FINITE GEOMETRIES:

A BRIEF STUDY

A THESIS

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CHAPTER I

INTRODUCTION

Introduction. The word "geometry" is derived from the Greek word for "earth measure".[1] Since in early times the earth was assumed to be flat, this led early geometers to consider all of the properties of measurement of line segments, angles, and other figures as being in a plane. Gradually the meaning of "geometry" was extended beyond this sense to also include the study of spaces based upon other properties. For example, the study of lines and planes is an extension of the concept of the word "geometry", whether in the ordinary space of solids, or as in analytical plane geometry, where points are represented by sets of numbers and lines by sets of points which satisfy linear equations. It is in this broader sense that the word "geometry" will be used in this paper.

Statement of the problem. In many geometry texts, one will find that when the author introduces the ideas of an axiomatic system he will often give for an example the axioms of a finite geometry. In almost every case he will leave the

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[1] B. E. Meserve, Fundamental Concepts of Geometry (Cambridge: Addison-Wesley Publishing Company, Inc., 1955), p. 1.

development of this system as an exercise for the student. In this study, some finite geometries will be investigated and developed to show what can be done with them.

Importance of the study. Recent interest in finite geometries has arisen from their connection with designs used in experiments.[2] Using as an example the finite geometry with seven lines and seven points, suppose one wanted to test seven different varieties of seeds by planting. One might have seven different plots of ground. Let these plots correspond to the lines of the geometry and the varities of seeds correspond to the points. A seed then would be planted in a given plot of ground if the corresponding point occurred on the corresponding line. Thus, there would be a desirable symmetry of the treatment since each seed would be planted in three different plots of ground, each plot of ground would contain three different varieties of seed, and each seed would be competing with each other seed in a plot of ground exactly once.

Sources of information. In obtaining the material for this study the writer has used books and magazines as sources of information. Of particular assistance were the books dealing on the subject of projective geometry.

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[2] Burton W. Jones, "Minature Geometries," The Mathematics Teacher, LII (February, 1959), p. 71.

Organization. This thesis is divided into six chapters. The second chapter outlines some of the properties of an axiomatic system. The third chapter presents a finite seven point geometry. In the fourth chapter we investigate a finite nine point geometry. The fifth chapter presents a finite ten point geometry, and the sixth chapter summarizes the material of the thesis.

CHAPTER II

PROPERTIES OF AN AXIOMATIC SYSTEM

In any logical development of any geometry there must be some terms that remain undefined. It is desirable to keep the number of these undefined terms to a minimum. Also, these terms usually have some intuitive significance or some interpretation in which the axioms seem to be plausible.

It is not the purpose of this study to introduce an extensive geometry, but rather to use as few undefined terms as possible and see how far the geometry can be developed or to what the undefined terms will lead. In this thesis, point, line, and incidence will be taken as undefined terms.

The concept of picturing a point as the idealized limit of smaller and smaller dots is about as good an interpretation as can be given to a point. However, the concept of a line will not refer to the concept of "straight line" which is usually defined as the shortest distance between two points. Instead, a line will be defined as a path connecting certain points. The path will make no difference as long as it passes through all of the points belonging on the line and does not pass through any points which do not belong on the line. When a point lies on a line or a line lies on a point, the point and the line are said to be incident. A line passing through two points is called their join, and a point lying on two lines is called their intersection. The word "on" will be used to mean either join when referring to points, or intersection when referring to lines. As will be seen later, this will tend to clear up some confusion that arises from the use of the words "join" and "intersection".

Besides having undefined terms, it is also necessary to have some undefined propositions or relations between the undefined terms. These propositions are usually called axioms or postulates. Formally, it is always required that a set of axioms be consistent. Also, while it is not necessary, it is desirous that they be few in number, involve only a few accepted undefined terms, be independent, and that they be categorical.

A set of axioms may be said to be consistent if no two of the axioms are contrary or if no two theorems which may be deduced from the axioms are contrary. Since an irrational universe would invalidate all the physical sciences, it is generally assumed that the universe is subject to the laws that govern the reasoning of men.[3] Under this assumption, a set of axioms and the associated science may be proved to be consistent by finding a single concrete representation of the axioms, i.e., by finding in the physical universe an interpretation of the undefined elements and relations of the

[3] Meserve, op. cit., p. 14.

science such that the statements of the science are valid for this interpretation.

A set of axioms is said to be independent of no one of the axioms can be deduced from any combination of the other axioms. In other words, if an axiom is removed from the set of axioms, there is then no way in which it can be obtained from the remaining axioms.

A set of axioms is said to be categorical if there is essentially only one system for which the axioms are valid. In other words, if all the concrete representations of a set of axioms may be shown to be simply isomorphic, then the set of axioms is said to be categorical.

Another property which a set of axioms may or may not have is the property of duality. The principle of duality (in two dimensions) means that every definition remains significant, and every theorem remains true when the two pairs of concepts point and line, and join and intersection are interchanged whenever they appear in a theorem, axiom, or definition. Since the word "on" is used to mean either join or intersection, this makes the writing of the dual of a theorem, axiom, or definition much easier.

CHAPTER III

A SEVEN POINT GEOMETRY

Now that we have investigated the properties which an axiomatic system must have, and also the properties which we would like the system to have, let us construct a set of axioms and investigate them. The axioms will be denoted by a capital letter A followed by the number of the axiom. Thus, the first axiom would be denoted by A.1. Our axioms are as follows.

A.1. Two points determine a line.

A.2. Two points determine at most one line.

A.3. Any two lines have at least one point in common.

A.4. There exists at least one line.

A.5. There exist at least three points on a line.

A.6. Not all points are on the same line.

A.7. No line is on more than three points.

We should first investigate to see if this system is consistent, since it is formally required that all axiomatic systems possess this property. Looking at the model

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A B C D E F G
B C D E F G A
D E F G A B C
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where each letter represents a point and each column represents a line, we see that our axioms are all satisfied and therefore our system does possess the property of being consistent. Next we look at the axioms and see that they are independent, because it would be impossible to obtain any of them from the other axioms.

Now let us see if the system possesses the property of duality. One way of doing this is to prove as theorems the dual of every axiom. This is the method we will now use. We will denote the dual of an axiom by the letter D followed by the number of the axiom of which it is the dual. Thus, the dual of the first axiom would be denoted D.1. This dual is stated and proved in the next paragraph.

D.1. Two lines determine a point. This is just a restatement of A.3. and its proof is thereby established.

D.2. Two lines determine at most one point. By A.3. we know they have at least one point in common. If we assume that they have a second point in common then, by A.2., we know the two lines are the same line. Since this is a contradiction, we know that our dual must be true.

D.3. Any two points have at least one line in common. This is just a restatement of A.1. and therefore its proof is established.

D.4. There exists at least one point. We know by A.4. that there exists a line and by A.5. we know that there are at least three points on this line. Therefore, we know that there must exist at least three points, and thus our dual must be true.

Before we prove our next dual, it would be helpful to introduce some notation. Throughout the next two chapters we will use single digits to denote points. Also, we will denote a line by listing the three points that belong on it. Thus the line which is on the points 1, 2, and 3 would be denoted 123.

D.5. There exist at least three lines on a point. By A.4. we know there exists a line, and by A.5. and A.7. we know this line contains three and only three points, say points 1, 2, and 3. Then, by A.6. we know there exists at least one other point, say point 4, not on this line. Points 1 and 4 determine a line by A.1. In a like manner, points 2 and 4, and also points 3 and 4 determine lines by A.1. Therefore, point 4 has at least three lines on it. In the same manner we can prove that any point not on line 123 has at least three lines on it. Choosing any point on line 123, say point 1, we know that it has at least one line on it. However, points 3 and 4 determine a line by A.1. and we know by A.5. and A.7. that this line has a third point, say point 5, on it. Then points 1 and 5 determine a line by A.1. Also, points 1 and 4 determine a line by A.1. Thus, point 1 has at least three lines on it. In a like manner, we can prove that each of the other two points on line 123 has at least three lines on it. Thus, since every point on line 123 and every point not on line 123 has at least three lines on it, we can conclude that our dual is true.

D.6. Not all lines are on the same point. By A.4. we know there exists a line and by A.5. and A.7. we know this line contains three points, say points 1, 2, and 3. By A.6. we know there exists at least one more point, say point 4, not on this line. By A.1. points 1 and 4 determine a line. We now have two lines which have as their only point in common point 1. Points 2 and 4 also determine a line by A.1. This line cannot pass through point 1, since that would mean line 123 would meet the line containing points 2 and 4 in two points. This is a contradiction of D.2., which we have already proved. Therefore, not all points are on the same line, and we have proved our dual.

D.7. No point is on more than three lines. By A.4. we know there exists a line and by A.5. and A.7. we know that there are exactly three points on the line, say points 1, 2, and 3. By A.6. we know that there exists at least one more point, say point 4, not on this line. Assume point 4 has four distinct lines on it. Then each of these lines must intersect line 123, by A.3. This implies that either there are four points on line 123 or that more than one of the lines intersects line 123 in the same point. The first part of this implication is a contradiction, because we know line 123 has no more than three points, by A.7. The last part of the implication implies that point 4 and one of the points on line 123 determine more than one distinct line. This is a contradiction of A.2. and therefore our assumption that there are four lines on point 4 is false. In a like manner for any number greater than three and for any point not on line 123, we can prove that the assumption leads to a contradiction. Therefore, for any point not on line 123 we know that it is on no more than three lines. Looking at the points on line 123, let us assume that one of these points, say point 1, is on four or more distinct lines. Point 3 and point 4 determine a line by A.1., and by A.5. and A.7. we know that this line contains a third point, say point 5. The four or more lines on point 1 each intersect line 345, by A.3. This implies that either there are four or more points on line 345 or that two or more of the distinct lines on point 1 intersect line 345 in the same point. Once again, both parts of this implication lead to a contradiction and thus we can conclude that our assumption that there were four or more lines on point 1 is false. In a like manner, we can prove that the remaining points on line 123 are not on four or more lines. Thus, since every point not on line 123 and also since every point on line 123 cannot be on more than three lines, we can conclude that our dual is true.

We have now proved the duals of all seven of our axioms and we can conclude that this axiomatic system possesses the property of duality. Now that we have our axioms and have investigated their properties, we are ready to develop and prove some theorems. A theorem will be denoted by the letter T followed by the number of the theorem. Thus, the first theorem will be denoted T.1.

T.1. Two points determine one and only one line. By A.1. we know that two points determine a line, and by A.2. we know that two points determine at most one line. Therefore, we can conclude that two points determine one and only one line, and thus we know our theorem is true.

T.2. Two distinct lines determine one and only one point. By A.3. we know that two lines determine at least one point. If we then assume that the two lines have a second point in common, we have by T.1. that these two lines are the same line. This is a contradiction, and the two lines must have only one point in common. Thus, we know that our theorem is true.

T.3. Every line contains three and only three points. By A.5. we know that there exist at least three points on a line. By A.7. we know that there are no more than three points on a line. Combining these two, we know that every line contains exactly three points, and thus our theorem is true.

T.4. Every point is on three and only three lines. By D.5. we know that there exist at least three lines on a point. By D.7. we know that there are no more than three lines on a point. Combining these two, we know that every point is on exactly three lines, and thus our theorem is true.

T.5. There exist three points not all on the same line. By A.4. we know there exists a line and by T.3. we know this line contains exactly three points, say points 1, 2, and 3. By A.6. we know there is at least one point not on this line, say point 4. Point 4 and point 1 determine one and only one line by T.1. Point 2 or point 3 cannot be on this line, since by T.1. two points determine one and only one line. Therefore, the points 1, 2, and 4 or points 1, 3, and 4 are not on the same line. Thus, our theorem is true.

T.6. There exist three lines not all on the same point. By A.4. we know there exists a line and by T.3. we know this line contains three and only three points, say points 1, 2, and 3. By A.6. we know there exists at least one point, say point 4, not on this line. Points 1 and 4 determine a line by A.1. and points 2 and 4 also determine a line by A.1. By T.3. we know that each of these lines contains a third point. Let us call these point 5 and point 6, respectively. We now have the three lines 123, 145, and 246. Looking at any two of these three lines, we see that they meet in a point that is not on the third. For example, if we take the first two lines, we see that they meet in the point 1, which is a point not on the third line. Therefore, we have proved our theorem.

T.7. There exist at least seven points. By A.4. we know there exists at least one line and by T.3. we know this line contains exactly three points, say points 1, 2, and 3. By A.6. we know there is at least one other point, say point 4, not on this line. Points 1 and 4 determine a line by A.1. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 5. Points 2 and 4 determine a line by A.1. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 6. Points 3 and 4 also determine a line by A.1. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 7. We have now shown that there must exist at least seven points and our theorem is true.

T.8. There exist at least seven lines. By A.4. we know there exists at least one line and by T.3. we know that this line contains exactly three points, say points 1, 2, and 3. By A.6. we know there is at least one other point, say point 4, not on this line. Point 1 and point 4 determine a line by A.1. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 5. Point 2 and point 4 determine a line by A.l. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 6. Point 3 and point 4 also determine a line by A.l. and this line contains a third point by T.3. This point cannot be any of the existing points by T.2. Let us call this point 7. Listing the lines we have so far, we find we have the lines 123, 145, 246, and 347, so that we have at least four lines. However, we know point 1 and point 6 must determine a line by A.l. and that this line must contain a third point by T.3. By T.2. we know that this point cannot be any of the points 2, 3, 4, or 5 since all of these points are already on a line with either point 1 or point 6. This leaves of the seven points only the point 7 to be on this line and thus gives us the line 167 for our fifth line. Using A.l. again, we know that point 3 and point 6 must also determine a line and that by T.3. this line must also contain a third point. By T.2. we know that this point cannot be any of the points 1, 2, 4, or 7 since all of these points are already on a line with either point 3 or point 6. This leaves of the seven points only the point 5 to be on this line, and thus gives the line 356 for our sixth line. Finally, using A.l. and point 2 and point 5, we know that these two points must also determine a line and by T.3. this line must contain a third point. By T.2. we know that this point cannot be any of the points 1, 3, 4, or 6. This leaves of the seven points only the point 7 to be on this line, and thus gives the line 257 for our seventh line. We can therefore conclude that our theorem is true.

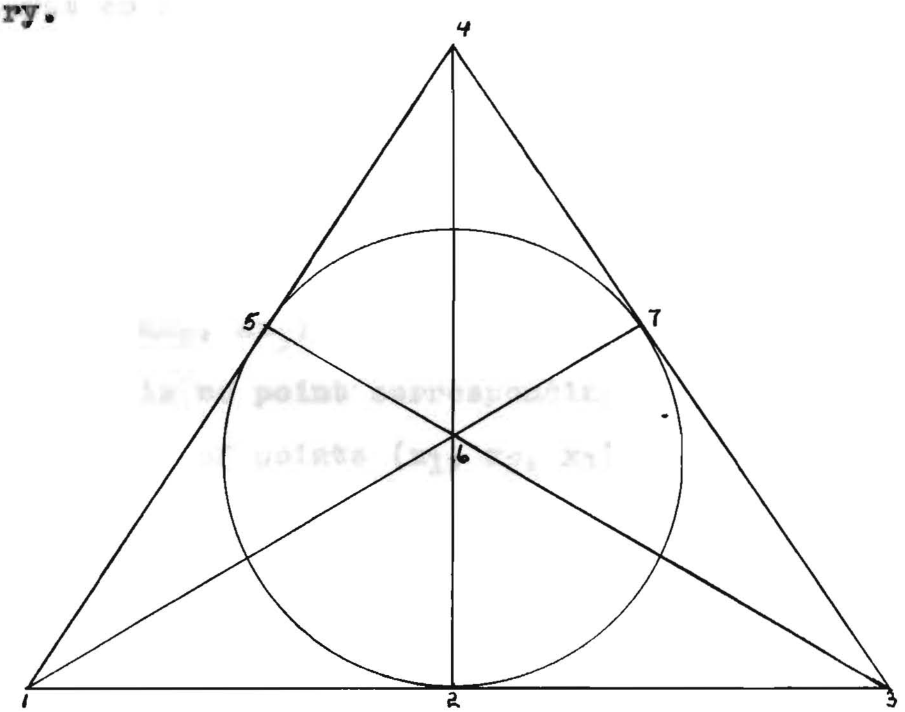
T.9. There exist no more than seven points. By T.7. we know that there exist at least seven points and by T.8. we know that these seven points determine at least seven lines. Listing the seven lines as they are listed in T.8., they are lines 123, 145, 246, 347, 167, 356, and 257. Let us show that there can be no more than seven points by assuming that an eighth point, say point 8, does exist. Then by A.1. point 8 and point 1 determine a line. Listing the lines which are now on point 1, we see they are the lines 123, 145, 167, and the line joining point 8 to point 1. This means that point 1 is on four lines, which is a contradiction of D.7. Since our assumption lead to a contradiction, our theorem must be true.

T.10. There exist no more than seven lines. By T.8. we know that there exist at least seven lines. Listing these as they are listed in T.8., they are the lines 123, 145, 246, 347, 167, 356, and 257. Let us show that there can be no more than seven lines by assuming that an eighth line does exist. Since every point listed above is already on three lines, then by D.7. there can be no more lines on any of these points. By T.3. we know this line must contain three points, and since we have shown that it cannot contain any of the existing seven points, there must be more than seven points. This a contradiction of T.9. and thus, our assumption is false and our theorem must be true.

T.11. There exist exactly seven points. By T.7. we know that there exist at least seven points and by T.9. we know that there exist no more than seven points, so we can conclude that our theorem is true.

T.12. There exist exactly seven lines. By T.8. we know that there exist at least seven lines and by T.10. we know that there exist no more than seven lines, so we can conclude that our theorem is true.

These last two theorems are very important, since they are the proof that we have a finite geometry. If we follow the steps in the proof of T.8. and use them as the steps involved in a construction, we obtain the following model for our geometry.



The reader will remember that earlier in this paper there was another model exhibited for this set of axioms. That was the model

A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C

where each letter represented a point and each column represented a line. Now let us see if this set of axioms is categorical. By letting the letter A and the number 1 represent the same point and likewise for the pairs B and 2, C and 4, D and 3, E and 6, F and 7, and G and 5, we see that these two models are isomorphic and, for at least these two concrete models, our axioms can be said to be categorical.

So far we have been able to determine if a point was on a line only by listing all of the points belonging to each line. Then, by inspection, we were able to tell if a specific point was on a specific line. Now, instead of representing points by numbers, let us represent them by homogeneous coordinates or number triples (x_1, x_2, x_3) subject to the following conditions.

A. It is a modular system.

B. $(kx_1, kx_2, kx_3) = (x_1, x_2, x_3)$ when $k \neq 0$.

C. There is no point corresponding to $(0, 0, 0)$.

The totality of points (x_1, x_2, x_3) such that $u_1x_1 + u_2x_2 + u_3x_3 = 0$, where at least one u_j $(j = 1, 2, 3)$ is different from zero, is called a line and is indicated by $[U_1, U_2, U_3]$. When $k \neq 0$, we write $[U_1, U_2, U_3] = [kU_1, kU_2, kU_3]$.

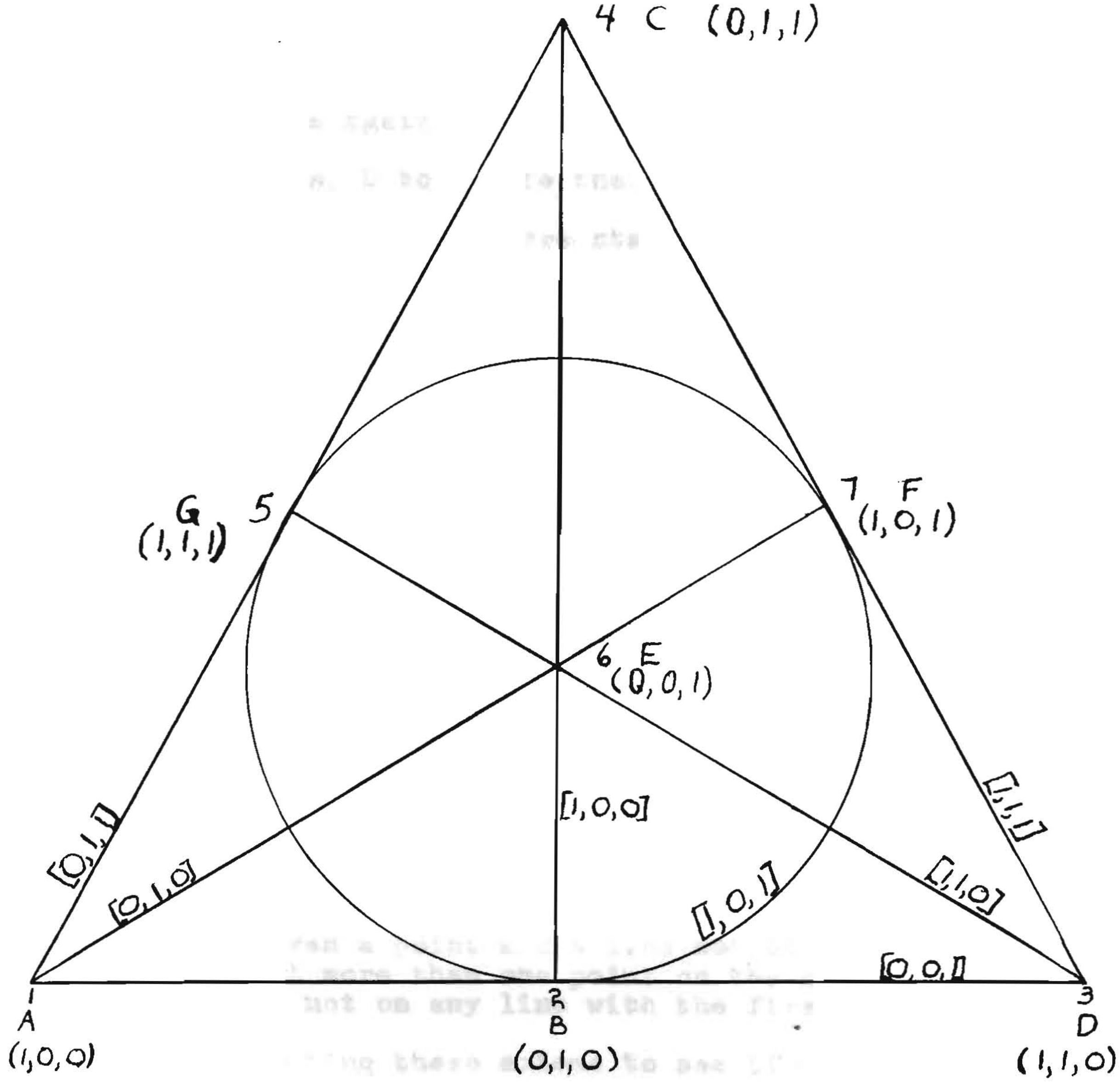
Now let us consider the possible points and lines from the modular system Mod 2. In this system we have available the numerals 1 and 0 to be placeholders in number triples. Listing all possible combinations, we have (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), and (1, 1, 1) as all the possibilities. Counting up, we see that there are eight of these combinations. However, one of our restrictions is that there is no point corresponding to (0, 0, 0), so that we have exactly seven points. Similarly, there are exactly seven lines, which may be written: [0, 0, 1], [0, 1, 0], [0, 1, 1], [1, 0, 0], [1, 0, 1], [1, 1, 0], and [1, 1, 1].

Now it is possible to determine, given any line, whether a point is on this line. Choosing for an example the line [1, 0, 1], we know that the point (1, 0, 0) is not on this line since (1)(1) + (0)(0) + (1)(0) is not equal to zero. However, the point (1, 0, 1) is on the line [1, 0, 1] since (1)(1) + (0)(0) + (1)(1) is equal to two, which is equivalent to zero in a Mod 2 system. In the same manner, we can find out which lines lie on each point. Completing all of the computations involved, we find that each line is on three points and each point is on three lines. In the list that follows the points listed in each column are on the line listed at the base of the column.

$$\begin{array}{c}(1,\ 0,\ 0)\\(0,\ 1,\ 0)\\(1,\ 1,\ 0)\\\hline[0,\ 0,\ 1]\end{array}\quad\begin{array}{c}(1,\ 0,\ 0)\\(0,\ 0,\ 1)\\(1,\ 0,\ 1)\\\hline[0,\ 1,\ 0]\end{array}\quad\begin{array}{c}(1,\ 0,\ 0)\\(0,\ 1,\ 1)\\(1,\ 1,\ 1)\\\hline[0,\ 1,\ 1]\end{array}\quad\begin{array}{c}(0,\ 1,\ 0)\\(0,\ 0,\ 1)\\(0,\ 1,\ 1)\\\hline[1,\ 0,\ 0]\end{array}$$

$$\begin{array}{c}(0,\ 1,\ 0)\\(1,\ 0,\ 1)\\(1,\ 1,\ 1)\\\hline[1,\ 0,\ 1]\end{array}\quad\begin{array}{c}(1,\ 1,\ 0)\\(0,\ 0,\ 1)\\(1,\ 1,\ 1)\\\hline[1,\ 1,\ 0]\end{array}\quad\begin{array}{c}(1,\ 1,\ 0)\\(1,\ 0,\ 1)\\(0,\ 1,\ 1)\\\hline[1,\ 1,\ 1]\end{array}$$

This arrangement can be shown to be isomorphic to the arrangement of the seven letters by letting the letter A and the number triple (1, 0, 0) represent the same point. In the same manner, let B and (0, 1, 0), C and (0, 1, 1), D and (1, 1, 0), E and (0, 0, 1), F and (1, 0, 1), and G and (1, 1, 1) each represent the same point. Thus, the geometry of number triples is isomorphic to the seven point geometry about which we have been talking. Combining all of these representations and using one model to demonstrate all of them, we have the model on the following page.



4 C (0,1,1)
G
5
(1,1,1)
7 F
(1,0,1)
6 E
(0,0,1)
[1,0,0]
[0,1,1]
[0,1,0]
[1,0,1]
[1,1,0]
[1,1,1]
[0,0,1]
1
A
(1,0,0)
2
B
(0,1,0)
3
D
(1,1,0)

CHAPTER IV

A NINE POINT GEOMETRY

Now that we have investigated a seven point geometry, let us construct another set of axioms and investigate their properties. Once again, we will use the capital letter A to denote our axioms, D to denote the dual of an axiom, and T to denote a theorem. Since we are starting with another system, we will once again start our numbering with 1.

A.1. There exists at least one line.

A.2. Every line is on exactly three points.

A.3. Not all points are on the same line.

A.4. Two distinct points determine at most one line.

A.5. Given a line and a point not on it, there exists a line on the given point which has no point in common with the first line.

A.6. Given a line and a point not on it, there exists not more than one line on the given point which has no point in common with the first line.

A.7. Given a point and a line not on it, there exists a point on the given line which is not on any line with the first point.

A.8. Given a point and a line not on it, there exists not more than one point on the given line which is not on any line with the first point.

Investigating these axioms to see if they are consistent, we look at the model

A	B	C	D	E	F	G	A	B
G	H	I	G	H	I	H	C	D
B	C	D	E	F	A	I	E	F

where each letter represents a point and each column represents a line. We see that our axioms are satisfied and we can conclude that our axioms are consistent. After inspecting our axioms, we can also conclude that they are independent. Before trying to prove the duals of our axioms, let us first prove some theorems.

T.1. Given a line and a point not on it, there exists exactly one line on the given point which has no point in common with the first line. By A.5. we know there exists a line which has this property, and by A.6. we know there exists not more than one line which has the property. Therefore, by combining the two, we know that there exists exactly one line with the property, and our theorem is true.

T.2. Given a point and a line not on it, there exists exactly one point on the given line which is not on any line with the first point. By A.7. we know there exists a point which has this property, and by A.8. we know that there exists not more than one point which has the property. Therefore, by combining the two, we know that there exists exactly one line with the property, and thus the theorem is proved.

T.3. Given a point and a line not on it, there exist at least two lines joining the point to points on the line. By A.2. we know that every line contains three points, say points 1, 2, and 3. By T.2. we know the given point, say point 4, and exactly one of the points on the line, say point 3, are not joined. Since only one point on the line is not joined to point 4, we can conclude that the other two points on the line are joined to point 4. Since points 1 and 4 are joined, they lie on a line by the definition of join, and likewise points 2 and 4 lie on a line. Now it is necessary to show that these two lines are distinct. Let us assume that the two lines are not distinct. Then we have that the points 1, 2, and 4 are all on the same line. Since by A.4. two points determine at most one line, this must then be the same line as the line determined by the points 1, 2, and 3. By A.2. we know every line contains exactly three points and we can conclude that point 4 and point 3 are the same point. This means point 4 is on the line 123, which is a contradiction of our assumption. Therefore, our assumption that the two lines are not distinct is false. Thus the two lines are distinct, and our theorem is true.

T.4. Given a point and a line not on it, there exist no more than two distinct lines joining the point to points on the line. To prove this theorem, let us start by assuming that there are more than two distinct lines joining the point to the line. Let us assume that there are three lines and show this leads to a contradiction. Let us call the given line 123 and the given point 4. Then, by our assumption, there are three distinct lines joining point 4 with line 123. Therefore points 1 and 4 must determine a line, as must points 2 and 4 and points 3 and 4, by A.4. However, by T.2. we know there is exactly one point on line 123 which is not joined to point 4. However it cannot be points 1, 2, or 3, since they are already joined to point 4. Therefore, there must be a fourth point on line 123. This is a contradiction of A.2. Since our assumption leads us to a contradiction, we can conclude that it is false. Thus, since there cannot be more than two distinct lines joining the point to points on the line, our theorem is true.

T.5. Given a point and a line not on it, there exist exactly two distinct lines joining the point to points on the line. By T.2. we know there exist at least two lines possessing this property. By T.4. we know that there exist no more than two lines possessing this property. Therefore, we can combine this knowledge and say that there are exactly two lines which possess this property. Thus, we can conclude that our theorem is true.

Before proving any more theorems, let us establish that this geometry possesses the property of duality by proving the duals of our axioms.

D.1. There exists at least one point. By A.1. we know there exists a line and by A.2. we know that this line is on exactly three points. Therefore, we know that there exist at least three points and we have proved our dual.

D.2. Every point is on exactly three lines. By A.1. we know there exists a line and by A.2. we know this line contains three points. Let us call these points 1, 2, and 3. By A.3. we know that there must exist at least one point, say point 4, not on this line. By T.1. we know that there exists exactly one line on point 4 which does not have any point in common with line 123. By T.5. we know that there exist exactly two lines on point 4 which have points in common with line 123. Therefore we can conclude that there are exactly three lines on point 4. In a like manner, we can prove that there are exactly three lines on any point not on line 123. Now let us prove that there are exactly three lines on the points of line 123 by choosing any point, say point 1 on this line. By T.2. we know that there exists only one point on line 123 not joined to point 4. Thus, at least one of the two remaining points must be joined to point 4. Let us assume that this is the point 2. Then the points 2 and 4 determine a line and point 1 is not on this line, since this would lead to a contradiction of A.4. Therefore, by T.1. there exists exactly one line on point 1 which does not have any point in common with the line joining the point 2 to the point 4. Thus we can conclude that there are exactly three lines on point 1. In a like manner, we can prove that there are exactly three lines on every point on line 123. Therefore we can conclude that every point is on exactly three lines, and that our dual is true.

D.3. Not all lines are on the same point. By A.1. we know there exists a line. By A.3. we know that there exists at least one point not on this line. By T.1. we know that there exists exactly one line on this point which has no point in common with the first line. Thus we have the existence of two lines which have no points in common and we can conclude that our dual is true.

D.4. Two distinct lines determine at most one point. Let us assume that two lines determine at least two distinct points, which is the same as assuming that they determine more than one point. By A.4. two distinct points determine at most one line. Therefore, our two distinct lines must be the same line. This, however, is a contradiction and we can conclude that our assumption is false and that our dual is true.

Instead of writing out the duals of the last four axioms it will be sufficient to point out to the reader that the dual of A.5. is A.7. and that the dual of A.6. is A.8. Thus the proof of each of these duals would be a trivial exercise. Before proving any more theorems, let us define another term. Two lines will be called parallel if and only if they have no point in common. As will be seen a little later, this will make the writing and proving of some of our theorems a little easier.

T.6. Given a line and a point not on it, exactly two points on the line are joined to the first point. By T.5. we know that there are exactly two distinct lines on the point joined to points on the line. By A.4. we know that two distinct points determine at most one line. Therefore, at least two points on the line must be joined to the original point. By A.2. we know that there are exactly three points on the line and by T.2. we know that there is exactly one point on this line which is not joined to the original point. From this we can conclude that no more than two of the points on the line can be joined to the original point. By combining the established facts that at least two and at the same time no more than two points of the line are joined to the original point, we can conclude that exactly two points on the line are joined to the original point, and thus our theorem is proved.

T.7. Every line has at least two distinct lines parallel to it. By A.1. we know there exists at least one line and by A.2. we know that this line has exactly three points, say points 1, 2, and 3, on it. By A.3. we know that there exists at least one point, say point 4, not on line 123. By T.6. we know that at least one of the points on line 123 is joined to point 4, say point 2. By A.2. we know the line joining points 2 and 4 has exactly one more point on it, say point 5. Now by T.1. there is exactly one line on point

4 which is parallel to line 123. In a like manner, there is exactly one line on point 5 which is parallel to line 123. Now if these two lines are distinct, we will have proved our theorem. Let us prove that the two lines are distinct by assuming that they are the same line. Then this line is on the points 4 and 5. However, since by A.4. we know that two distinct points cannot determine more than one line, this line must be the line 245 used earlier in the proof. But line 245 is not parallel to line 123, since they both have the point 2 in common. Thus our assumption that the two parallel lines were the same line leads to a contradiction. This, then, leads us to the conclusion that our theorem is true.

T.8. If two lines are parallel to a third line, they are parallel to each other. Let us start by assuming that the two lines which are parallel to the third are not parallel to each other. Then by the definition of parallel lines, these two lines must have a point in common, say point X. Then, since each of the two lines on point X is parallel to the third line, point X has two lines on it which have no point in common with the third line. This is a contradiction of T.1. Since our assumption leads us to a contradiction, we can assume that it is false and that our theorem is true.

T.9. The six points on two parallel lines determine exactly six lines. Let us start by naming our two parallel lines 123 and 456, respectively. Then by T.5. there are exactly two lines on point 1 joining point 1 to points on line 456. In the same manner, there are exactly two lines on point 2 and two lines on point 3 joining these points to points on line 456. Now, adding up we see that each of the three points on line 123 is on exactly three lines and by D.2. we know there can be no more lines on points 1, 2, and 3. Thus, since we have six lines and there can be no more, we can consider our theorem proved.

Before investigating any more theorems, let us define some more terms. The first term that we will define will be a hexagon. A hexagon is the figure determined by the six points of two parallel lines and the six lines joining these six points, with each of the six points being a vertex and each of the six lines being a side. A hexagon will be denoted by listing the six points which are its vertices in the order one would come upon them when proceeding around the perimeter of the hexagon. Thus, a hexagon could be denoted 123456 if this was the order in which you came upon the vertices of it as you proceeded around the perimeter. One should note that 654321 is just another name for the hexagon 123456.

The next term we are interested in is opposite sides of a hexagon. Since the reader will probably already have an intuitive idea of what this term means, probably the easiest way to define it is to tell how to determine the opposite sides of a hexagon. Starting on any side between vertices of a hexagon and proceeding around the perimeter, counting each vertex as we pass through it, the side between and on the third and fourth vertices is the side opposite the side from which you started. For an example, consider the hexagon 123456. To find the side opposite side 12, start on that side and proceed around the perimeter counting each vertex as you pass through it. The first vertex you would pass through in one case would be the point 2, the second would be the point 3, the third would be the point 4, and the fourth would be the point 5. Thus, the side opposite side 12 would be the side on the points 4 and 5. In the second case, the first vertex you would pass through would be the point 1, the second would be the point 6, the third the point 5, and the fourth the point 4. Therefore, the side opposite side 12 would be the line on the points 5 and 4. Since according to A.4. two points determine at most one line, this side is the same as the side obtained in the first case. We will now use these definitions and prove some additional theorems.

T.10. The opposite sides of a hexagon intersect in a point. Let us call the two parallel lines 123 and 456, and consider the hexagon 153426 which they determine. By our definition of opposite sides we know the side opposite the side 15 is the side 24. By A.2. we know that the line which is side 24 has a third point on it, say point 7. Consider the point 1, which is not on the line 247. Point 4 has three lines on it, these being the lines 247, 456, and the line which is the side 34 of the hexagon. Point 1 cannot be on the line which is the side 34, since if it were, points 1 and 3 would determine two lines, and this is a contradiction of A.4. Therefore, since we know by D.2. that there are exactly three lines on every point and point 1 is not on any of the three lines on point 4, we can conclude that point 1 is not joined to point 4. Since by T.2. there is exactly one point on a line not joined to a point not on the line, we can conclude that point 7 on line 247 must be joined to point 1. However, point 1 is already on three lines, these being the line 123, the line which is the side 16 of the hexagon, and the line which is the side 15 of the hexagon. By D.2. we know that there are exactly three lines on a point and we can conclude that point 7 is on one of the lines listed above as already being on point 1. Since line 123 already has three points on it, we know by A.2. that point 7 cannot be on this line. Looking at the point 6, we see that it is joined to the point 4 by the line 456 and that it is also joined to the point 2 by the line which is the side 26 of the hexagon. Therefore, since point 6 is already joined to two points of the line 247, we know by T.2. that point 7 cannot be joined to the point 6. Thus the point 7 cannot be on the line which is the side 16 of the hexagon. Since there is no other line on point 1 which point 7 can be on, it must be on the line which is the side 15 of our hexagon. We have now proved that the lines which are the sides 15 and 24 of our hexagon, (and are by definition opposite sides of a hexagon), have the point 7 in common. In a like manner, we can prove that any pair of opposite sides of a hexagon intersect in a point, and therefore we can conclude that our theorem is true.

T.11. The points formed by the intersection of the opposite sides of a hexagon are distinct. Consider once again the hexagon 153426 and two pairs of opposite sides, these being the sides 15 and 24, and 35 and 26. We know by T.10. that the lines which are sides 15 and 24 intersect in a point, say point 7. We also know by T.10. that the lines which are the sides 35 and 26 intersect in a point. Let us assume that this is the same point which sides 15 and 24 have in common, which is the point 7. Then point 7 would have four lines on it, these being the lines which are the sides 15, 24, 35, and 26 of the hexagon. This is a contradiction of D.2., and we can conclude that our assumption is false. In a like manner, we can prove that any two pairs of opposite sides of a hexagon cannot have the same point as the intersection, and we can conclude that our theorem is true.

T.12. The three points determined by the intersection of the opposite sides of a hexagon are all on the same line. Let us consider the two parallel lines 123 and 456, which determine the hexagon 153426. By T.10. we know that the lines which are the opposite sides 15 and 24, 16 and 34, and 26 and 35 each have a point in common, and by T.11. we know that each of these points is distinct. Let the point 7 lie on the lines which are the sides 15 and 24 of the hexagon, the point 8 lie on the lines which are the sides 16 and 34 of the hexagon, and the point 9 lie on the lines which are the sides 26 and 35 of the hexagon. Listing all of the lines which we have so far, we see that we have the lines 123, 456, 157, 247, 348, 168, 269, and 359. Now looking at the point 7 and the line 348, we see that point 3 is on the lines 123, 348, and 359. Since point 7 is not on any of these lines, we can conclude that point 3 is not joined to point 7. By T.2. we know that only one point on a line is not joined to a point not on that line. Since point 3 is not joined to point 7, then points 4 and 8 on the line 348 must be joined to the point 7. Of course, we already know that the points 4 and 7 lie on the line 247. However, points 7 and 8 do not lie on any of the existing lines, and there must be another line joining these two points. By A.2. we know that the line joining the point 7 to the point 8 must have another point, say point X, on it. This gives us the line 78X. Now, looking at the point 7 and the line 359, we remember that it has already been shown that the point 3 is not joined to the point 7. By T.2. we know that only one point on a line is not joined to a point not on that line. Since the point 7 is not joined to the point 3 on the line 359, then it must be joined to the points 5 and 9. We already know that the points 7 and 5 are on the line 157. However, none of the lines which we have listed as existing so far have the points 7 and 9 on them. Therefore, there must be another line which joins the point 7 to the point 9. By A.2. we know that every line has exactly three points on it and thus there must be another point, say point Y, on this line. This gives us the line 79Y. Now either the lines 79Y and 78X are the same line or they are not. Assume that the two lines are distinct. Then the point 7 has the four lines 157, 247, 79Y, and 78X on it. This is a contradiction of D.2. Therefore, the two lines 79Y and 78X must be the same line. Since by A.2. we know that every line has exactly three points on it, the three points on this line must be the points 7, 8, and 9. These three points are also the intersections of the opposite sides of our hexagon, and thus we have proved that our theorem is true.

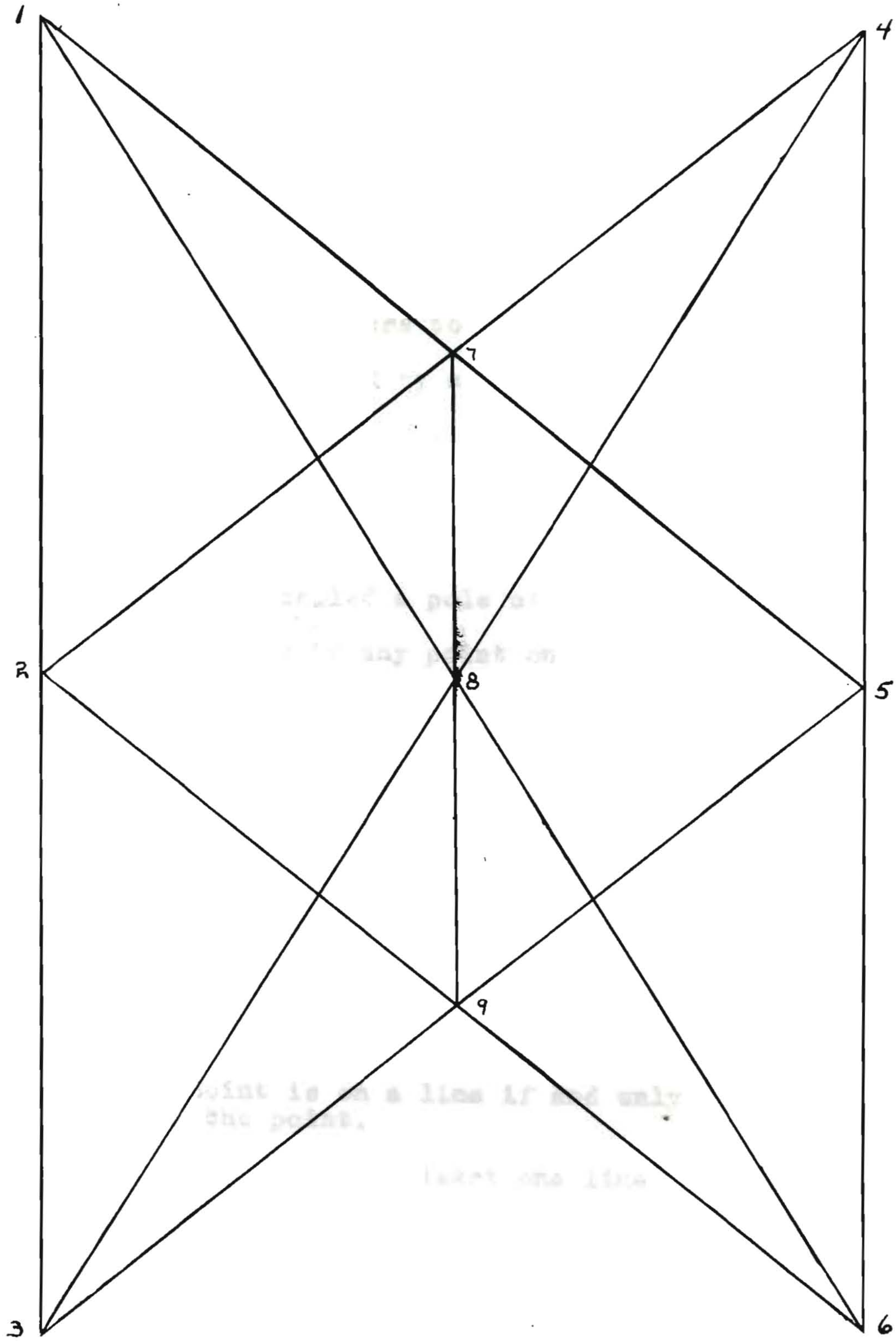
T.13. There exist exactly nine points. By A.1. we know there exists at least one line, and by A.2. we know that there are exactly three points, say points 1, 2, and 3, on it. By T.7. we know that there is at least one line parallel to line 123, and by A.2. we know that there are exactly three points, say points 4, 5, and 6, on it. By the definition of a hexagon we know that these six points determine a hexagon, say hexagon 153426. By T.10. we know that the opposite sides of a hexagon intersect in a point, and by T.11. we know that the points of intersection for each pair of opposite sides of a hexagon are distinct. Since there are three pairs of opposite sides in a hexagon, this gives us three more points, say points 7, 8, and 9, where point 7 is on the lines which are the sides 15 and 24, point 8 is on the lines which are the sides 34 and 16, and point 9 is on the lines which are the sides 35 and 26. By T.12. we know that the three points 7, 8, and 9 are all on the same line. Looking at what we have so far, we see we have the nine points 1, 2, 3, 4, 5, 6, 7, 8, and 9 and the nine lines 123, 456, 157, 247, 348, 168, 359, 269, and 789. We now know that there exists at least nine points. Let us show that we cannot have more than nine points by assuming that there is a tenth point, say point 0. By D.3. we know that there must be at least one line which point 0 is not on, say line 123. By T.6. exactly two points on line 123 are joined to the point 0. However, every point on line 123 is already on three lines. Due to this fact, joining point 0 to any point on the line 123 would cause that point to have four lines on it. This is a contradiction of D.2. and therefore our assumption must be false. Since we have shown that there must be at least nine points and also that there can be no more than nine points, we can conclude that our theorem which states that there are exactly nine points is true.

T.14. There exist exactly nine lines. By A.1. we know that there exists at least one line, and by A.2. we know that there are exactly three points, say points 1, 2, and 3, on it. By T.7. we know that there is at least one line parallel to line 123, and by A.2. we know that there are exactly three points, say points 4, 5, and 6, on it. By T.9. we know that these six points determine exactly six lines, and by the definition of a hexagon we know that these six lines are the sides of the hexagon. Let us call this the hexagon 153426. By T.10. we know that the opposite sides of a hexagon intersect in a point, and by T.11. we know that the points of intersection are distinct. Since there are three pairs of opposite sides in a hexagon, this gives us three more points, say points 7, 8, and 9, where point 7 is on the lines which are the sides 15 and 24, point 8 is on the lines which are the sides 34 and 16, and point 9 is on the lines which are the sides 35 and 26. By T.12. we know that the points 7, 8, and 9 all lie on one line. Looking at what we have so far, we see that we have the nine lines 123, 456, 359, 157, 247, 348, 168, 269, and 789. Thus we know that there must be at least nine lines. Let us now show there can be no more than nine lines by assuming that a tenth line exists. We know by A.2. that there are exactly three points, say points X, Y, and Z, on this line. By A.3. we know that there is at least one point, say point 1, which is not on this line. By T.5. we know that point 1 is joined to exactly two points on line XYZ. However, point 1 is already on three lines, and joining it to any of the points on line XYZ would cause there to be more than three lines on point 1. This is a contradiction of D.2., and we can conclude that our assumption is false. Therefore, since we know that there exist at least nine lines and at the same time that there cannot be any more than nine lines, we can conclude that there must be exactly nine lines and that our theorem is true.

T.15. There are no more than two distinct lines parallel to a third line. Let us start by assuming that there are three lines, which is the same as saying that there are more than two lines parallel to our given line. By A.2. we know that our given line is on exactly three points, say the points 1, 2, and 3. Let us call the three lines parallel to line 123 line a, line b, and line c. By T.8. we know that line a and line b cannot have a point in common. In the same manner, we can show that the line b and line c, and line a and line c can have no point in common. Therefore, we can conclude that the three lines a, b, and c have no points in common. By A.2. we know line a must be on exactly three points, line b must be on exactly three points, and line c must be on exactly three points. Since these three lines have no points in common, then there must be a total of nine points on them. These nine points plus the three points on line 123 make a total of twelve points. This is a contradiction of T.13., and our assumption is therefore false and our theorem is true.

T.16. There are exactly two distinct lines parallel to a third line. By T.7. we know that every line has at least two distinct lines parallel to it, and by T.15 we know that every line has no more than two distinct lines parallel to it. Thus, we can conclude that every line has exactly two lines parallel to it, and that our theorem is true.

If the reader will follow the proof of T.12. and use the steps involved as the steps in a construction, he will obtain the model which is on the following page. It might also be interesting to note that this theorem is a famous theorem in projective geometry. This is the theorem known as Pascal's theorem, and appears in the following form:
If the alternate vertices of a hexagon lie on two straight lines, then the points formed by the intersection of the opposite sides are collinear.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CHAPTER V

A TEN POINT GEOMETRY

In the last two geometries we have used numbers to represent points and groups of numbers to represent lines. In the geometry which we are about to investigate we are going to use capital letters to represent points. A line will either be represented by a group of letters listing the points on the line or by a lower case letter. Also, before setting up our axiomatic system it will be necessary to define two terms. These are the terms pole and polar.

A point P is called a pole of a line p if and only if point P is not joined to any point on line p by a line.

A line p is called a polar of a point P if and only if line p does not meet any line on point P in a point.

The reader will note the use of the same letter to denote the pole of a line or the polar of a point. This practice will be continued throughout this chapter. With the two additional terms and the different notation, we are now ready to set up our axiomatic system.

A.1. A point is on a line if and only if the line is on the point.

A.2. There exists at least one line.

A.3. There are exactly three distinct points on every line.

A.4. Every line has exactly one pole.

A.5. Every point has exactly one polar.

A.6. If a point P is not on line x, but is joined to a point on line x by a line m, then point P is joined to exactly one other point on line x by a line n distinct from line m.

Investigating to see if these axioms are consistent, we look at the model

A	B	C	D	E	F	G	H	I	J
B	E	D	G	F	I	H	J	A	C
C	H	I	B	D	J	F	A	G	E

where once again the letters represent points and the columns represent lines and see that our axioms are satisfied. Inspection will also show us that our axioms are independent. Before trying to prove the duals of our axioms, let us prove some theorems that will make the proving of our duals easier.

T.1. A point P is a pole of a line p if and only if the line p is a polar of the point P. Since this is an if and only if theorem, the proof must be divided into two parts. To prove a point is a pole of a line if the line is a polar of the point, we assume the line is not a polar of the point. Then by our definition of a polar, the point and a point on the line are joined by a line. This implies that the point is not the pole of the line. Since this is a contradiction of our condition we can conclude that our assumption was false, and that if a point is a pole of a line the line is a polar of the point. The second part of our proof reads, if a line is a polar of a point, then the point is a pole of the line. Once again we assume that the point is not a pole of the line. Then by our definition of pole, the point must be joined to a point on the line. Since the point and line are joined, the line is not a polar of the point by our definition of polar. This is also a contradiction, and our assumption is false. By proving both parts of our theorem true, we can now conclude that our entire theorem is true.

T.2. If a point P is not joined to two points on a line p, the point P is a pole of line p. Once again, we start by assuming that point P is not a pole of line p. Then point P is joined to some point, say point X on line p by the definition of a pole. Point P then has two possibilities. Either it is on line p or it is not on line p. Starting with the first possibility, we know by A.3. that there must be a third point, say point Y, on line p. Then point P is joined to point Y, since they are on the same line. This contradicts our assumption, and for the first of our cases the theorem is established. If point P is not on line p, we already know it is connected to one point on line p. However, by A.6. we know point P must be connected to a second point on line p. This contradicts our assumption, and as we have proved it for both cases, our theorem must be true.

T.3. Two points are not joined by more than one line. Assume that there are two points, say points A and B, which determine two or more lines. Then there are at least two lines on these two points by our assumption, say lines m and n. By A.4. we know that line m has a distinct pole, say point M. However, since point M is not joined to points A and B of line n, it must be a pole of line n by T.2. Then, by T.1. lines m and n are polars of point M. Since lines m and n are distinct, this is a contradiction of A.5. Therefore, since our assumption leads us to a contradiction, it must be false, and thus our theorem is true.

T.4. If a point P is a pole of line p, and a point Q is a pole of line q, and if point P is on line q, then point Q is on line p. We know from our hypotheses that point P is on line q. From A.3. we know that there exist exactly two more points on line q, say points A and B. Also from our hypotheses, we know that point P is a pole of line p. Therefore by A.4. point A cannot be a pole of line p. Hence, point A is joined to some point on line p, say point X, by our definition of the pole of a line. Also, by A.6. point A is joined to one more point on line p, say point Y. By A.3. we know there is a third point on line p, say point Z. Since point A is already joined to two points other than Z on line p, we know by A.6. that point A is not joined to point Z. From our definition of the pole of a line we know that point P is not joined to point Z. Thus by T.2. we know that point Z must be a pole of line q. By A.4. we know that every line has exactly one pole and point Z must be the same point as the point Q. Since point Z is on line p, we can conclude that point Q is on line p, and we have proved our theorem.

T.5. If points A, B, and C are on line l, then the polars of points A, B, and C are on point L which is the pole of line l. By our hypothesis we know that points A, B, and C are on line l and also that point L is the pole of line l. Then by T.4. we know that the polar of point A is on point L. In a like manner, we can prove that the polar of point B and the polar of point C are also on point L. Therefore, we can conclude that the polars of points A, B, and C are all on point L, and that our theorem is true.

Now we are ready to write and prove the duals of our axioms. However, before we can write the duals we must determine what the dual of a pole and the dual of a polar is. If one writes the dual of the definition of a pole, he will discover that it is the definition of a polar. Thus we can conclude that these two words are the dual of each other, and we can now write and prove our duals.

D.1. A line is on a point if and only if the point is on the line. This is just a restatement of A.1., and thereby we know that it is true.

D.2. There exists at least one point. By A.2. we know there exists at least one line and by A.3. we know that there are exactly three points on every line. Therefore, since we know that there must exist at least three points, we can conclude that our theorem is true.

D.3. There are exactly three distinct lines on every point. Let us call the point given in our hypothesis point L. Then by A.5. we know that point L has one polar, say line l. By A.3. we know that line l has exactly three points, say points A, B, and C, on it. By T.5. we know that the polars of these three points, say lines a, b, and c, are on point L. Suppose that there is a fourth distinct line, say line d, on point L. Then by T.4. we know that the pole of line d, say point D, is on line l. If we assume that point D and point A are the same point, then A.5. tells us that line d and line a are the same line. This is not true according to our assumption. Therefore, we can conclude that point A and point D are not the same point. In the same manner, we can prove that the points B and D, and the points C and D are not the same point. This, then, infers that line l has on it four distinct points. Since this is a contradiction of A.3. we can conclude that our assumption that there was a fourth distinct line on point L is false. Since we have proved that there are at least three lines on point L and at the same time there can be no more than three lines on point L, we can conclude that there are exactly three lines on point L. In a like manner we can prove that every point is on exactly three lines, and thus our theorem is true.

D.4. Every point has exactly one polar. Since this is just a restatement of A.5., we know that it is true.

D.5. Every line has exactly one pole. This is an exact restatement of A.4. and thus needs no further proof.

D.6. If a line p is not on a point X, but meets a line on X on a point M, then line p meets exactly one other line on point X on a point N distinct from point M. We know that point X is not on line p and also that the point X is joined to point M, which is on line p by our assumption. By our definition of join, the points M and X must lie on some line, say line s. Therefore, by A.6. point X is joined to exactly one other point on line p by another line, say line t, distinct from line s. Let us call this the point N. Point M and point N are distinct or we would have a contradiction of T.3. Therefore, we can conclude that our dual is true.

Now that we have proved the duals of all our axioms, we can conclude that our system possesses the property of duality. Before proving any more theorems, let us define additional terms. The first of these terms will be a triangle. A triangle consists of three points not all on the same line and three lines not all on the same point such that each of the points is on exactly two lines and each of the lines is on exactly two points. The points are called vertices and the lines are called sides. A triangle will be denoted by the three vertices in parentheses. Thus, the triangle whose vertices are the points A, B, and C would be denoted (ABC). A triangle is said to be perspective from a point C if and only if its three vertices are on the three lines on point C, with exactly one vertex on each line. A triangle is said to be perspective from a line c if and only if its three sides are on the three points on line c, with exactly one side on each point.

T.6. If two distinct points are not joined by a line, then each lies on the polar of the other. Let us call the two distinct points point P and point Q and the lines which are the polars of these two line p and line q, respectively. Let us first show that point Q is on line p by assuming that it is not, and that this assumption leads to a contradiction. From our hypothesis we know that point P is a pole of line p. By A.4. point Q cannot be a pole of line p. Then, by the definition of pole, point Q must be joined to some point, say point R, on line p by a line, say line x. By T.2. we can conclude that point P is a pole of line x, since it is not joined to either point Q or point R on this line. Since point Q is on line x and point Q is not on line p, then line x and line p are two distinct lines. By T.1. line x is a polar of point P and line p is a polar of point P. This gives us two distinct lines which are polars of the same point. This is a contradiction of A.5. Thus, we can conclude that point Q is on line p. In the same manner, we can prove that point P is on line q. Therefore, we have proved that our theorem is true.

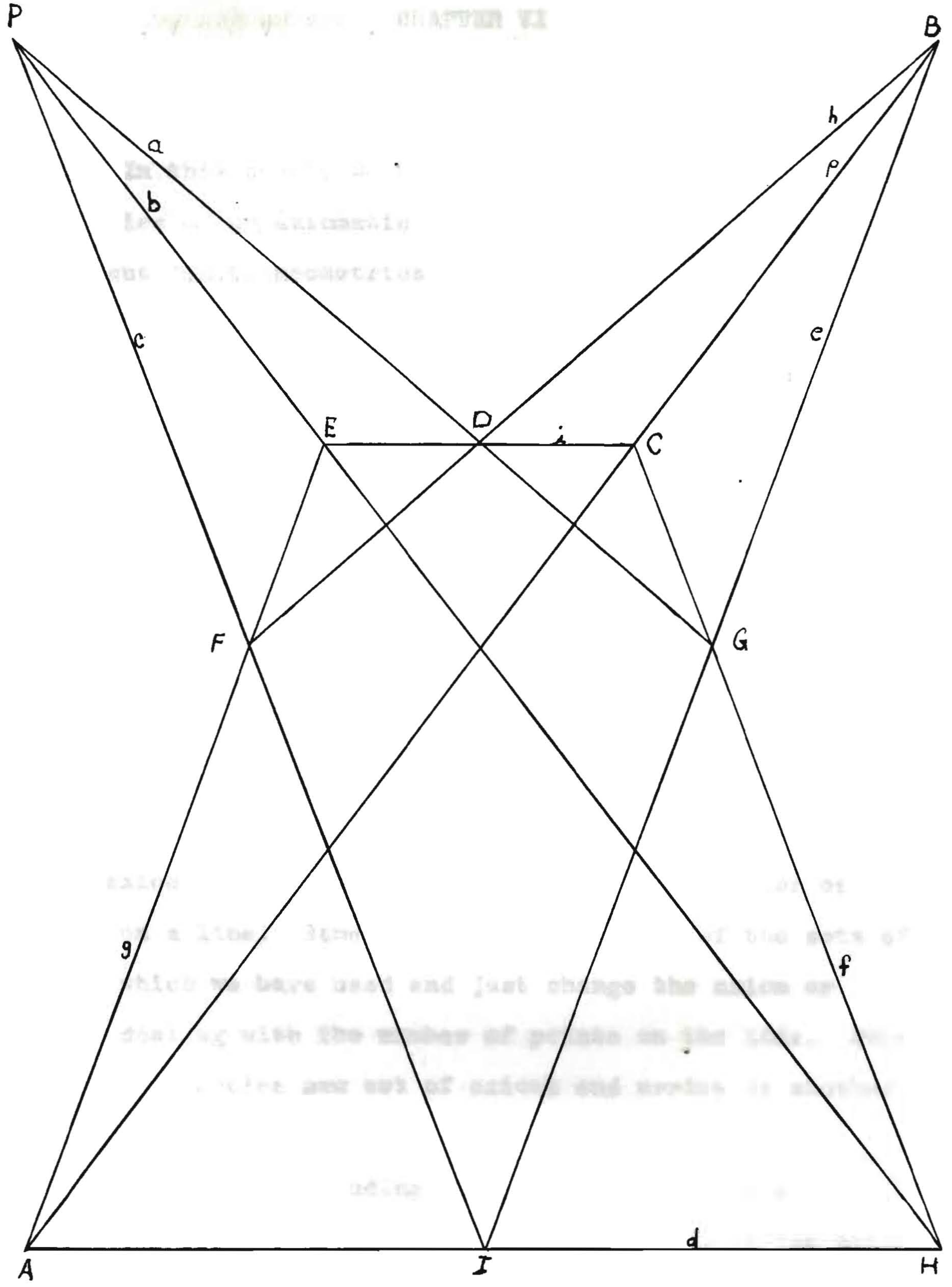
T.7. Every point has exactly two triangles perspective from it. By D.2. we know that there exists at least one point, say point P. By D.3. we know that there are exactly three lines, say lines a, b, and c, on point P. By A.3. we know that line a has two other points on it, say points D and E. Since point D is on line a, and line a and line b are joined at point P, then by A.6. we know that point D must be joined to exactly one other point, say point F on line b. In the same manner, we can show that point D is joined to exactly one other point, say point G on line c. Also in the same manner, we can show that point F is joined to exactly one other point, say point H, on line c. We would now like to show that point G and point H are the same point. Let us do this by assuming that they are not the same point. Now we have the three lines line b, the line joining the point D to the point E, and the line joining the point F to the point H, all on the point F. Let us show that these are three distinct lines. If the line joining the point D to the point F is the same as line b, then point D is on line b and by T.3. we know line a and line b are the same line. This is a contradiction and therefore, we know that line b and the line joining point D to the point F are not the same line. In the same manner, we can show that the line joining the point F to the point H is not the same as line b. Finally, if the line joining the point F to the point H is the same as the line joining the point D to the point F, then the point P is joined to all three points on this line. This is a contradiction of A.6., and these two lines must also be distinct. By D.3. we know that there are exactly three lines on every point and thus the three distinct lines we have just been discussing are the only lines on point F. Now we know that point F is joined to point P and point F is joined to point H. By A.6. point F cannot be joined to point G. Since point F is not joined to point G, then by T.6. point F must be on the polar of point G. But point G is joined to the points D, P, and H, and these three points are each on one of the three lines on point F. Therefore, by the definition of a polar of a point, none of the three lines on point F can be the polar of the point G. Thus we have a contradiction and our assumption that the point G and the point H were distinct is false. Now we must show that the three points, D, F, and G are distinct. Assume point D and point F are the same point. Then, since line a and line b have the point P in common, by T.3. they must be the same line. This is a contradiction and thus point D and point F are not the same point. In the same manner, we can show that the point D and the point G, and the point F and the point G are also distinct. Let us assume that the line joining the point D to the point G and the line joining the point D to the point F are the same line. Then point P is joined to all three points on this line. This is a contradiction of A.6. and therefore these two lines must be distinct. In the same manner, we can show that the line joining the point D to the point G and the line joining the point F to the point G are distinct. Therefore, the points D, F, and G are the vertices of a triangle. To get the second triangle perspective from the point P, the reader will remember that there is a third point, point E, on line a. Following the above proof through again will give us the second triangle perspective from the point P. Thus, we can see that our theorem is true.

T.8. If two triangles are perspective from a point, they are perspective from a line. By D.2. we know that there exists at least one point, say point P, and by D.3. we know that there are exactly three lines, say lines a, b, and c, on point P. By T.7. we know that there are two triangles, say triangles (DEF) and (GHI), with points D and G on line a, points E and H on line b, and points F and I on line c. Point I is joined to point P and also to point G, both on line a. Therefore, by A.5. we know that point I is not joined to point D on line a. Also point I is joined to point P and to point H on line b and, therefore, by A.5. we know that it is not joined to point E. Thus, by T.2. we know that point I is a pole of the line joining point D to point E. In the same manner, we can show that point F is a pole of the line joining the point G to the point H. Let us call the pole of line c point C and the polar of point P line p. Then by T.5., since points P, F, and I are on line c, the line p, the line which joins the point G to the point H, and the line which joins the point D to the point E are all on the point C. In the same manner, we can show that if point B is the pole of line b, then the line p, the line which joins the point D to the point F and the line which joins the point F to the point I are all on point B. Also, we can show that if the point A is the pole of line a, then the line p, the line which joins the point E to the point F, and the line which joins the point H to the point I are all on point A. Now since point A, point B, and point C are all on line p, we could conclude that the two triangles are perspective from a line if these three points are distinct. Let us show that they are distinct by assuming that two of the points, say point A and point B are the same point. Then we know that this point is on the line p, the line which joins point E to point F, the line which joins point H to point I, the line which joins the point D to the point F, and the line which joins point G to point I. Thus, the point is on five distinct lines. This is a contradiction of D.3. Therefore, the point A and the point B cannot be the same point. In the same manner, we can prove that all three of the points are distinct. Thus, we can conclude that the two triangles are perspective from a line and that our theorem is true.

T.9. There exist exactly ten points and ten lines. By D.2. we know that there exists at least one point, say point P, and by D.3. we know that there are exactly three lines, say lines a, b, and c, on point P. By T.7. we know that point P has two triangles, say triangles (DEF) and (GHI), perspective from it, with points D and G on line a, points E and H on line b, and points F and I on line c. From T.8. we know that the line which joins point D to point E and the line which joins point G to point H have the point C in common, where point C is the pole of line c and is on the line p, which is the polar of point P. Similarly, the line which joins the point E to the point F and the line which joins the point H to the point I have the point A in common, and the line which joins the point D to the point F and the line which joins the point G to the point I have the point B in common. We see that we now have the ten points A, B, C, D, E, F, G, H, I, and P and also the ten lines PDG, PEH, PFI, AHI, BGI, CHG, AEF, BDF, CDE, and ABC. We now know that there must be at least ten points and ten lines. Let us show that there can be no more than ten points by assuming that there is an eleventh point, say point Q. Since every one of the ten lines that we have so far already has three points on it, then by A.3. we know that point Q cannot be on any of these lines. Since by A.5. point Q has only one polar, we know that point Q must be joined to at least one of the lines, say line ABC, that already exist. Let the point A be the point to which line ABC is joined to point Q. Then point A has the lines ABC, AEF, AHI, and the line joining point A to point Q on it. This is a contradiction of D.3. Thus, we know that there cannot exist more than ten points. Let us now show that there can be no more than ten lines by assuming that there exists an eleventh line, say line q. Since every one of the ten points is already on three lines, this line cannot be on any of the ten points. By A.4., since every line has only one pole, then at least one of the points, say point A, must be joined to line q. Thus, point A has the lines ABC, AEF, AHI, and the line joining point A to the line q on it. This is a contradiction of D.3. As we have shown that there must be at least ten points and at the same time there cannot be more than ten points, we can conclude that there must be exactly ten points. Likewise, since we have shown that there must be at least ten lines and at the same time that there cannot be more than ten lines, we can conclude that there must be exactly ten lines. Therefore, we can conclude that our theorem is true.

If the reader will follow the steps used in proving T.8., using them as the steps involved in a construction, he

will obtain the model which is on the following page. It might also be interesting to note that this theorem is a famous theorem in projective geometry known as Desargues' theorem. It is usually stated in the following manner:
If two triangles are perspective from a point, then they are perspective from a line.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CHAPTER VI

SUMMARY

In this thesis we have presented some of the properties of an axiomatic system and then investigated three different finite geometries in the light of these properties. Each geometry has been developed to the point where it was proven to be finite. However, it did not seem feasible to go much beyond that point.

The reader will remember that each of the geometries possessed the properties of being consistent, categorical, and independent. Also, in each case there were three points on a line and three lines on a point, so that the property of duality was present.

It should be noted that there are an infinite number of finite geometries and that these geometries are not limited to having three points on a line. There are many other axiomatic systems involving a different number of points on a line. Some of these may use one of the sets of axioms which we have used and just change the axiom or axioms dealing with the number of points on the line. Others will use an entire new set of axioms and arrive at another finite geometry.

By way of concluding our work with the three geometries that we have investigated, let us list the axioms followed by all of the duals and theorems which we have proved to be true for each of our axiomatic systems.

THE SEVEN POINT GEOMETRY

A. 1. Two points determine a line.

A. 2. Two points determine at most one line.

A. 3. Any two lines have at least one point in common.

A. 4. There exists at least one line.

A. 5. There exist at least three points on a line.

A. 6. Not all points are on the same line.

A. 7. No line is on more than three points.

D. 1. Two lines determine a point.

D. 2. Two lines determine at most one point.

D. 3. Any two points have at least one line in common.

D. 4. There exists at least one point.

D. 5. There exist at least three lines on a point.

D. 6. Not all lines are on the same point.

D. 7. No point is on more than three lines.

T. 1. Two points determine one and only one line.

T. 2. Two distinct lines determine one and only one point.

T. 3. Every line contains three and only three points.

T. 4. Every point is on three and only three lines.

T. 5. There exist three points not all on the same line.

T. 6. There exist three lines not all on the same point.

T. 7. There exist at least seven points.

T. 8. There exist at least seven lines.

T. 9. There exist no more than seven points.

T.10. There exist no more than seven lines.

T.11. There exist exactly seven points.

T.12. There exist exactly seven lines.

THE NINE POINT GEOMETRY

A. 1. There exists at least one line.

A. 2. Every line is on exactly three points.

A. 3. Not all points are on the same line.

A. 4. Two distinct points determine at most one line.

A. 5. Given a line and a point not on it, there exists a line on the given point which has no point in common with the first line.

A. 6. Given a line and a point not on it, there exists not more than one line on the given point which has no point in common with the first line.

A. 7. Given a point and a line not on it, there exists a point on the given line which is not on any line with the first point.

A. 8. Given a point and a line not on it, there exists not more than one point on the given line which is not on any line with the first point.

D. 1. There exists at least one point.

D. 2. Every point is on exactly three lines.

D. 3. Not all lines are on the same point.

D. 4. Two distinct lines determine at most one point.

D. 5. Given a point and a line not on it, there exists a point on the given line which has no line in common with the first point.

D. 6. Given a point and a line not on it, there exists not more than one point on the given line which has no line in common with the first point.

D. 7. Given a line and a point not on it, there exists a line on the given point which is not on any point with the first line.

D. 8. Given a line and a point not on it, there exists not more than one line on the given point which is not on any point with the first line.

T. 1. Given a line and a point not on it, there exists exactly one line on the given point which has no point in common with the first line.

T. 2. Given a point and a line not on it, there exists exactly one point on the given line which is not on any line with the first point.

T. 3. Given a point and a line not on it, there exist at least two lines joining the point to points on the line.

T. 4. Given a point and a line not on it, there exist no more than two distinct lines joining the point to points on the line.

T. 5. Given a point and a line not on it, there exist exactly two distinct lines joining the point to points on the line.

T. 6. Given a line and a point not on it, exactly two points on the line are joined to the first point.

T. 7. Every line has at least two distinct lines parallel to it.

T. 8. If two lines are parallel to a third line, they are parallel to each other.

T. 9. The six points on two parallel lines determine exactly six lines.

T.10. The opposite sides of a hexagon determined by two parallel lines intersect in a point.

T.11. The points formed by the intersection of the opposite sides of a hexagon are distinct.

T.12. The three points determined by the intersection of the opposite sides of a hexagon are all on the same line.

T.13. There exist exactly nine points.

T.14. There exist exactly nine lines.

T.15. There are no more than two distinct lines parallel to a third line.

T.16. There are exactly two distinct lines parallel to a third line.

THE TEN POINT GEOMETRY

A. 1. A point is on a line if and only if the line is on the point.

A. 2. There exists at least one line.

A. 3. There are exactly three distinct points on every line.

A. 4. Every line has exactly one pole.

A. 5. Every point has exactly one polar.

A. 6. If a point P is not on line x but is joined to a point on line x by a line m, then point P is joined to exactly one other point on line x by a line n distinct from line m.

D. 1. A line is on a point if and only if the point is on the line.

D. 2. There exists at least one point.

D. 3. There are exactly three distinct lines on every point.

D. 4. Every point has exactly one polar.

D. 5. Every line has exactly one pole.

D. 6. If a line p is not on a point X, but is joined to a line on X by a point M, then line p is joined to exactly one other line on point X by a point N distinct from point M.

T. 1. A point P is a pole of a line p if and only if the line p is a polar of the point.

T. 2. If a point P is not joined to two points on a line p, the point P is a pole of line p.

T. 3. Two points are not joined by more than one line.

T. 4. If a point P is a pole of line p, and a point Q is a pole of line q, and if point P is on line q, then point Q is on line p.

T. 5. If points A, B, and C are on line l, then the polars of points A, B, and C are on point L, which is the pole of line l.

T. 6. If two distinct points are not joined by a line, then each lies on the polar of the other.

T. 7. Every point has exactly two triangles perspective from it.

T. 8. If two triangles are perspective from a point, they are perspective from a line.

T. 9. There exist exactly ten points and ten lines.

It is interesting to note, in conclusion, that many of the theorems which were proved in the finite geometries are also true in an infinite geometry. For example, Pascal's Theorem in the nine point geometry and Desargues' Theorem in the ten point geometry are two theorems which are true in projective geometry. Also in the nine point geometry, we have the theorem which states that if two lines are parallel to a third line, then they are parallel to each other. Since in most cases we live in a finite world rather than an infinite one, it is possible that many of the ideas used in a finite space could be used in our everyday world.

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