SUPPLEMENTARY MATERIAL FOR THE JUNIOR HIGH SCHOOL MATHEMATICS TEACHER

A THESIS

SUBMITTED TO THE DEPARTMENT OF

MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE

TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

by

James Madison Alderman, III

August 1961

Approved for the Major Department

John M. Burger

Approved for the Graduate Council

James C. Boylan

ACKNOWLEDGMENTS

To Professor Lester E. Laird, Department of Mathematics of the Kansas State Teachers College of Emporia, who generously gave of his time and effort to the writing of this thesis, the writer is indebted and wishes to express his sincere appreciation and gratitude.

To Dr. John M. Burger, Head of the Department of Mathematics of the Kansas State Teachers College of Emporia, who willingly and cheerfully offered suggestions for the improvement of this thesis as second reader, the writer wishes to offer his thanks.

Jim Alderman

TABLE OF CONTENTS

CHAPTER		PAGE
I.	THE THESIS	1
	1.1 Introduction to the thesis	1
	1.2 Purpose of the thesis	2
	1.3 Limitations of the thesis	2
	1.4 Organization of remainder of the thesis .	2
	1.5 Instructions on how to use the supplementary material	4
II.	THE REASONS FOR REVOLUTION IN MATHEMATICS	6
	2.1 Introduction to this chapter	6
	2.2 A list of reasons	6
	2.3 Report by Dr. G. Baley Price	7
	2.4 A report that summarizes the revolution .	10
III.	ANSWERS TO QUESTIONS CONCERNING THE REVOLUTION IN MATHEMATICS	11
	3.1 Introduction	11
	3.2 A list of questions	11
	3.3 Answers to the list of questions in section 3.2	12
	3.4 Summary of Chapter III	17
IV.	NAME AND PROPERTIES OF NUMBERS	19
	4.1 Number	19
	4.2 Definitions of the complex numbers	19

CHAPTER PAGE

4.3 Questions over numbers 23

4.4 Constructing the number system 24

4.5 Questions over constructing the number systems 28

V. CONCEPTS OF LARGE NUMBERS 31

5.1 Definitions of concept 31

5.2 Powers of two 33

5.3 Powers of two, problems 38

5.4 A million 39

5.5 Questions about a million 40

5.6 Dividing by zero 41

VI. CONCEPTS OF SMALL NUMBERS 45

6.1 Introduction 45

6.2 Exponents of ten 45

6.3 Questions over the exponents of ten . . . 48

6.4 Scientific notation 48

6.5 Questions over scientific notation 49

6.6 Powers of two 50

6.7 Problems on the powers of two 53

VII. THE NUMBER LINE 54

7.1 Introduction 54

7.2 How to solve problems concerned with whole numbers 55

CHAPTER PAGE

7.3 Problems over the whole numbers 57

7.4 How to solve problems concerned with signed numbers 57

7.5 Problems on signed numbers 60

7.6 How to solve multiplication problems concerned with signed numbers 61

7.7 Problems over multiplying signed numbers . 63

7.8 Fractions used with the number line . . . 63

7.9 Problems over fractions 65

7.10 Irrational numbers used with the number line 65

VIII. SYSTEMS OF NUMERATION 67

8.1 Introduction 67

8.2 Base Six--Senary System 68

8.3 Problems over the Senary System 73

8.4 Base Five--Quinary System 73

8.5 Problems over the Quinary System 76

8.6 Base Four--Quaternary System 76

8.7 Problems over the Quaternary System . . . 79

8.8 Base Two--Binary System 80

8.9 Problems over the Binary System 83

IX. SUMMARY AND CONCLUSION 85

9.1 Summary 85

CHAPTER PAGE

9.2 Conclusion 86

X. KEY TO THE PROBLEMS 87

BIBLIOGRAPHY 93

LIST OF FIGURES

FIGURES PAGE

5.2. Farmer Jones' Checker Board 34

8.80. Base Two, Multiplication and Addition Tables 80

8.81. Chart for a Game Using the Base Two 82

CHAPTER I

THE THESIS

1.1 Introduction to the thesis. Ten years ago the traditional secondary mathematics teacher was very content and happy; he was watching electronic computers solve astounding mathematical problems for industry; he was reading newspapers concerning atomic energy which was replacing manpower and improving production at a tremendous rate; and he was watching some of his own graduate students earn names for themselves in this new era of electronic computers and atomic energy.

Now, 1961, that same teacher finds himself in a revolution of mathematics brought on by: (1) the atomic era; (2) the broad variety of mathematical models and situations in which they are used; (3) an increasing recognition of an attempt to deal with the group behavior characteristics of the modern world; and (4) the rapid development of the computing machine.[1]

What is the traditional secondary mathematics teacher to do about the revolution of mathematics?

---

[1]Report of the Secondary-School Curriculum Committee of the National Council of Teachers of Mathematics, "The Secondary Mathematics Curriculum," Mathematics Teacher, May, 1959, p. 393.

1.2 Purpose of the thesis. The purpose of this thesis is twofold: first, it will give some reasons why the great revolution in mathematics and suggested solutions to this problem of a change; and second, it contains supplementary material based on parts of these changes that a traditional junior high school teacher could initiate in his mathematics classes.

1.3 Limitations of the thesis. The emphasis of this thesis is on the seventh, eighth, and ninth grade levels. The supplementary materials found in chapters IV through VIII have been carefully selected, and the chapters contain only material that is basic to the different experimental programs carried on at the present time. However, classes as low as the fifth grade and as high as the twelfth grade have been known to use many of the ideas available in the following chapters.

1.4 Organization of remainder of the thesis. Chapter II lists many of the reasons for a revolution in mathematics. One list of reasons was made by the U.S. Department of Health, Education, and Welfare.[2] Dr. G. Baley

---

[2]Report of the conference held March 17-19, 1960, at the U.S. Office of Education on "Inservice Education of Teachers of Secondary School Mathematics," U.S. Department of Health, Education, and Welfare, Office of Education, Washington 25, D. C.

Price, Executive Secretary, Conference Board of the Mathematical Sciences,[3] states some of the reasons as he sees them.

Chapter III attempts to answer some of the more important questions that are being asked by many of todays traditional mathematics teachers. Dr. Kenneth E. Brown, Specialist for Mathematics, U.S. Department of Health, Education, and Welfare;[4] Dr. Eugene Ferguson, Head of the Mathematics Department, Newton High School, Newtonville, Massachusetts;[5] and The Secondary-School Curriculum Committee[6] write in general about the situation.

Chapter IV presents the properties of numbers and operations. Definitions for such terms as commutative, associative, closure, and distributive will be presented. A method of constructing the numbers will also be outlined.

---

[3]Dr. G. Baley Price, Professor of Mathematics, University of Kansas and Executive Secretary of the Conference Board of the Mathematical Sciences, in an address "Progress in Mathematics and its Implications for the Secondary School," December 1, 1960, to the Regional Orientation Conference in Mathematics held at Topeka, Kansas.

[4]Kenneth E. Brown, "Projects to Improve School Mathematics," <u>Aids for Mathematics Education</u>, (U.S. Department of Health, Education, and Welfare--Office of Education--Washington 25, D.C., July, 1960).

[5]Dr. W. Eugene Ferguson, Head, Department of Mathematics, Newton High School, Newtonville, Massachusetts, in an address "Implementing the New Mathematics Program in Your School," December 1, 1960, to the Regional Orientation Conference in Mathematics held at Topeka, Kansas.

[6]The Secondary-School Curriculum Committee, op. cit.

Chapter V and Chapter VI give the reader a way to visualize numbers both large and small. The word concept will also be defined.

Chapter VII presents the number line. Elementary teachers that use the number line realize it explains properties of numbers better than any other tool.

Chapter VIII explains how systems of numeration are used. The study of these systems (bases six, five, four, and two) enables the student to understand more effectively our own decimal system (base ten). Base two is the system on which the electronic computers operate.

Chapter IX summarizes the material already given. Answers to the problems in the thesis are found in Chapter X and a bibliography follows Chapter X.

1.5 Instructions on how to use the supplementary material. This thesis deals with the content rather than the philosophy or technique of teaching mathematics. The teacher may adopt the material to the interests and needs of the students.

This author uses the supplementary material in his traditional classroom. It is done on Friday of each week. The problems do not require completion in class, but are given to the students for extra credit.

Most of the sections in the supplementary material are labeled for a certain Friday of the school year. Each of these sections are followed by a group of problems. A single section takes from ten to thirty minutes to present. The series of questions after each section takes from five to sixty minutes to complete.

The traditional teacher may select only the chapters he desires to use in the classroom and in any sequence. Most of the material has been organized in the sequence it may appear in the newer contemporary textbooks.

The emphasis of the supplementary material in these chapters is on the numbers.

CHAPTER II

THE REASONS FOR REVOLUTION IN MATHEMATICS

2.1 Introduction to this chapter. Mathematics is "the fastest growing most rapidly changing of the sciences." It does rely on the "time tested validity" of the traditional subject content to give it the strength it needs to keep abreast with modern times.[1] Electronic computers and space missiles are only a few of the by-products of scientific knowledge of the 20th century; this is accredited to the revolutionary advance in both the development and the use of mathematics.[2]

2.2 A list of reasons. In a short report of the conference at the U.S. Office of Education on "Inservice Education of Teachers of Secondary School Mathematics," held March 17-19, 1960, the following list of reasons was published for changes brought about by the recognition of a need for better mathematics.[3]

> (1) The subject matter of mathematics is growing in both the area of advanced mathematics and elementary mathematics.

---

[1] *Ibid.*, p. 389.

[2] The report on "Inservice Education of Teachers of Secondary School Mathematics," *op. cit.*, p. 2.

[3] *Ibid.*, p. 2.

(2) Mathematics is being called on to meet a wide variety of needs of which a few years ago people did not dream existed.

(3) The emphasis in mathematics is changing significantly. It is moving away from human computation to an understanding and construction of symbolic representation of factors that relate to scientific or social situations.

(4) To give better understanding of the subject, new mathematical language and symbolism are being introduced.

(5) There is a need for a jointly planned program between secondary mathematics and college mathematics.

(6) There is mathematics being taught in high schools that is obsolete and should be replaced by more significant subject matter.

(7) A number of experimental classrooms in the modern curriculum have demonstrated the advantages of teaching new topics in junior and senior high school mathematics courses.

(8) Several national groups of mathematics educators have made extensive and detailed studies of possible curriculum changes with specific recommendations.

2.3 Report by Dr. G. Baley Price. Dr. G. Baley Price, in an address to the Regional Orientation Conference in Mathematics, December 1, 1960, presented three main reasons for the need of a modern revolution in the mathematics curriculum:

(1) To keep up in research study. A mathematics book used until 75 years ago was adapted from Euclid's writings. Euclid lived about 300 B.C. Research teams have improved this to the degree of adding three dimensions--the field of solid geometry, the field of topology, the field of

non-Euclid geometry and other fields of modern geometry. Dr. Price, in acquiring his doctoral degree, did not have to study "Abstract Algebra," "Topology," "Probability and Statistics," "Lebesque Measure," "Hilbert Spaces," and "Functional Analysis"; as the research teams had not put this material into the college curriculum. Now, these are gaining status for anyone obtaining a doctoral degree and in some cases a master's degree in mathematics.

Some of the important subjects that have developed from the research teams and are very important to the progress of modern mathematics are:

(a) Probability and Statistics.

(b) Theory of Games (strategy). The first widespread use of the theory of games was during the war years between 1944 and 1945.

(c) Linear Programing. This should, in the opinions of many, be taught in advanced algebra.

(d) Operation Research. This again developed during the war years and was introduced first in England. It undertakes to supply quantatative data to aid in making decisions. Some examples of early use of the methods of operational research were: in the accuracy of a new bomb sight in an airplane, and the accuracy of the firing of a new type gun.

(e) Quality Control. This was started in 1929 in the field of economics. Until World War II this was not very effective or put to use. It provides means by which industry would determine the inefficiency in producing quality merchandise from an assembly line.

(2) To keep up with the Automation Revolution. This is the concept of machines replacing manpower, even to the

extent of machines controlling machines. Some examples are: Automatic pilots on an airplane, telephone hookups, missiles guided by electronic computers, and computers to control other machines. The mathematics that is needed in this field of study is as follows:

(a) to eliminate the trial and error method of building a good product;

(b) to eliminate inductive methods of reasoning and replace with deductive methods of reasoning;

(c) to decide deductively how a machine will perform before building it.

(3) To keep up with the concept of modern large scale digital computers. The Bible stated the value of π was equal to three "3." An Englishman named Shanks spent twenty years working out the value of π to 707 decimal places. Later it was discovered that he made a mistake on the 528th digit. An electronic computer in 1943 was able to find the value of π to 2,000 decimal places in 20 hours time. The machine was called "Eniac." Later, a machine did the same problem in 13 minutes to 3,000 decimal places. Recently (1957) a machine carried the value of π to 10,000 decimal places. The electronic computer, which was developed to a large degree since World War II, has been known to reach speeds of tens of thousands of operations per second with high reliability.[4]

---

[4]Dr. G. Baley Price, *op. cit.*

These machines have tended to reduce the need for semi-skilled and unskilled workers; thus the need for professional and managerial skills to improve the plant, product, and processes to run these machines will be in great demand in the immediate future; and professional and managerial skills will continue these trends at an accelerated pace.[5]

2.4 A report that summarizes the revolution.

> The biologist is applying mathematical theory to the study of inheritance; industry is using mathematics in scheduling production and distribution; the social scientist is using ideas from modern statistics; the psychologist is using mathematics of game theory. In fact, the logic of mathematical models shows promise as the basis for developing teaching machines for all areas of knowledge. The new uses of mathematics require less manipulation of formulas and equations but greater understanding of the structure of mathematics and mathematical systems. There is less emphasis on human computation than on computation by machines, and more emphasis on the construction of mathematical models and symbolic representation of ideas and relationships. Because of these new uses, mathematics is being firmly woven into the fabric of the national culture. The role of mathematics is not only to grind out answers to engineering problems, but to produce mathematical models (prototypes) that forecast the outcome of social trends and even the behaviorial changes of the group. Such important new uses and interpretations of mathematics require that students have a program with a greater depth than the classical program designed for the 19th century education. The demands of society require a thorough revision of our present secondary school mathematics curriculum.[6]

---

[5] The Mathematics Teacher, May, 1959, op. cit., p. 391.

[6] The report on "Inservice Education of Teachers of Secondary School Mathematics," op. cit., p. 2.

CHAPTER III

ANSWERS TO QUESTIONS CONCERNING THE REVOLUTION IN MATHEMATICS

3.1 Introduction. Both lay opinion and professional opinion have expressed quite a concern over the curriculum content and teaching practices which characterize the modern mathematics program in schools of our nation.[1] The serious nature of a modern mathematics program are found by questions which teachers and administrators are asking. Section 3.2 will give some questions which have been asked, in turn, section 3.3 will give some answers to these questions. The questions are made up by the author of this thesis.

3.2 A list of questions.

(1) What is wrong with traditional mathematics courses?

(2) What will be some of the major changes in the curriculum?

(3) What are some of the books and periodicals one may read, so he may become familiar with the newer ideas in mathematics?

(4) What college courses should one take so he may study the newer ideas in mathematics?

(5) How can one set up such a program in his own school system?

---

[1] The Mathematics Teacher, May, 1959, op. cit., p. 389.

(6) What projects are in progress to improve school mathematics?

3.3 Answers to the list of questions in section 3.2.

(1) What is wrong with traditional mathematics courses? The mathematics program for grades seven, eight, and nine has caused concern among both educators and parents for many years. Lack of interest in mathematics and science among students in senior high school and college may be due largely to unsatisfactory experiences while in junior high school.

Traditional courses at this level, teachers often report, offer little challenge to the upper 50 per cent of the students, and the lower 50 per cent usually lose interest because emphasis is placed on drill. New programs in mathematics for these grades are geared to give basic instruction for everyone.[2]

> Perhaps the most striking criticism that can be made of the present program in high schools is that it fails to capture the imagination of the students.[3]

(2) What will be some of the major changes in the curriculum? There is a great need for helping the student obtain knowledge of such mathematical concepts as functions

---

[2]*Ibid.*, p. 403.

[3]E. J. Cogan, "A New Approach to Mathematics," *The Mathematics Teacher*, May, 1947, p. 347.

and relations. Set theory (elementary) provides the terminology for describing models. Man's need to understand the nature of his world results in a demand for an early introduction of probability and statistics into the curriculum. The power of the electronic computers in solving linear equations has brought about the development of linear algebra on the elementary level.

> The unprecedented demands for mathematicians, their need for a new and more intensive training program, and the basic mathematical requirements of our technological age combine to pose a very difficult curriculum problem that demands the thoughtful attention of those whose concern it is to provide the most effective program in mathematics through-out secondary schools.[4]

Objectives that call for understanding of concepts and principles are being emphasized more than ever before. Highly specific terms such as coefficient, and quadrilateral, and converse have been defined and students have been "drilled" in stating the definitions. Today, there is increasing concern with a broader type of concept indicated by such terms as structure, relation, and mathematical system.

The content or subject-matter is being broadened and generalized by giving close attention to concepts which have until the present been largely "ignored."

There is more emphasis upon giving students certain types of learning experiences that serve not only to

---

[4]The Mathematics Teacher, May, 1959, op. cit., p. 393.

stimulate their understanding of mathematics, but also to "enliven" their interests and increase their appreciation. Experiences that lead to "discovery" of mathematical properties and relations are highly favored for this purpose.[5]

(3) What are some of the books and periodicals one may read, so he may become familiar with newer ideas in mathematics? A list of some of the better books and periodicals may be found in the bibliography of this thesis.

(4) What college courses should one take so he may study the newer ideas in mathematics? The Secondary School Curriculum Committee states:

> In view of current curriculum demands teachers of mathematics in grades seven through twelve will need to have competence in (1) analysis--trigonometry, plane and solid analytic geometry, and calculus; (2) foundations of mathematics--theory of sets, mathematical or symbolic logic, postulational systems, real and complex number systems; (3) algebra--matrices and determinants, theory of numbers, theory of equations, and structure of algebra; (4) geometry--Euclidean and non-Euclidean, metric and projective, synthetic and analytic; (5) statistics--probability and statistical inference; (6) applications--mechanics, theory of games, linear programing, and operations research.[6]

Emphasis is on the teacher to work for a master's degree. Teachers of mathematics at the seventh and eighth grade levels, as a minimum, should have at least eighteen

---

[5] *Ibid.*, p. 397.

[6] *Ibid.*,pp. 414-415.

semester hours, including six semester hours of calculus, in courses selected from the above areas.[7]

(5) <u>How can one set up such a program in his own school system</u>? Each school must develop its own program and since each school is "unique" in some respects, each program will vary to some degree in spite of a great many similarities in the current mathematics program.

There is no one mathematical curriculum that works for all schools, since there are factors such as: Schools varying in size; the percentage of college capable students varies from school to school; the teachers have varied amounts of credited training in mathematics, and community interest varies.[8]

Dr. Ferguson explains eight basic steps necessary to install the current mathematics into a school system.[9]

(a) Recognition by the school authorities of the need for a new mathematics program.

(b) Adequate preparation of teachers in the mathematics that is now being taught for the first time in secondary schools.

(c) Selection of a new program.

(d) Selection of students for the program.

---

[7] <u>Ibid</u>., p. 414.

[8] Dr. W. Eugene Ferguson, <u>op</u>. <u>cit</u>.

[9] <u>Ibid</u>.

(f) Informing other members of the school system about the new program and its implications for the mathematics program in kindergarten through grade 12.

(g) Continuation of teacher preparation for carrying the new program to higher and lower grades.

(h) Provisions for adequate time and compensation for carrying on the new program year after year.

(6) What projects are in progress to improve school mathematics? Dr. Kenneth Brown prepared a pamphlet for the U.S. Department of Health, Education, and Welfare on "Projects to Improve School Mathematics." The list included seventeen colleges and fifty-five junior and senior high schools that indicate an experimental program is being carried out in their school system. They give an indication of the experimentation that is in progress to improve school mathematics. It is not implied that this list is exhaustive. A survey in 1958 of randomly selected sample of 4,254 secondary schools showed that 40 per cent were conducting or planning a curriculum revision project in mathematics. Nine of the fifty-five high schools listed in Dr. Brown's report are teaching algebra in the eighth grade.[10]

The major projects in Dr. Brown's report are listed below. The name of the school or sponsoring agency is given first, followed by the name of the person who may be contacted for additional information.[11]

---

[10]Kenneth E. Brown, op. cit. [11]Ibid.

School Mathematics Study Group (SMSG), Drawer 2502A, Yale Station, New Haven, Conn.,--Professor E. G. Begle, Director.

University of Illinois Committee on School Mathematics (UICSM), University of Illinois, Urbana, Ill.,--Dr. Max Berberman, Director.

University of Maryland Mathematics Project, 1515 Massachusetts Ave., N.W., Washington 5, D.C., Dr. John R. Mayor, Director.

Boston College Mathematics Series, Chestnut Hill 67, Mass.,--Rev. Stanley Beruska, S.J., Director.

Ball State Teachers College Experimental Program. Ball State Teachers College, Muncie, Ind.,--Dr. Charles Brumfiel, Director.

Developmental Project in Secondary Mathematics, Southern Illinois University, Carbondale, Ill.,--Morton R. Kenner, Director.

The Syracuse University "Madison Project," Syracuse University, Syracuse, N.Y.,--Dr. Robert B. Davis, Director.

University of Illinois Arithmetic Project, 1207 West Stoughton Ave., Urbana, Ill.,--David A. Page, Director.

3.4 Summary of Chapter III. "Modern mathematics" for secondary schools is composed partially of new points of view toward traditional topics and partially of the replacement of a few traditional topics by new ones.[12]

For many teachers of the traditional mathematics that want to join the revolution, the most urgent need is to become familiar with the courses and terms mentioned in this chapter.

---

[12] The Mathematics Teacher, May, 1959, op. cit., p. 397.

Revision will be in an experimental stage for an indefinite time. Schools reshuffling their curriculums and confronting instructors with new material and new methods of approach. Some textbooks are coming out with modern mathematics material. One thing is certain; revision is needed. The question is how to attack the problems most effectively.[13]

---

[13] The Mathematics Teacher, May, 1957, op. cit., p. 347.

CHAPTER IV

NAME AND PROPERTIES OF NUMBERS

4.1 Number. According to Webster's New World Dictionary of the American Language, college edition, number means:[1]

> A symbol or word, or a group of either of these, showing how many or what place in a sequence: 1, 2, 3, 13, 23, 123 (one, two, three, thirteen, twenty-three, one hundred twenty-three) are called cardinal numbers; 1st, 2d, 3d, 4th, 24th, 100th, 124th (first, second, third, fourth, twenty-fourth, one hundredth, one hundred twenty-fourth) are called ordinal numbers. The symbol (#) is often used with a definite numeral, as in designating grade, size, rank, position, etc.

According to Webster's New Collegiate Dictionary number means:[2]

> The, or a total, aggregate, or amount of units. Abbr. No. or no. A figure or word, or a group of figures or words, representing graphically an arithmetical sum; a numeral; as, the number 45.

4.2 Definitions of the complex numbers. (First Friday) In mathematics the term natural number is taken as undefined. The concept of natural numbers may be understood more clearly if the logical structure of the natural numbers is postulated.

---

[1]Webster's New World Dictionary of the American Language (college edition), The World Publishing Co., Cleveland and New York, 1960, p. 1007.

[2]Webster's New Collegiate Dictionary, G. & C. Merriam Co., Springfield, Mass., 1956, p. 575.

Peano (1858-1932) set up probably what were the first postulates for the natural numbers.[3] He was looking for a set of basic facts. The postulates are:[4]

(1) The number one (1) is a natural number. This number is called unity.

(2) For each natural number (x) there exists one natural number called the successor of (x). The symbol S(x) means the successor of (x).

(3) One (1) is not a successor of any other natural number.

(4) If the successors of two numbers are equal then the numbers themselves are equal. Example of this statement follows: S(x) = S(y), therefore (x) = (y).

(5) Let "set M" be a set of natural numbers such that the number one (1) is contained in set M and such that if (x) is contained in set M implies that S(x) is contained in set M, then set M contains all the natural numbers.

Before using these postulates one must understand that one, natural number, and successor are the undefined terms used in the postulates. Postulates give relations between the undefined terms. Now that natural numbers have been postulated they will be used in defining whole numbers,

---

[3]The National Council of Teachers of Mathematics, Insights Into Modern Mathematics, 23rd yearbook, Washington, D.C., 1957, pp. 34 & 437.

[4]Prof. Lester E. Laird, Associate Professor of Mathematics, Kansas State Teachers College, Emporia, Kansas, in an address, "Constructing the Numbers" to a summer workshop in high school mathematics at Kansas State Teachers College, Emporia, Kansas, 1960.

integers, rational numbers, irrational numbers, real numbers, and complex numbers as follows:[5]

(1) Whole numbers. Whole numbers are the natural numbers with one as a successor of the element zero (0). Zero is not a successor of any other element in the set of whole numbers. Whole numbers are discrete. Discrete used here means there exists a space between each two neighboring members of the set of whole numbers. Webster's New Collegiate Dictionary defines it as "separate; individually distinct."[6] Examples of whole numbers are: 0, 1, 2, 3, 4, 5,..., the successor of 0 is 1; of 6 is 7; of 24 is 25.

The three dots when placed at the end of a set of whole numbers means "and so on endlessly" and shows that the set of numbers is unlimited.

(2) Integers. The integers consist of all the whole numbers either positive or negative. The term negative will be explained in detail in the next section (4.3). When a set of integers is written: ..., -3, -2, -1, 0, 1, 2, 3, ..., it indicates that the elements of the number system of integers is unlimited in both directions. Examples of integers are: 0, 5, -3, 1, and 467. Integers are also discrete.

---

[5]Edwin I. Stein, Supplementary Units in Contemporary Arithmetic and Elementary Algebra, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1960, p. 3S.

[6]Webster's New Collegiate Dictionary, op. cit., p. 237.

(3) Rational numbers. Rational numbers may be expressed as quotients of two whole numbers (fractions) with division by zero excluded. Dividing by zero is explained in section 5.4. The system of rational numbers includes zero and all the positive and negative whole numbers (the integers), common fractions and decimal fractions (both terminating and repeating decimals), mixed numbers and mixed decimals.

(a) One-fourth is equivalent to a terminating decimal .25.

(b) One-third is equivalent to a repeating decimal .3333....

It should be noted rational numbers are not discrete. The term dense may be used instead of the expression "not discrete." The term dense, as now used, means that between any two rational numbers another rational number may be found. Examples of rational numbers are: 9, 0, $\frac{1}{2}$, $-\frac{1}{4}$, -3, and 1.

(4) Irrational numbers. Real numbers that cannot be expressed as a quotient of two integers are said to be irrational numbers. Examples are the square roots of positive numbers other than perfect squares. Any number whose decimal representation is both non-terminating and non-repeating is an irrational number. Examples of irrational numbers are: $\sqrt{3}$, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{13}$ (algebraic numbers) and π and e (transcendental numbers).

(5) Real numbers. All rational and irrational numbers combined are called real numbers. A one-to-one correspondence of all the real numbers can be set up with the points on the number line. The system of real numbers includes zero, the positive and negative whole numbers, fractions and irrational numbers. Examples of real numbers are: 5, 0, $-\frac{1}{4}$, $\frac{1}{2}$, $\sqrt{3}$, -18, -3, 2, and π.

(6) Complex numbers. Complex numbers are the numbers which are written in the form a + bi where "a" and "b" are real numbers and "i" is the symbol indicating $\sqrt{-1}$ with $i^2 = -1$. A number like +3i is generally called an imaginary number. The complex number system includes all real and imaginary numbers.

4.3 Questions over numbers. (First Friday)

(1) Which of the following are natural numbers?

7 -3 $\frac{1}{2}$ 157 5 .89 .666... $4 + 2\sqrt{7}$

(2) Which of the following are integers?

-5 $\sqrt{17}$ 7/8 -81 $\sqrt{-5}$ 11 $\sqrt{4}$ .875 $-\frac{1}{4}$

(3) Which of the following are non-positive integers?

16 -3/5 -40 $\sqrt{63}$ -7 11 89 -62 $\sqrt{-7}$

(4) Which of the following are non-negative integers?

-5/6 $\sqrt{83}$ -29 .9 $-2\sqrt{3}$ 0 53 -167

(5) Which of the following are rational numbers?

.632 -4 3/8 $\sqrt{13}$ 1 52 -189 $4 + \sqrt{7}$

(6) Which of the following are irrational numbers?

$72 \quad -8 \quad \sqrt{23} \quad 0 \quad -.45 \quad -134 \quad \sqrt{-26} \quad 11/15$

(7) Which of the following are real numbers?

$3/7 \quad -.3 \quad 64i \quad -\sqrt{7} \quad 0 \quad \sqrt{-5} \quad -187 \quad .897$

(8) Which of the following are complex numbers?

$.999\ldots \quad -5/7 \quad 0 \quad 3 + 7\sqrt{-5} \quad \sqrt{-2} \quad 1{,}000 \quad -56$

(9) Which of the following are imaginary numbers?

$.92 \quad 0 \quad 23 \quad \sqrt{64} \quad 2 + \sqrt{5} \quad 7/11 \quad \sqrt{-9} \quad -1/3$

(10) Write all the one-digit natural numbers.

(11) Write all the one-digit integers.

(12) What is the successor of 13? of 543? of 1,999? of 19,099?

(13) Write four different names or numerical ways of expressing 9.

4.4 Constructing the number systems. (Second Friday) Mr. Lester E. Laird, Associate Professor of Mathematics, Kansas State Teachers College, Emporia, Kansas, addressed the 1960 Summer Workshop in High School Mathematics at Kansas State Teachers College, Emporia, Kansas, on the subject "Constructing the Number System."[7] Most of the information given in this section will be credited to his address.

---

[7]Laird, *op. cit.*

Each of the number systems described in the last section consists of a set of numbers, basic binary operations of addition and multiplication (subtraction is the inverse operation of addition and division is the inverse operation of multiplication), and properties concerning these numbers and operations.[8]

When adding 5 and 3, this is operating on two numbers at one time. This is called a binary operation. Multiplying 8 by 4, subtracting 7 from 9, and dividing 6 by 3 are binary operations.

A thing is said to be unique when it is the one and only one. When two numbers are added together the sum is a unique number because there should be the same answer every time the given numbers are added together. What is the unique sum of 27 and 15? The answer is 42. What is the unique product of 6 by 9? The answer is 54.

Closure, with respect to any operation, means there exists an answer that is a unique member of the set under discussion. Is the set of natural numbers closed with respect to addition and multiplication? The answer is yes, because the sum or product of two natural numbers is also a natural number. Example: 2 + 4 = 6, where 2, 4, and 6 are all natural numbers. Is the set of odd numbers closed under

---

[8]Stein, op. cit., p. 7.

addition? The answer is no, because two odd numbers when added together will give a sum that is even. Example: 7 + 3 = 10, where the sum (10) is not an odd number.

The six basic postulates of the set of natural numbers are:

(1) Closure property. The closure property with respect to the two operations addition (+) and multiplication (X).

(2) Commutative property. The commutative property of addition (or multiplication) means a change in the order of the two numbers will not affect the sum (or product).

(a) In addition if "a" and "b" each represent a natural number then for all "a" and for all "b": a + b = b + a.

(b) In multiplication if "a" and "b" each represent a natural number then: a X b = b X a or ab = ba.

(3) Associative property. Associative property is that the numbers may be grouped in pairs without affecting the sum (when adding) or the product (when multiplying).

(a) In addition if "a," "b," and "c" each represent a natural number then (a + b) + c = a + (b + c) = a + b + c.

(b) In multiplication if "a," "b," and "c" each are natural numbers then (a X b) X c = a X (b X c) = a X b X c.

(4) Multiplicative identity (unity). There exists an element (u) such that for every element a, a X (u) = (u) X a = a. In the case of natural numbers the multiplicative identity (unity) is one (1).

(5) Cancellation property. If (a) and (b) are elements of the set then c X (a) = c X (b) implies (a) = (b); or c + (a) = c + (b) then (a) = (b).

(6) Distributive property of multiplication with respect to addition. If "a," "b," and "c" are elements of the set then a X (b + c) = ab + ac; and (a + b) X c = ac + ab.

Notice this system lacks additive identity (0), multiplicative inverse (fraction) and additive inverse (negative number).

It is quite common today to use these properties and the multiplicative identity as postulates rather than the more basic and difficult Peano postulates.

When the additive identity (zero) is joined to the set of natural numbers, a new system is formed called the whole numbers.

(7) Additive identity. There exists an element (z) such that for every element a: a + (z) = (z) + a = a.

When a postulate concerning negative numbers is added to the above seven postulates the integers are formed.

(8) Additive inverse. If the sum of two numbers is zero (0) then each addend is said to be the additive inverse of the other. An additive inverse can be defined as an element (-a) such that (-a) + a = 0. (-a) is the additive inverse of (a); (a) is the additive inverse of (-a).

The set of elements that satisfies these eight postulates, make up what is called the integral domain.

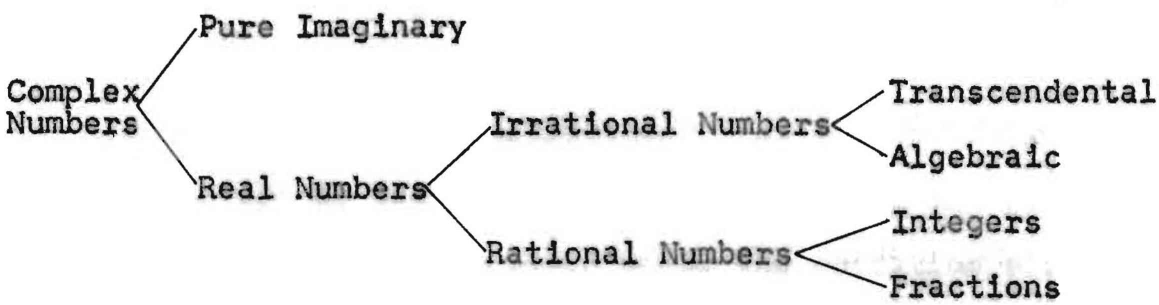
The next postulate is the multiplicative inverse (fraction). When it is added to the eight postulates already given the rational numbers are formed.

(9) Multiplicative inverse. If the product of two numbers is one (1) then each factor is called the

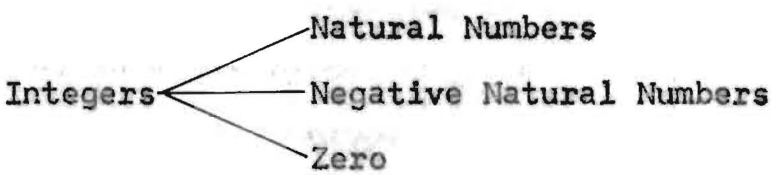
multiplicative inverse of the other. There exists an element (1/a) such that when multiplied times its inverse (a) the result is one. Example: (1/a) x (a) = 1, 6 is the inverse of (1/6) such that 6 x (1/6) = 1, or 2/3 is the inverse to 3/2 such that 2/3 x 3/2 = 1. Sometimes the multiplicative inverse 1/8 is also called the <u>reciprocal</u> of 8.

Any set with two operations for which all these nine postulates are true is a <u>field</u>.

A diagram showing relationships of systems of complex numbers which have been discussed is as follows:



The integers can still be broken down:



4.5 <u>Questions over constructing the number system</u>.

(Second Friday)

(1) What is the unique answer of the following:

(a) add 5 and 4. (b) subtract 15 from 30.

(c) divide 25 by 5.

(2) Which of the following are closed under addition?

(a) Natural numbers (b) Integers (c) Rational numbers

(3) Which of the following are closed under subtraction?

(a) Natural numbers (b) Integers (c) Real numbers

(4) Which of the following are closed under multiplication?

(a) Whole numbers (b) Integers (c) Rational numbers

(5) Which of the following are closed under division?

(a) Natural numbers (b) Integers (c) Real numbers

(6) If the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 form a set, is this set of numbers closed under the operation of:

(a) Addition? (b) Subtraction? (c) Multiplication?

(7) If the set of numbers consists of all even integers, is this set of numbers closed under the operation of:

(a) Addition? (b) Subtraction? (c) Division?

(8) When adding a column of three addends first in a down direction and check by adding in an up direction, what two properties are used?

(9) Does the commutative property hold true when baking a cake?

(10) What is another name for the additive identity?

(11) What is another name for the multiplicative inverse?

(12) What is the additive inverse of: (a) 14? (b) -5?

(c) ½? (d) -¼? (e) .75? (f) -.3333...?

(13) What is the multiplicative inverse of: (a) 14? (b) -5? (c) 4/5? (d) -7/8? (e) 1/8? (f) 1?

(14) What is the inverse operation of: (a) Addition? (b) Subtraction? (c) Multiplication?

(15) If the numbers 0, 0, 0, 1, and 1 form a set, is this set of numbers closed under the operation of: (a) Addition? (b) Subtraction? (c) Multiplication

(16) The set of numbers in problem 15 cannot be closed under the operation of division. Why?

(17) Show how the multiplicative identity 1 is used when changing fraction 3/9 to its lowest term.

(18) Show how the multiplicative identity 1 is used when dividing the fraction ½ by ¼. (½ ÷ ¼)

CHAPTER V

CONCEPTS OF LARGE NUMBERS

5.1 Definitions of concept. Before going into the chapter it would be wise to understand what is to be achieved first. If concept is defined it will be easier to understand what the author is trying to accomplish in this chapter.

The following are different definitions of the word concept:

(1) Psychologist definition. A so-called general notion or highly schematized idea, formerly supposed to emprise (undertake) all the attributes common to the individuals that make up a class; distinguished both from the fantasm or image and from the percept, which are individual and concrete, while the concept is general and relatively abstract.[1]

(2) Philosophy definition. An idea, as distinguished from a percept; esp., as orig., and idea representing the meaning of a universal term and comprehending the essential attributes of a class of logical species; now, chiefly, an idea that includes all that is characteristically associated with, or suggested by, a term; also, a mental image of an action or thing.[2]

(3) Logic definition. Concept is the synonym most frequently used for simple apprehension, regarded as the product of an act. Simple apprehension is an act by

---

[1]Funk and Wagnall's New Standard Dictionary, Funk & Wagnalls Company, New York and London, 1942, p. 547.

[2]Webster's New International Dictionary, second edition, G. & C. Merrian Company, Springfield, Mass., 1958, p. 552.

which the mind grasps the general meaning of an object without affirming or denying anything about it.[3]

(4) Brief definitions. Any notion in which elements are combined into the idea of an object.[4] A thought; an opinion.[5]

Concept is used often in a modern mathematics class; either by the teacher or in the textbook.

The concept of large numbers aids students in learning about some of the following:

(1) Astronomy;

(a) mass of the planets and stars,

(b) distances of the planets and stars,

(c) time in terms of the creation of the stars and planets.

(2) Money;

(a) compound interest,

(b) National debt (need the author say more?).

(3) Biology;

(a) combinations of heredity,

(b) reproduction of flies, rabbits, fish, and bacteria.

---

[3]Dr. Raymond J. McCall, Basic Logic, Barnes & Noble, Inc., New York, 1961, p. 6.

[4]Funk and Wagnall's New Standard Dictionary, op. cit., p. 547.

[5]Webster's New International Dictionary, op. cit., p. 552.

(4) Mathematics;

(a) tangent and cotangent curves,

(b) limits,

(c) recreations in mathematics.

This list is just a few of the things used with large numbers.

5.2 <u>Powers of two (2)</u>. (Third Friday) The expression (2^n) is used in most of the problems listed below. The "n" is called the exponent of 2. The exponent "n" means the number of times that 2 is written as a factor in obtaining the value of 2 to the nth power. The list of application of definitions used to explain the exponents of 2 follow:

$$2^0 = 1$$
$$2^1 = 2$$
$$2^2 = 2 \cdot 2 = 4$$
$$2^3 = 2 \cdot 2 \cdot 2 = 4 \cdot 2 = 8$$
$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 8 \cdot 2 = 16$$
$$2^{n-2} = 2^{n-3} \cdot 2$$
$$2^{n-1} = 2^{n-2} \cdot 2$$
$$2^n = 2^{n-1} \cdot 2$$

Here is a very popular problem concerning the powers of two: Farmer Jones has a bushel of wheat and a checker board. Like a curious mathematician, he takes a grain of wheat and puts it on the first square of the checkerboard;

two grains on the second; four on the third; and so on by doubling the number of grains on the preceding square. In figure 5.2 is a diagram of the checker board. Notice how the numbers keep doubling. Before finding the answer to the number of grains that Farmer Jones will put on the last square, let the students guess the number.

•	:	∷	8	16	32	64	188
256							

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7
2^8	...						
				...	2^{61}	2^{62}	2^{63}

FIGURE 5.2

FARMER JONES'

CHECKER BOARD

In figure 5.2 the numbers have been replaced by the corresponding powers of two (2). Let "n" stand for the number of the square. Since the first square has one grain, one (1) cannot correspond to 2^n if n stands for 1 as $2^1 = 2$. If the formula 2^{n-1} is used for the number of grains on each square the n will correspond to the number of that square as shown in figure 5.2. The first square (n = 1) contains a number of wheat obtained by the use of the formula $2^{n-1} = 2^{1-1} = 2^0 = 1$ grain of wheat; the second square (n = 2) contains $2^{n-1} = 2^{2-1} = 2^1 = 2$ grains of wheat; the third square ($n = 3$) contains $2^{n-1} = 2^{3-1} = 2^2 = 4$ grains of wheat; and finally the last square ($n = 64$) contains $2^{n-1} = 2^{64-1} = 2^{63} = 9,223,372,036,854,775,808$.

This number (9,223,372,036,854,775,808) of grains of wheat is so large it would take the world's wheat production for the next nine hundred and twenty-three years; assuming 5,000,000 grains of wheat per bushel and the world production of wheat averaging about 2,000,000,000 bushels a year.[6] This answer was for the last square alone.

While on this problem, how many grains of wheat would it take to put on all the squares of the board? Notice on the third square the answer is four which is one more than the sum of the first two squares (3); on the fifth square there are 16 grains of wheat which is one more than the sum of the first four squares (15). Empirically the process shows the last square to hold one more than the sum of that on the other 63 squares. Since the last square has 2^{63} then the 63 other squares have $2^{63} - 1$. The sum of the grains on all the squares is $2^{63} + 2^{63} - 1 = 2 \cdot 2^{63} - 1 = 2^{64} - 1$. $2^{64} - 1 = 18,446,744,073,709,551,615$. To make a general formula for finding the number of grains on a certain number of

---

[6]George Gamow, One, Two, Three, ..., Infinity, A Mentor Book, published by The American Library, 1959, p. 20.

squares: If the last square has 2^{n-1} then the sum of the remaining terms is $2^{n-1} - 1$; therefore, the sum of all the terms is $2^{n-1} + 2^{n-1} - 1 = 2 \cdot 2^{n-1} - 1 = 2^n - 1$.

The natural numbers have a property which is essential for many portions of mathematics. This property enables one to use a process called "Mathematical Induction" in proving theorems about natural numbers. To see whether the formula $2^n - 1$ can be proven correct by using the method known as mathematical induction, the following two steps must be performed:[7]

(1) Verify the statement for $n = 1$.

$n = 1$; $2^n - 1 = 2^1 - 1 = 2 - 1 = 1$, which is true.

(2) Assume the statement for $n = k$, and on this basis prove it for $n = k + 1$.

$1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$, let $k = n$;

$1 + 2 + 4 + 8 + \ldots + 2^{k-1} = 2^k - 1$, add 2^k to both sides;

$1 + 2 + 4 + 8 + \ldots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$

$1 + 2 + 4 + 8 + \ldots + 2^{k-1} + 2^k = 2 \cdot 2^k - 1$, finally;

$1 + 2 + 4 + 8 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1$, hence

the formula is true for $n = k + 1$. This is the form the first statement would take on if n was replaced by $k + 1$, since $n - 1$ would be equal to k.

---

[7]C. B. Allendoerfer and C. O. Oakley, Fundamentals of Freshman Mathematics, McGraw-Hill Book Co., New York, 1959, p. 32.

Therefore, by mathematical induction the formula $2^n - 1$ is equal to the sum of all the powers of 2 up to 2^{n-1}.

If Farmer Jones was using a bushel of wheat in putting the grains on the board it would take twenty-two squares to use all the grain in his bushel basket.

Problems dealing with compound interest work on the same principle. If a man had invested a penny in stock about 0 B.C.-A.D. and that stock increased in value at a rate equal to 6% compounded annually, that stock would be worth now (1961) about $429,450,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000.00. The formula for the amount of principle after the principle (A) has been invested at a rate (r) for (n) number of years compounded annually is $A(1 + r)^n$. Instead of two (2) as the base in this problem 1.06 is the base. If the world population were about 3 billion then that money could be given to each of the people in this world with each person getting about $143,000,000,000,000,000,000,000,000,000,000,000,000. If <u>each</u> grain of sand on this earth cost $1.00, this amount (143 undecillion dollars) would more than pay for all the sand.

5.3 Powers of two, problems. (Third Friday)

(1) Jack, a very fine golfer likes to take side bets with anyone who plays with him. He bets 1¢ on the first hole, 2¢ on the second, 4¢ on the third, and keeps doubling the bet. If he plays two rounds of golf (18 holes), what amount of money will each golfer have at stake on the 18th hole? Remember to use the formula 2^{n-1}.

(2) If Jack had won on each hole, how much did he win?

(3) Bill, hearing about Farmer Jones and the wheat problem, decided to work in his dad's grocery store for thirty days. Each day his father would double his wages starting the first day at 1¢. If his father agreed to the proposition how much would the father have to pay Bill for the last day's wages alone?

(4) In problem three (3) how much were Bill's wages for all 30 days?

(5) Zeus, father of the Olympian Gods in Greek mythology, owns ten-quadrillion planets that he wants to sell as it is nearly inventory time and he does not want to count these again. These are on sale for one-trillion dollars apiece. A smart mathematician asks if he may help Zeus for one hundred days clean some of these planets up and get them ready for the great sale. The mathematician will start to work for 1¢ the first day and double his

wages each day after that. Zeus thinking this would be cheap wages, agrees to the proposition. Sure, the mathematician is able to buy all the planets; but what is the bill he handed Zeus for his wages?

(6) After the mathematician bought all the planets in problem five (5) how much money did he have left to live on?

5.4 A million. (Fourth Friday) A million seems small when talking about the public national debt (285 billion dollars in 1959), but try to visualize a million. It will take a bigger and better imagination than you think.

If a person had to walk a million steps starting at the eastern edge of Kansas walking west, how far would he get across the State? It would take about 428 miles to walk across the State of Kansas. One million steps (providing one walking step takes 1 yard) would equal about 568 miles. The answer is: The person would walk clear across the State of Kansas and still have about 140 miles to walk.

To visualize a million by walking 568 miles is too hard on one's feet; so try writing a million strokes with a pencil. If you make one stroke a second it would take about 278 hours, or 11 days and 14 hours of nonstop writing. (There are 3,600 seconds in an hour; then there are

1,000,000/3,600 hours in a million seconds. This is equal to about 278 hours.)[8] With this information, by counting a star per second, how long would it take Zeus to count his quadrillion stars? The answer is about 32 billion years, providing of course for nonstop counting. The best estimates of the age of the earth range from 3 to 10 billion years old; so Zeus would still be counting.

It is said that the word "million" is just a little more than four hundred years old. About eight years after Columbus discovered America, it first appeared in Italy, (about 1500 A.D.). Since then only about 242 million minutes have elapsed, there being about 526,000 minutes in a year.[9]

5.5 Questions about a million. (Fourth Friday)

(1) A thimble of water weighs about a gram. What would be the weight of a million thimbles of water in pounds units? (453.6 grams equal a pound.)

(2) A million days is equal to how many years?

(3) A mosquito is about 3/8 inch long. How big would this mosquito be one million times enlarged?

---

[8] Aaron Bakst, Mathematics, Its Magic and Mastery, D. Van Nostrand Co., 250 Fourth Avenue, New York 3, N. Y., 1941, p. 52.

[9] Ibid., pp. 51-52.

(4) A man smokes, on the average, a pack of cigarettes a day (a pack contains 20 cigarettes). How many years would it take him to smoke a million cigarettes?

5.6 Dividing by zero. (Fifth Friday) Teachers have been asked by their students, "Why can't zero be divided into a number?" The teachers usually become irritated especially if they don't know why themselves.

It can be proven that division by zero is impossible by using the axioms and two postulates: any number times zero is equal to zero ($X \cdot 0 = 0$) and zero does not equal one ($0 \neq 1$). Assume $1/0 = X$:

$1/0 = X$ (assume)

$(1/0) \cdot 0 = X \cdot 0$ (equals may be multiplied by equals and the products are equal)

$1 = X \cdot 0$, but $X \cdot 0 = 0$ (postulate)

$1 = 0$ (is a contradiction to postulate, $1 \neq 0$)

If $1/0 = X$, X does not exist and $1/0$ is meaningless. This is also true of any integer divided by zero and shows why zero was not allowed as a denominator in the definition of rational numbers.

Consider what happens to the fraction $1/X$ as X approaches 0. Substitute some values for X: $X = 5$, $X = 2$, $X = 1$, $X = 1/2$, $X = 1/10$, $X = 1/1{,}000$, and $X = 1/1{,}000{,}000$.

Notice what happens to the value of the fraction $1/X$ as X approaches zero.

let $X = 5$, then $1/X = 1/5 = .200$;

let $X = 2$, then $1/X = 1/2 = .500$;

let $X = 1$, then $1/X = 1/1 = 1.00$;

let $X = 1/2$, then $1/X = 1/(1/2) = 2.00$;

let $X = 1/10$, then $1/X = 1/(1/10) = 10$;

let $X = 1/1{,}000$, then $1/X = 1/(1/1{,}000) = 1{,}000$;

let $X = 1/1{,}000{,}000$, then $1/X = 1/(1/1{,}000{,}000) = 1{,}000{,}000$.

The smaller X becomes the larger the value of the fraction (1/X) gets. Choosing smaller and smaller values of X, the successive values of 1/X get larger and larger. As X approaches 0, 1/X increases indefinitely.

Now a new term involving the word infinity may be introduced at this time. By "$\lim_{X \to 0} 1/X = \infty$" is meant the limit of 1/X as X approaches 0 is infinity.

Webster's New Collegiate Dictionary defines infinity as:[10]

> (1) The quality of being infinite; also, that which is infinite; unlimited extent of time, space, or quantity; eternity; boundlessness. (2) An indefinitely great number or amount. (3) *Math.* An infinite denoted by ∞. In geometry; that region of a line, plane, or space, which is infinitely distant from the finite region regarded.

---

[10] *Webster's New Collegiate Dictionary*, *op. cit.*, p. 429.

Webster's New Collegiate Dictionary defines infinite as:[11]

> (1) Without limits of any kind; undetermined or indeterminate. (2) Without end; boundless; immeasurable. (3) Indefinitely large, extensive, or numerous; hence, vast; immense; also, inexhaustible. (4) *Math.* Greater than any assignable quantity of the same kind or equivalent to some proper part of itself.

If zero was divided into a number then $1 = 2$. Axioms and number properties will be used to prove $1 = 2$ as follows:

$a = b$ (given)

$aa = ab$ (equals may be multiplied by equals and the products will be equal)

$a^2 = ab$ ($aa = a^2$ by definition)

$a^2 - b^2 = ab - b^2$ (equals may be subtracted by equals and the differences will be equal)

since $a^2 - b^2 = a^2 - ab + ab - b^2$ (additive identity: $-ab + ab = 0$ and $X + 0 = X$)

then $a^2 - b^2 = a(a - b) + b(a - b)$ (distributive property)

and $a^2 - b^2 = (a - b)(a + b)$ (distributive property)

therefore $(a - b)(a + b) = ab - b^2$ (equals may be substituted for equals)

$(a - b)(a + b) = b(a - b)$ (distributive property: $ab - b^2 = b(a - b)$)

---

[11]*Ibid.*

$(a - b)(a + b)/(a - b) = b(a - b)/(a - b)$ (equals may be divided by equals and the quotients are equal)

$a + b = b$ (unity element: $a \cdot X/X = a \cdot 1 = a$)

$b + b = b$ (equals may be substituted for equals: $a = b$)

$2b = b$ ($b + b = 2b$ by definition)

$2b/b = b/b$ (equals may be divided by equals and the quotients are equal)

therefore $2 = 1$. (Unity element: $a \cdot X/X = a \cdot 1 = a$)

The fallacy of this proof is that both sides of the equation were divided by 0. Since $a = b$ then $a - b = 0$ and $(a - b)(a + b)/(a - b) = b(a - b)/(a - b)$.

CHAPTER VI

CONCEPTS OF SMALL NUMBERS

6.1 Introduction. How small is small? The breadth of a hair is about three-thousandths of an inch. Industries pride themselves in the fact that some parts are measured to a ten-thousandth of an inch. These small quantities mentioned are large in size compared to germs, molecules, atoms and electrons.

Until the microscope was invented man had no suspicion of the existence of bacteria; and even the most powerful microscopes are inadequate to detect certain viruses.[1]

It is impossible to answer the question, "How small is small?" An attempt will be made to use a concept to dwell in the land of the small, the smaller, and the smallest (if it is possible).

6.2 The exponents of ten. (Sixth Friday) In section 5.2 by definition: $2 \cdot 2 = 2^2$, $2 \cdot 2 \cdot 2 = 2^3$, $2 \cdot 2 \cdot 2 \cdot 2 = 2^4$, etc. The powers of ten work on the same order: $10 \cdot 10 = 10^2$, $10 \cdot 10 \cdot 10 = 10^3$, etc. The number above the base ten is called the exponent (indicator). Example: a^x is called the

---

[1] Bakst, op. cit., p. 65.

expression, "a" is called the base and "x" is called the exponent. The powers of ten follow:

10,000 is equal to 10•10•10•10 and written as 10^4,

1,000 is equal to 10•10•10 and written as 10^3,

100 is equal to 10•10 and written as 10^2,

10 is equal to 10 and written as 10^1,

1 is equal to 1 and written as $10^?$.

In section 5.2, $2^0 = 1$ by definition. Why is $2^0 =$ 1 rather than equal to 0? One basic law of exponents is needed to show why $X^0 = 1$ is necessary in order to be consistent: $X^a \cdot X^b = X^{a+b}$. Example: $2^2 \cdot 2^3 = (2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^{2+3} = 2^5$. The argument follows:

If $X^a \cdot X^b = X^{a+b}$ is to be true for the exponent zero, then

$X^0 \cdot X^n = X^{0+n} = X^n$ (definition and additive identity), then

$X^0 \cdot X^n = X^n$, and then

$X^0 \cdot X^n/X^n = X^n/X^n$ (equals may be divided by equals and the quotients are equal), whereby

$X^0 = 1$.

An additional member now is added to the set of numbers which may be used as exponents, zero. With this definition, $10^0 = 1$. Watch the pattern:

$10,000$ is written as 10^4,

$1,000$ is written as 10^3,

100 is written as 10^2,

10 is written as 10^1,

1 is written as 10^0,

$.1$ $(1/10)$ is written as $10^?$.

Can decimal fractions be expressed by the exponents of 10? $1/10 = 10^?$. The additive inverse property will be useful in obtaining a consistent definition for this type of exponent:

$1/10 = 10^?$ (What is the value of the exponent?)

Consider:

$(1/10) \cdot 10^1 = 1$ (multiplicative inverse property: 1/10 is the multiplicative inverse of 10), that is:

$(1/10) \cdot 10^1 = 10^0$ (equals may be substituted for equals: $10^0 = 1$) Thus:

$10^? \cdot 10^1 = 10^0$ (equals may be substituted for equals: $10^? = 1/10$), or

$10^{?+1} = 10^0$ (equals may be substituted for equals: $10^? \cdot 10 = 10^{?+1}$), and

$? + 1 = 0$, therefore $? = -1$ (additive inverse property).

Therefore for consistency, $1/10 = 10^{-1}$.

An additional definition now is added to the system of exponents ($1/X^n = X^{-n}$). The exponents of 10 for decimal fractions are as follows:

$1,000 = 10^3$

$100 = 10^2$

$10 = 10^1$

$1 = 10^0$

$.1 = 1/10 = 10^{-1}$

$.01 = 1/100 = 10^{-2}$

$.001 = 1/1000 = 10^{-3}$

The multiplicative inverse of 10^n is 10^{-n}.

6.3 <u>Questions over the exponents of ten.</u> (Sixth Friday)

(1) $100,000,000. = 10^?$ (?) = _____

(2) $100 \cdot 1,000 = 10^2 \cdot 10^3 = 10^?$ (?) = _____

(3) $1/100 = 10^?$ (?) = _____

6.4 <u>Scientific notation.</u> (Seventh Friday)

$2.42 \times 100 = 2.42 \times 10^2$, the expression 2.42×10^2 is called the <u>scientific notation</u> for 242. Scientific notation saves writing time and space when it comes to numbers like: The total number of atoms in the universe (measured as far as the largest telescope can penetrate) is 300,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000; in scientific notation this number would be written 3×10^{74}.[2] A small number like .00006 in scientific notation is 6×10^{-5}.

The rule for scientific notation is:[3]

> Count the number of places the decimal point must be moved to produce a number between 1 and 10. This number of places is the exponent of the base 10. It is positive if the decimal point is moved to the left, negative if the decimal point is moved to the right.

Some examples:

$2,876 = 2.876 \times 10^3$

$2,000,000,000 = 2 \times 10^9$

$983.78 = 9.8378 \times 10^2$

$.666 = 6.66 \times 10^{-1}$

$.00567 = 5.67 \times 10^{-3}$

$.00001 = 1 \times 10^{-5}$

6.5 Questions over scientific notation. (Seventh Friday)

(1) $100 = 10^?$ (?) = ____

(2) $1000 \times 10,000 \times 1,000,000 = 10^?$ (?) = ____

(3) $26,000,000 = 2.6 \times 10^?$ (?) = ____

(4) $5,826,000 = 5.826 \times 10^?$ (?) = ____

(5) $1/1,000,000 = 10^?$ (?) = ____

---

[2] Gamow, *op. cit.*, p. 16.

[3] Virgil S. Mallory & Kenneth C. Skeen, *Second Algebra*, Benj. H. Sanborn & Co., Chicago, 1952, p. 330.

(6) Complete: The scientific notation for .0367 is________

(7) Complete: The scientific notation for 36,000 is________

(8) $1/10{,}000 \cdot 1/1{,}000 = 10^{-4} \cdot 10^{-3} = 10^{-7} = 1/?$ (?) = ____

(9) The distance across the Milky Way is 1,000,000,000,000,000,000 miles. This distance is also expressed by $10^?$ miles. (?) = ____

6.6 Powers of two. (Eighth Friday) If an exponent of the base 2 is decreased then the expression decreases in value:

$2^2 = 4$

$2^1 = 2$

$2^0 = 1$

$2^{-1} = 1/2 = .5$

$2^{-2} = 1/4 = .25$

$2^{-3} = 1/8 = .125$

$2^{-4} = 1/16 = .0625$

In section 5.2 Farmer Jones continued doubling the number of grains of wheat on the squares of his checker board until he found himself with some very large numbers. This section shows how numbers may be very small using negative exponents of two.

Farmer Jones finishes reading his county newspaper; fascinated by the wheat problem and its doubling, he decides to do just the opposite by cutting the newspaper in half (½).

He takes a half of the paper and cuts it in half keeping a fourth (¼), he takes this fourth of the paper and cuts it in half keeping an eighth, etc. He finally has a piece of paper so small that he is unable to cut it in half with ordinary scissors. How many cuts did he make before the final piece of cut-up newspaper was too small to continue cutting?

The newspaper Farmer Jones had was 22" by 16"; 352 sq. in. in area. The scheme is as follows:

After no cut $2^0(352) = 1(352) = 352$ sq. in. is left of the paper.

After the first cut $2^{-1}(352) = (1/2)(352) = 176$ sq. in. is left of the paper.

After the second cut $2^{-2}(352) = (1/4)(352) = 88$ sq. in. is left of the paper.

After the third cut $2^{-3}(352) = (1/8)(352) = 44$ sq. in. is left of the paper.

After the fourth cut $2^{-4}(352) = (1/16)(352) = 22$ sq. in. is left of the paper.

After the "n"th cut $2^{-n}(352) = (1/2^n)(352)$ sq. in. is what is left of the paper.

The formula for the remainder of the newspaper after "n" cuts is $(2^{-n})(352$ sq. in.$)$. If Farmer Jones finally gave up cutting after the 20th time, the area would be .0003366160 sq. in. (about the size of a period "."). By substituting 20 for n; the formula yields $(1/2^{20})$(352 sq. in.) = (1/1,048,576)(352 sq. in.) = .000000955(352) = .0003366160 sq. in. This is the area of a square .01752 inches on a side.

Of course the period at the end of this sentence would be a giant compared to the cross section of a water molecule (.0000000000000001068 or 1.068×10^{-16} sq. in. on a two-dimensional surface). The period would cover about 3,160,000,000,000 (3.16×10^{12}) times the area of the water molecule.

If Farmer Jones were able to cut the newspaper by halves 90 times he would find the remainder of the paper about the size of a water molecule. The water molecule is a giant compared to the hydrogen nucleus. Atoms may be regarded as miniature solar systems in which the electrons play the role of the planets and the nucleus corresponds to the sun.[4]

It is difficult to compare units of matter. The following size scheme is such an attempt where the next object is about 250,000 times larger in diameter than the preceding one:[5]

> Nucleus--Atom--A speck of dust--An average home--The earth's sphere--The solar system--The distance to the Polar Stars--The Milky Way Galaxy.

---

[4]Baskt, p. 77.

[5]*Ibid.*, p. 78.

This same scheme could be set up as a sequence of proportions: The electron is to the atom as the atom is to a speck of dust; the atom is to a speck of dust as the speck of dust is to an average home; etc.

6.7 <u>Problems on the powers of two.</u> (Eighth Friday)

(1) Which is larger in value? (a) 2^{2^2} (b) 222
(c) 2/22 (d) 2^{22} (e) 22 X 2

(2) Which is larger in value? (a) 2,222 (b) 222^2
(c) 2^{22^2} (d) $2^{2^{22}}$

(3) Write the largest number using five two's.

(4) Farmer Jones newspaper is about .0082 in. thick. He cuts the paper in half and puts one pile on top the other to obtain a stack of paper twice as thick (.0164 in.); he then cuts these two pieces in half and stacks the four pieces together making a stack .038 in. thick. If Farmer Jones were able to cut his paper in this manner 50 times, what would be the total thickness in miles of the pile of papers? Hint: 2^{50} = 1,125,899,906,842,624 and one mile = 63,360 inches.

(5) Which is smaller in value? (a) 2/22 . (b) $2/(2)^2$
(c) 2^{-22} (d) 2^{-2^2}

CHAPTER VII

THE NUMBER LINE

7.1 Introduction. The line contains an infinite number of points.[1] The number of points on a line segment (a line with definite length) is the continuum (C) and the number of elements in the set of all real numbers is also the continuum. With this reasoning a number line was developed to show a one-to-one correspondence of the points on a line with the elements in the set of real numbers.

A line that has its points labeled with numbers is called a number line.[2]

Some objectives of the use of the number line in the classroom are:

(1) To strengthen a person's mental image, thereby developing the ability to compute mentally.[3]

(2) To aid in determining the order of relative size of numbers.[4]

(3) To make the real numbers more meaningful.

(4) To make the four operations: addition, subtraction multiplication, and division more understandable.

---

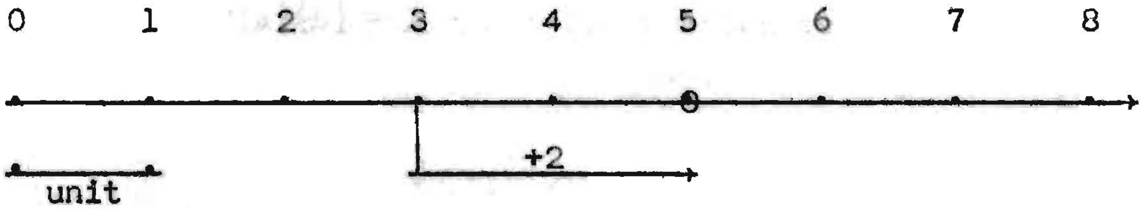
[1]Wayne Peterson, "A Case in Point," The Arithmetic Teacher, Vol. 8, no. 1, January 1961, p. 12.

[2]Stein, op. cit., p. 59S.

[3]Robert B. Ashlock, "The Number Line in the Primary Grades," The Arithmetic Teacher, Vol. 8, no. 2, February 1961, p. 75.

[4]Peterson, op. cit., p. 12.

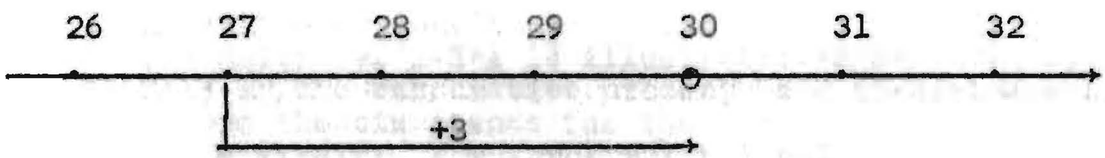
7.2 How to solve problems concerned with whole numbers. (Ninth Friday) To construct the whole numbers onto the number line, mark a starting point "0" on a straight line and establish a unit of length that will allow marking off the points corresponding to the whole numbers:



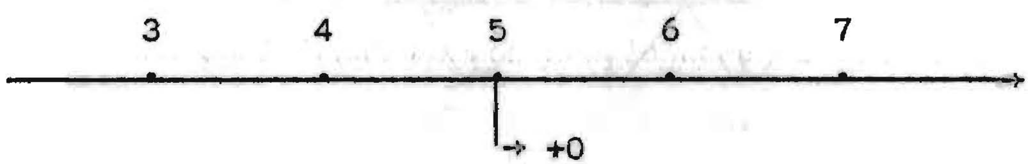
The method of adding two units (binary operation) is as follows:

(1) In the figure above add 3 and 2: Start with the point that corresponds to 3 (representing 3 units from 0) and add 2 units. The addition of 2 is shown by the arrow and the sum is seen to be 5. A circle is drawn around the answer.

(2) Add 27 and 3: Start with the point that corresponds to 27 (representing 27 units from 0) and add 3 units. The diagram below shows the answer to be 30:



(3) Add 5 and 0; start with the point that corresponds to 5 and add 0 units. The sum of course will show no change from the starting point 5:

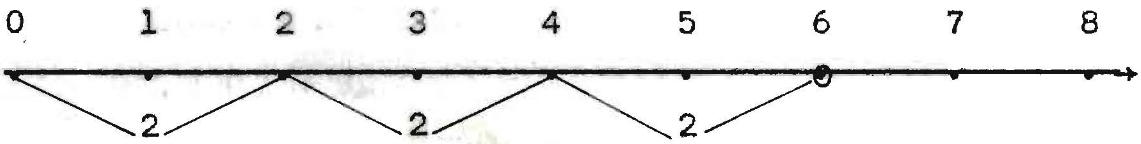


Before attempting to illustrate multiplication examples on the number line a definition must be given: (a)(b) means ab where a is the number of times b is added. Example: (3)(2) means 2 + 2 + 2. The sum of the three twos is six. In a general form:

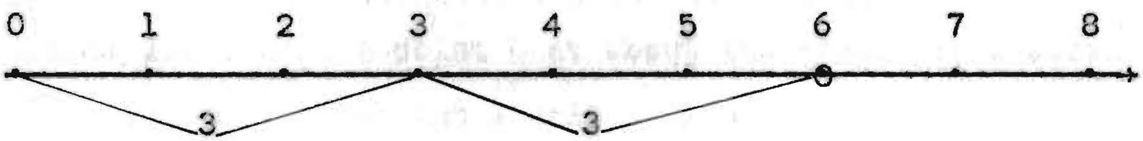
$$(a)(b) = \underbrace{b + b + b + b + \ldots + b}_{\text{a units of b}} = ab.$$

The method of multiplying two numbers is as follows:

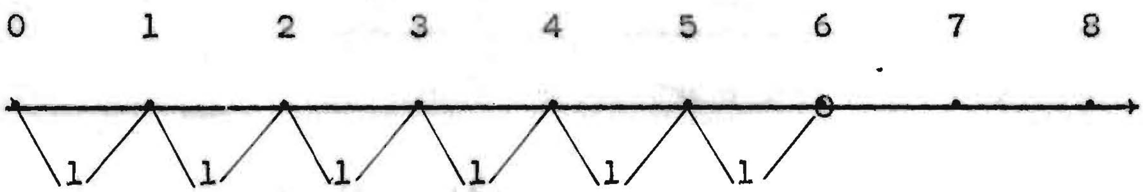
(1) Multiply, (3) X (2) is illustrated--the 3 stands for 3 steps, each step 2 units long:



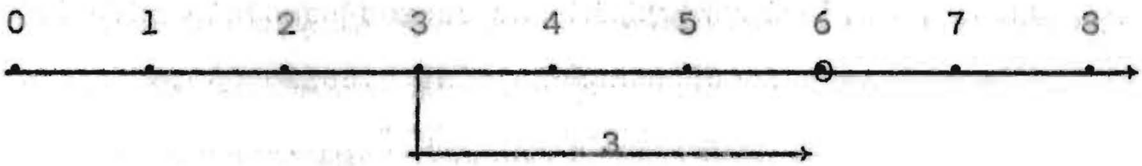
(2) Multiply, (2) X (3) is illustrated--the 2 stands for 2 steps, each step 3 units long:



(3) Multiply, 2•3•1 is illustrated--2•3•1 = (2•3)•1 = 2•(3•1) by the associative property and (2•3)•1 = 6•1. Therefore the six stands for the number of times 1 is added to itself: 1 + 1 + 1 + 1 + 1 + 1 = 6:



7.3 Problems over the whole numbers. (Ninth Friday) With a number line like the figure given below add 3 + 3 (example):



(1) Add 5 and 2.

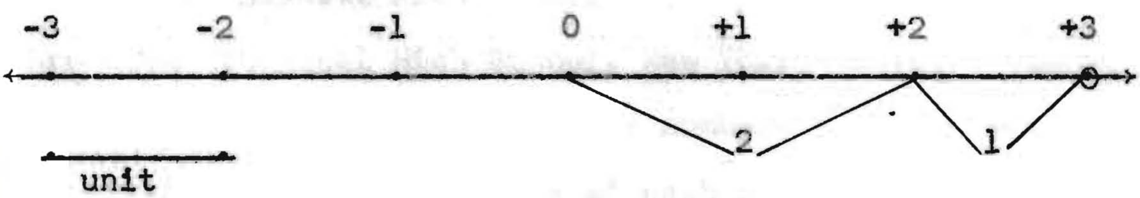
(2) Add 2 and 5. Is this the same answer as problem 1? Why?

(3) Multiply 2 X 4.

(4) Multiply 2 X 2 X 2.

(5) Add 2 + 1 + 3.

7.4 How to solve problems concerned with signed numbers. (Tenth Friday) To construct the integers onto the number line, mark a point 0 at about the middle of a straight line segment. Establish a unit of length that allows for marking off the points corresponding to the integers as in the following figure:

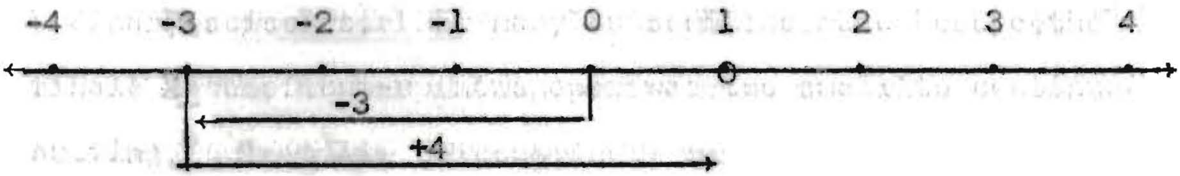


The following problems are illustrated:

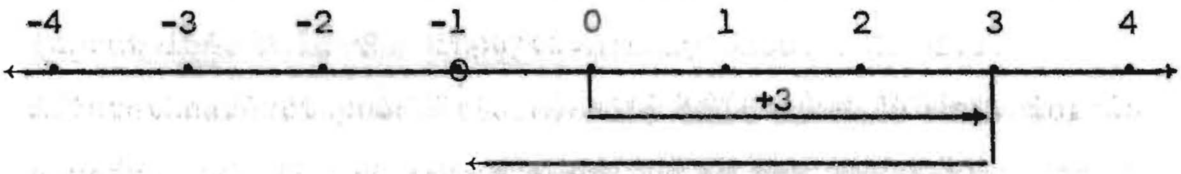
(1) On the figure given above, add 2 + 1 (if no sign is in front of a number (2) it is accepted as being positive); from the starting point "0" go 2 units to

the right, then go 1 unit again to the right. The answer is +3 representing 3 units to the right of 0 as the sum.

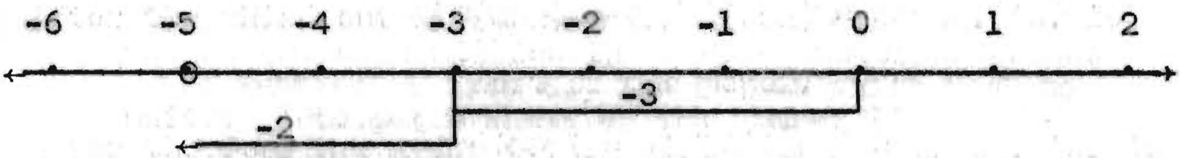
(2) Add (-3) + (+4): The -3 means the start is 3 units to the left of zero, then go 4 units to the right ending at +1 (1 unit to the right of 0). +1 is the sum:



(3) Add (+3) and (-4): The +3 means starting from 0, going 3 units to the right, and then to add the -4 going 4 units to the left ending at -1 (one unit to the left of 0):



(4) Add (-3) and (-2): The -3 means starting from 0 and going 3 units to the left, then the -2 going 2 more units to the left ending at -5 (5 units to the left of the starting point):



Dr. Barnett Rich, Chairman, Department of Mathematics, Brooklyn Technical High School, New York,[5] says: "Signed numbers are positive or negative numbers used to represent quantities that are opposite of each other." Signed numbers

---

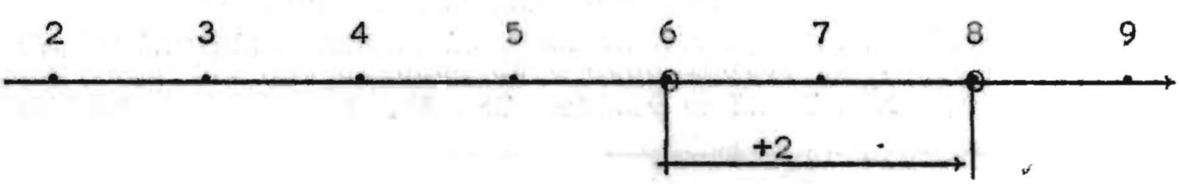
[5]Barnett Rich, Schaum's Principles and Problems of Elementary Algebra, Schaum Publishing Company, New York, 1960, p. 36.

expressing opposites may be used in other ways: "to the right of (+)" and "to the left of (-)," "above (+)" and "below (-)," "credit (+)"and "debit (-)," "true (+)" and "false (-)," etc.

The numerical value of a signed number (absolute value) is the number which remains when the sign is removed.[6] Example: Which of the two has the larger numerical value, +8 or -10? The answer is -10 as 10 is greater than (>) 8.

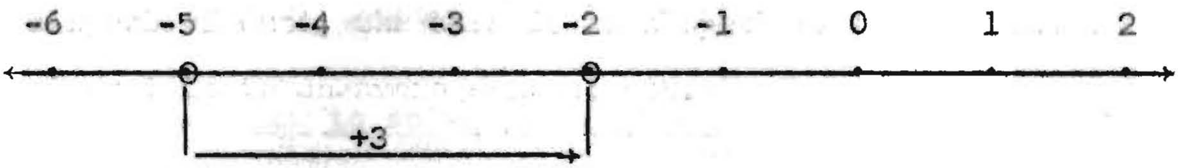
In the expression, "Subtract 6 from 8," 6 is the subtrahend, 8 is the minuend and the answer is the difference. To find an answer to this example with the number line, the starting point is at the subtrahend and the answer is the number that must be added to get the minuend--the direction from the subtrahend to the minuend determines the sign of the answer. Examples:

(1) Subtract 6 from 8. The number which must be added to 6 to get 8 shows on the number line to be 2 units to the right (+2). +2 is the answer:

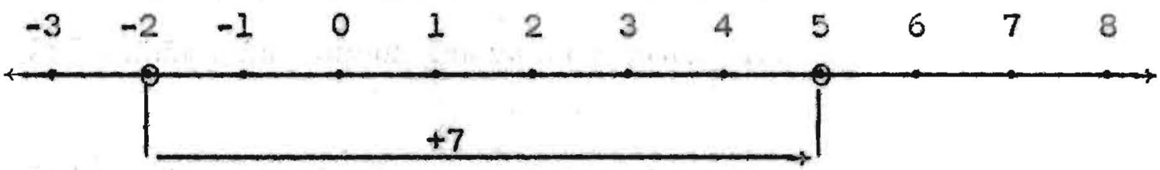


[6]Ibid., p. 36.

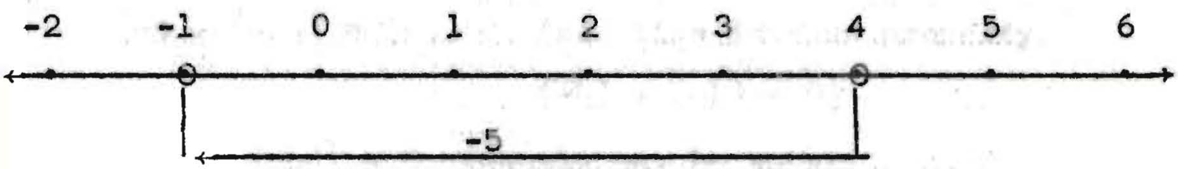
(2) Subtract -5 from -2. The number that must be added to -5 to get -2 on the number line shows 3 units traveled to the right (+3). <u>+3</u> is the answer:



(3) Subtract -2 from 5. The number added to -2 to get 5 on the number line shows 7 units traveled to the right (+7). <u>+7</u> is the answer:

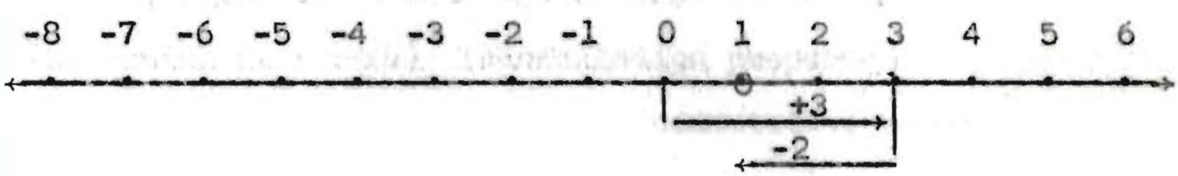


(4) Subtract 4 from -1. Starting from the subtrahend (4) and going left to -1 shows on the number line that the distance traveled is 5 units to the left (-5). <u>-5</u> is the answer:



7.5 <u>Problems on signed numbers</u>. (Tenth Friday)

With a number line like the figure given below add +3 and -2. Example:



(1) Add +1 and +2.

(2) Add +6 and -4.

(3) Subtract -3 from 5.

(4) Find the number when added to -2 will give -5.

(5) Add +3, +2, -4, -1, 0 and +3.

(6) Add +2, +2, and +2.

7.6 How to solve multiplication problems concerned with signed numbers. (Eleventh Friday) By definition in section 7.2, (a)(b) = ab. To multiply two positive signed numbers, multiply their absolute values and mark the product positive. What does (+a)(-b) equal, +ab or -ab? $0 \cdot x$ (x being an element of the integers) stands for no x's added together which is 0. $0 \cdot x = 0$. With this definition accepted then (+a)(-b) = -ab. The proof follows:

(+a)(-b) = ? (+ab or -ab)

(+a)[(+b + (-b)] = (+a)·0 (additive inverse property: (+b) + (-b) = 0)

(+a)[(+b) + (-b)] = 0 (equals may be substituted for equals)

(+a)(+b) + (+a)(-b) = 0 (distributive property: a(b + c) = ab + ac))

(+ab) + (+a)(-b) = 0 (Definition: (+a)(+b) = +ab)

(+ab) + (-ab) = 0 (additive inverse property)

then (+a)(-b) = (-ab) (cancellation property)

(+a)(-b) = (-b)(+a) = -ab (commutative property: $x \cdot y = y \cdot x$)

Thus the rule: To multiply two signed numbers with unlike signs, multiply their absolute values and mark the product negative. What does (-a)(-b) equal, +ab or -ab? If the theorem (+a)(-b) = -ab is accepted, then (-a)(-b) equals +ab. The proof follows:

(-a)(-b) = ? (ab or -ab)

(-a)[(+b) + (-b)] = (-a)(0) (additive inverse property: (+b) + (-b) = 0)

(-a)(+b) + (-a)(-b) = 0 (distributive property)

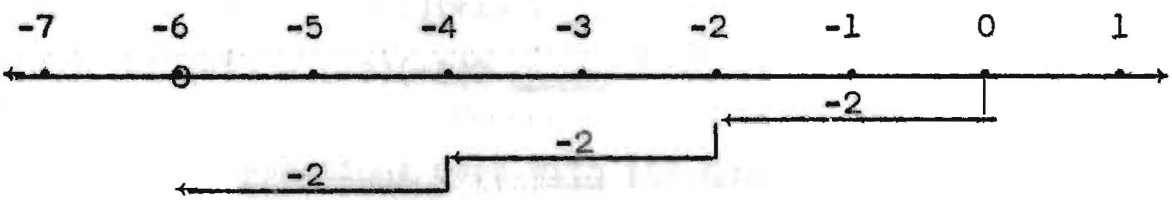
(-ab) + (-a)(-b) = 0 ((-a)(+b) = -ab)

(-ab) + (+ab) = 0 (additive inverse property)

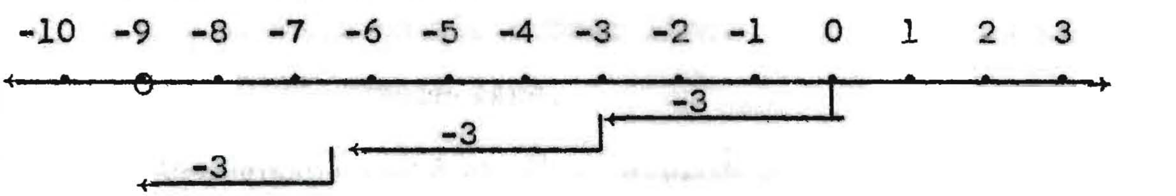
then (-a)(-b) = (+ab) (cancellation property)

Thus the rule: To multiply two negative-signed numbers, multiply their absolute values and mark the product positive. The method of multiplying two numbers on the number line follows:

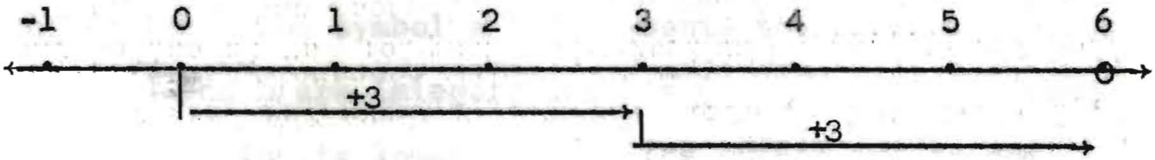
(1) Solve (+3)(-2). The 3 stands for the number of times the -2 is to be added. (-2) + (-2) + (-2) = -6:



(2) Solve (-3)(+3). By the commutative property, (-3)(+3) = (+3)(-3), letting the +3 represent the number of times the -3 is to be added. (-3) + (-3) + (-3) = -9:



(3) Solve (-2)(-3). By the proof presented above, (-2)(-3) = +2·3. The +2 stands for the number of times +3 is to be added. (+3) + (+3) = +6:



7.7 <u>Problems over multiplying signed numbers</u>. (Eleventh Friday) With a number line solve the following problems:

(1) (-2)(+3)

(2) (+1)(-4)

(3) (0)(+4)

(4) (+1)(-2)(-3)

Answer the following problems with a true or false.

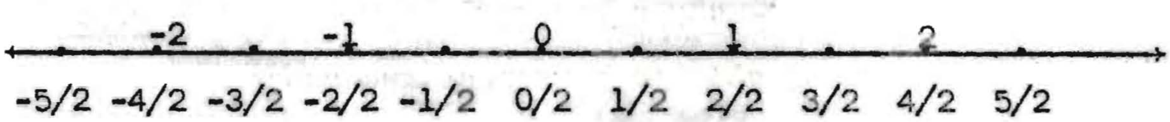
(5) (-2)(+3) = (+3)(-2)? ____

(6) (+1)(-1)(-1) = (-1)(-1)(-1)? ____

(7) (-1)(+40) = (-40)(+1) ? ____

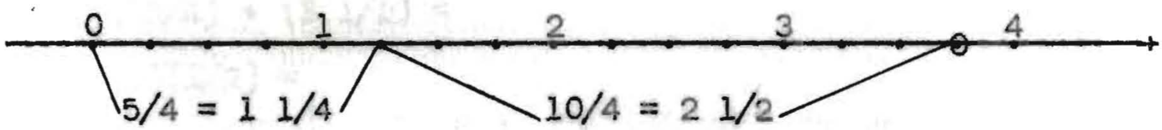
(8) (+1)(+2) = (-2)(-1)? ____

7.8 <u>Fractions used with the number line</u>. (Twelfth Friday) Since the interval between each whole number may be divided into halves, thirds, fourths, etc., the fractions may be represented on the number line. The halves are shown on the following number line.

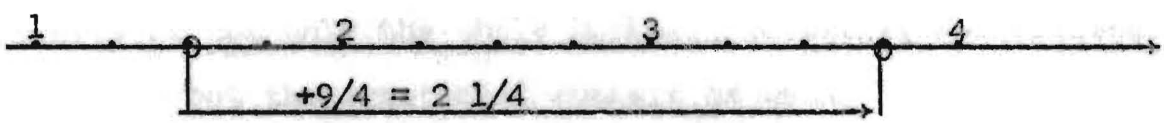


In the set of rational numbers many different fractions may represent the same number. To avoid confusion: The symbol a/b represents the rational number where a and b are integers and b ≠ 0. a/b is considered to be reduced to its lowest term. The number 2/4 is not reduced to its lowest term but its <u>representative</u>, 1/2, is. The following are problems dealing with fractions:

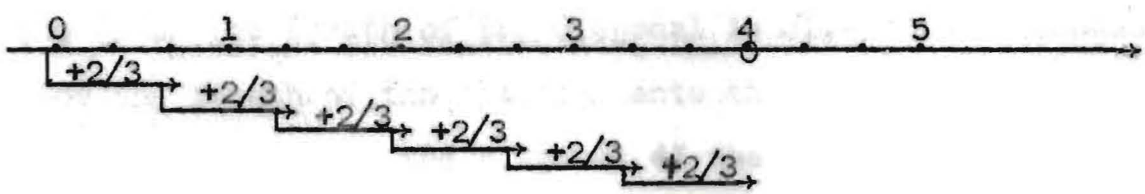
(1) Add 1 1/4 + 2 1/2. 1 1/4 equals 5/4 and 2 1/2 equals 5/2. The common denominator of 5/4 and 5/2 is 4 therefore 5/2 is changed to 10/4. The number line should be marked to represent the fourths; then 5/4 + 10/4 on the number line will equal 15/4 or 3 3/4:



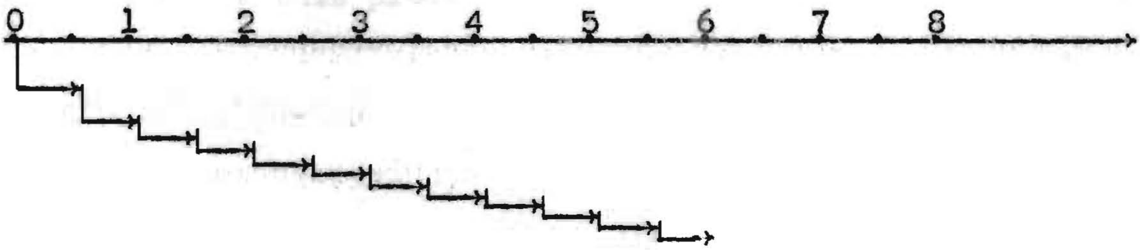
(2) Subtract 1 1/2 from 3 3/4. 1 1/2 equals 3/2 and 3 3/4 equals 15/4. The comon denominator of 3/2 and 15/4 is 4 therefore 3/2 is changed to 6/4. The number line should be marked to represent the fourths; then the number added to 6/4 to get 15/4 on the number line will be 9/4 or 2 1/4:



(3) Multiply 6 X (2/3). Since the common denominator is 3 the number line should be marked to represent the thirds. The 6 means 2/3 is to be added 6 times. 2/3 + 2/3 + 2/3 + ... + 2/3 = 12/3 = 4:



(4) Divide 6 by 1/2. The problem asks how many half units does it take to make 6 units. The number line shows 12 steps of one-half unit each. 12 is the answer:



7.9 <u>Problems over fractions</u>. (Twelfth Friday) With a number line solve the following problems:

(1) (3 1/2) + (-1/4) =

(2) (-3/4) + (3 1/2) =

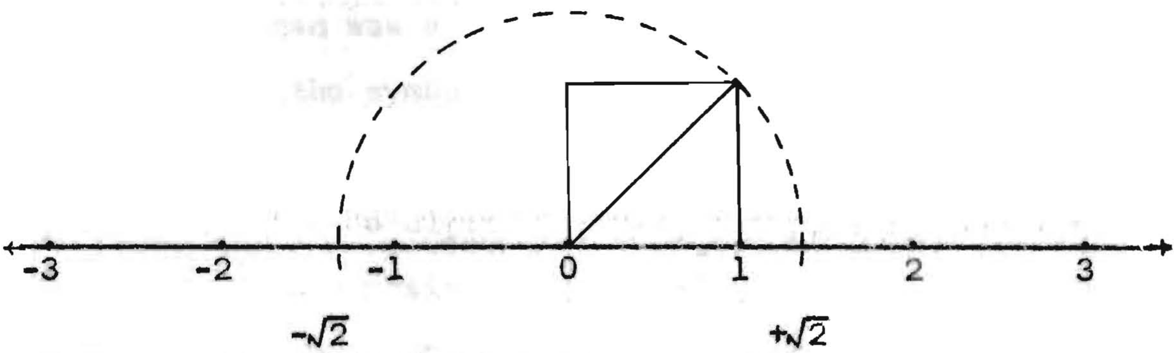
(3) (3)(1/2) =

(4) 4 ÷ (1/2) =

(5) (1/2) ÷ (1/4) =

7.10 <u>Irrational numbers used with the number line</u>. (Thirteenth Friday) The number line may be used to represent not only the whole numbers, integers, and rational numbers; but the irrational numbers as well. Example:

To locate a point on the number line that corresponds to the irrational number $\sqrt{2}$, construct on the number line a square with the side measuring the unit length one (1). The length of the diagonal is $\sqrt{2}$. With a compass copy the length of the diagonal onto the number line. Use 0 as the center and the diagonal of the square as the radius to swing an arc with the compass onto the number line as shown in the figure below. Two points, $+\sqrt{2}$ and $-\sqrt{2}$ may be located by this procedure.



The number line is just partly seen because the line is endless. Numbers can be compared according to size by the farther to the right a number is on the number line, the larger it is.[7]

Decimal fractions (tenths, hundredths, and even thousandths) may be represented on the number line.

> Though one of the least expensive yet easily made visual aids, the number line is one of the most useful in teaching primary arithmetic. By using it properly, we will increase the effectiveness of our teaching in the primary grades--<u>and in the levels which follow</u>.[8]

---

[7]Stein, op. cit., p. 59S.

[8]Ashlock, op. cit., p. 76.

CHAPTER VIII

SYSTEMS OF NUMERATION

8.1 *Introduction*. Probably the greatest invention ever made by man was a numeration system. Although it is simple to use the symbol "25" to represent the how many of twenty-five objects, it took man centuries to build the system in which the digits 2 and 5, arranged in that order, stand for the numerosity of twenty-five things.[1] The symbols used to represent numbers are called *numerals*. The twenty-five objects were represented by the numeral "25." In other words a numeral is a name or symbol used to stand for numbers, a numeration system is formed. The numeration system used most in this country is the *decimal* system of notation. Other names for it are Hindu-Arabic system and place value system with *base ten*.

The numeration system used today was not the first one developed by man. Here are some of the ways that 25 has been written in the past:[2]

(1) Egyptian ∩∩||||| (∩ denoted 10 and | denoted 1),

(2) Babylonian <<VVVVV (< denoted 10 and V denoted 1),

---

[1]Donovan A. Johnson and William H. Glenn, *Understanding Numeration Systems*, Webster Publishing Company, St. Louis, 1960, p. 2.

[2]Bask, *op. cit.*, pp. 4-5.

(3) Roman XXV (X denotes 10 and V denotes 5),

(4) Chinese ═══ ||||| (═══ denotes 20 and | denotes 1).

The number of symbols in the system, their grouping, and the place value are necessary to all systems of notation.[3]

The base of a numeration system is the number it takes in any one place to make 1 in the next higher place.[4] Example: The base 3 (ternary system) uses three symbols; 0, 1, and 2. The number three is represented by the next higher order numeral "10." These symbols are called digits.

If a man would have had only eight fingers our system would probably have been an octonary system (base eight) instead of one with base ten.

If other numeration systems, besides our own, are taken into consideration and studied, they will help give meaning and a better understanding of our numerals, how our computation processes work, and how new systems are being used in science.[5]

8.2 Base Six--Senary System. (Fourteenth Friday) The five fingers on each hand of a person could be used as a very handy counting device to explain the senary system.

---

[3]Stein, op. cit., p. 13S. [4]Ibid.

[5]Johnson and Glenn, op. cit., p. 3.

Since there are two hands to each person, there will only be two positions. The fingers on the right hand represent the units; the fingers on the left hand represent the sixes. The senary system has six symbols (0, 1, 2, 3, 4, and 5) which the fingers represent. A person could count from zero to thirty-five as follows:

(1) Zero, is represented by holding up both palms showing no fingers.

(2) One, two, three, four, and five, are represented by the number of fingers showing on the right hand.

(3) The left hand, is used to represent the sixes. One finger on the left hand, no fingers showing on the right hand represents the number six.

(4) Seven, eight, nine, ten, or eleven are represented by one finger on the left hand showing, 6, plus the 1, 2, 3, 4, or 5 fingers, respectively, showing on the right hand.

(5) Twelve, thirteen, fourteen, fifteen, sixteen, and seventeen are represented by two fingers on the left hand showing, 12, plus 0, 1, 2, 3, 4, and 5 fingers, respectively, on the right hand.

(6) The rest of the numbers would be represented in the same manner until all five fingers on the right hand were open, 5, and all five fingers on the left hand were open, 5 X 6. This summation would be thirty-five, (5x6) + 5 = 30 + 5 = 35.

The senary number 10 means "one unit of the higher order" and corresponds to the decimal number 6.

Fifty elements would, in the senary system, be represented by the numeral "$122_{(6)}$." This can be seen by grouping the elements as follows:

xxxxxx xxxxxx xxxxxx xxxxxx xxxxxx xxxxxx	xxxxxx xxxxxx xx

The fifty elements (x's) were grouped in sixes. The boxed-in group represents six-sixes or thirty-six. Therefore the numeral $122_{(6)}$ means $[(1 \times 36) + (2 \times 6) + (2 \times 1)]_{(10)} = 36 + 12 + 2 = 50_{(10)}$.

The six digits, 0, 1, 2, 3, 4, and 5 in the first place to the left of the decimal point represent the units ($6^0 = 1$); the digits in the second place represent the number of sixes ($6^1 = 6$); the digits in the third place represent the number of thirty-sixes ($6^2 = 36$); the digits in the fourth place would represent the number of two hundred sixteens ($6^3 = 216$); and so on. The six digits in the first place to the right of the decimal point represent the number of one-sixths ($6^{-1} = 1/6$); the digits in the second place to the right of the decimal point represent the number of one thirty-sixths ($6^{-2} = 1/36$) and so on.

What base ten number is equivalent to $2145_{(6)}$? The base of the numeral 2145 is indicated by the subscript 6 written at the lower right of the numeral. If the numeral has no subscript, it is assumed to be a number belonging to the decimal system.

$2145_{(6)} = [(2 \times 216)+(1 \times 36)+(4 \times 6)+(5 \times 1)]_{(10)}$

$= 532 + 36 + 6 + 5$

$= 597_{(10)}$ is the answer.

2	x	216	=	532
1	x	36	=	36
4	x	6	=	6
5	x	1	=	5
				597

It becomes a little more difficult when changing a base ten numeral into a base six numeral. Two methods are shown as follows:

What is the numeral $672_{(10)}$ equivalent to in the base six system?

(1) Quotients method. (This will apply to all systems.) Divide the largest possible power of the base into the given number. Then divide the remainder by the next lower power of the base, continuing in this manner until the divisor is the base itself. The quotient of each division will give the digit for the corresponding position in the required base number. The final remainder will indicate the number of ones.[6]

$6^3 = 216$
$6^2 = 36$
$6^1 = 6$
$6^0 = 1$

$216\overline{)672} = 3$
$\underline{648}$
$36\overline{)\ 24} = 0$
$\underline{0}$
$6\overline{)\ 24} = 4$
$\underline{24}$
$1\overline{)\ 0} = 0$

The answer is $3040_{(6)}$.

(2) Remainder method. (This will apply to all systems.) Divide the base into the given number, then divide the base into the quotient, then divide the base into the new quotient, continuing until the quotient is zero. The remainders in these divisions will give the required digits with the final remainder used as the digit for the greatest place value and the first remainder as the digit for the ones' place.[7]

---

[6] Stein, op. cit., p. 148.

[7] Ibid.

$6\overline{)672} = 112$ with a remainder of 0

$6\overline{)112} = 18$ with a remainder of 4

$6\overline{)18} = 3$ with a remainder of 0

$6\overline{)3} = 0$ with a remainder of 3

The answer is $3040_{(6)}$.

The method of changing a numeral in the base six to another base system (other than the base ten) requires two different steps. Again this method will apply to all systems. Change the given base numeral to a base ten numeral. Then change the base ten numeral to the required base numeral. The two steps are illustrated as follows:

(1) Change $2513_{(6)}$ to a base nine numeral.

$$2513_{(6)} = [(2x216) + (5x36) + (1x6) + (3x1)]$$
$$= 432 + 180 + 6 + 3$$
$$= 621_{(10)}$$

$$\begin{array}{r} 2 \times 216 = 432 \\ 5 \times 36 = 180 \\ 1 \times 6 = 6 \\ 3 \times 1 = \underline{3} \\ 621 \end{array}$$

(2) Change $621_{(10)}$ to a base nine numeral.

(a) Quotients method:

$9^2 = 81$
$9^1 = 9$
$9^0 = 1$

$$\begin{array}{r} 81\overline{)621} = 7 \\ \underline{567} \\ 9\overline{)54} = 6 \\ \underline{54} \\ 1\overline{)0} = 0 \end{array}$$

The answer is $760_{(9)}$

(b) Remainder method:

$9\overline{)621} = 69$ remainder 0

$9\overline{)69} = 7$ remainder 6

$9\overline{)7} = 0$ remainder 7

The answer is $760_{(9)}$.

8.3 Problems over the Senary System. (Fifteenth Friday)

(1) Write the first forty base six numerals beginning with 0.

(2) Fill in the unfilled spaces for the tables below for addition and multiplication.

+	0	1	2	3	4	5
0		1	2			5
1					5	10
2					10	
3		4	5	10	11	
4	4		10			
5	5	10	11			14

X	0	1	2	3	4	5
0	0	0		0	0	
1		1	2			5
2			4			
3		3	10	13		
4	0				24	32
5	0	5	14			44

(3) Change $1325_{(6)}$ to a base ten numeral.

(4) Change $3962_{(10)}$ to a base six numeral.

(5) Change $4132_{(6)}$ to a base four numeral.

8.4 Base Five--Quinary System. (Fifteenth Friday)

Many times the quinary system is used by students, teachers, housewives, and even salesmen, without their realizing it.

John and Mary ran for president of the mathematics club. Jack, a fellow student, kept track of the votes as the ballots were read by the sponsor. The blackboard Jack had kept score on, after all the ballots were read, looked something like this:

John 𝍸 𝍸 ||| Mary 𝍸 ||

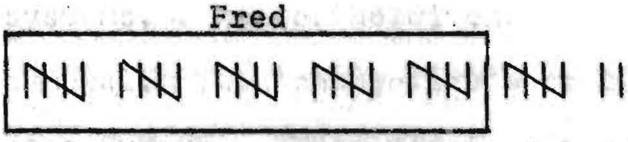
Both John and Mary's tallies were grouped by fives. John had ((2x5) + (3x1)) = 13 votes to Mary's ((1x5) +

(2x1)) = 7 votes. The numbers 13 and 7 could have been represented by the numerals $23_{(5)}$ and $12_{(5)}$.

To show an example of bigger numbers in the base five; Bill ran against Fred for student council president. The final tally sheet showed:

Bill

Fred



Bill won the election 36 votes to Fred's 32. The symbols used in the base five system are 0, 1, 2, 3, and 4. The tallies grouped in fives are too many to be represented by a two digit numeral in the base five, therefore, a numeral of higher order is needed to represent five. In section 8.2 it was shown that the groups could be grouped; five of the seven groups of fives tallied for Bill are boxed in and five of the six groups of fives tallied for Fred are boxed in representing $100_{(5)}$ or 25 = five-fives. $10_{(5)}$ is the next lower order representing five. The final results therefore could be read, "Bill won with $121_{(5)}$ votes to Fred's $112_{(5)}$ votes."

$[(1x25) + (2x5) +(1)] = 36$ $[(1x25) + (1x5) + (2)] = 32$

Addition and subtraction in numeration systems such as the base five follows.

(1) Add $2432_{(5)}$, $1433_{(5)}$, and $4324_{(5)}$.

$$\begin{array}{rrrrr} & (2) & (1) & (1) & \\ & 2 & 4 & 3 & 2 \\ & 1 & 4 & 3 & 3 \\ & 4 & 3 & 2 & 4 \\ \hline 1 & 4 & 2 & 4 & 4 \end{array}$$

In the one's column $2 + 3 + 4 = 14_{(5)}$. The $14_{(5)}$ must be used as adding is done in one system only. Write the 4, and carry the 1.

In the five's column 1 + 3 = 4; 4 + 3 = 12; 12 + 2 = 14. Write 4, and carry the 1.

In the twenty-five's column 1 + 4 = 10; 10 + 4 = 14; 14 + 3 = 22. Write the 2, and carry a 2.

In the one hundred twenty-five's column 2 + 2 = 4; 4 + 1 = 10; 10 + 4 = 14. Write 14.

The answer is $14244_{(5)}$.

(2) Subtract $432_{(5)}$ from $2233_{(5)}$.

$$\begin{array}{rrrr} & (5) & (5) & (5) \\ 1 & 1 & 2 & \\ 2 & 2 & 3 & 3 \\ - & 4 & 3 & 4 \\ \hline 1 & 2 & 4 & 4 \end{array}$$

In the one's column 4 cannot be subtracted from the 3. One of the five's is borrowed from the top of the second column. (5 + 3) - 4 = 4. Write the 4. In the five's column 3 cannot be subtracted from the remaining 2. One of the twenty-five's is borrowed from the next column. Twenty-five is equal to 5 five's. 5 five's + 2 five's - 3 five's = 4 five's. Write a 4.

In the twenty-five's column 4 cannot be subtracted from 1. One of the one hundred twenty-five's is borrowed from the next column. One hundred twenty-five is equal to 5 twenty-five's. 5 twenty-five's + 1 twenty-five - 4 twenty-five's = 2 twenty-five's. Write the 2 and bring down the remaining 1 in the one hundred and twenty-five's column.

The answer is $1244_{(5)}$.

8.5 Problems over the Quinary System. (Fifteenth Friday)

(1) Write the first fifty base five numerals beginning with 0.

(2) Fill in numbers for the multiplication and addition tables below.

+	0	1	2	3	4
0					
1					
2					
3					
4					

X	0	1	2	3	4
0					
1					
2					
3					
4					

(3) Add $1034_{(5)}$, $342_{(5)}$, and $441_{(5)}$.

(4) Subtract $1034_{(5)}$ from $31422_{(5)}$.

(5) What is the numeral $444_{(5)}$ in the base ten?

(6) What is the numeral $324_{(5)}$ in the base four?

8.6 Base Four--Quaternary System. (Sixteenth Friday)

An eccentric old mathematician, when he died, left some unpublished papers behind. His friends sorting some of them out came across the following statement:[8]

[8] Bakst, op. cit., p. 9.

I graduated from high school when I was 101 years old. Four years later I, 111 years old, graduated from college. Three years later, I, a 120 year old man, married a young, 103 year old, girl. Since the difference in our ages was only 11 years, we had many common interests and hopes. One year later my wife, now 110 years of age, had her first child; later we had 11 children, 3 boys and 2 girls. I had a job teaching mathematics in college and my salary was only $21,130 a month, just barely enough to support my family.

Was the mathematician a mental case? Let's see:

Only the symbols 0, 1, 2, and 3 were used to represent numbers. Moreover, when 2 was added to 3 the answer was 11. The system used by the dead mathematician was of another system of numeration than the decimal system. In the decimal system when 1 is added to 9 the answer is 10. In the system used by the mathematician 3 did the work of 9, that is 1 + 3 = 10. Just as 10 (the next higher order) is not a digit of the decimal system, 4 is not a digit of the mathematician's system, four being written as 10.

The mathematician therefore was using the quaternary system. The first position to the left of the decimal point is the units place, the next place on the left is the four's position, then comes the sixteen's position, then the sixty-four's position, then the two hundred fifty-six's position, and so on.

Applying the base four numeral to the autobiography of the old mathematician:

$$101_{(4)} = 1x16 + 0 + 1 = 17_{(10)}$$

$$10_{(4)} = 1x4 + 0 = 4_{(10)}$$

$$111_{(4)} = 1x16 + 1x4 + 1 = 21_{(10)}$$

$$120_{(4)} = 1x16 + 2x4 + 0 = 24_{(10)}$$

$$103_{(4)} = 1x16 + 0 + 3 = 19_{(10)}$$

$$11_{(4)} = 1x4 + 1 = 5_{(10)}$$

$$110_{(4)} = 1x16 + 1x4 + 0 = 20_{(10)}$$

$$21130_{(4)} = 2x256 + 1x64 + 1x16 + 3x4 + 0 = 604_{(10)}.$$

Thus the puzzle may be translated as follows:

I graduated from high school when I was 17 years old. Four years later I, 21 years old, graduated from college. Three years later, I, a 24 year old man, married a young, 19 year old, girl. Since the difference in our ages was only 5 years, we had many common interests and hopes. One year later my wife, now 20 years of age, had her first child; later we had 5 children, 3 boys and 2 girls. I had a job teaching mathematics in college and my salary was only $604 a month, just barely enough to support my family.

Multiplication and division in the numeration systems with the base four follows:

(1) Multiply $132_{(4)}$ by $201_{(4)}$.

```
  132          132 x 1 = 132
 x201          132 x 0 = 000
  132          132 x 2 = 132 + 132 = 330
 000
330
33132
```

The answer is $33{,}132_{(4)}$.

(2) Divide $1323_{(4)}$ by $31_{(4)}$.

```
     213
31)13323
   122
    112
     31
     213
     213
       0
```

$133_{(4)}$ divided by $31_{(4)}$ is 2 with a remainder of $11_{(4)}$. Write the 2 in the quotient. $2_{(4)} \times 31_{(4)} = 122_{(4)}$. $122_{(4)}$ divided by $31_{(4)}$ is 1 with a remainder of $21_{(4)}$. Write the 1 in the quotient. Bring the three down with the 21 and divide the $213_{(4)}$ by $31_{(4)}$ which is equal to 3. Write the 3 in the quotient. The answer is $213_{(4)}$.

8.7 Problems over the Quaternary System. (Sixteenth (Friday)

(1) Add $3213_{(4)}$, $3321_{(4)}$, and $223_{(4)}$.

(2) Subtract $23_{(4)}$ from $210_{(4)}$.

(3) Multiply $31_{(4)}$ by $133_{(4)}$.

(4) Divide $11330_{(4)}$ by $210_{(4)}$.

(5) Change the number $3221_{(4)}$ to the base ten.

(6) Change the number $212_{(4)}$ to the base two.

(7) Count from 0 to 50, inclusive, using the quaternary system.

8.8 Base Two--Binary System. (Seventeenth Friday) One of the simplest and most important numeration systems is the base two.[9] It uses only the two digits 0 and 1.

To understand any numeration system, one must be able to count. It also is necessary to know how to add or multiply any two digits together.[10] This is one of the main reasons why the binary system is simple to learn; it has only four entries each to learn in the multiplication and addition tables. These are shown in figure 8.80.

+	0	1
0	0	1
1	1	$10_{(2)}$

X	0	1
0	0	0
1	0	1

FIGURE 8.80

BASE TWO, MULTIPLICATION AND ADDITION TABLES

Every integer can be written as a binary integer. Thus the binary 0 corresponds to the decimal 0, and the binary unit 1 corresponds to the decimal unit. The binary 10 means "one unit of higher order" and corresponds to the decimal number two (not ten). The binary numeral 100 means $1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 1 \times 4 + 0 + 0 = 4$.

---

[9] Richard V. Andree, Selections from Modern Abstract Algebra, Henry Holt & Company, New York, June 1959, p. 32.

[10] John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson, Introduction to Finite Mathematics, Prentice-Hall, Inc., 1957, p. 70.

In general, if $b_n b_{n-1} b_{n-2} \ldots b_2 b_1 b_0$ is a binary numeral, where each digit, b_0, b_1, b_2, ..., is either 0 or 1, then the corresponding decimal integer, I, is given the formula:[11]

$$I = b_n \cdot 2^n + b_{n-1} \cdot 2^{n-1} + \ldots + b_2 \cdot 2^2 + b_1 \cdot 2^1 + b_0.$$

Thus the binary numeral 10101 corresponds to $2^4 + 2^2 + 2^0 = 16 + 4 + 1 = 21_{(10)}$.

In this chapter ideas concerning how to add, subtract, multiply, divide, and changing a number from a numeral in one base to another, have been given. Now some ideas for basinal fractions are given:

(1) What is the numeral $111.111_{(2)}$ in the base ten?

$((1x4) + (1x2) + 1 + (1x1/2) + (1x1/4) +$
$1x(1/8)) = 4 + 2 + 1 + 1/2 + 1/4 + 1/8 =$
$4 + 2 + 1 + 4/8 + 2/8 + 1/8 =$
7 7/8 or 7.875.

Answer: 7 7/8 or 7.875.

(2) Find the numeral $.6_{(10)}$ in the base two.

.6 (1's) = 1.2 (1/2's) represented by .1 and
.2 1/2's remainder,

.2 (1/2's)= .4 (1/4's) represent .00,

.4 (1/4's) = .8 (1/8's) represent .000,

.8 (1/8's) = 1.6 (1/16's) represent .0001,

---

[11] *Ibid*., p. 72.

.6 (1/16's) = 1.2 (1/32's) represent .00001, and

.2 (1/32's) = .4 (1/64's) represent .000000.

Answer: $.100110..._{(2)}$.

A game, using the binary system, that intrigues both young and old is as follows:

A (1)	B (2)	C (4)	D (8)
1	2	4	8
3	3	5	9
5	6	6	10
7	7	7	11
9	10	12	12
11	11	13	13
13	14	14	14
15	15	15	15

FIGURE 8.81

CHART FOR A GAME USING THE BASE TWO

Pick any number from 1 to 15, inclusive. Find the columns in which the given number appears in figure 8.81; add the numbers corresponding to each of the columns and the sum will be the given number. Example: The number 14 appears in columns B, C, and D; add the corresponding numbers to each of the columns; 2 + 4 + 8 = 14, the given number.

The game uses the numbers 1, 2, 4, and 8. The numbers 1 to 15, inclusive, may be formed as summation of one or more of these given numbers 1, 2, 4, and 8. The number 1 is listed in column A; the number 2 is listed in column B; the number 3 is listed by columns B and A; and so on. If the words "yes" and "no" correspond to digits 1 and 0; each arrangement of the numbers in figure 8.81 match the arrangement of the digits in the binary scale.[12] By taking the rows backward, say yes, 1, or no, 0, if the number appears or not in the column, the numeral in the decimal system has been changed into the binary system. Example: Is 13 in column D? Yes. Is 13 in column C? Yes. Is 13 in column B? No. Is 13 in Column A? Yes, is the answer. The words "yes, yes, no, yes" corresponds to the digits 1101 which is the numeral for 13 in the base two.

8.9 Problems over the Binary System. (Seventeenth Friday)

(1) Write the first twenty base two numerals beginning with 0.

(2) What symbols are used in the binary system?

(3) Perform the indicated operations: (a) $1 + 1 + 1 = ?$ (b) $1 + 10_{(2)} + 111_{(2)} = ?$ (c) $1 - 1 = ?$

(4) Multiply: $11101_{(2)} \times 111_{(2)} = ?$

(5) Divide: $111_{(2)} \overline{)111111_{(2)}} = ?$

---

[12] Irving Adler, Magic House of Numbers, A Signet Key Book, The New American Library, New York, 1960, p. 77.

(6) Divide: $101_{(2)}\overline{)100011_{(2)}}$

(7) Change $100111011_{(2)}$ to the base twelve system. (Hint: the base twelve system, duodecimal system, uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, and e.)

CHAPTER IX

SUMMARY AND CONCLUSION

9.1 Summary. The junior high school mathematics teacher's job is a difficult one. His students range in achievement from primary level through senior high level; in some cases even above senior high level. To keep his students interested the teacher must make his subject interesting and he must have materials at hand for all of these achievement levels.[1] Constantly the teacher is seeking new ways to "add spice" to those necessary phases of learning needed to develop skill and competency in using mathematical ideas.[2]

October 4, 1957, the Russians launched the first earth satellite. This is a date long to be remembered by every person engaged in operating the American educational system. That day marks a turning point in the attitude of the great body of citizens toward their schools. The mathematics and science teachers stand in the spotlight.[3]

---

[1] Margaret F. Willerding, "Stimulating Interest in Junior High Mathematics," The Mathematics Teacher, Vol. LII, No. 3, March, 1959, p. 197.

[2] E. Glenadine Gibb, "As We Read," editorial, The Arithmetic Teacher, Vol. 8, No. 5, May, 1961, p. 209.

[3] Ray C. Maul, "Lets Look at the New Mathematics and Science Teacher," The Mathematics Teacher, Vol. LI, No. 7, November, 1958, p. 531.

From all this turmoil the mathematics teacher has a chance to benefit--The results of sober thinking are beginning to emerge; financial aid by the government is helping the teacher to receive more and better education in the field of mathematics, and he has an opportunity to co-ordinate more and closely the elementary with the secondary schools and so on through the institutions of higher education.[4]

The material contained in this thesis tries to point out a need for a change in the traditional mathematics program and the supplementary material is intended to be of value to the teacher of junior high school mathematics who has had little or no formal training in modern mathematics.

9.2 Conclusion. Most traditional junior high school mathematics teachers are looking for a sunrise in this mathematical turmoil. This thesis is an attempt to bring some of that light to them.

---

[4] Ibid.

CHAPTER X

KEY TO THE PROBLEMS

Section 4.3 (First Friday)

(1) 7, 157, 5.

(2) -5, -81, 11.

(3) -40, -7, -62.

(4) 0, 53.

(5) .632, -4, 3/8, 1, 52, -189.

(6) 23.

(7) 3/7, -.3, -7, 0, -187, .897.

(8) .999..., -5/7, 0, $3 + 7\sqrt{-5}$, -2, 1,000, -56.

(9) -9.

(10) (1, 2, 3, 4, 5, 6, 7, 8, 9)

(11) (-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

(12) 14, 544, 2000, 19, 100.

(13) 5+4, 81, 18/2, 3X3, IX, etc.

Section 4.5 (Second Friday)

(1) 'a' 9, 'b' 15, 'c' -5.

(2) a, b, c.

(3) b, c.

(4) a, b, c.

(5) c.

(6) 'a' no, 'b' no, 'c' no.

(7) 'a' yes, 'b' yes, 'c' no.

(8) commutative and associative properties.

(9) not necessarily.

(10) zero.

(11) reciprocal.

(12) 'a' -14, 'b' +5, 'c' -2, 'd' +1/4, 'e' -.75, 'f' + .3333....

(13) 'a' 1/14, 'b' -1/5, 'c' 5/4, 'd' -8/7, 'e' 8, 'f' 1.

(14) 'a' subtraction, 'b' addition, 'c' division.

(15) 'a' no, 'b' no, 'c' yes.

(16) cannot divide by zero.

(17) $3/9 \times 1 = \frac{3 \times 1/3}{9 \times 1/3} = 1/3$.

(18) $\frac{1/2}{1/4} \times 1 = \frac{1/2 \times 4}{1/4 \times 4} = 2/1 = 2$

Section 5.3 (Third Friday)

(1) 2^{17} = \$1,310.72.

(2) $2^{18} - 1$ = \$2,621.43.

(3) 2^{30} = \$5,368,709.12.

(4) $2^{30}-1$ = \$10,737,418.23.

(5) $2^{100}-1$ = \$12,676,509,002, 282,294,014,967,032,053.75.

(6) \$2,676,506,002,282,294, 014,967,032,053.75.

Section 5.5 (Fourth Friday)

(1) about 2,200#.

(2) 2,700 years.

(3) 31,250 feet or 5.92 miles.

(4) 50,000 days or 137 years.

Section 6.3 (Sixth Friday)

(1) 8.

(2) 5.

(3) -2.

Section 6.5 (Seventh Friday)

(1) 2.

(2) 13.

(3) 7.

(4) 6.

(5) -6.

(6) 3.67×10^{-2}.

(7) 3.6×10^4.

(8) 10^7.

(9) 10^{18}.

Section 6.7 (Eighth Friday)

(1) 2^{22}.

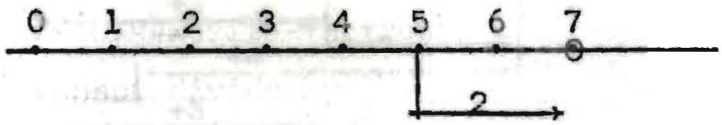
(2) $2^{2^{22}}$.

(3) $2^{2^{2^{22}}}$.

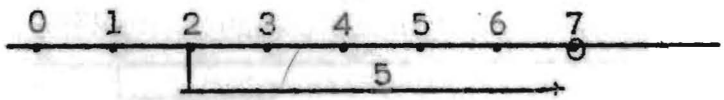
(4) about 145,900,000 miles.

(5) 2^{-22}.

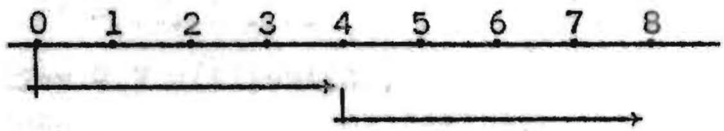
Section 7.3 (Ninth Friday)

(1) 

The answer is 7.

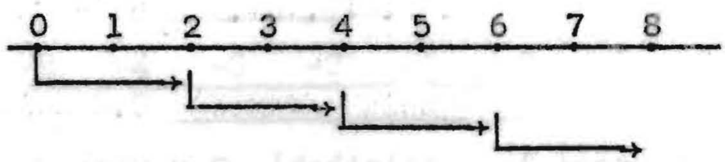
(2) 

The answer is 7; Yes, because the commutative property holds.

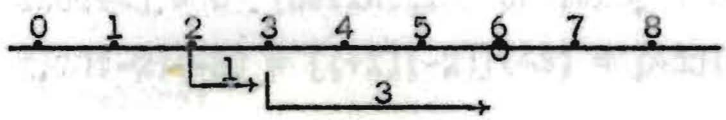
(3) 

The answer is 8.

(4) 2x2x2 = (2x2)x2 = 4x2 therefore,

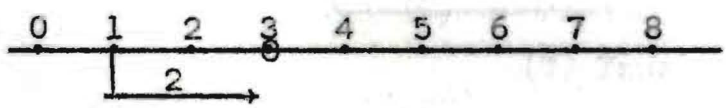


The answer is 8.

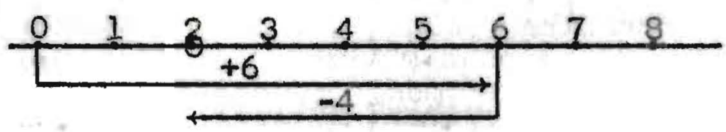
(5) 

The answer is 6.

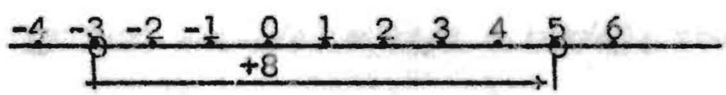
Section 7.5 (Tenth Friday)

(1) 

The answer is 3.

(2) 

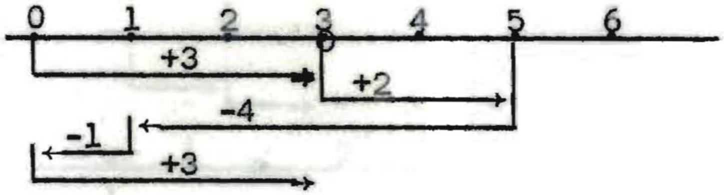
The answer is 2.

(3) 

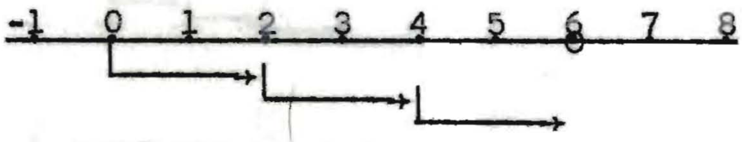
The answer is 8.

(4) 

The answer is -3.

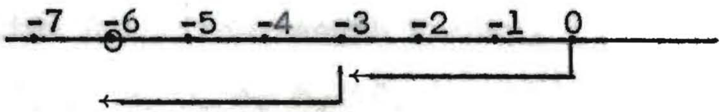
(5) 

The answer is 3.

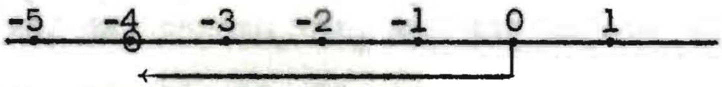
(6) 

The answer is 6.

Section 7.7 (Eleventh Friday)

(1) 

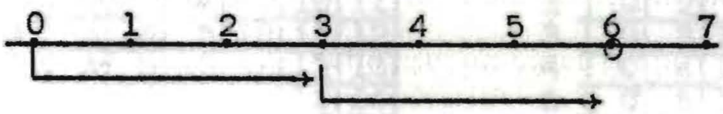
The answer is -6.

(2) 

The answer is -4.

(3) (0)(+4) = 0 (definition of zero, X•0 = 0)

(4) (+1)(-2)(-3) = [(+1)(-2)](-3) = (-2)(-3) = (+2x3); therefore:



The answer is 6.

(5) True.

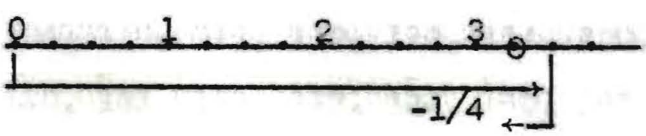
(6) False.

(7) True.

(8) True.

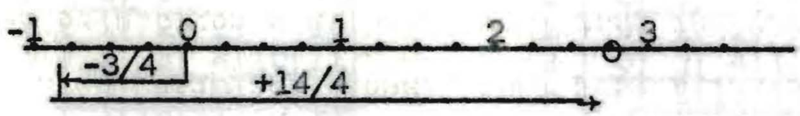
Section 7.9 (Twelfth Friday)

(1) (3 1/2) + (-1/4) = 14/4 + (-1/4); therefore:

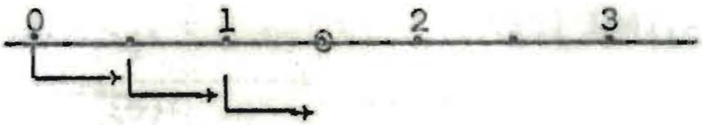


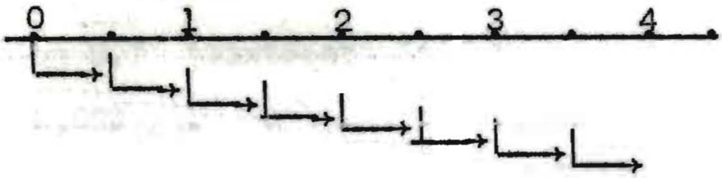
The answer is 3 1/4.

(2) (-3/4) + (3 1/2) = -3/4 + 14/4; therefore:



The answer is 2 3/4.

(3)  The answer is 1½.

(4)  The answer is 8.

(5)  The answer is 2.

Section 8.3 (Fourteenth Friday)

(1) 0, 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 30, 31, 32, 33, 34, 35, 40, 41, 42, 43, 44, 45, 50, 51, 52, 53, 54, 55, 100, 101, 102, 103.

(2)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

X	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	44

(3) $341_{(10)}$ (4) $3022_{(6)}$ (5) $32120_{(4)}$.

Section 8.5 (Fifteenth Friday)

(1) 0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44, 100, 101, 102, 103, 104, 110, 111, 112, 113, 114, 120, 121, 122, 123, 124, 130, 131, 132, 133, 134, 140, 141, 142, 143, 144.

(2)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

(3) $2422_{(5)}$.

(4) $30333_{(5)}$.

(5) $124_{(10)}$.

(6) $1121_{(4)}$.

Section 8.7 (Sixteenth Friday)

(1) $20,023_{(4)}$.

(2) $121_{(4)}$.

(3) $12,103_{(4)}$.

(4) $22_{(4)}$ remainder $110_{(4)}$.

(5) $233_{(10)}$.

(6) $100110_{(2)}$.

(7) 0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103, 110, 111, 112, 113, 120, 121, 122, 123, 130, 131, 132, 133, 200, 201, 202, 203, 210, 211, 212, 213, 220, 221, 222, 223, 230, 231, 232, 233, 300, 301, 302.

Section 8.9 (Seventeenth Friday)

(1) 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011.

(2) 0, 1.

(3) 'a' $11_{(2)}$, 'b' $1010_{(2)}$, 'c' 0.

(4) $11001011_{(2)}$.

(5) $1001_{(2)}$.

(6) $111_{(2)}$.

(7) $277_{(12)}$.

BIBLIOGRAPHY

BIBLIOGRAPHY

A. BOOKS

Adler, Irving. Magic House of Numbers. New York: The New American Library, 1960.

Allendoerfer, C. B., and G. O. Oakley. Fundamentals of Freshman Mathematics. New York: McGraw-Hill Book Company, Inc., 1959.

Andree, Richard V. Selections from Modern Abstract Algebra. New York: Henry Holt and Company, 1959.

Bakst, Aaron. Mathematics, Its Magic and Mastery. New York: D. Van Nostrand Co., 1941.

Gamow, George. One Two Three ... Infinity. New York: The New American Library, 1959.

Johnson, Donovan A. and William H. Glenn. Understanding Numeration Systems. St. Louis: Webster Publishing Company, 1960.

Kemeny, John G., J. Laurie Snell, and Gerald L. Thompson Introduction to Finite Mathematics. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958.

Mallory, Virgil S. and Kenneth C. Skeen. Second Algebra. Chicago: Benj. H. Sanborn and Co., 1952.

McCall, Raymond J. Basic Logic. New York: Barnes and Noble, Inc., 1961.

Rich, Barnett. Schaum's Principles and Problems of Elementary Algebra. New York: Schaum Publishing Company, 1960.

Stein, Edwin I. Supplementary Units in Contemporary Arithmetic and Elementary Algebra. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1960.

B. PUBLICATIONS OF THE GOVERNMENT AND LEARNED ORGANIZATIONS

Brown, Kenneth E. "Projects to Improve School Mathematics," _Aids for Mathematics Education_. Washington, D. C.: U. S. Dept. of Health, Education, and Welfare--Office of Education, 1960.

National Council of Teachers of Mathematics. _Insights into Modern Mathematics_. Twenty-third Yearbook of the National Council of Teachers of Mathematics. Washington, D.C., 1957.

U. S. Office of Education. _Inservice Education of Teachers of Secondary School Mathematics_. Washington, D. C.: U. S. Dept. of Health, Education, and Welfare--Office of Education, 1960.

C. PERIODICALS

Ashlock, Robert B. "The Number Line in the Primary Grades." _The Arithmetic Teacher_, February, 1961.

Cogan, E. J. "A New Approach to Mathematics," _The Mathematics Teacher_, May, 1957.

Gibb, E. Glenadine. "As We Read," editorial, _The Arithmetic Teacher_, May, 1961.

Laird, Lester E. "Constructing The Numbers," _Reports: Workshop in High School Mathematics, Sixth Year_, Emporia, Kansas: Kansas State Teachers College, 1959, pp. 7-11.

Maul, Ray C. "Lets Look at the New Mathematics and Science Teacher," _The Mathematics Teacher_, November, 1958.

Peterson, Wayne. "A Case in Point," _The Arithmetic Teacher_, January, 1961.

Secondary-School Curriculum Committee of The National Council of Teachers of Mathematics. "The Secondary Mathematics Curriculum," _The Mathematics Teacher_, May, 1959.

Willerding, Margaret F. "Stimulating Interest in Junior High Mathematics Teacher," _The Mathematics Teacher_, March, 1959.

D. ADDRESSES

Ferguson, W. Eugene. "Implementing The New Mathematics Program in Your School," Address: The Regional Orientation Conference in Mathematics. Topeka, Kansas: December 1, 1960.

Price, G. Baley. "Progress in Mathematics and Its Implications for the Secondary School," Address: The Regional Orientation Conference in Mathematics. Topeka, Kansas: December 1, 1960.

E. DICTIONARIES

Funk and Wagnall's New Standard Dictionary. New York and London: Funk and Wagnalls Company, 1942. p. 547.

Webster's New Collegiate Dictionary. Springfield, Mass.: G. & C. Merrian Company, 1958. p. 552.

Webster's New International Dictionary; second edition. Springfield, Mass.: G. & C. Merrian Company, 1958, p. 552.

Webster's New World Dictionary of The American Language; College edition. Cleveland and New York: The World Publishing Co., 1960.