$C_2 = \frac{e_1^2}{2x_1} + \frac{e_1^3}{2x_1^2} + \frac{5e_1^4}{8x_1^3} + \frac{7e_1^5}{8x_1^4} + \frac{21e_1^6}{16x_1^5} + \frac{33e_1^7}{16x_1^6} + \cdots$$

$$\frac{1\cdot 3\cdot 5\cdots(2n - 3)\, e_1^n}{n!\; x_1^{n-1}} \; .$$

If this equation is compared with the approximation of the error of $x_2$ used in Chapter III, immediately evident is the fact that only the first term of the right member has been used.

5.3. Use of the correction factors. A logical way to estimate the value of $C_2$ would be to take the sum of the first $n$ terms. Using only one term has worked very well and the next step to improve the process would be to use one more term. This would give

$$\sqrt{N} \approx x_2 - \frac{e_1^2}{2x_1} - \frac{e_1^3}{2x_1^2}$$

The quantity $\frac{e_1^3}{2x_1^2}$ would be relatively easy to calculate since it may be written in the form $(e_1^2/2x_1)(e_1/x_1)$. This method was used to calculate Table II from Table I.

This may be extended to the use of the first three terms, the first four terms, and any first $n$ terms. But

TABLE II

THE EXTENDED CORRECTION FORMULA
APPLIED TO THE RESULTS OF
TABLE I

| N | $x_2 - \frac{e_1^2}{2x_1}$ | $\frac{e_1^3}{2x_1^2}$ | $x_2 - \frac{e_1^2}{2x_1} - \frac{e_1^3}{2x_1^2}$ * |
|---|---|---|---|
| 1 | 1.000000 | ----- | ----- |
| 2 | 1.4375 | .012 | 1.42 |
| 3 | 1.734 | .0012 | 1.7328 |
| 4 | ----- | ----- | ----- |
| 5 | 2.2375 | .0012 | 2.2362 |
| 6 | 2.459 | .0064 | 2.45 |
| 7 | 2.6485 | .0014 | 2.647 |
| 8 | 2.8285 | .0001 | 2.8284 |
| 9 | ----- | ----- | ----- |
| 10 | 3.1626 | .0002 | 3.1624 |
| 11 | 3.319 | .0012 | 3.317 |
| 12 | 3.469 | .0037 | 3.465 |
| 13 | 3.608 | .0015 | 3.606 |
| 14 | 3.7422 | .00045 | 3.7417 |
| 15 | 3.87306 | .000061 | 3.87299 |
| 16 | ----- | ----- | ----- |
| 17 | 4.12382 | .00005 | 4.1237 |
| 18 | 4.2471 | .0004 | 4.245 |
| 19 | 4.360 | .0012 | 4.3588 |
| 20 | 4.475 | .0025 | 4.4725 |
| 21 | 4.584 | .0013 | 4.5827 |
| 22 | 4.691 | .00054 | 4.69046 |
| 23 | 4.796 | .00016 | 4.79584 |
| 24 | 4.899 | .00002 | 4.89898 |
| 25 | ----- | ----- | ----- |
| 26 | 5.09904 | .00002 | 5.09902 |
| 27 | 5.1963 | .0001 | 5.1962 |
| 28 | 5.292 | .0004 | 5.2916 |
| 29 | 5.386 | .001 | 5.385 |
| 30 | 5.48 | .0016 | 5.478 |

*Note: All the digits accurate except the last one.

TABLE II (continued)

| N | $x_2 - \frac{e_1^2}{2x_1}$ | $\frac{e_1^3}{2x_1^2}$ | $x_2 - \frac{e_1^2}{2x_1} - \frac{e_1^3}{2x_1^2}$ |
|---|---|---|---|
| 31 | 5.5686 | .00098 | 5.5678 |
| 32 | 5.6575 | .00033 | 5.657 |
| 33 | 5.7448 | .0002 | 5.7446 |
| 34 | 5.831033 | .000072 | 5.83096 |
| 35 | 5.916088 | .000003 | 5.916085 |
| 36 | ----- | ----- | ----- |
| 37 | 6.08277 | .000005 | 6.082765 |
| 38 | 6.16443 | .00002 | 6.16441 |
| 39 | 6.2452 | .00018 | 6.245 |
| 40 | 6.3251 | .0004 | 6.3247 |
| 41 | 6.4036 | .0005 | 6.4031 |
| 42 | 6.482 | .00126 | 6.48074 |
| 43 | 6.5583 | .0006 | 6.5577 |
| 44 | 6.6338 | .00045 | 6.6333 |
| 45 | 6.7085 | .00024 | 6.70826 |
| 46 | 6.7821 | .0001 | 6.7824 |
| 47 | 6.85575 | .0001 | 6.85565 |
| 48 | 6.92820 | .0000035 | 6.92820 |
| 49 | ----- | ----- | ----- |
| 50 | 7.07107 | .0000036 | 7.07106 |
| 51 | 7.1415 | .000026 | 7.14147 |
| 52 | 7.2116 | .00009 | 7.2115 |
| 53 | 7.2804 | .0002 | 7.2802 |
| 54 | 7.349 | .0004 | 7.3486 |
| 55 | 7.4175 | .00078 | 7.4168 |
| 56 | 7.4844 | .0009 | 7.4835 |
| 57 | 7.5504 | .00055 | 7.54985 |
| 58 | 7.6163 | .00036 | 7.6159 |
| 59 | 7.6815 | .00024 | 7.6812 |
| 60 | 7.7461 | .00012 | 7.74598 |
| 61 | 7.8103 | .00005 | 7.81025 |
| 62 | 7.8741 | .00001 | 7.87409 |
| 63 | 7.9372547 | .0000009 | 7.9372538 |
| 64 | 8.0000 | ----- | ----- |
| 65 | 8.06228 | .0000016 | 8.06227 |

TABLE II (continued)

| N | $x_2 - \frac{e_1^2}{2x_1}$ | $\frac{e_1^3}{2x_1^2}$ | $x_2 - \frac{e_1^2}{2x_1} - \frac{e_1^3}{2x_1^2}$ |
|---|---|---|---|
| 66 | 8.12412 | .0000064 | 8.1241 |
| 67 | 8.1854 | .00004 | 8.18536 |
| 68 | 8.2464 | .00009 | 8.2463 |
| 69 | 8.3070 | .00018 | 8.3068 |
| 70 | 8.3669 | .0003 | 8.3666 |
| 71 | 8.4265 | .0004 | 8.4261 |
| 72 | 8.4861 | .0005 | 8.4856 |
| 73 | 8.5448 | .0005 | 8.5443 |
| 74 | 8.6027 | .0003 | 8.6024 |
| 75 | 8.6606 | .0002 | 8.6604 |
| 76 | 8.7179 | .00011 | 8.71779 |
| 77 | 8.77507 | .00008 | 8.77499 |
| 78 | 8.8318 | .00003 | 8.83177 |
| 79 | 8.88821 | .000011 | 8.888199 |
| 80 | 8.94427444 | .0000007 | 8.944273 |
| 81 | ----- | ----- | ----- |
| 82 | 9.05385 | .0000007 | 9.055385 |
| 83 | 9.11046 | .000007 | 9.110453 |
| 84 | 9.1654 | .00002 | 9.1653 |
| 85 | 9.2197 | .00005 | 9.2196 |
| 86 | 9.27367 | .00007 | 9.2736 |
| 87 | 9.3277 | .0002 | 9.3275 |
| 88 | 9.38084 | .0003 | 9.3806 |
| 89 | 9.4351 | .00036 | 9.434 |
| 90 | 9.4875 | .000625 | 9.4869 |
| 91 | 9.9539875 | .00045 | 9.5394 |
| 92 | 9.592 | .0003 | 9.5916 |
| 93 | 9.643875 | .00021 | 9.6436 |
| 94 | 9.6955 | .00012 | 9.6954 |
| 95 | 9.746875 | .000075 | 9.74679 |
| 96 | 9.798 | .00004 | 9.79796 |
| 97 | 9.848875 | .000016 | 9.84885 |
| 98 | 9.8995 | .000005 | 9.899495 |
| 99 | 9.949875 | .0000006 | 9.9498744 |
| 100 | ----- | ----- | ----- |

after the first two terms, the labor becomes greater and the rewards for the labor are smaller. Hence any further extension of this is omitted. In Chapter VI, a much faster method is developed for use where a greater accuracy is desired.

5.4. Results of Table II. Table II is merely an extension of Table I. A second correction factor is subtracted from $x_2$. In each case the accuracy is extended farther with the greater accuracy near the perfect squares and with the larger denominators. A general statement can be made that this end result gives four-digit accuracy in almost all cases and five-digit accuracy in many.

# CHAPTER VI

## IMPROVED CORRECTIONS BASED ON $X_2$

6.1. Introduction. The methods developed in Chapter III and Chapter V work very well for most calculations and do not require a lot of work. In this chapter another extension of the correction formulas is shown which speeds up the process much more.

6.2. Use of $x_2$. In all of the calculation of $x_2$, the formula

$$C_2 = \frac{C_1^{\ 2}}{2x_1}$$

has been used as a starting point. This is the case in this chapter also.

The first step necessary to develop the desired formula is to find two equivalent statements for $C_1$. If $x_1$ is taken greater than $\sqrt{N}$, then $C_1 = x_1 - \sqrt{N}$, and using the definition $x_1 - x_2 = e_1$, then $C_1 = e_1 + C_2$. This was the basic premise of Chapter V.

When $x_1 > \sqrt{N}$, then $\frac{N}{x_1} < \sqrt{N}$. Using $\frac{N}{x_1}$ as the estimate, then

$$C_1 = \sqrt{N} - \frac{N}{x_1}$$

and

$$C_1 = e_1 - C_2 \quad .$$

If each of the two values of $C_1$ are substituted into the error formula, the two equations

$$C_2 = \frac{(e_1 + C_2)^2}{2x_1} \quad \text{and} \quad C_2 = \frac{(e_1 - C_2)^2}{2(\frac{N}{x_1})}$$

will result. Multiply each by its denominator and expand each of the numerators to obtain

$$2x_1C_2 = e_1{}^2 + 2e_1C_2 + C_2{}^2 \text{ and } 2(\tfrac{N}{x_1})C_2 = e_1{}^2 - 2e_1C_2 + C_2{}^2.$$

Dividing each side by 2 in each equation and combining them by addition gives

$$(x_1 + \tfrac{N}{x_1})C_2 = e_1{}^2 + C_2{}^2 \quad .$$

Since $2x_2 = x_1 + \frac{N}{x_1}$ , this equation becomes

$$2x_2C_2 = e_1{}^2 + C_2{}^2$$

or

$$C_2 = \frac{e_1{}^2 + C_2{}^2}{2x_2} \quad . \text{ [1]}$$

It can be seen that the right member of the equation represents the value of $C_2$, but it contains a $C_2$ as part of itself. By substituting the expression and expanding each place that $C_2$ is contained in the right member, it becomes apparent immediately that this is an infinite series not in the proper form to be expanded.

---

[1] Statement by Dr. Oscar J. Peterson, personal interview. Permission to use granted.

Rewrite the equation with the right member zero.

$$C_2^2 - 2x_2C_2 + e_1^2 = 0.$$

Regard $e_1^2$ as a constant and complete the square of the quadratic equation in terms of $C_2$ and $x_2$. Thus

$$C_2^2 - 2x_2C_2 + x_2^2 = x_2^2 - e_1^2$$

and

$$(C_2 - x_2)^2 = x_2^2 - e_1^2 ,$$

$$C_2 - x_2 = \pm(x_2^2 - e_1^2)^{\frac{1}{2}}$$

$$C_2 = x_2 \pm (x_2^2 - e_1^2)^{\frac{1}{2}}$$

This equation will then give an infinite series the sum of which equals the error of the second approximation in the square root process. Only the positive value of this expression can be valid as the $x_2 > \sqrt{N}$. In this equation the plus sign will yield an infinite series of negative numbers and the minus sign will yield an infinite series of positive numbers. Hence the equation would become

$$C_2 = x_2 - x_2(1 - \frac{e_1^2}{x_2^2})^{\frac{1}{2}}$$

This quite readily expands into

$$C_2 = x_2 - x_2 + \frac{e_1^2}{2x_2} + \frac{e_1^4}{8x_2^3} + \frac{e_1^6}{16x_2^5} + \frac{5e_1^8}{128x_2^7} + \cdots$$

$$+ \frac{1\cdot3\cdot5\cdot7\cdots(2n - 3) \quad e^{2n}}{2^n\, n!\, x^{2n - 1}}$$

This is much better than the correction formula obtained in Chapter V as each term increases by powers of two and will thus converge more rapidly. This is desirable.

It can be noted that the term $\frac{e_1^2}{2x_2}$ can be factored out of each term in the equation after the first two terms are added together to give zero. If this expression is factored from each term, it will be found in increasing powers as the terms progress. Using the definition

$$E = \frac{e_1^2}{2x_2}$$

the expanded equation becomes

$$C_2 = E + \frac{E^2}{2x_2} + \frac{E^3}{2x_2^2} + \frac{5E^4}{8x_2^3} + \frac{7E^5}{8x^4} + \cdots$$

6.3. <u>Advantages and disadvantages</u>. Using these correction terms to compute square root has both advantages and disadvantages. It converges very rapidly due to the increase of the power of each term by two each time. But it requires working with figures to several decimal places. So this will give a highly accurate square root approximation with fewer calculations, but working with more digits. In most cases, this becomes a machine operation for use when a calculator is available. Also, since more accurate approximations are being obtained and the convergence is very rapid, care must be taken to carry each operation out the desired

TABLE III

EXTRACTION OF SQUARE ROOTS USING CORRECTIONS BASED ON $X_2$

| N | E | $x_2 - E$ | $E^2/2x_2$ | $x_2 - E - E^2/2x_2$ | $E^3/2x_2{}^2$ | $x_2$ corrected |
|---|---|---|---|---|---|---|
| 2 | .08333333333 | 1.4166666666 | .002314 | 1.41351 | .0000128 | 1.4142 |
| 6 | .05 | 2.45 | .0005 | 2.4495 | .00001 | 2.449490 |
| 12 | .035714 | 3.464286 | .00001822 | 3.464104 | .000002 | 3.464102 |
| 20 | .02777777777 | 4.4722222222 | .0000863511 | 4.472135710 | .0000001 | 4.4721356 |
| 30 | .02272727272 | 5.4772727272 | .000046957 | 5.477225771 | .00000016 | 5.4772256 |
| 42 | .019230169 | 6.480769231 | .000028445 | 6.480740786 | .00000008 | 6.4807406 |
| 56 | .01666666666 | 7.483333333 | .000018518 | 7.483314815 | .00000002 | 7.4833146 |
| 72 | .014705882 | 8.485294118 | .0000127111 | 8.485281407 | .00000001 | 8.48528139 |
| 90 | .013157895 | 9.486842105 | .00000910118 | 9.486833004 | .000000009 | 9.4868330031 |

number of digits. Without a machine, this process becomes very involved in computations. A machine minimizes the labor necessary and the results are very fruitful. It is a very accurate process.

6.4. Interpretation and explanation of Table III. Table III is a short table of values for which the square root has been calculated by this method. The values were purposely picked to give the worst possible cases between 1 and 100. These are the least accurate values the process will give. Even the worst is good.

This process is more accurate than the one in Chapter V. All the digits in the table are accurate. Other values will give even better accuracy than these examples. The preliminary steps are not included in the table. An $x_2$ is obtained with its accompanying $e_1$ and all.

# CHAPTER VII

## PROCEDURE FOR EXTRACTING SQUARE ROOTS

7.1. Introduction. The final step in developing any useable process is to describe the procedure for its use. This should be the final step because a procedure evolves from the understanding of the process involved. This chapter merely puts forward the material already discussed in the preceding chapters. It is a review and restatement of what new tools are available for extracting square roots as the result of this study. Actually, the results are in two sets of information. That is, a procedure for extracting square roots without a calculating machine and a procedure with a calculating machine. Examples are given for each.

7.2. General statements. There are certain steps that need to be taken regardless of whether the calculation is to be done with or without a calculating machine.

First, determine the number of significant digits wanted. There is no need to carry a calculation any farther than what is needed. Doing unnecessary work is a type of inefficiency.

Second, put the N in the form ($N_o \cdot 10^{2a}$) where $N_o$ equals some number between 1 and 100 and a is an integer. This makes it fairly simple to find a first estimate. Calculate $\sqrt{N_o}$ and multiply by $10^a$ to find $\sqrt{N}$.

Third, take a first estimate of $\sqrt{N}$, $x_1$, and calculate $x_2$ by the iteration formula, $x_2 = \frac{1}{2}(x_1 + N/x_1)$. The work required for this is negligible and in many cases can be done mentally.

Fourth, find $e_1 = x_1 - x_2$. Usually this may be done mentally. $x_1$ is the larger of $x_1$ or $\frac{N}{x_1}$.

Up to this point, no really time consuming work has been encountered. This is true regardless of the value of N. These four things are done before it is really necessary to start to work. In the following sections, these four steps are to be taken before taking the steps discussed there.

7.3. Procedure without a calculating machine. One of the aims in developing this method for computing square roots is to minimize the amount of work necessary. It will usually be easier to divide by $x_1$ than to divide by $x_2$. Using the concepts gained in Chapter III and Chapter V gives the method for calculating square roots without a machine.

First, complete the four steps of section 7.2. If the desired accuracy is less than five digits, then

$$\sqrt{N} \approx x_2 - \frac{e_1^{\,2}}{2x_1} - \frac{e_1^{\,3}}{2x_1^{\,2}}$$

will probably give an adequate answer. If N is close to a perfect square, this may give an answer to more digits. When more digits of accuracy are needed, an $x_3$ may be calculated using the corrected $x_2$ term. This will double the number of correct digits and using two correction factors will extend this even farther.

For example, calculate $\sqrt{61.94}$ to five significant digits.

$$x_1 = 8$$

$$\frac{N}{x_1} = 7.7425$$

$$x_2 = \tfrac{1}{2}(8 + 7.7425) = 7.87125$$

$$e_1 \approx .13$$

$$\frac{e_1^2}{2x_1} = .0010$$

$$\frac{e_1^3}{2x_1^2} = .000016$$

Subtracting

$$\begin{array}{r} 7.87125 \\ -.001016 \\ \hline 7.870234 \end{array}$$

Looking at the second correction factor tells that the first digit of error can occur in the fifth decimal place and hence this number is accurate at least to five significant digits.

7.4. Procedure with a calculating machine. The procedure involving a calculating machine should be the quickest possible method without regard to the amount of labor involved in dividing by a denominator of more than one digit. In Chapter III and Chapter VI a method was discussed which converges very rapidly. This was brought about because the powers of the numerator and denominator of the terms of the series increased by two from one term to the next. The letter E was then used to represent the first term and the resulting formula for the method becomes

$$\sqrt{N} \approx x_2 - E - \frac{E^2}{2x_2} - \frac{E^3}{2x_2^{\,2}} \quad \text{where } E = \frac{e_1^{\,2}}{2x_2}$$

After the four steps in section 7.2 are taken, probably without the machine, E is calculated and subtracted from $x_2$. Then $E^2/2x_2$ is calculated and subtracted from the result of $x_2 - E$. And last, $E^3/2x_2^{\,2}$ is calculated and subtracted. The process is stopped whenever the desired accuracy is reached. If the desired accuracy is not reached by the final step of the process, the most efficient thing to do would be to calculate an $x_3$ using the most accurate estimate.

For an illustration, N = 61.94 which for the purposes here is regarded as an exact number.

$$x_1 = 8$$

$$\frac{N}{x_1} = 7.7425$$

$$x_2 = \tfrac{1}{2}(8 + 7.7425) = 7.87125$$

$$E = .0010529791$$

$$x_2 - E = 7.8701970209$$

$$E^2/2x_2 = .0000000704$$

$$x_2 - E - \frac{E^2}{2x_2} = 7.8701969505$$

$$E^3/2x_2^{\,2} = .000000000013$$

$x_2$ corrected = 7.870196950487 correct to at least ten digits.

This is quite an improvement in the accuracy over the same example in section 7.3. It is quite unnecessary to find an approximation to this many digits if 61.94 is only an approximate number, and also unnecessary if only a fewer number of correct digits are needed.

# CHAPTER VIII

## CONCLUSION

8.1. Summary of the thesis. It was the purpose of this study to develop and investigate the use of the error formula to compute square roots, extend the formula concept, and develop a procedure for the use of the findings of the thesis. This has been accomplished in the seven preceding chapters.

Chapter VII sets forth the procedure based on the use of error formulas to compute square roots.

8.2. Suggestions for further study. One immediate suggestion for further study is to investigate the possibility of extending the use of error formulas to the extraction of cube roots. An iteration method is well known for this as it is for square roots.

Further investigation might be suggested in developing an error formula for finding the nth root of a number. An iteration formula is well known for this too.

And finally, the concept of an error formula might be investigated for the solution of equations other than pure quadratics.

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