THE LATTICES OF ABSTRACT GROUPS OF ORDER TWENTY-FOUR

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

by

Harold W. Bohm

August 1961

Approved for the Major Department

John M. Burger

Approved for the Graduate Council

Laurence C. Boylan

ACKNOWLEDGMENT

To Dr. John M. Burger, Head of the Department of Mathematics, and Mr. Lester E. Laird, Associate Professor of Mathematics, of the Kansas State Teachers College of Emporia, who generously gave of their time in directing the writing of this thesis, the writer wishes to express his cordial appreciation. The writer is also indebted to his wife for her patience during this time.

Harold W. Bohm

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

1.1. <u>Introduction</u>. The subject of group theory is indeed an interesting area. The study of finite groups is but a small segment of the general theory of groups, however group theory has its place in all pure and applied mathematics.

Through the study of abstract finite groups of a given order, it is of interest to determine the subgroups which may be contained in the groups under consideration. This thesis is concerned with the abstract groups of order 24.

1.2. <u>Statement of the problem</u>. The purpose of this study is to construct the lattices of the abstract groups of order 24. To construct the lattice of each group of order 24, the subgroups contained in each of the abstract groups of order 24 will be determined. From the lattice of each group it will be possible to determine the number of elements of each order and the order of each element contained in a particular abstract group of order 24.

1.3. <u>Limitations of the problem</u>. As was previously stated, this thesis will deal only with the abstract groups of order 24. The purpose of the lattices of groups of order

2, 3, 4, 6, 8, and 12 will be to aid in determining the subgroups which may be contained in the abstract groups of order 24.

There are fifteen distinct abstract groups of order 24.[1] Although there are many groups of order 24, each is isomorphic or a faithful representation of one of the abstract groups of order 24 which will be considered in this thesis.

1.4. Importance of the study. Through the study of the abstract groups of a given order, one can gain a better understanding of the theory of groups. The contents of this thesis could be utilized in analyzing groups whose orders include 24 as a factor. This thesis could also be used to analyze groups whose orders include the factors of 24 as factors.

1.5. Organization of the thesis. Chapter II contains a brief history of group theory and mentions some of the men who have contributed extensively to the development of the theory of groups. This chapter also contains some of the theorems discovered by these men.

Chapter III defines the group concept and the axioms of group theory. Also contained in this chapter are the

---

[1] W. Burnside, Theory of Groups of Finite Order ([New York]: Dover Publications, Inc., 1955), pp. 157-161.

definitions of some of the terms used in the remainder of this thesis. Chapter IV presents the lattices of groups whose orders are factors of 24.

Chapters V through IX constitute the core of the thesis. Chapters V through VII deal with the groups of order 24 whose subgroups of order 8 are Abelian. Chapter V considers the groups of order 24 which contain cyclic subgroups of order 8. Chapter VI presents the groups of order 24 whose subgroups of order 8 are defined by $C_4 \times C_2$. The groups of order 24 considered in Chapter VII contain subgroups of order 8 which are defined by $C_2 \times C_2 \times C_2$.

Chapters VIII and IX constitute groups of order 24 whose subgroups of order 8 are non-Abelian. Chapter VIII discusses the groups of order 24 whose subgroups of order 8 are dicyclic. The groups of order 24 which contain dihedral subgroups of order 8 are presented in Chapter IX.

Chapter X contains a brief summary of the study. This chapter also contains some conjectures of the writer pertaining to abstract finite groups. Table III, showing the number of elements of each order contained in a particular group, is contained in this chapter.

CHAPTER II

HISTORICAL BACKGROUND OF GROUP THEORY

2.1. History of group theory. Evarsite Galois (1811-1832), a French mathematician, was the first to use the term "group" in a technical sense.[1] He proved the theorem that every invariant subgroup gives rise to a quotient group which exhibits many properties of the group. Galois also showed that groups may be divided into simple and compound groups.[2]

Some concepts of group theory had been anticipated and explored by J. L. Lagrange (1736-1813) and Paolo Ruffini (1765-1822), but Galois revealed the complete concept of the theory of groups. Galois, on the eve of his fatal duel, wrote a summary of his discoveries pertaining to the theory of equations. This summary included his thoughts and ideas on the theory of groups, the key to modern algebra, and modern geometry.[3]

---

[1]James R. Newman, The World of Mathematics (New York: Simon and Schuster, Inc., 1956), III, p. 1534.

[2]Florian Cajori, A History of Mathematics (New York: The Macmillan Company, 1924, 2nd ed.), p. 351.

[3]Dirk J. Struik, A Concise History of Mathematics (New York: Dover Publications, Inc., 1948), II, pp. 223-225.

N. H. Abel (1802-1829), a Norwegian mathematician, persued Galois' ideas pertaining to group theory. Commutative groups are called Abelian groups which is indicative of Abel's interest and prominence in the area of group theory.[4]

Prior to 1854, the theory of groups of finite order originated from the writings of Lagrange, Ruffini, Abel, and Galois. The theory of groups of finite order sprang from an analysis of the theory of algebraic equations and the theory of numbers.[5] A. L. Cauchy (1789-1857) is credited as being the founder of the theory of groups of finite order. In 1844 Cauchy proved the theorem which is now known as Cauchy's Theorem. This theorem had previously been stated by Galois but not proven.[6]

The founding of the theory of abstract groups is usually credited to A. Cayley (1821-1895), an English mathematician. L. Kronecker (1823-1891), and H. Weber (1842-1913) later gave the formal definitions for abstract groups.[7]

Ludwig Sylow (1832-1918), a Norwegian mathematician, obtained a theorem which was first proposed by Cauchy. Out

---

[4] Ibid., pp. 226-227.

[5] Cajori, op. cit., p. 353.

[6] Ibid., p. 352.

[7] Ibid., pp. 352-353.

of the study of this theorem, Sylow discovered the theorem which is now known as Sylow's Theorem.[8]

Many mathematicians, through the centuries, have studied group theory. However, the men referred to in this chapter must certainly stand out as pioneers in the area of group theory. The remainder of this chapter is devoted to some of the more significant theorems pertaining to the theory of groups. The proofs of these theorems may be found in any reputable textbook dealing with group theory. Hence, the proofs will not be included in this thesis.

Galois' Theorem:

When $n > 4$, A_n is a simple group.[9]

Lagrange's Theorem:

The order of a subgroup of a finite group G is a factor of the order of G.[10]

Cayley's Theorem:

Every group is isomorphic to a permutation group of its own elements.[11]

---

[8] Ibid., p. 354.

[9] Walter Ledermann, Introduction to the Theory of Finite Groups (New York: Interscience Publishers, Inc., 1957, 3rd ed.), p. 120.

[10] Robert D. Carmichael, Introduction to the Theory of Groups of Finite Order (Boston: Ginn and Company, 1937), p. 44.

[11] Marshall Hall, Jr., The Theory of Groups (New York: The Macmillan Company, 1959), p. 9.

Cauchy's Theorem:

If p is a prime factor of the order of a group G, then G contains at least one element of order p.[12]

Sylow's Theorem:

Every group whose order is divisible by p^m, but not by p^{m+1}, p being a prime number, contains 1+kp subgroups of order p^m.[13]

Other Theorems:

The order of an element of G is a factor of the order of G.[14]

A group of prime order has no proper subgroups and is necessarily cyclic.[15]

All subgroups of a cyclic group are cyclic. If $\{A\}$ is a cyclic group of order g, then corresponding to every divisor h of g there exists one, and only one, subgroup of order h, which may be generated by $A^{g/h}$.[16]

---

[12]Ledermann, op. cit., p. 129.

[13]Cajori, op. cit., p. 354.

[14]Carmichael, op. cit., p. 45.

[15]Ledermann, op. cit., p. 38.

[16]Ibid.

CHAPTER III

THE AXIOMS OF GROUP THEORY

3.1. Axioms of group theory. To form a group, a set of elements must obey the following axioms:

> Definition 1. A set G of a finite or infinite number of elements, for which a law of composition ("multiplication") is defined, forms a group if the following conditions are satisfied:
>
> (I) Closure: to every ordered pair of elements A, B of G there belongs a unique element C of G, written
>
> $$C = AB$$
>
> which is called the product of A and B.
>
> (II) Associative law: if A, B, C are any three elements of G, which need not be distinct, then
>
> $$(AB)C = A(BC)$$
>
> so that either side may be denoted by ABC.
>
> (III) Unit Element: G contains an element I, called the unit element or identity such that for every element A of G
>
> $$AI = IA = A.$$
>
> (IV) Inverse or reciprocal element: corresponding to every element A of G, there exists in G an element A^{-1} such that
>
> $$AA^{-1} = A^{-1}A = I.$$[1]

3.2. Definition of terms. This section will define some of the terms used in this thesis.

---

[1]Walter Ledermann, Introduction to the Theory of Finite Groups (New York: Interscience Publishers, Inc., 1957, 3rd ed.), pp. 2-3.

Subgroup. If there is a set, H, of elements in a group G which, by themselves form a group; then H is a subgroup of G.[2]

Improper subgroup. Every group contains two improper subgroups; namely, the subgroup composed of the unit element alone and the subgroup which contains all the elements of the group.[3]

Proper subgroup. All subgroups of a group G except the two improper subgroups are proper subgroups.[4] In this thesis the term subgroup implies proper subgroup unless otherwise stipulated.

Common subgroup. A subgroup which is contained in two or more subgroups of a given group, G, is called a common subgroup. That is, if A is a subgroup contained in subgroups B and C, then A is a common subgroup of B and C.

Abelian group. A group which has the property that every element commutes with every other element is an Abelian group. That is, in a given group, if AB = BA for any A or B then the group is Abelian.[5]

---

[2]Richard V. Andree, Selections from Modern Abstract Algebra (New York: Henry Holt and Company, 1958), p. 91.

[3]Ledermann, op. cit., p. 31.

[4]Ibid.

[5]Ibid., p. 3.

Non-Abelian group. If in a given group, some element is not commutative with some other element then the group is non-Abelian.

Order of a group. In a finite group, the number of elements contained in the group is the order of the group.[6]

Order of an element. If A^g equals the unit element where g is the least positive integer for which this condition is true, then the order of A is g.[7]

Cyclic group. A cyclic group of order g is a group which contains at least one element, A, of order g. The element A is said to generate the group.[8]

Dicyclic group. The dicyclic group is defined by the following relations,

$$G_{4n} = \{A,B\}, \text{ where } A^{2n} = I,\ A^n = (AB)^2 = B^2,\ (n>1),$$

and is of order 4n.[9]

Dihedral group. The group defined by

$$G_{2m} = \{A,B\}, \text{ where } A^m = B^2 = (AB)^2 = I$$

is the dihedral group of order 2m.[10]

---

[6]Andree, *loc. cit.*

[7]*Ledermann*, op. cit., p. 21.

[8]*Ibid.*, p. 24.

[9]Robert D. Carmichael, *Introduction to the Theory of Groups of Finite Order* (Boston: Ginn and Company, 1937), p. 183.

[10]*Ibid.*

Isomorphic groups. By an isomorphism between two groups $G = \{A,B,\ldots\}$ and $G' = \{A',B',\ldots\}$ is meant a one-to-one correspondence $A \rightleftharpoons A'$ between their elements which preserves group multiplication--that is, which is such that if $A \rightleftharpoons A'$ and $B \rightleftharpoons B'$, then $AB \rightleftharpoons A'B'$.

Defining relations. A set of generating elements and their relationship from which a group may be derived is called the defining relations of the group. That is,

$$G = \{A,B\}, \text{ where } A^4 = B^2 = I,\ AB = BA,$$

completely defines an Abelian group of order 8. The eight elements contained in this group are I, A, A^2, A^3, B, AB, A^2B, and A^3B.

3.3. The lattice. A lattice is a diagram which reveals the structure of a given group. The lattice discloses the subgroups contained in this group. The letter G in a lattice indicates a group or subgroup. The numerical subscripts of a certain G indicate the order of that group or subgroup. The alphabetical subscript of a group or subgroup, G, is used to indicate a particular group or subgroup. For example G_{2a} and G_{4b} are groups or subgroups of orders 2 and 4 respectively. G_{6a} and G_{6b}, two subgroups or groups of order 6, which may or may not be isomorphic, are nevertheless distinct subgroups or groups. In a lattice, the lines extending downward from a given group, G, to various subgroups indicate the subgroups which are contained in the group G.

Figure 3.1 is the lattice of a group of order 8. G_8 indicates that this group is of order 8. G_8 contains three distinct subgroups of order 4. G_{2a}, a subgroup of order 2, is common to the three subgroups of order 4. Hence, G_{2a} is also contained in G_8. The unit element, I, is contained in G_8 and all subgroups of G_8.

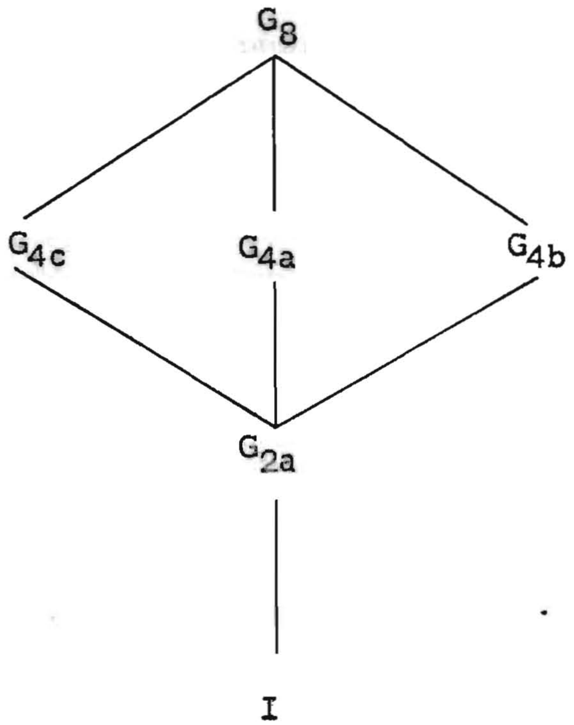

Figure 3.1. Lattice of a group of order 8.

CHAPTER IV

LATTICES OF GROUPS WHOSE ORDERS ARE FACTORS OF TWENTY-FOUR

4.1. Groups of order one. Since the number one is prime, all groups of order one are cyclic. Hence, there is only one type of group whose order is one. The group of order one is composed of only the unit element which is always of order one. The lattice of such a group is trivial and therefore is not shown. The defining relations of this group are:

$$G_1 = \{A\}, \text{ where } A = I.$$

4.2. Groups of order two. The groups of order 2 are cyclic. Since two is a prime number, the groups of order 2 do not contain any subgroups. All groups of order 2 are isomorphic and may be represented by the lattice in Figure 4.1. The defining relations of the group of order 2 are as follows:

$$G_2 = \{A\}, \text{ where } A^2 = I.$$

Figure 4.1. Lattice of group of order 2.

4.3. Groups of order three. Three is a prime number; hence, all groups of order 3 are cyclic and do not contain any subgroups. All groups of order 3 are isomorphic and are represented by the lattice in Figure 4.2. Since the order of the elements must be factors of the order of the group and the unit element must be unique, this group contains two elements of order 3 and the unit element of order one. The defining relations for the group of order 3 are:

$$G_3 = \{A\}, \text{ where } A^3 = I.$$

Figure 4.2. Lattice of group of order 3.

4.4. Groups of order four. There are two distinct groups of order 4, both of which are Abelian.[1] As the factors of 4 are 1, 2, and 4, the subgroups contained in a group of order 4 must be of order 2. The cyclic group of order 4 contains two elements of order 4, an element of order 2, and the unit element. This group contains a subgroup of order 2 as shown in the lattice in Figure 4.3.

---

[1]Walter Ledermann, Introduction to the Theory of Finite Groups (New York: Interscience Publishers, Inc., 1957, 3rd ed.), p. 48.

The defining relations for the cyclic group of order 4 are as follows:

$$G_4 = \{A\}, \text{ where } A^4 = I.$$

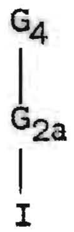

Figure 4.3. Lattice of cyclic group of order 4.

The second group of order 4 contains three elements of order 2 and the unit element. This group is referred to as the four-group or quadratic group and contains three subgroups of order 2. Figure 4.4. shows the lattice of the quadratic group. The defining relations of such a group are as follows:

$$G_4 = \{A,B\}, \text{ where } A^2 = B^2 = (AB)^2 = I.$$

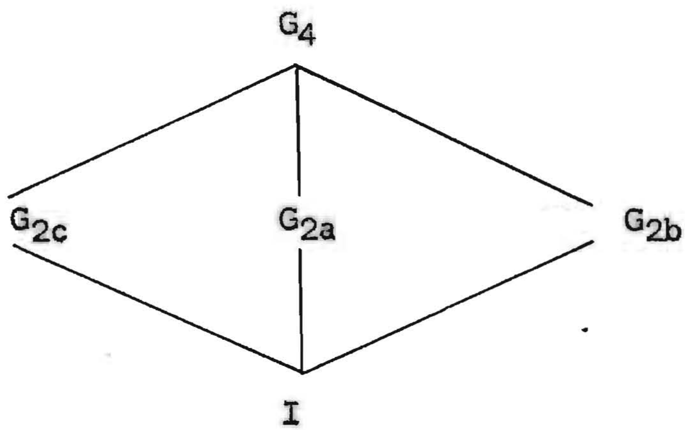

Figure 4.4. Lattice of quadratic group of order 4.

4.5. Groups of order six. There are two distinct groups of order 6; one is cyclic and the other non-Abelian.[2] The cyclic group of order 6 contains two elements of order 6, two elements of order 3, one element of order 2, and the unit element. Figure 4.5 is the lattice of such a group and shows the subgroups of orders 2 and 3. The cyclic group of order 6 is defined by the following relations:

$$G_6 = \{A\}, \text{ where } A^6 = I.$$

Figure 4.5. Lattice of cyclic group of order 6.

The non-Abelian group of order 6 is shown in Figure 4.6. This group is composed of two elements of order 3, three elements of order 2, and the unit element. There are three subgroups of order 2 and a subgroup of order 3

---

[2] Ibid., p. 49.

contained in this group. The defining relations of this group are as follows:

$$G_6 = \{A,B\}, \text{ where } A^3 = B^2 = (AB)^2 = I.$$

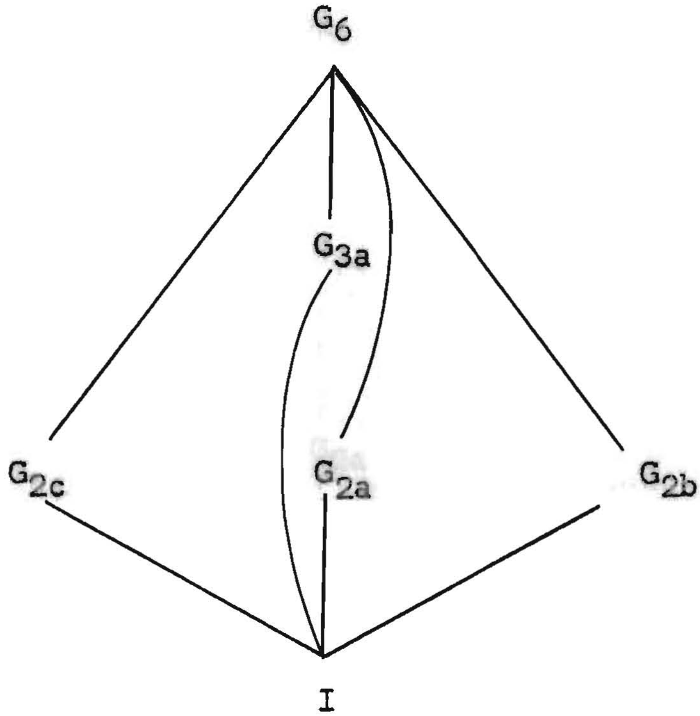

Figure 4.6. Lattice of non-Abelian group of order 6.

4.6. Groups of order eight. The five groups of order 8 consist of three Abelian and two non-Abelian groups.[3] The cyclic group of order 8 is represented by the lattice in Figure 4.7. This group contains four elements of order 8, two elements of order 4, one element of order 2, and the unit element. There are two subgroups contained in this group; a cyclic subgroup of order 4 and a subgroup of order 2.

---

[3] Ibid., p. 51.

The cyclic group of order 8 is defined by the following relations:

$$G_8 = \{A\}, \text{ where } A^8 = I.$$

Figure 4.7. Lattice of cyclic group of order 8.

Another Abelian group of order 8 contains seven elements of order 2 and the unit element. This group is the direct product of three groups, each of which is of order 2. The group contains seven subgroups of order 4 and seven subgroups of order 2. The subgroups of order 4 are quadratic subgroups. Figure 4.8 is the lattice of this Abelian group of order 8. The defining relations for the group are as follows:

$$G_8 = \{A,B,C\}, \text{ where } A^2 = B^2 = C^2 = I,$$
$$AB = BA,\ AC = CA,\ BC = CB.$$

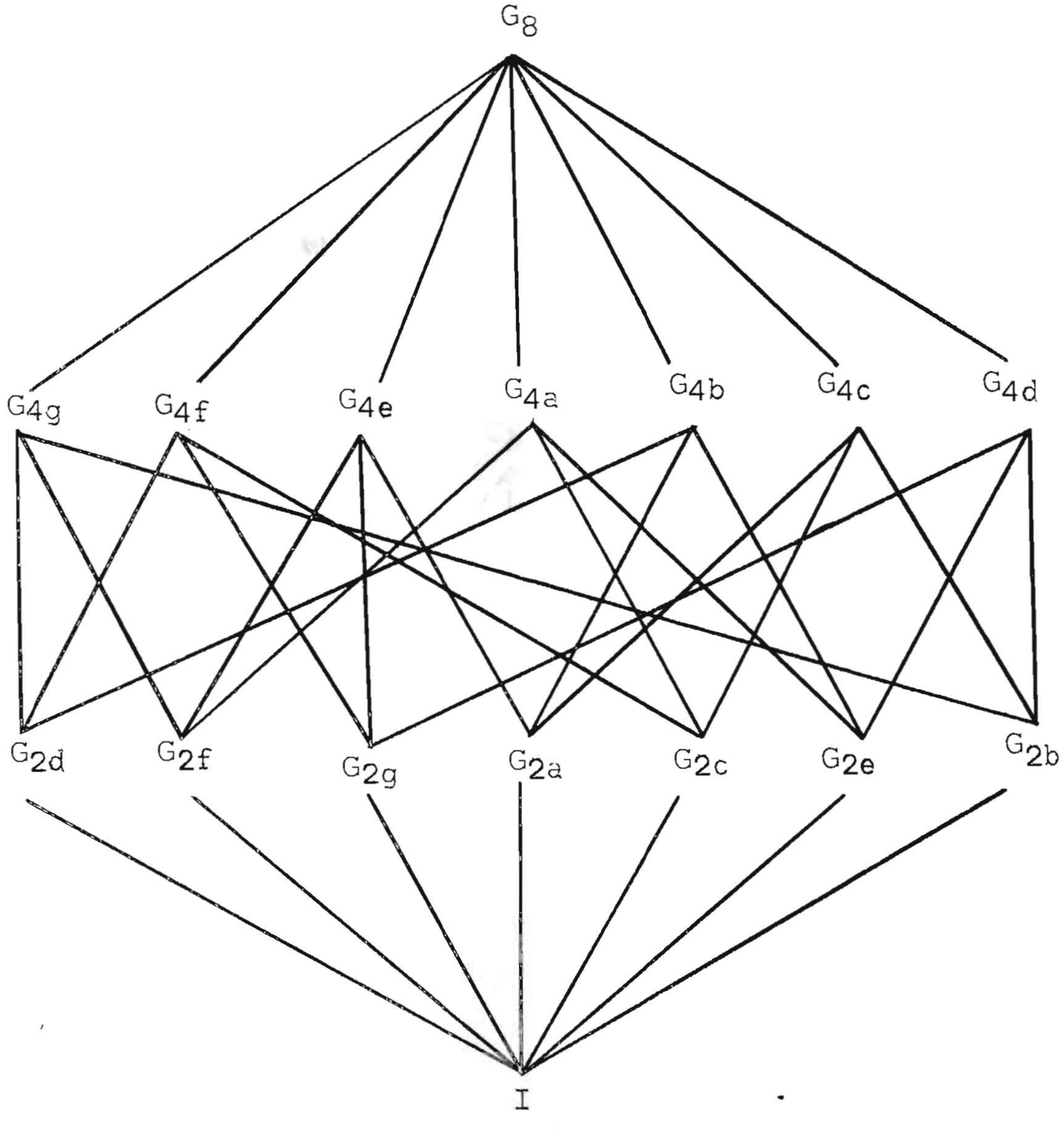

Figure 4.8. Lattice of group of order 8 of type $C_2 \times C_2 \times C_2$.

The third Abelian group of order 8 is the direct product of a cyclic group of order 4 and a group of order 2. This group is composed of four elements of order 4, three elements of order 2, and the unit element. The group contains two cyclic subgroups and one quadratic subgroup of order 4. There are also three subgroups of order 2. The lattice of this group is represented in Figure 4.9. The group is defined by the following relations:

$$G_8 = \{A,B\}, \text{ where } A^4 = B^2 = I, AB = BA.$$

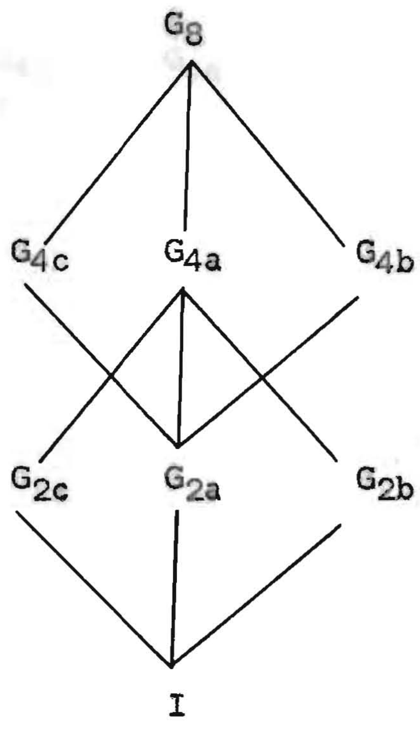

Figure 4.9. Lattice of group of order 8 of type $C_4 \times C_2$.

The dihedral group of order 8 is non-Abelian and is composed of two elements of order 4, five elements of order 2, and the unit element. This group contains one cyclic subgroup and two quadratic subgroups of order 4. There are five subgroups of order 2 contained in this group of order 8. Figure 4.10 is the lattice of the dihedral group of order 8 which is defined by the following relations:

$$G_8 = \{A,B\}, \text{ where } A^4 = B^2 = (AB)^2 = I.$$

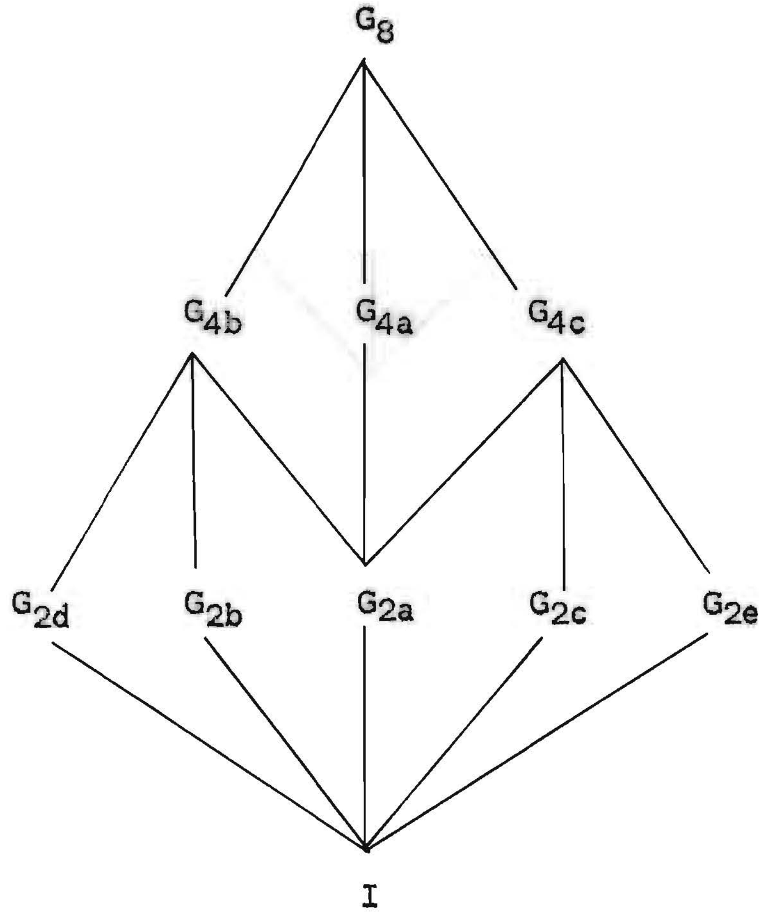

Figure 4.10. Lattice of dihedral group of order 8.

The fifth group of order 8 is also non-Abelian and is the dicyclic group of order 8. This group is represented by the lattice in Figure 4.11, and contains six elements of order 4, an element of order 2, and the unit element. The group contains three cyclic subgroups of order 4 and a subgroup of order 2. The defining relations for the dicyclic group are:

$$G_8 = \{A,B\}, \text{ where } A^4 = B^4 = I, A^2 = B^2 = (AB)^2.$$

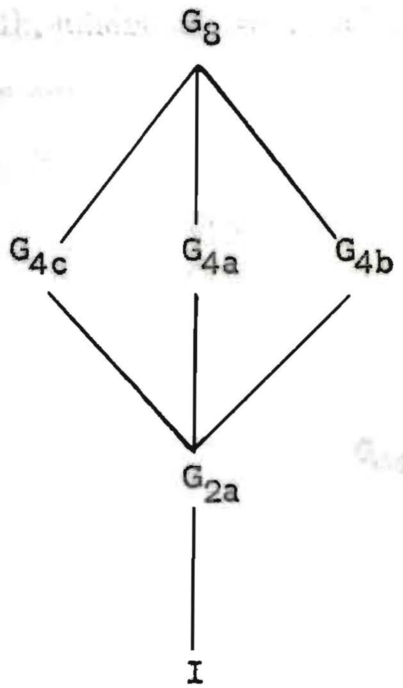

Figure 4.11. Lattice of dicyclic group of order 8.

4.7. Groups of order twelve. There are five distinct groups of order 12.[4] Two of the groups are Abelian and three are non-Abelian. The cyclic group of order 12 is composed of four elements of order 12, two elements of order 6, two elements of order 4, two elements of order 3, an element of order 2, and the unit element. This group

---

[4]Robert D. Carmichael, Introduction to the Theory of Groups of Finite Order (Boston: Ginn and Company, 1937), p. 69.

contains a cyclic subgroup of order 6. The subgroup of order 6 contains a subgroup of order 3 and a subgroup of order 2. The group of order 12 also contains a cyclic subgroup of order 4 which contains a subgroup of order 2. The lattice of the cyclic group of order 12 is shown in Figure 4.12. This group is defined by the following relations:

$$G_{12} = \{A\}, \text{ where } A^{12} = I.$$

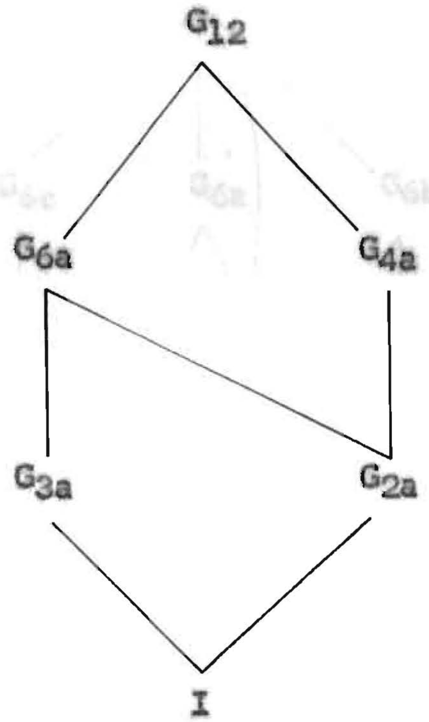

Figure 4.12. Lattice of cyclic group of order 12.

The remaining Abelian group of order 12 contains six elements of order 6, two elements of order 3, three elements of order 2, and the unit element. This group is the direct product of the quadratic group of order 4 and a group of order 3. Three cyclic subgroups of order 6 are contained in this group. The group also contains one quadratic subgroup of order 4, a subgroup of order 3, and three subgroups of order 2. The group is represented by the lattice in Figure 4.13 and is defined as follows:

$$G_{12} = \{A,B,C\}, \text{ where } A^2 = B^2 = C^3 = I,$$
$$AB = BA,\ AC = CA,\ BC = CB.$$

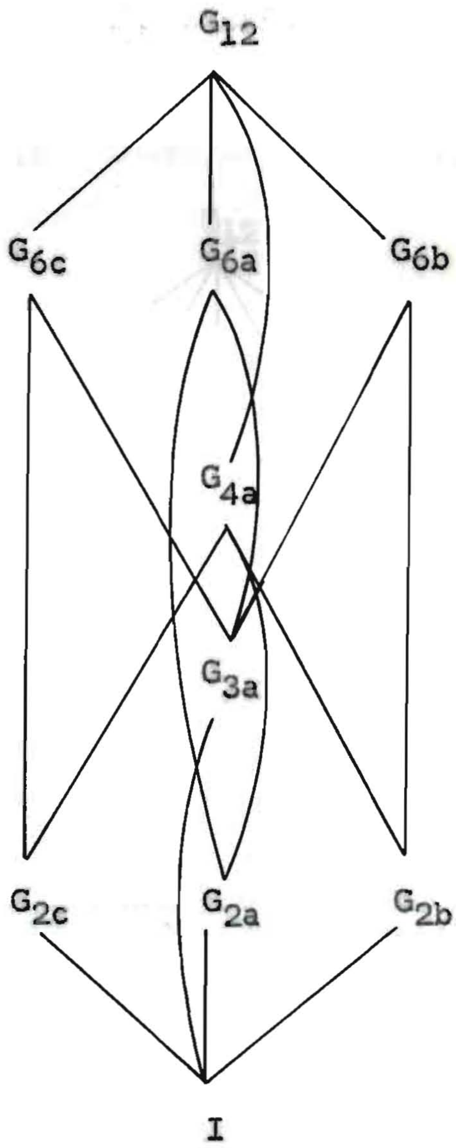

Figure 4.13. Lattice of Abelian group of order 12.

The alternating group of order 12 is composed of eight elements of order 3, three elements of order 2, and the unit element. This group contains one quadratic subgroup of order 4, and therefore contains three subgroups of order 2. The group also contains four subgroups of order 3. The lattice of the alternating group is shown in Figure 4.14. The defining relations for this group are as follows:

$$G_{12} = \{A,B\}, \text{ where } A^3 = B^3 = (AB)^2 = I.$$

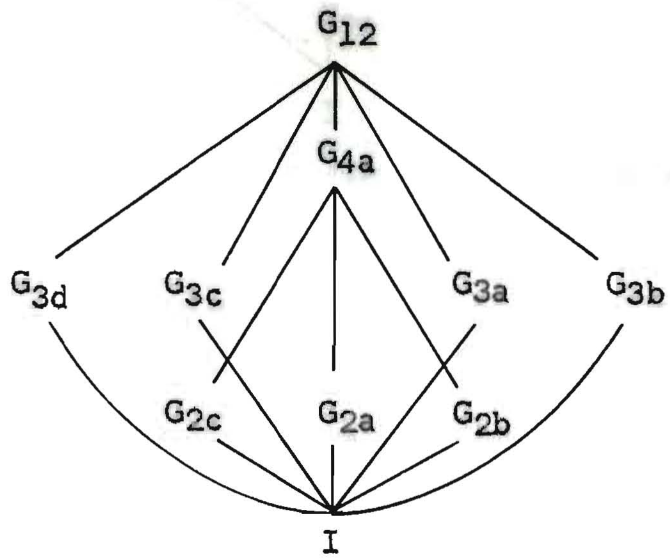

Figure 4.14. Lattice of alternating group of order 12.

The dicyclic group of order 12 is represented by the lattice in Figure 4.15. Two elements of order 6, six elements of order 4, two elements of order 3, an element of order 2, and the unit element constitute this group. Three cyclic subgroups of order 4 are contained in the group. The group also contains a cyclic subgroup of order 6 which has a subgroup of order 3 and a subgroup of order 2 which is common to the subgroups of order 4. The defining relations of this group are:

$$G_{12} = \{A,B\}, \text{ where } A^6 = I, A^3 = B^2 = (AB)^2.$$

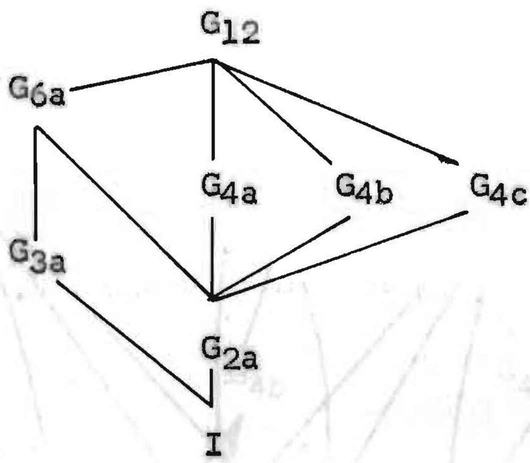

Figure 4.15. Lattice of dicyclic group of order 12.

The lattice in Figure 4.16 represents the dihedral group of order 12. This group is composed of two elements of order 6, two elements of order 3, seven elements of order 2, and the unit element. There are three subgroups of order 6 contained in the dihedral group. One of the subgroups of order 6 is cyclic. However, all of the subgroups of order 6 contain a common subgroup of order 3. There are three quadratic subgroups of order 4 and each of these subgroups contain three subgroups of order 2. One of the subgroups of order 2 is common to the cyclic subgroup of order 6 and to all quadratic subgroups of order 4. The following relations define this group:

$$G_{12} = \{A,B\}, \text{ where } A^6 = B^2 = (AB)^2 = I.$$

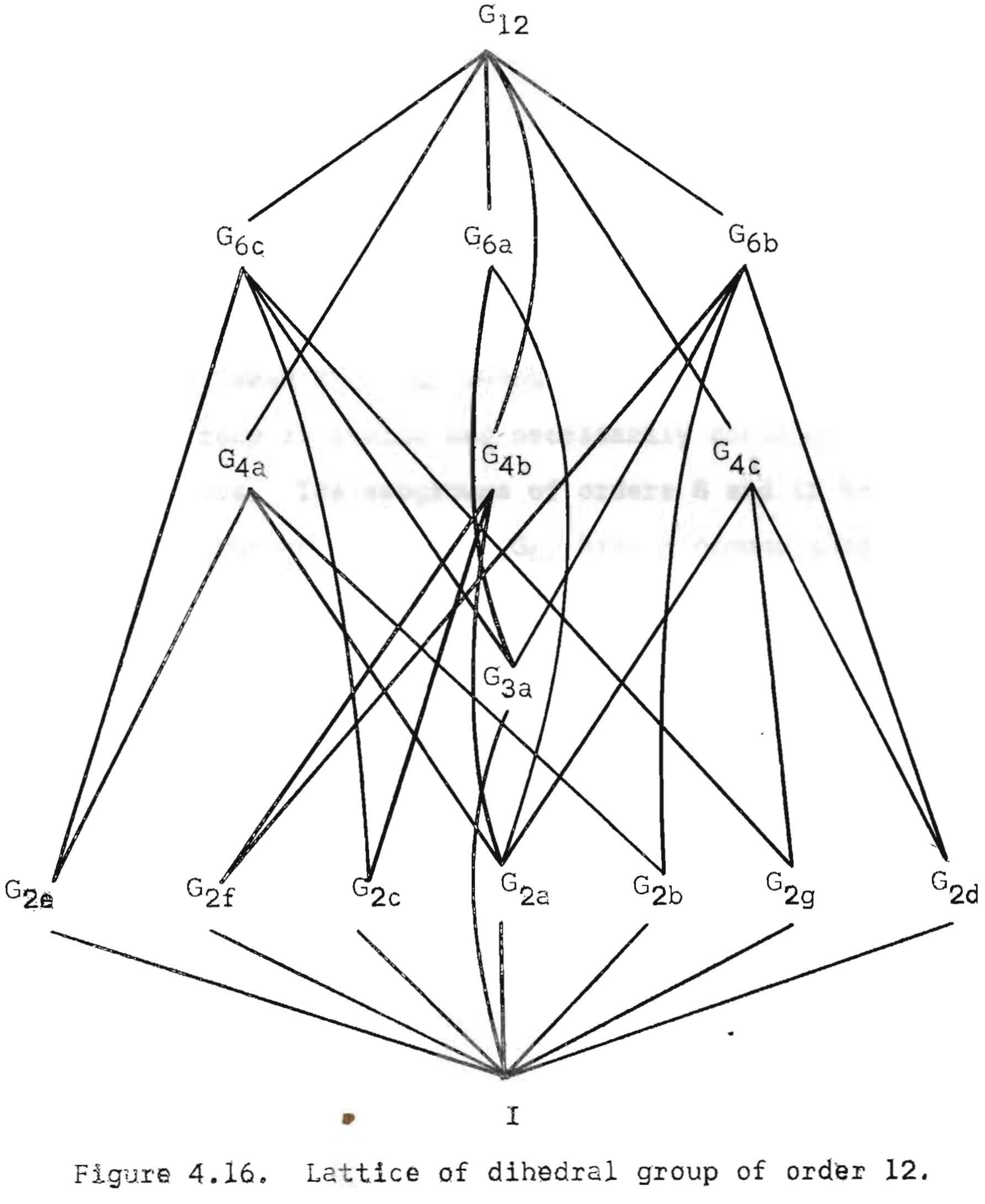

Figure 4.16. Lattice of dihedral group of order 12.

CHAPTER V

G_{24} WITH CYCLIC SUBGROUPS OF ORDER EIGHT

5.1. Type one. This group of order 24 is Abelian and is defined by the following relations:

$$G_{24} = \{A,B\}, \text{ where } A^8 = B^3 = I,\ AB = BA.$$

Figure 5.1 is the lattice of this group. The number of elements of each order is shown in Table III.

This group is cyclic and necessarily contains only cyclic subgroups. The subgroups of orders 8 and 12 have G_{4a} as a common subgroup. G_{12a} and G_{6a} have a common subgroup of order 3. The subgroup G_{2a} is common to all the subgroups whose orders are multiples of two.

5.2. Type two. The group defined by

$$G_{24} = \{A,B\}, \text{ where } A^8 = B^3 = I,\ A^{-1}BA = B^{-1},$$

is non-Abelian and is shown in Figure 5.2. Table III shows the number of elements of each order in this group.

All subgroups of this group are cyclic. The three subgroups of order 8 and the subgroup of order 12 have G_{4a} as a common subgroup. A subgroup of order 6 is contained in G_{12a}. The subgroups G_{6a} and G_{12a} contain a common subgroup of order 3. G_{2a} is again contained in all subgroups whose orders are multiples of two.

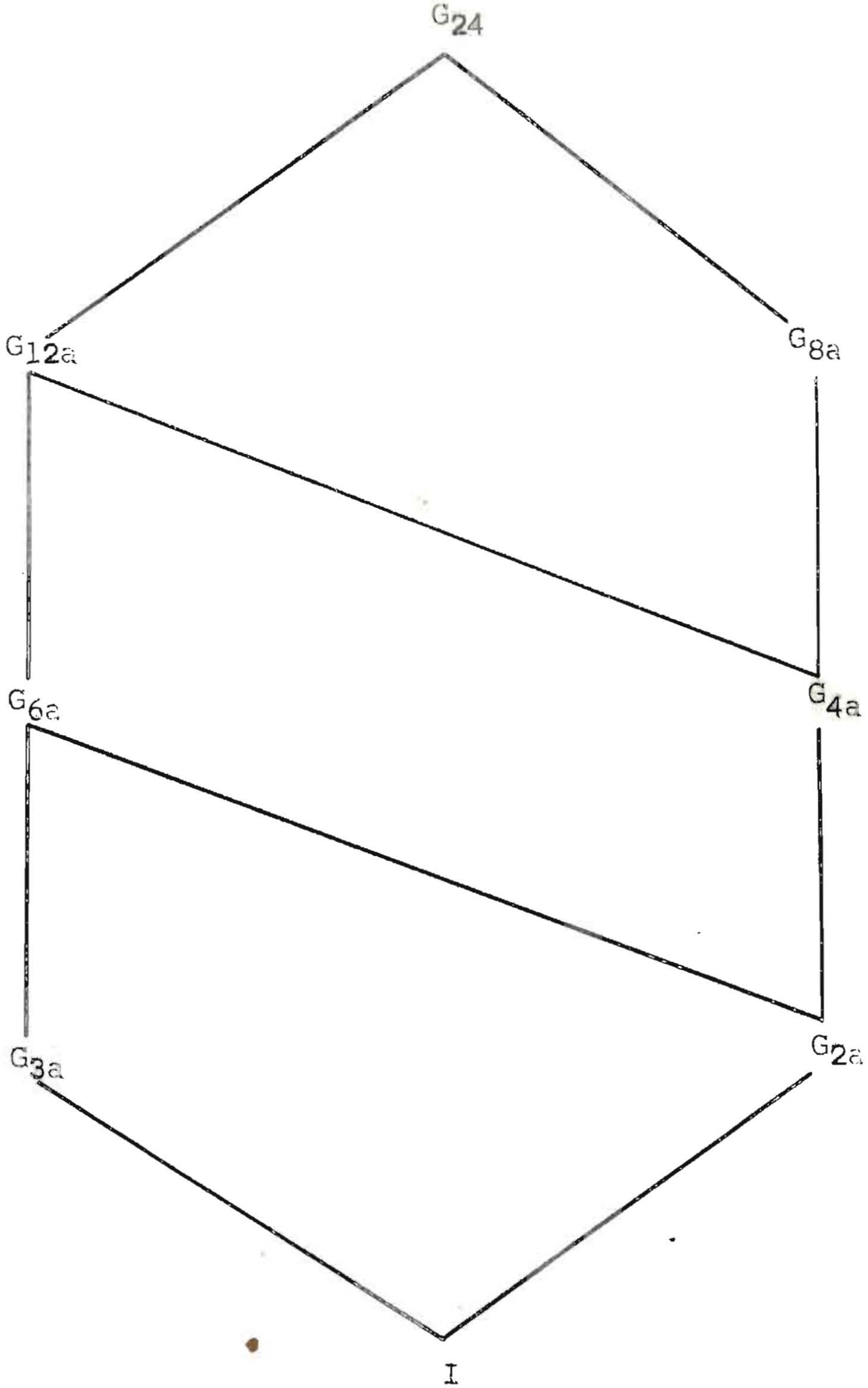

Figure 5.1. Lattice of G_{24} of type one.

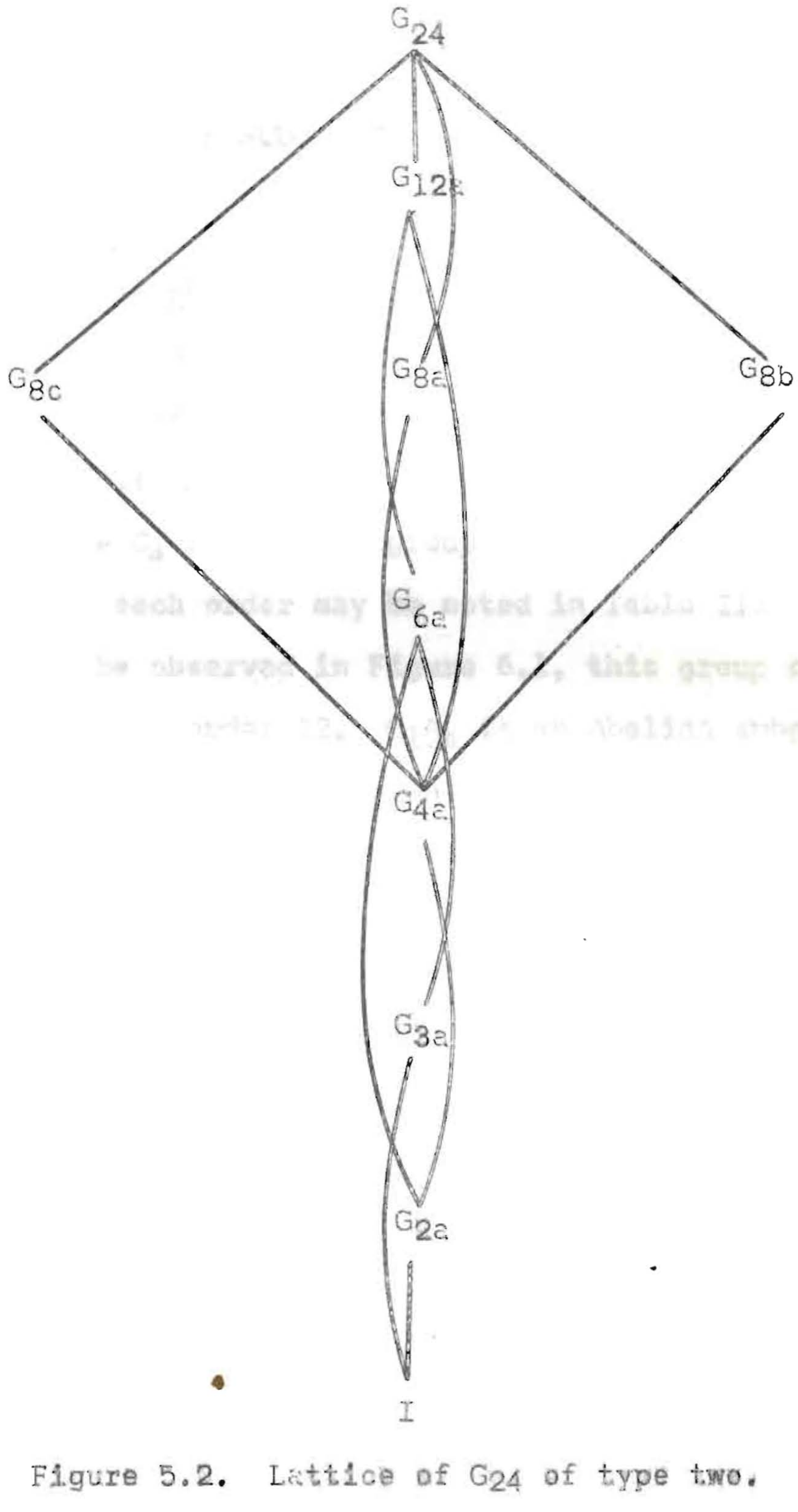

Figure 5.2. Lattice of G_{24} of type two.

CHAPTER VI

G_{24} WITH SUBGROUPS OF ORDER EIGHT OF TYPE $C_4 \times C_2$

6.1. Type three. The group of order 24 defined by

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$
$$AB = BA,\ AC = CA,\ BC = CB,$$

is Abelian. This group is the direct product of a group of order 8 of type $C_4 \times C_2$ and a group of order 3. The number of elements of each order may be noted in Table III.

As may be observed in Figure 6.1, this group contains three subgroups of order 12. G_{12a} is an Abelian subgroup and contains three cyclic subgroups of order 6 which have G_{3a} as a common subgroup. The quadratic subgroup contained in G_{12a} contains three subgroups of order 2.

The two cyclic subgroups of order 12 are G_{12b} and G_{12c}. These subgroups are isomorphic and contain G_{6a} as a common subgroup of order 6. Each subgroup contains a cyclic subgroup of order 4. G_{4a}, G_{4b}, and G_{6a} contain a common subgroup; G_{2a}. The subgroup G_{6a} also contains G_{3a}; a subgroup of order 3.

The subgroup of order 8 is Abelian and is isomorphic to the group in Figure 4.9. This subgroup contains two cyclic groups of order 4 and a quadratic subgroup of order 4. The three subgroups of order 4 have a common subgroup

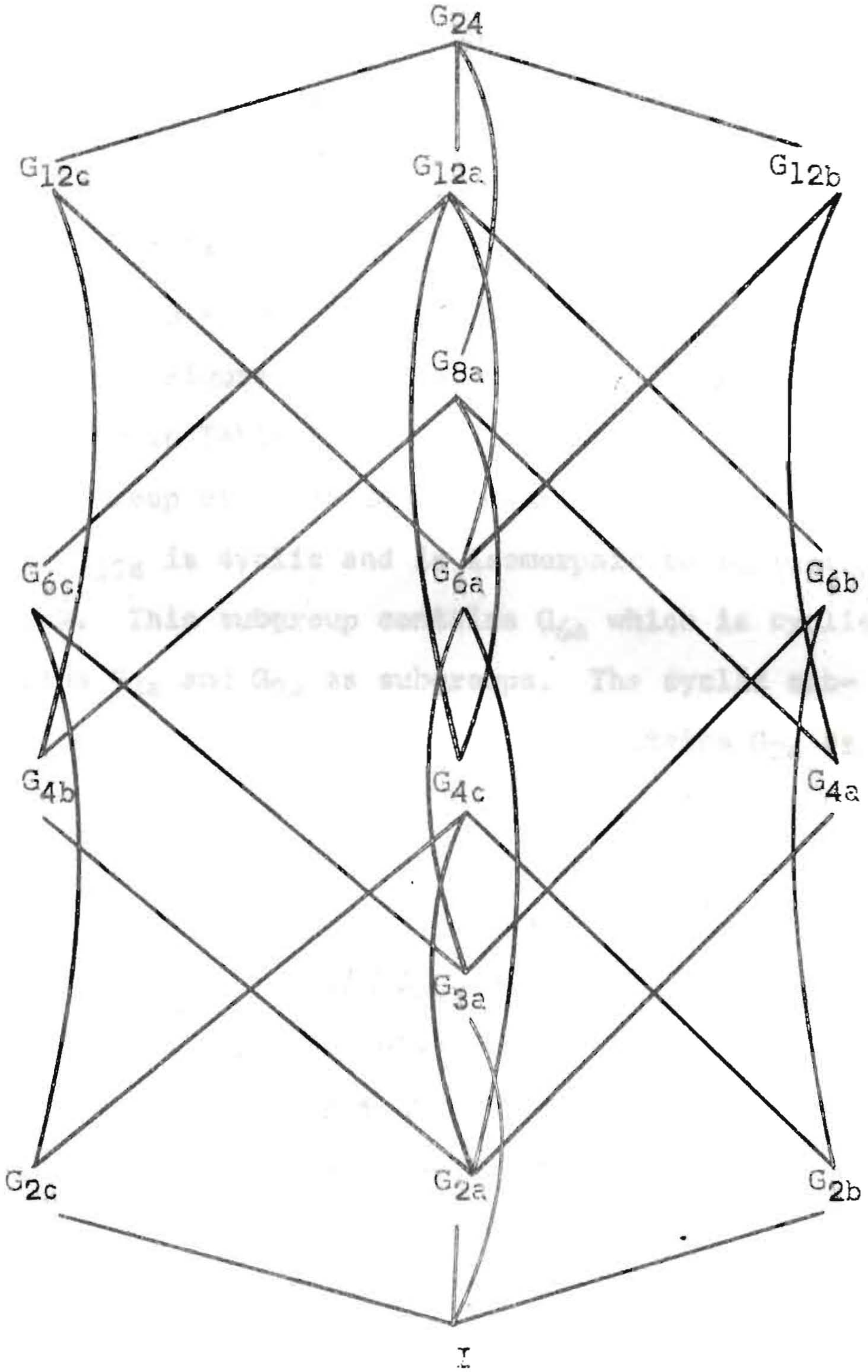

Figure 6.1. Lattice of G_{24} of type three.

whose order is 2. This common subgroup is G_{2a}. The quadratic subgroup contains the three subgroups of order 2.

6.2. Type four. This group is defined by

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$
$$AB = BA,\ BCB = C^{-1},\ A^{-1}CA = C,$$

and is shown in Figure 6.2. The number of elements of each order is shown in Table III.

This group of order 24 contains three subgroups of order 12. G_{12a} is cyclic and is isomorphic to the group in Figure 4.12. This subgroup contains G_{6a} which is cyclic. G_{6a} contains G_{3a} and G_{2a} as subgroups. The cyclic subgroup of order 4 contained in G_{12a} also contains G_{2a} as a subgroup.

G_{12c} is a dihedral subgroup of order 12 and contains one cyclic and two non-Abelian subgroups of order 6. The subgroups of order 6 contain G_{3a} as a common subgroup of order 3. G_{6b} and G_{6c} each contain three subgroups of order 2. G_{12c} also contains three quadratic subgroups of order 4, all of which contain G_{2a} as a common subgroup.

The third subgroup of order 12 is dicyclic and is denoted by G_{12b}. This subgroup contains three cyclic subgroups of order 4 and a cyclic subgroup of order 6. The subgroups of orders 4 and 6 contain G_{2a} as a common subgroup of order 2. The subgroup of order 6 is the same subgroup which is contained in G_{12a}.

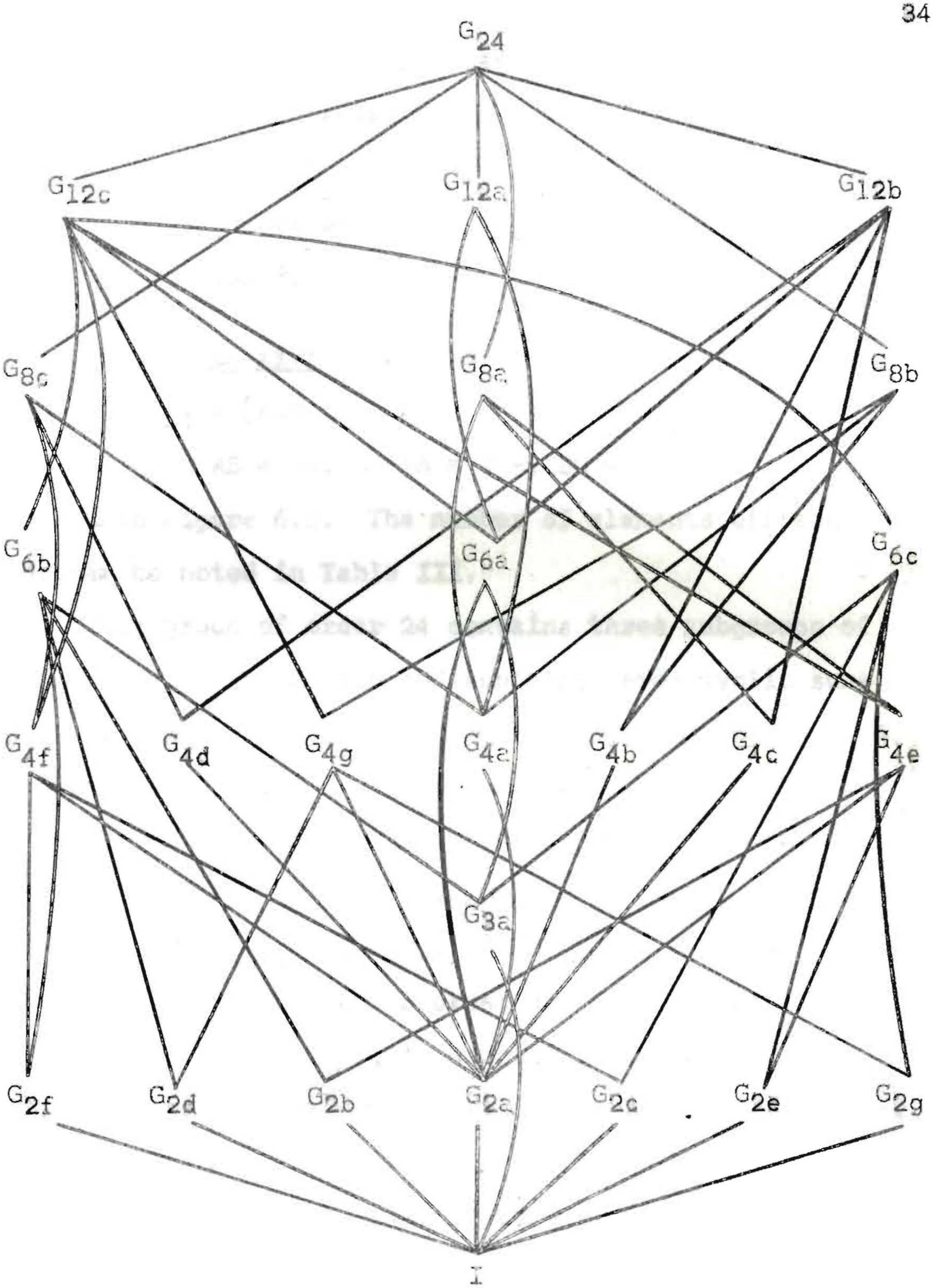

Figure 6.2. Lattice of G_{24} of type four.

The three subgroups of order 8 are isomorphic and each one may be represented by the group in Figure 4.9. G_{8a}, G_{8b}, and G_{8c} each contain two cyclic subgroups and a quadratic subgroup of order 4. G_{4a} is common to all of the subgroups of order 8.

6.3. Type five. The group of order 24 defined by

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$
$$AB = BA,\ A^{-1}CA = C^{-1},\ BC = CB,$$

is shown in Figure 6.3. The number of elements of each order may be noted in Table III.

This group of order 24 contains three subgroups of order 12. G_{12a} is Abelian and contains three cyclic subgroups of order 6 and G_{4a}, a quadratic subgroup of order 4. The subgroups of order 6 contain G_{3a} as a common subgroup. The three subgroups of order 2 contained in the subgroups of order 6 are all contained in G_{4a}.

G_{12b} and G_{12c} are dicyclic subgroups of order 12 and contain a common subgroup of order 6. G_{12b} contains G_{4b}, G_{4c}, and G_{4d} which are cyclic subgroups of order 4. The cyclic subgroups of order 4 contained in G_{12c} are G_{4e}, G_{4f}, and G_{4g}. All the subgroups of orders 4 and 6 contained in G_{12b} and G_{12c} contain a common subgroup of order 2, which is G_{2b}.

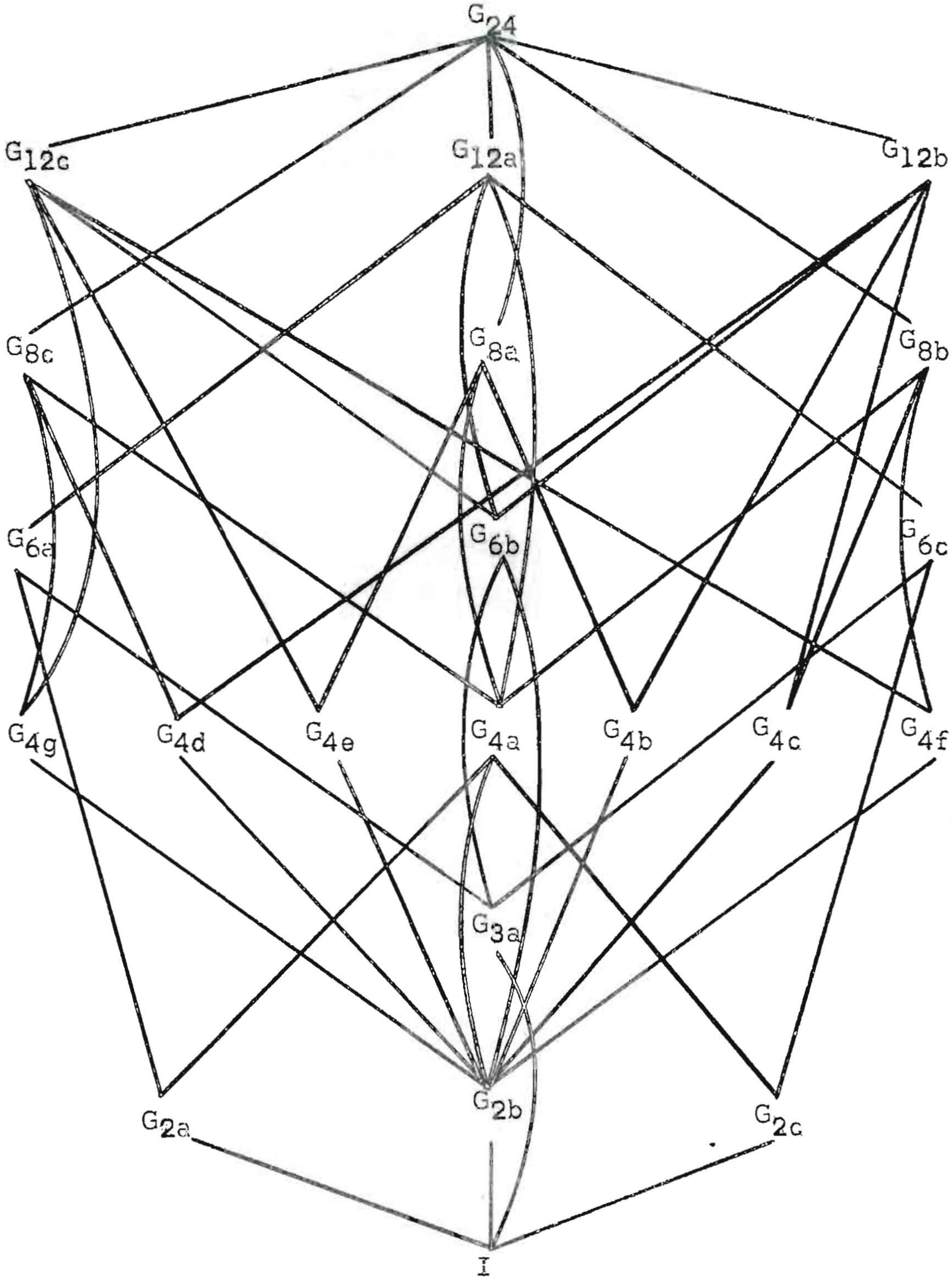

Figure 6.3. Lattice of G_{24} of type five.

The three subgroups of order 8 contained in this group of order 24 are Abelian. These subgroups are isomorphic and contain G_{4a} as a common subgroup. Each subgroup of order 8 also contains two cyclic subgroups of order 4. The seven subgroups of order 4 contain a common subgroup; G_{2b}.

CHAPTER VII

G_{24} WITH SUBGROUPS OF ORDER EIGHT OF TYPE $C_2 \times C_2 \times C_2$

7.1. Type six. This group of order 24 is the direct product of a group of order 3 and an Abelian group of order 8 of type $C_2 \times C_2 \times C_2$. The defining relations of this group are:

$$G_{24} = \{A,B,C,D\}, \text{ where } A^2 = B^2 = C^2 = D^3 = I,$$
$$AB = BA,\ AC = CA,\ BC = CB,$$
$$D^{-1}AD = A,\ D^{-1}BD = B,\ D^{-1}CD = C.$$

The lattice of this group is shown in Figure 7.1. The number of elements of each order may be observed in Table III.

G_{24} contains seven Abelian subgroups of order 12. The seven subgroups are isomorphic and each may be represented by Figure 4.13. These subgroups each contain a quadratic subgroup of order 4 and three cyclic subgroups of order 6. Each of the quadratic subgroups contain three subgroups of order 2; hence, three subgroups of order 2 are contained in each G_{12}. In a given subgroup of order 12, each of the subgroups of order 6 contain a subgroup of order 2 which is common to the quadratic subgroup. Each of the seven subgroups of order 6 is common to some three of the seven subgroups of order 12. G_{3a} is common to all subgroups of order 6 and therefore is common to all subgroups of order 12.

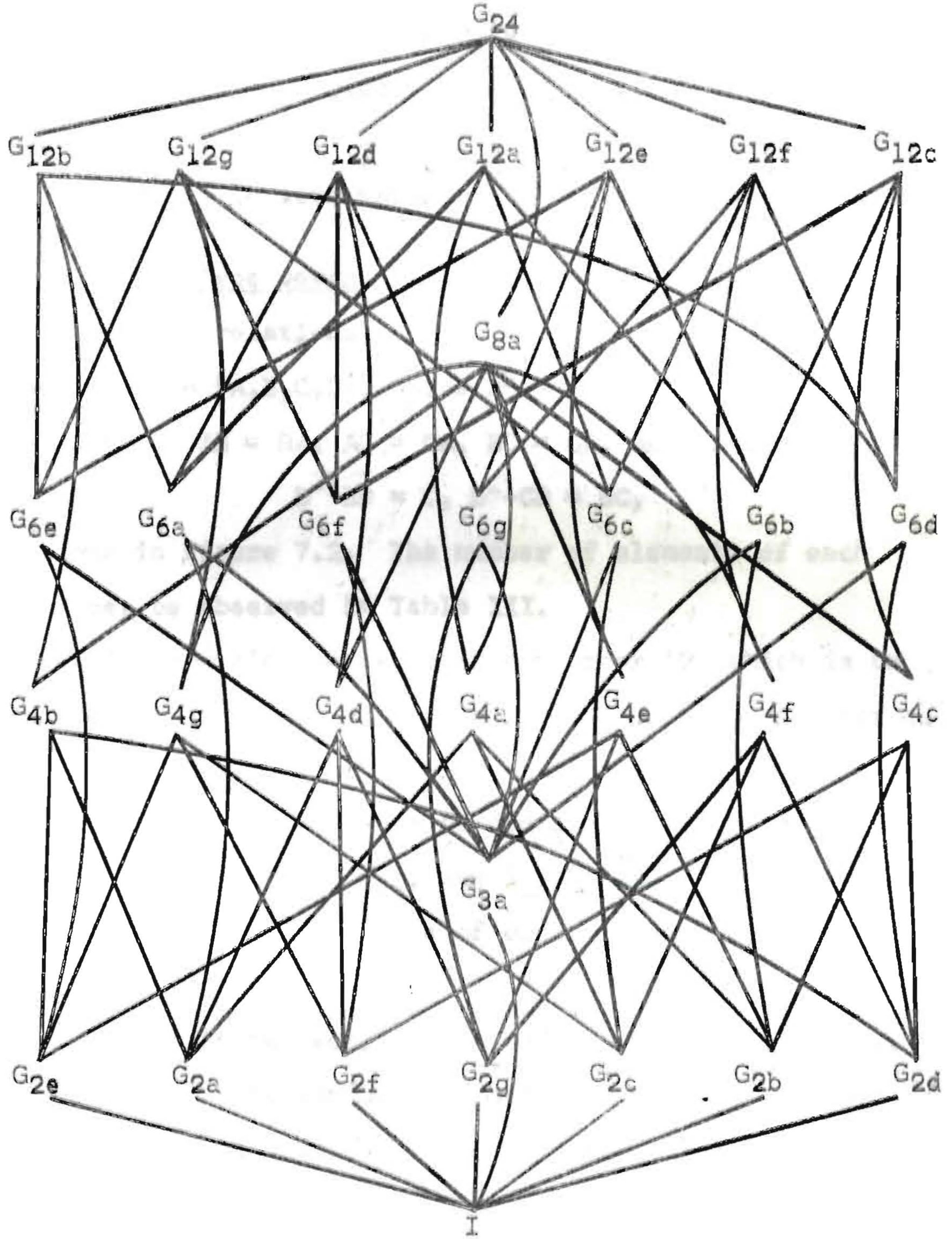

Figure 7.1. Lattice of G_{24} of type six.

The subgroup of order 8 is Abelian and isomorphic to the group shown in Figure 4.8. G_{8a} contains seven quadratic subgroups of order 4. Each of the seven subgroups of order 2 is contained in some three of the quadratic subgroups.

7.2. Type seven. The group of order 24 defined by the following relations,

$$G_{24} = \{A,B,C,D\}, \text{ where } A^2 = B^2 = C^2 = D^3 = I,$$
$$AB = BA, AC = CA, BC = CB, AD = DA,$$
$$D^{-1}BD = C, D^{-1}CD = BC,$$

is shown in Figure 7.2. The number of elements of each order may be observed in Table III.

G_{24} contains one subgroup of order 12, which is an alternating subgroup. G_{12a} contains a subgroup of order 4, four subgroups of order 3, and three subgroups of order 2. G_{4g} is a quadratic subgroup of order 4. This quadratic subgroup contains G_{2d}, G_{2e}, and G_{2f}, which are subgroups of order 2. The four subgroups of order 3 are G_{3a}, G_{3b}, G_{3c}, and G_{3d}.

There is one subgroup of order 8 contained in G_{24}. G_{8a} contains seven quadratic subgroups of order 4. Each of the quadratic subgroups contain three subgroups of order 2. As is indicated in the lattice, each subgroup of order 2 is common to some three of the seven quadratic subgroups.

The four subgroups of order 6 are cyclic. G_{2a}, a subgroup of order 2, is common to the four subgroups.

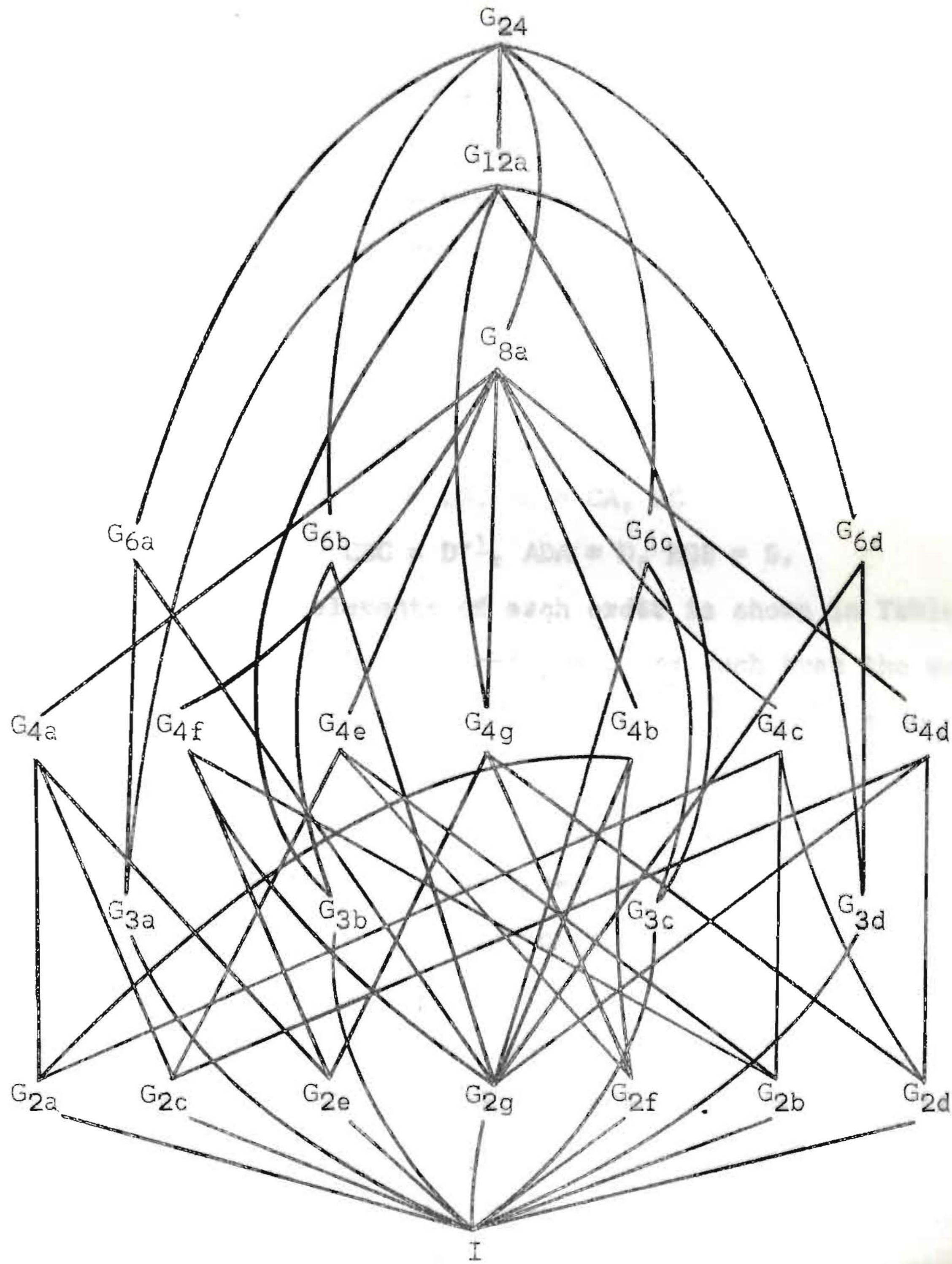

Figure 7.2. Lattice of G_{24} of type seven.

Since each of the subgroups of order 6 is unique, each must contain a different subgroup of order 3. G_{6a} contains G_{3a}; a subgroup of order 3. G_{3b} is the subgroup of order 3 which is contained in G_{6b}. The subgroups G_{6c} and G_{6d} contain G_{3c} and G_{3d} respectively.

7.3. Type eight. This group is defined by the following relations:

$$G_{24} = \{A,B,C,D\}, \text{ where } A^2 = B^2 = C^2 = D^3 = I,$$

$$AB = BA,\ AC = CA,\ BC = CB,$$

$$CDC = D^{-1},\ ADA = D,\ BDB = D.$$

The number of elements of each order is shown in Table III.

The structure of this group is such that the complete lattice if shown on one figure would be difficult to interpret. Therefore, Tables I and II are included as a guide to the structure of the group. Figures 7.3 and 7.4 show the two types of subgroups of order 12 which are included in G_{24}. The structure of each of the three subgroups of order 8 is shown in Figure 7.5.

G_{12a}, an Abelian subgroup of order 12, is shown in Figure 7.3. This subgroup contains three cyclic subgroups of order 6 and G_{4a}; a quadratic subgroup of order 4. G_{6a}, G_{6b}, and G_{6c}, the cyclic subgroups of order 6, each contain a subgroup of order 2 which is common to the quadratic subgroup. G_{3a}, a subgroup of order 3, is common to the three subgroups of order 6.

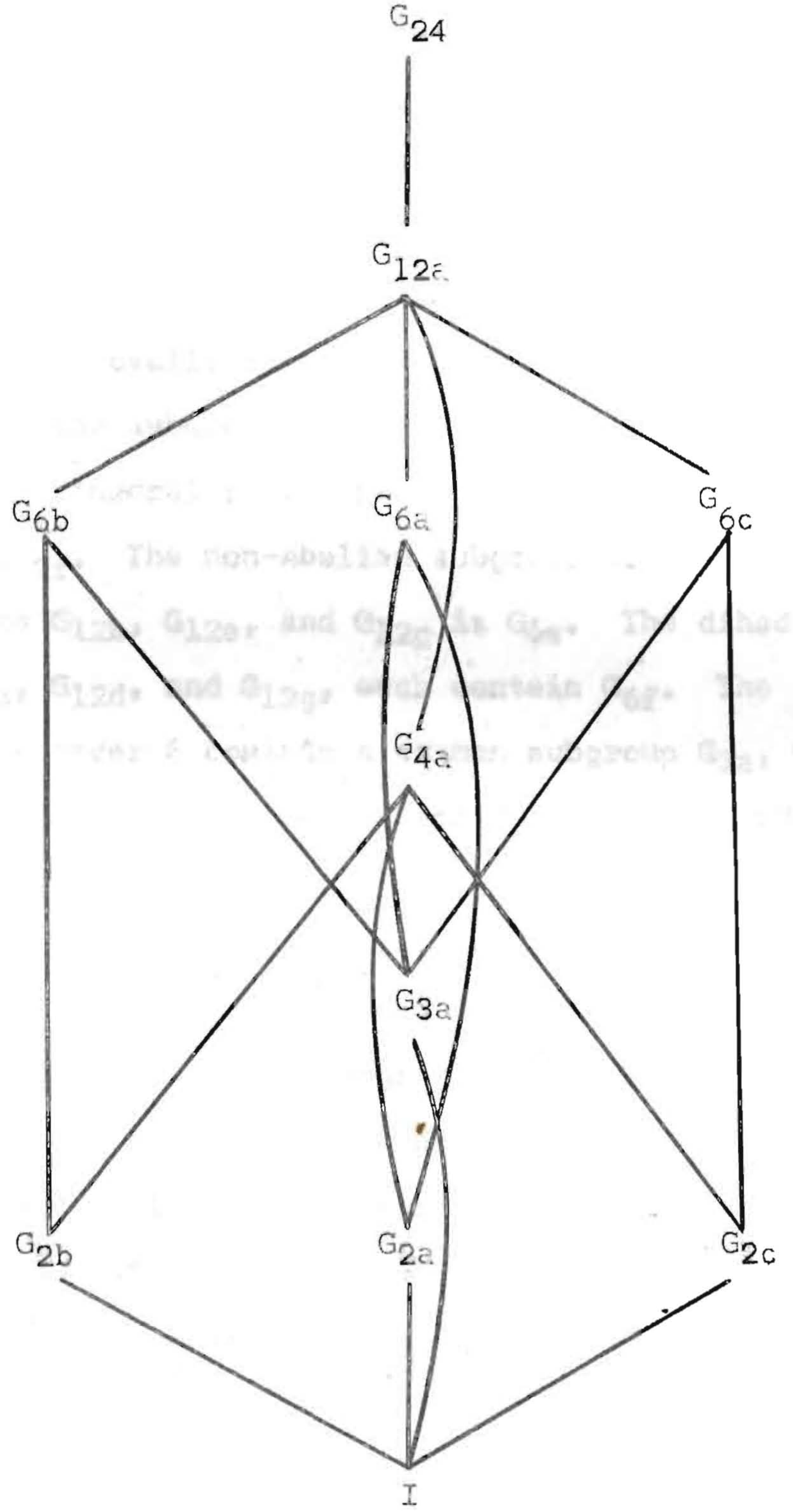

Figure 7.3. Lattice of Abelian subgroup of order 12.

Figure 7.4 and Table I are used as a guide in analyzing the six dihedral subgroups of order 12. Each of the subgroups contain a cyclic subgroup of order 6. G_{12b} and G_{12c} contain G_{6a}. The cyclic subgroup of order 6 which is common to G_{12d} and G_{12e} is G_{6b}. G_{12f} and G_{12g} contain G_{6c} as a common cyclic subgroup of order 6. Each of the four non-Abelian subgroups of order 6 is contained in some three of the dihedral subgroups. G_{6d} is common to G_{12b}, G_{12d}, and G_{12f}. The non-Abelian subgroup of order 6 which is common to G_{12b}, G_{12e}, and G_{12g} is G_{6e}. The dihedral subgroups G_{12c}, G_{12d}, and G_{12g}, each contain G_{6f}. The seven subgroups of order 6 contain a common subgroup G_{3a}, which is of order 3. Each of the fifteen subgroups of order 2 is contained in a subgroup of order 6.

With the exception of G_{4a}, each of the quadratic subgroups of order 4 is contained in a dihedral subgroup of order 12. There are seven subgroups of order 2 contained in each dihedral subgroup. One of these subgroups of order 2 is contained in the three quadratic subgroups and the cyclic subgroup of order 6. Each of the six remaining subgroups of order 2 is common to a quadratic subgroup and a non-Abelian subgroup of order 6.

Since it would be repetitious to show the lattice of each dihedral subgroup, only the lattice of G_{12b} is shown.

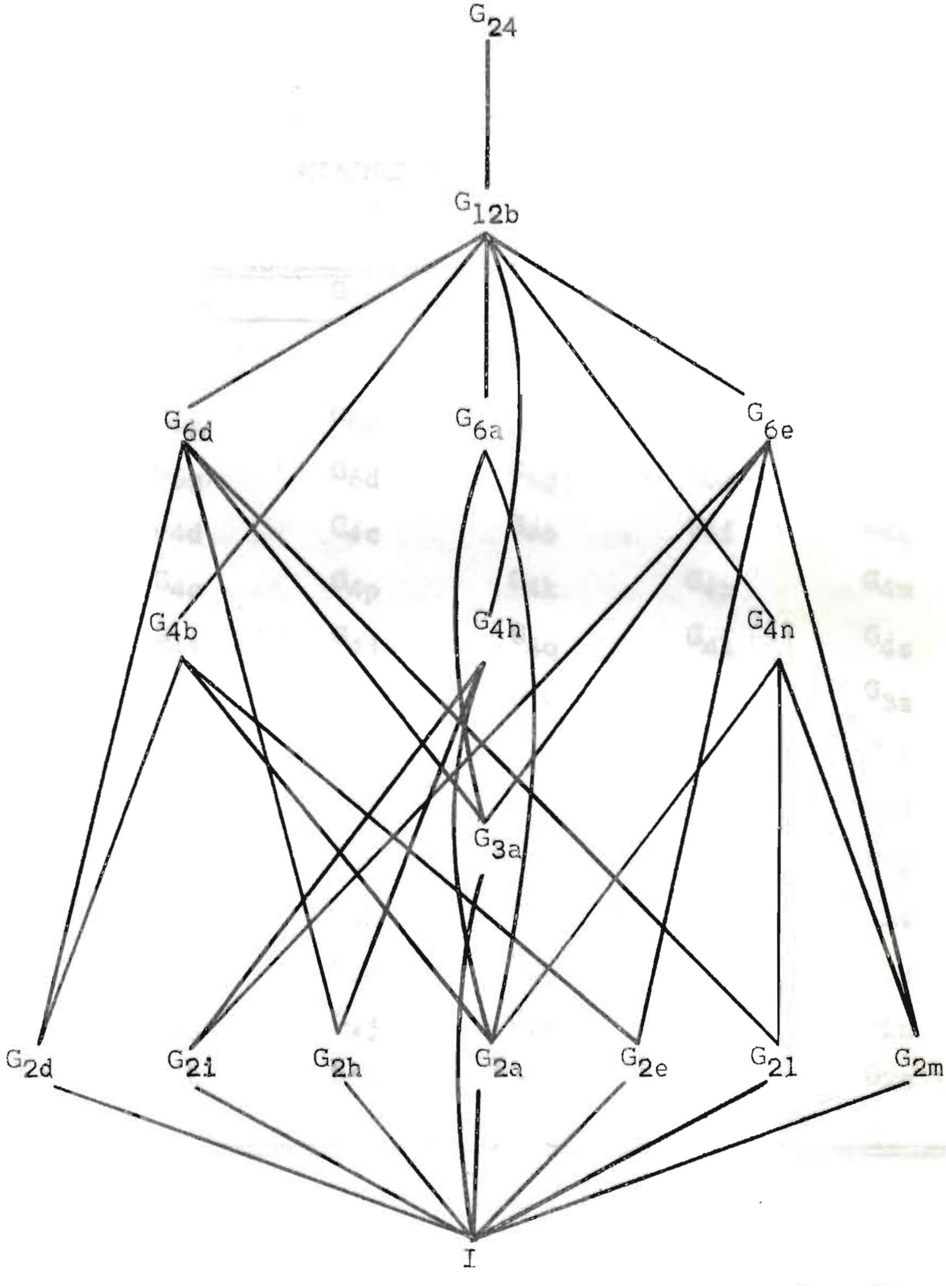

Figure 7.4. Lattice of dihedral subgroup of order 12.

TABLE I

DIHEDRAL SUBGROUPS OF ORDER TWELVE
CONTAINED IN TYPE EIGHT

G_{12b}	G_{12c}	G_{12d}	G_{12e}	G_{12f}	G_{12g}
G_{6d}	G_{6f}	G_{6f}	G_{6e}	G_{6d}	G_{6f}
G_{6a}	G_{6a}	G_{6b}	G_{6b}	G_{6c}	G_{6c}
G_{6e}	G_{6g}	G_{6d}	G_{6g}	G_{6g}	G_{6e}
G_{4b}	G_{4d}	G_{4c}	G_{4e}	G_{4f}	G_{4g}
G_{4h}	G_{4o}	G_{4p}	G_{4k}	G_{4r}	G_{4m}
G_{4n}	G_{4i}	G_{4j}	G_{4q}	G_{4l}	G_{4s}
G_{3a}	G_{3a}	G_{3a}	G_{3a}	G_{3a}	G_{3a}
G_{2d}	G_{2f}	G_{2f}	G_{2e}	G_{2d}	G_{2f}
G_{2i}	G_{2o}	G_{2l}	G_{2k}	G_{2o}	G_{2i}
G_{2h}	G_{2n}	G_{2n}	G_{2i}	G_{2l}	G_{2j}
G_{2a}	G_{2a}	G_{2b}	G_{2b}	G_{2c}	G_{2c}
G_{2e}	G_{2g}	G_{2d}	G_{2g}	G_{2g}	G_{2e}
G_{2l}	G_{2j}	G_{2j}	G_{2m}	G_{2h}	G_{2n}
G_{2m}	G_{2k}	G_{2h}	G_{2o}	G_{2k}	G_{2m}

Figure 7.4 is the lattice of this subgroup. Table I enumerates the subgroups contained in each of the six dihedral subgroups. For example, column one lists the subgroups contained in G_{12b}. A system of substitution is utilized in determining the lattice of each of the remaining dihedral subgroups. To obtain the lattice of a given dihedral subgroup, each subgroup in column one of Table I is replaced by the subgroup in the same row of the dihedral subgroup being considered. Thus, to obtain the lattice of G_{12d}, G_{12b} is replaced by G_{12d}; G_{6d} is replaced by G_{6f}; G_{6a} is replaced by G_{6b}; and similarly for the remainder of the column.

The three subgroups of order 8 contained in G_{24} are Abelian. Each of the subgroups contain seven quadratic subgroups of order 4 and seven subgroups of order 2. G_{8a}, G_{8b}, and G_{8c} contain a common subgroup; G_{4a} of order 4. Therefore, G_{2a}, G_{2b}, and G_{2c} are common to the three subgroups of order 8.

Since the subgroups of order 8 are isomorphic, it would be redundant to show the lattice structure of each. Hence, only the lattice of G_{8a} is shown. Figure 7.5 is the lattice of this subgroup. Table II lists the subgroups contained in each subgroup of order 8. For example, column one enumerates the subgroups contained in G_{8a}. As in the dihedral subgroups of order 12, a system of substitution is employed to determine the lattice of the two remaining subgroups of order 8. To obtain the lattice of a given subgroup of order 8, each subgroup in column one is replaced by a certain subgroup which is contained in the subgroup of order 8 under consideration. For instance, to determine the lattice of G_{8b}, G_{8a} is replaced by G_{8b}; G_{4c} is replaced by G_{4k}; G_{4f} is replaced by G_{41}; and similarly for the remainder of the column.

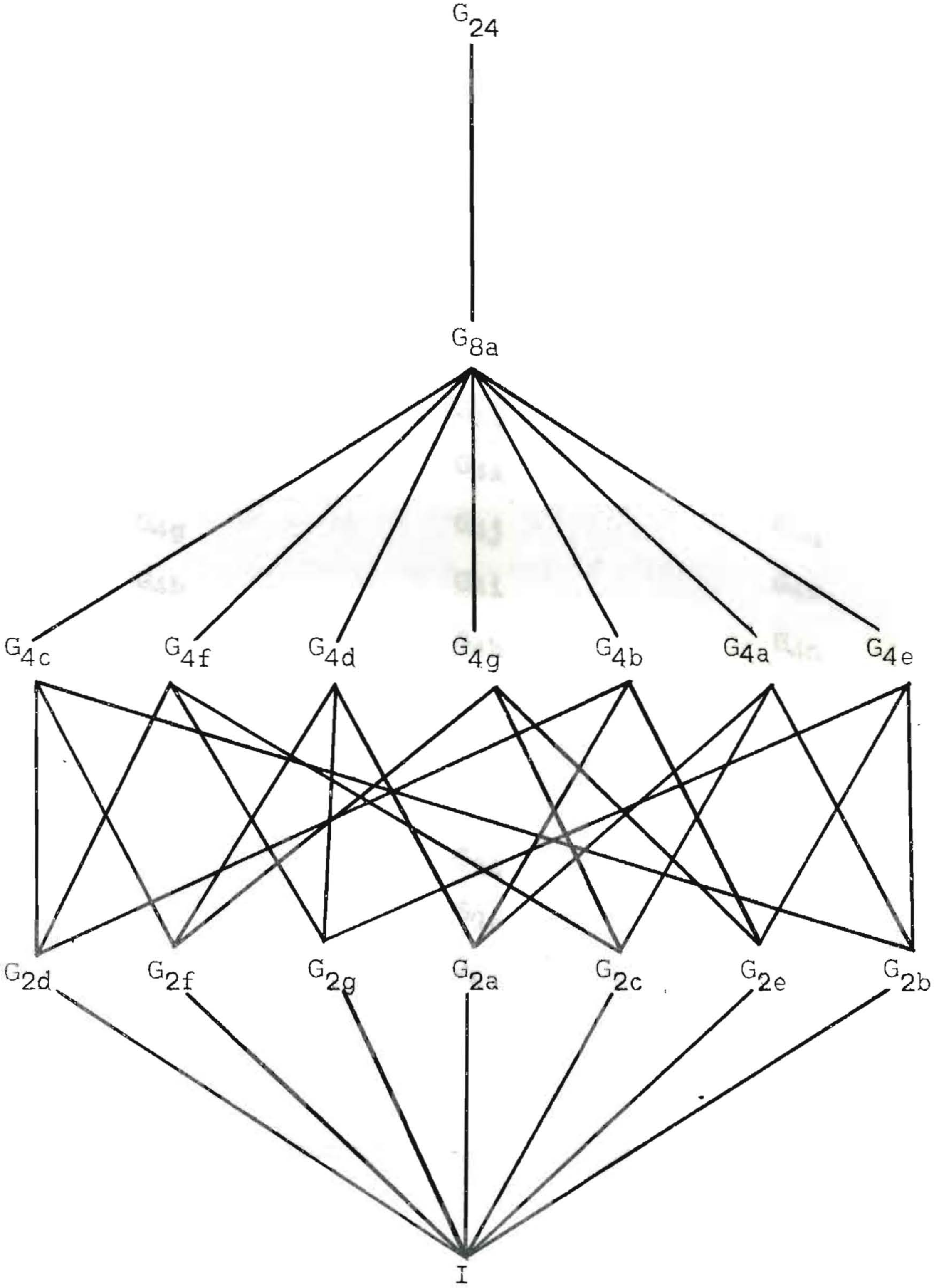

Figure 7.5. Lattice of Abelian subgroup of order 8.

TABLE II

SUBGROUPS OF ORDER EIGHT

CONTAINED IN TYPE EIGHT

G_{8a}	G_{8b}	G_{8c}
G_{4c}	G_{4k}	G_{4p}
G_{4f}	G_{4l}	G_{4s}
G_{4d}	G_{4a}	G_{4a}
G_{4g}	G_{4j}	G_{4q}
G_{4b}	G_{4i}	G_{4o}
G_{4a}	G_{4h}	G_{4n}
G_{4e}	G_{4m}	G_{4r}
G_{2d}	G_{2k}	G_{2n}
G_{2f}	G_{2b}	G_{2b}
G_{2g}	G_{2c}	G_{2c}
G_{2a}	G_{2a}	G_{2a}
G_{2c}	G_{2h}	G_{2m}
G_{2e}	G_{2j}	G_{2o}
G_{2b}	G_{2i}	G_{2l}

CHAPTER VIII

G_{24} WITH DICYCLIC SUBGROUPS OF ORDER EIGHT

8.1. Type nine. This group of order 24 is defined by the following relations,

$$G_{24} = \{A,B,C\},\ \text{where } A^4 = B^4 = C^3 = I,\ A^2 = B^2,$$
$$B^{-1}AB = A^{-1},\ C^{-1}AC = A,\ C^{-1}BC = B,$$

and is shown in Figure 8.1. The group is the direct product of a dicyclic group of order 8 and a group of order 3. The number of elements of each order is shown in Table III.

This group contains three subgroups of order 12. The subgroups of order 12 are cyclic and contain a common subgroup of order 6 which is cyclic. The subgroups of orders 2 and 3 are contained in G_{6a} and therefore are contained in the three subgroups of order 12. G_{12a}, G_{12b}, and G_{12c} contains G_{4a}, G_{4b}, and G_{4c} respectively as the cyclic subgroups of order 4.

One subgroup of order 8 is contained in this group of order 24. G_{8a} is dicyclic and contains three cyclic subgroups of order 4. The three subgroups of order 4 contain G_{2a}; a subgroup of order 2.

8.2. Type ten. The following defining relations define this group:

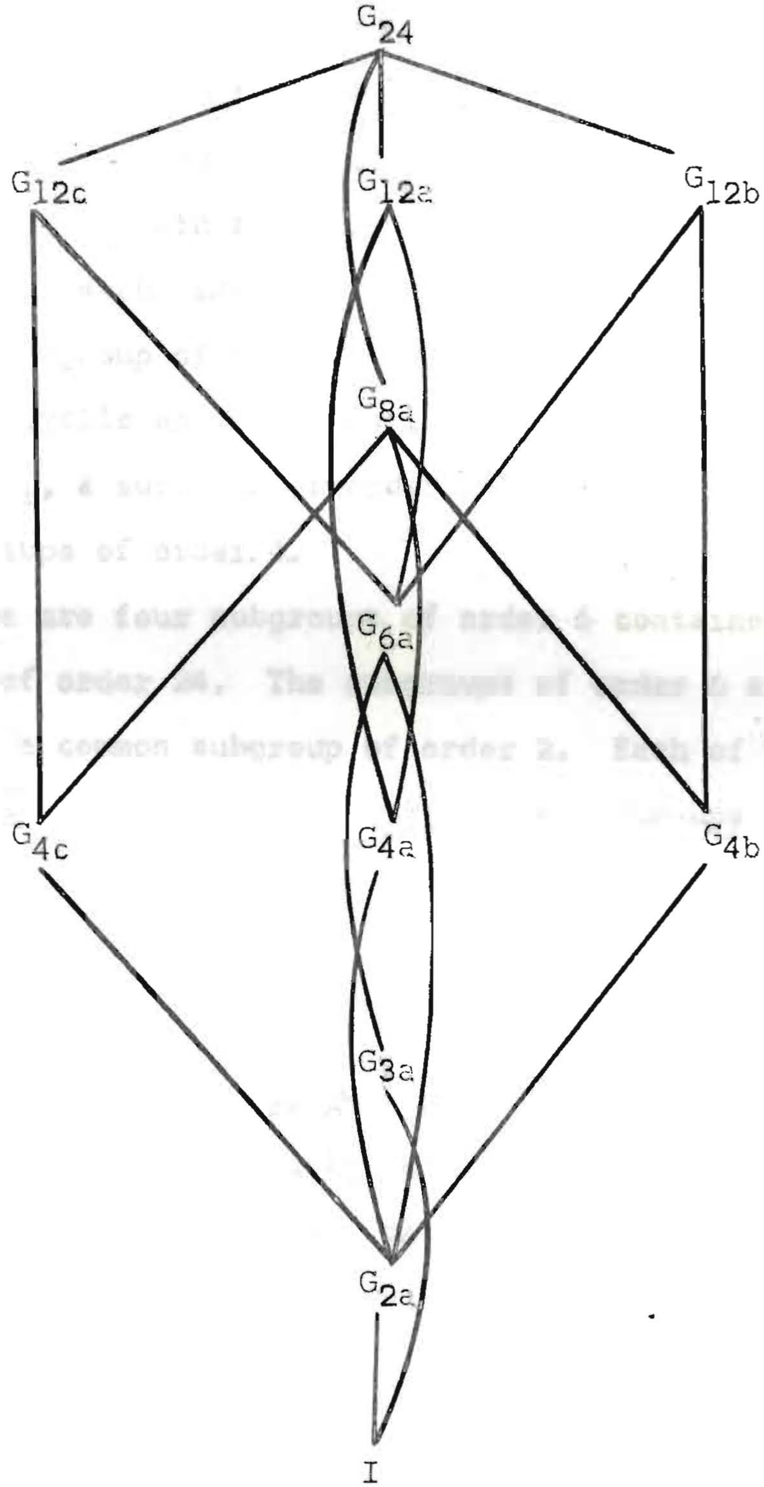

Figure 8.1. Lattice of G_{24} of type nine.

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^4 = C^3 = I,\ A^2 = B^2,$$
$$B^{-1}AB = A^{-1},\ C^{-1}AC = B,\ C^{-1}BC = AB.$$

As may be observed from the lattice in Figure 8.2, this group does not contain a subgroup of order 12. The number of elements of each order is shown in Table III.

The subgroup of order 8, which is contained in this group, is dicyclic and contains three cyclic subgroups of order 4. G_{2a}, a subgroup of order 2, is common to the three subgroups of order 4.

There are four subgroups of order 6 contained in this group of order 24. The subgroups of order 6 are cyclic and contain a common subgroup of order 2. Each of the subgroups of order 6 contains one of the subgroups of order 3.

8.3. Type eleven. The group of order 24 with the following defining relations,

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^4 = C^3 = I,\ A^2 = B^2,$$
$$B^{-1}AB = A^{-1},\ A^{-1}CA = C,\ B^{-1}CB = C^{-1},$$

is the dicyclic group of order 24 and is shown in Figure 8.3. The number of elements of each order is shown in Table III.

This group contains one cyclic and two dicyclic subgroups of order 12. The three subgroups contain G_{6a}; a cyclic subgroup of order 6. G_{2a} and G_{3a} also are common to the subgroups of order 12. G_{12a} contains G_{4a}; a cyclic

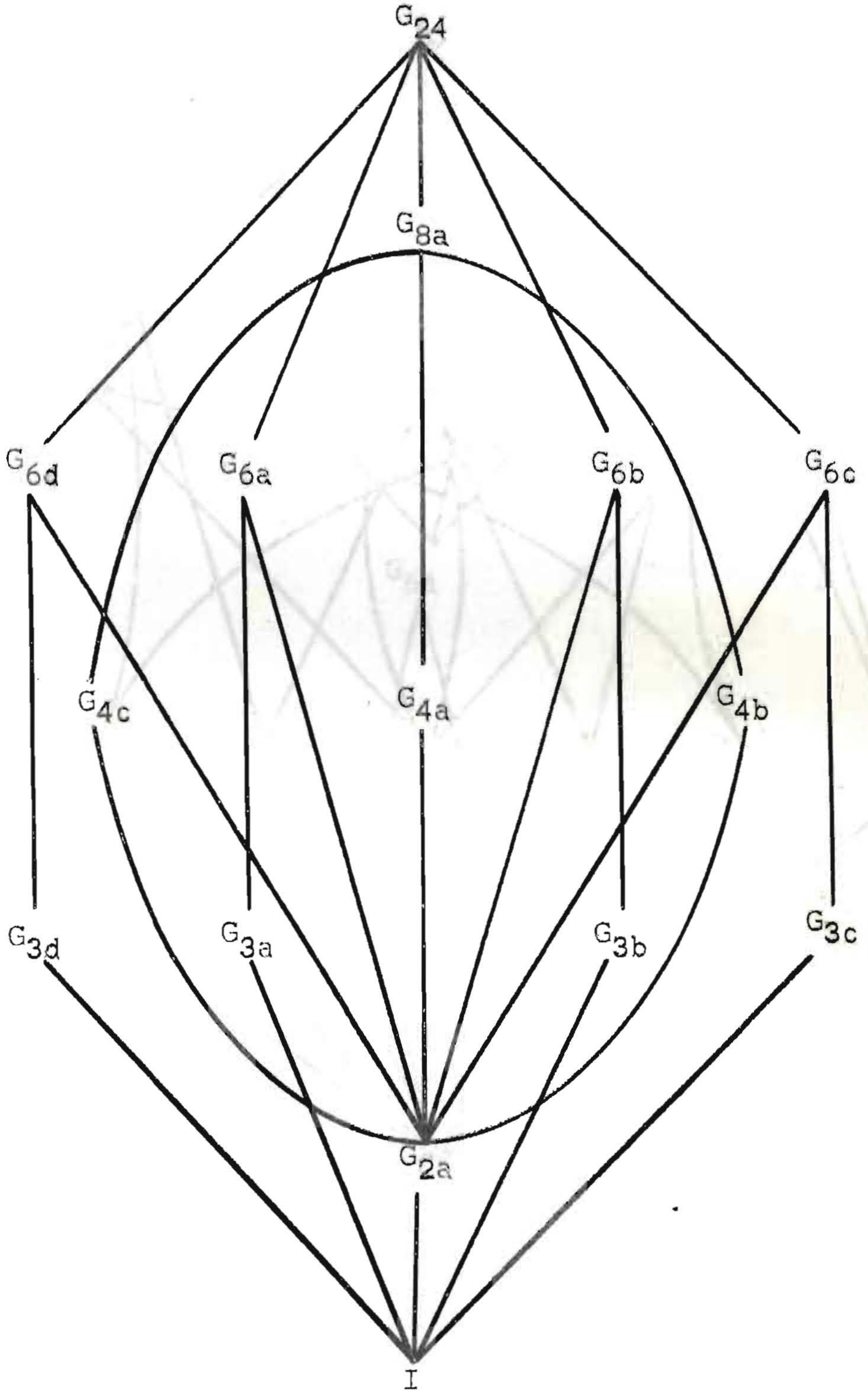

Figure 8.2. Lattice of G_{24} of type ten.

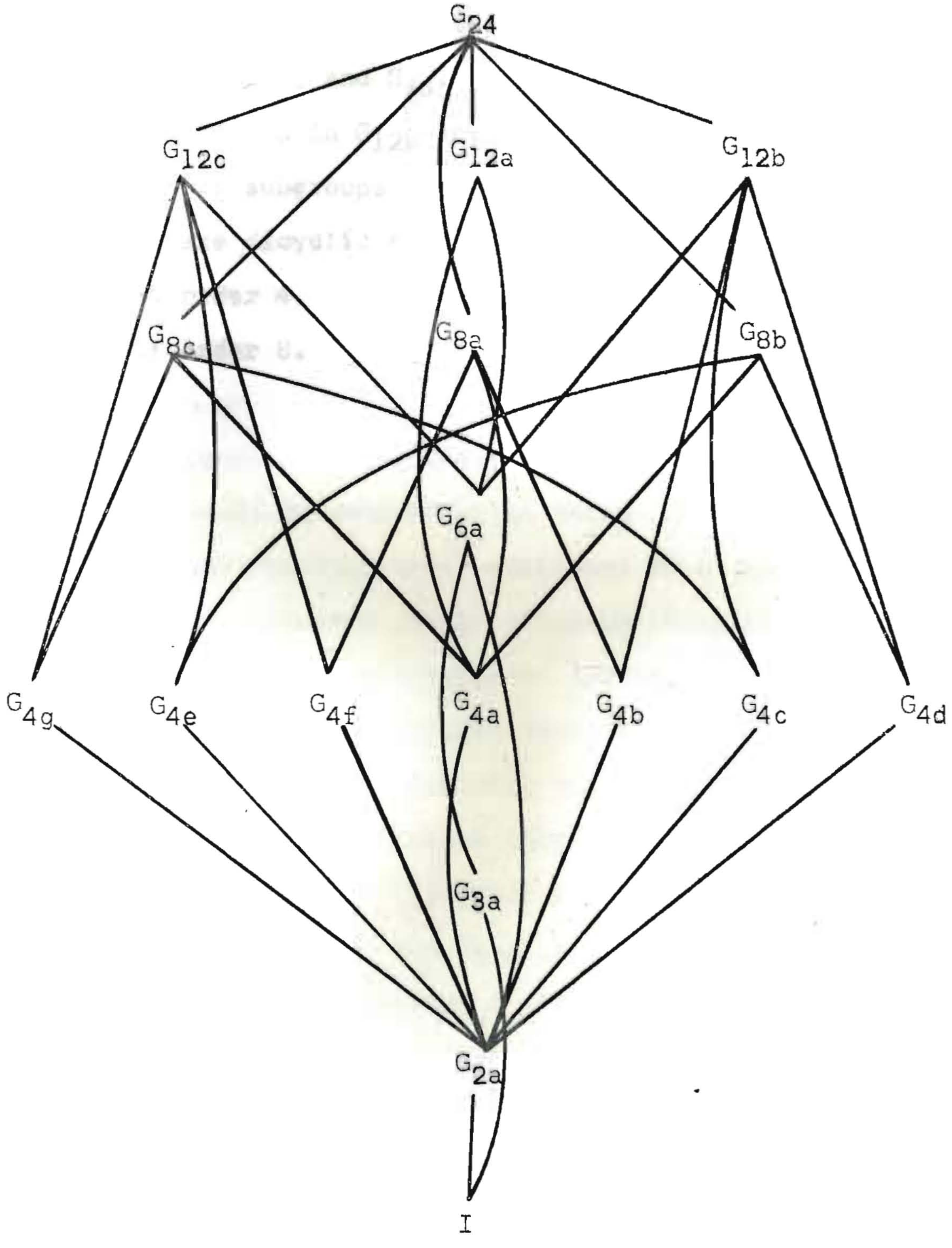

Figure 8.3. Lattice of G_{24} of type eleven.

subgroup of order 4. G_{12b} contains three cyclic subgroups of order 4; G_{4b}, G_{4c}, and G_{4d}. The three cyclic subgroups of order 4 contained in G_{12c} are G_{4e}, G_{4f}, and G_{4g}.

The three subgroups of order 8 are isomorphic. G_{8a}, G_{8b}, and G_{8c} are dicyclic and each contain three cyclic subgroups of order 4. G_{4a} and G_{2a} are common to the three subgroups of order 8.

CHAPTER IX

G_{24} WITH DIHEDRAL SUBGROUPS OF ORDER EIGHT

9.1. Type twelve. The group of order 24 defined by

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$

$$BAB = A^{-1},\ C^{-1}AC = A,\ C^{-1}BC = B,$$

is shown in Figure 9.1. This group is the direct product of the dihedral group of order 8 and a group of order 3. The number of elements of each order is shown in Table III.

The group contains three subgroups of order 12. The three subgroups contain a common subgroup of order 6. G_{3a} and G_{2a} are also common subgroups of the three subgroups of order 12. G_{12a}, which is cyclic, contains G_{4a}; a cyclic subgroup of order 4. The remaining two subgroups of order 12 are Abelian and each contains three cyclic subgroups of order 6. G_{4b} and G_{4c} are quadratic subgroups of order 4 and are contained in G_{12b} and G_{12c} respectively.

This group also contains a dihedral subgroup of order 8. G_{8a} contains one cyclic and two quadratic subgroups of order 4. The three subgroups of order 4 contain G_{2a} as a common subgroup of order 2.

The five subgroups of order 6 are cyclic and contain a common subgroup of order 3. The five subgroups of order 2

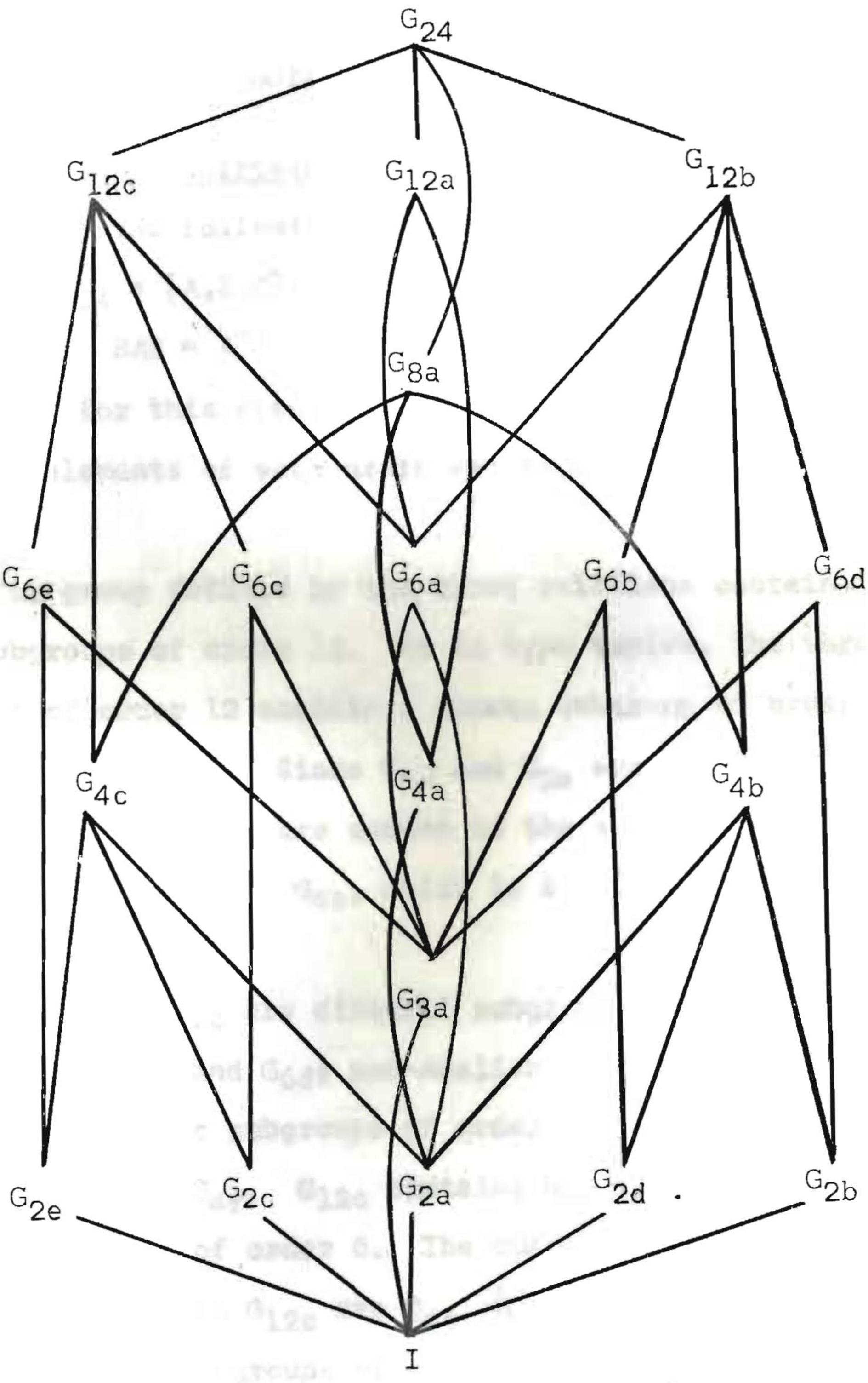

Figure 9.1. Lattice of G_{24} of type twelve.

are contained in the subgroups of order 6. However, no two subgroups of order 6 contain a common subgroup of order 2.

9.2. <u>Type thirteen</u>. The dihedral group of order 24 is defined by the following relations:

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$
$$BAB = A^{-1},\ A^{-1}CA = C,\ BCB = C^{-1}.$$

The lattice for this group is shown in Figure 9.2. The number of elements of each order may be observed in Table III.

The group defined by the above relations contains three subgroups of order 12. As in type twelve, the three subgroups of order 12 contain a common subgroup of order 6; G_{6a}, which is cyclic. Since G_{3a} and G_{2a} are subgroups of G_{6a}, the two subgroups are common to the subgroups of order 12. G_{12a} also contains G_{4a}, which is a cyclic subgroup of order 4.

G_{12b} and G_{12c} are dihedral subgroups of order 12. G_{12b} contains G_{6b} and G_{6d}; non-Abelian subgroups of order 6. The three quadratic subgroups of order 4 contained in G_{12b} are G_{4b}, G_{4d}, and G_{4f}. G_{12c} contains G_{6c} and G_{6e} as non-Abelian subgroups of order 6. The quadratic subgroups of order 4 contained in G_{12c} are G_{4c}, G_{4e}, and G_{4g}.

The three subgroups of order 8 are isomorphic and each may be represented by the dihedral group in Figure 4.10.

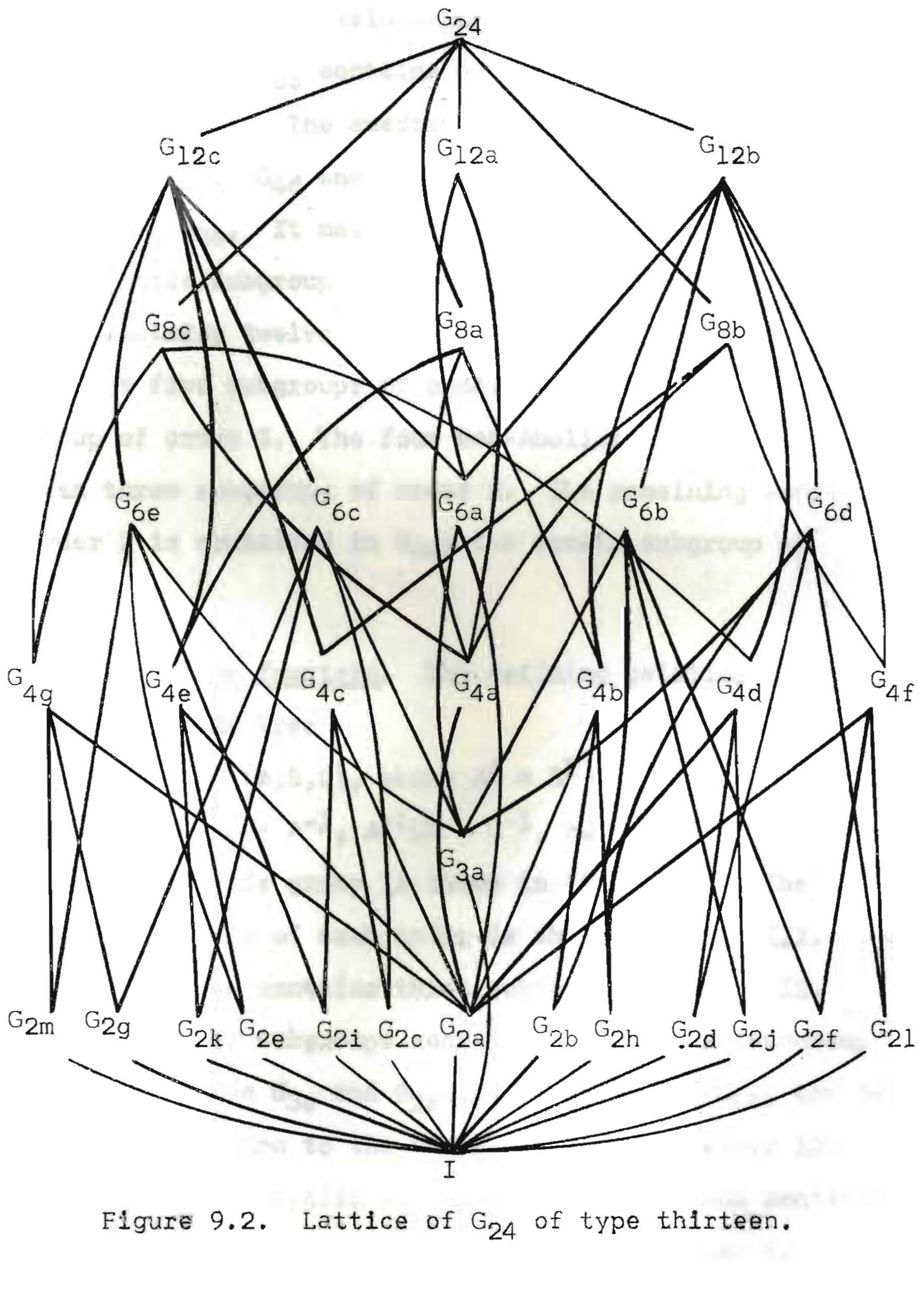

Figure 9.2. Lattice of G_{24} of type thirteen.

The three subgroups contain a common subgroup of order 4, which is cyclic. G_{8a} contains G_{4b} and G_{4e} as quadratic subgroups of order 4. The quadratic subgroups contained in G_{8b} are G_{4c} and G_{4f}. G_{4d} and G_{4g} are the quadratic subgroups contained in G_{8c}. It may be observed from the lattice that each quadratic subgroup of order 4 contains G_{2a} and some two of the remaining twelve subgroups of order 2.

The five subgroups of order 6 contain a common subgroup of order 3. The four non-Abelian subgroups each contain three subgroups of order 2. The remaining subgroup of order 2 is contained in G_{6a}; the cyclic subgroup of order 6.

9.3. <u>Type fourteen</u>. The defining relations for this group of order 24 are:

$$G_{24} = \{A,B,C\}, \text{ where } A^4 = B^2 = C^3 = I,$$
$$BAB = A^{-1},\ A^{-1}CA = C^{-1},\ BCB = C.$$

The lattice of this group is shown in Figure 9.3. The number of elements of each order is shown in Table III.

This group contains three subgroups of order 12. Each of the three subgroups contain G_{6a}; a cyclic subgroup of order 6. Since G_{3a} and G_{2a} are subgroups of G_{6a}, the two subgroups are common to the three subgroups of order 12.

G_{12a} is a dicyclic subgroup of order 12 and contains a subgroup of order 6 and three subgroups of order 4. The

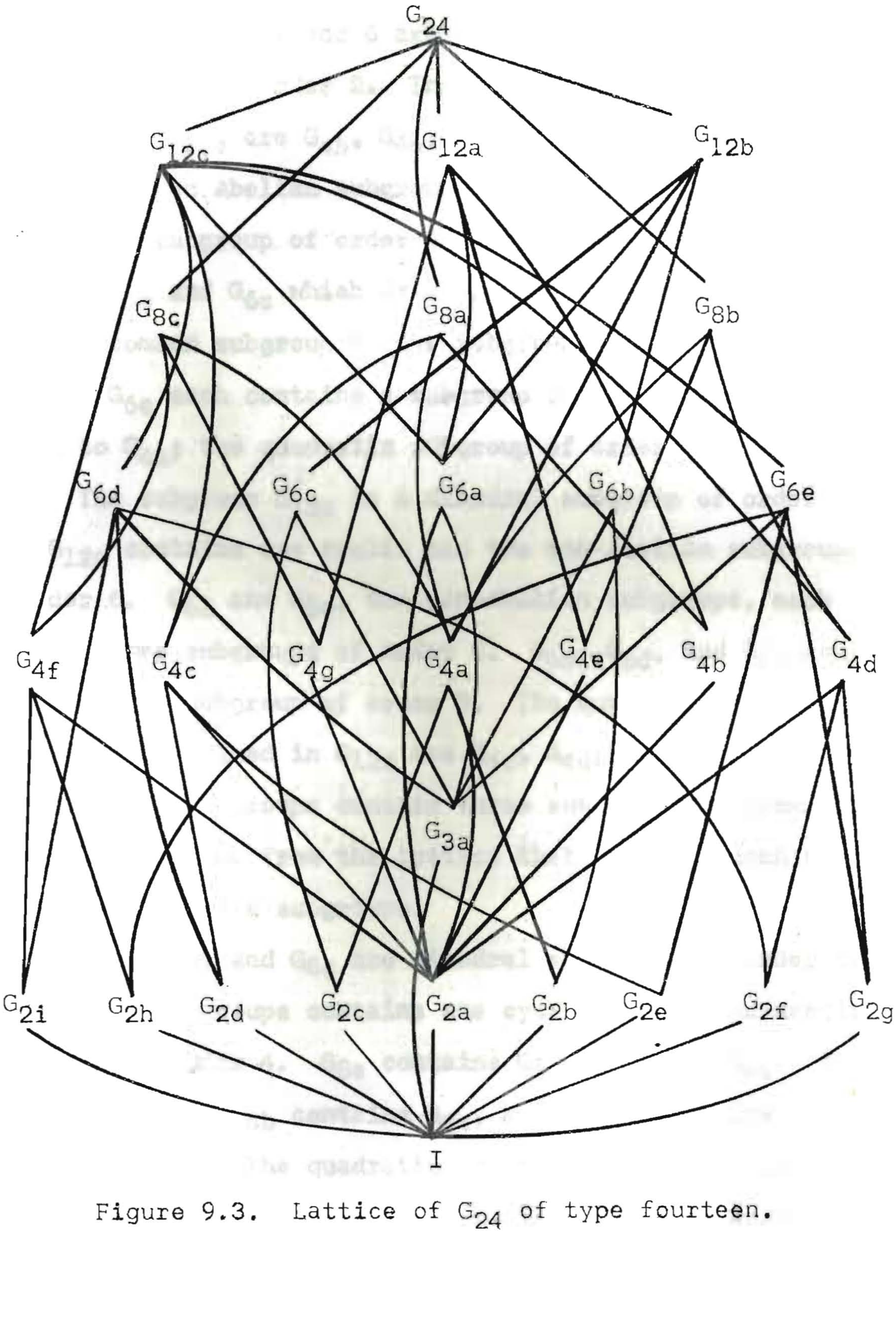

Figure 9.3. Lattice of G_{24} of type fourteen.

subgroups of orders 4 and 6 are cyclic and each contains G_{2a}; a subgroup of order 2. The three subgroups of order 4 contained in G_{12a} are G_{4b}, G_{4e}, and G_{4g}.

G_{12b}, an Abelian subgroup of order 12, contains G_{4a}; a quadratic subgroup of order 4. In addition to G_{6a}, G_{12b} contains G_{6b} and G_{6c} which are cyclic subgroups of order 6. G_{3a} is a common subgroup to the subgroups of order 6. G_{6a}, G_{6b}, and G_{6c} each contains a subgroup of order 2, which is common to G_{4a}; the quadratic subgroup of order 4.

The subgroup G_{12c} is a dihedral subgroup of order 12. G_{12c} contains one cyclic and two non-Abelian subgroups of order 6. G_{6d} and G_{6e}, the non-Abelian subgroups, each contain three subgroups of order 2. G_{6a}, G_{6d}, and G_{6e} contain a common subgroup of order 3. The quadratic subgroups of order 4 contained in G_{12c} are G_{4c}, G_{4d}, and G_{4f}. Each of the quadratic subgroups contain three subgroups of order 2. It may be observed from the lattice that G_{2a} is common to the three quadratic subgroups.

G_{8a}, G_{8b}, and G_{8c} are dihedral subgroups of order 8. Each of the subgroups contains one cyclic and two quadratic subgroups of order 4. G_{8a} contains G_{4a}, G_{4b}, and G_{4c}. In addition to G_{4e}, G_{8b} contains G_{4a}, and G_{4d}, which are quadratic subgroups. The quadratic subgroups contained in G_{8c} are G_{4a} and G_{4f}. G_{8c} also contains G_{4g}; a cyclic subgroup of order 4. G_{4a}, a quadratic subgroup, is contained in each of the subgroups of order 8. G_{2a}, a subgroup of order 2, is contained in all seven subgroups of order 4.

9.4. Type fifteen. The group defined by the following relations,

$$G_{24} = \{A,B\}, \text{ where } A^4 = B^3 = (AB)^2 = I,$$

is the symmetric group of order 24. The lattice of this group is shown in Figure 9.4. The number of elements of each order is shown in Table III.

This group contains one subgroup of order 12. G_{12a} is an alternating subgroup and contains subgroups whose orders are 2, 3, and 4. The subgroup of order 4, a quadratic subgroup, contains three subgroups of order 2. The four subgroups of order 3, which are contained in G_{12a}, necessarily contain only the unit element in common.

The three subgroups of order 8 are isomorphic and each may be represented by Figure 4.10. Each of the subgroups of order 8 contain one cyclic and two quadratic subgroups of order 4. G_{4g}, a quadratic subgroup, is common to the three subgroups of order 8. G_{2g}, a subgroup of order 2, is common to the three subgroups of order 4 which are contained in G_{8a}. G_{8b} contains G_{4c}, G_{4f}, and G_{4g}. G_{2i} is a common subgroup to these subgroups of order 4. The three subgroups of order 4 contained in G_{8c} are G_{4b}, G_{4e}, and G_{4g}. G_{2h} is a common subgroup to the three subgroups of order 4, which are contained in G_{8c}.

The four subgroups of order 6 are non-Abelian. Each contains three subgroups of order 2 and a subgroup of order 3. Any two of the subgroups of order 6 contain a common subgroup of order 2. G_{2a} is common to G_{6a} and G_{6d}. The subgroup of order 2 contained in G_{6a} and G_{6b} is G_{2b}. As a common subgroup of order 2, G_{6d} and G_{6b} contain G_{2f}. G_{6d} and G_{6c} contain G_{2d}, a subgroup of order 2. The subgroup which is common to G_{6a} and G_{6c} is G_{2e}. G_{6b} and G_{6c} contain G_{2c} as a common subgroup of order 2.

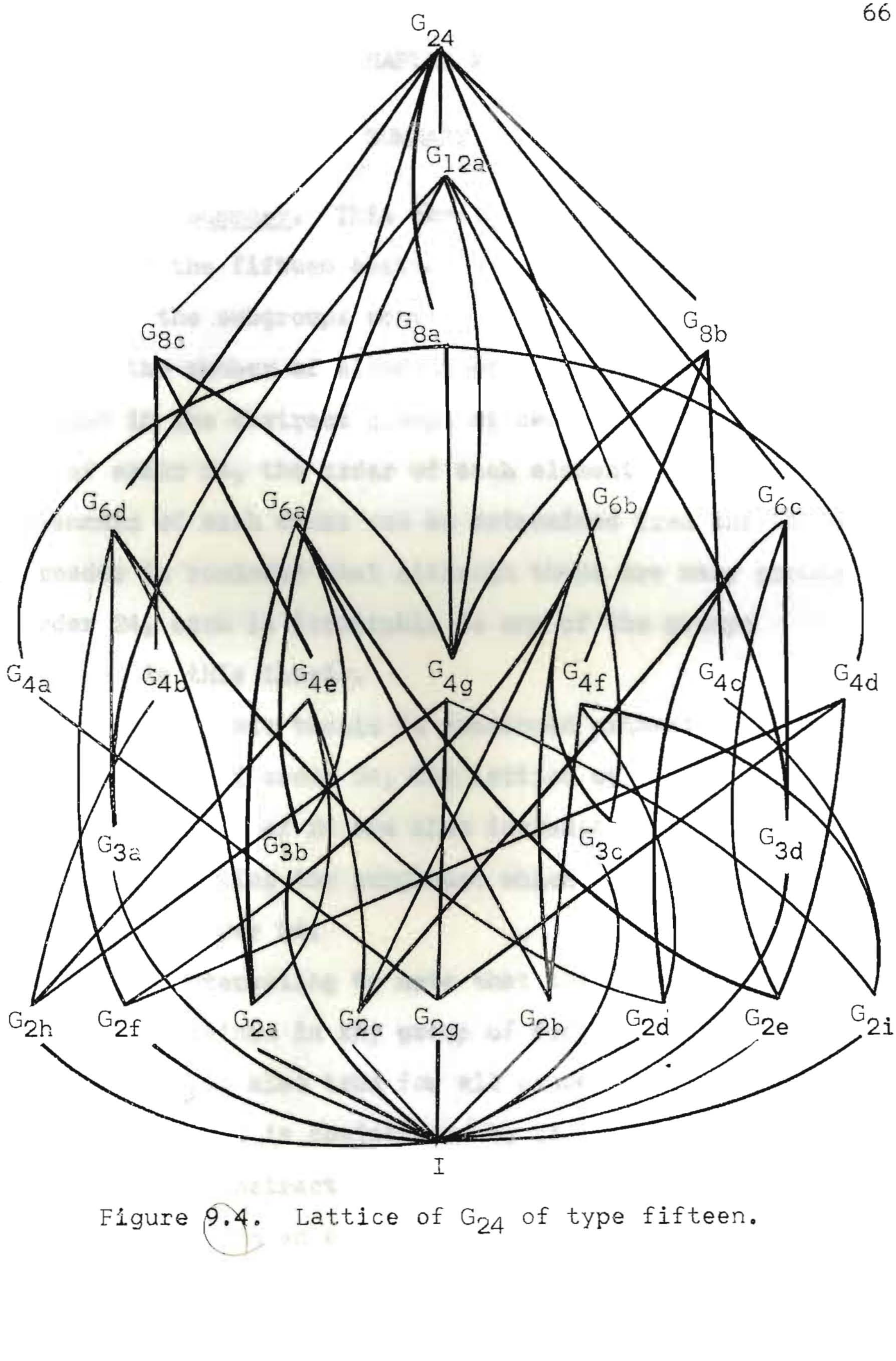

Figure 9.4. Lattice of G_{24} of type fifteen.

CHAPTER X

SUMMARY

10.1. Summary. This thesis contains the defining relations of the fifteen abstract groups of order 24 and an analysis of the subgroups contained in each group. Table III shows the number of elements of each order which are contained in the abstract groups of order 24. For a given group of order 24, the order of each element and the number of elements of each order can be determined from the lattice. The reader is reminded that although there are many groups of order 24, each is isomorphic to one of the groups considered in this thesis.

Although this thesis is concerned primarily with the abstract groups of order 24, the lattice of groups whose orders are factors of 24 are also included. These lattices aided in determining the subgroups which are contained in the groups of order 24.

It is interesting to note that the number of elements of order 2 contained in any group of order 24 is always odd. This condition is also true for all groups whose orders are factors of 24. It is conjectured by the writer of this thesis that any abstract finite group whose order is an even number will contain an odd number of elements of order 2.

10.2. Suggestions for further study. A study of the finite groups of any chosen order would be of interest. However, it is suggested that a study similar to this study be conducted on the abstract groups of order 16. Such a study would certainly lead to further verification of the conjecture pertaining to the elements of order 2.

Through the study of finite groups whose order is 24x, where x is some small prime number say 2 or 3, it would be of interest to determine if each type of group of order 24 is present in the groups of order 24x. That is, it is a conjecture of the writer that each type of group of order 24 will reveal itself in at least one group of order 24x.

Another proposed study would be the classification of the groups of order 24 according to technical types, that is, which groups are nilpotent, which groups are super-sovable, which groups are Hamiltonian, and so on for other technical types.

TABLE III

NUMBER OF ELEMENTS OF EACH
ORDER CONTAINED IN THE GROUPS
OF ORDER TWENTY-FOUR

Groups of order twenty-four	Number of elements of each order							
TYPE	1	2	3	4	6	8	12	24
one	1	1	2	2	2	4	4	8
two	1	1	2	2	2	12	4	-
three	1	3	2	4	6	-	8	-
four	1	7	2	8	2	-	4	-
five	1	3	2	12	6	-	-	-
six	1	7	2	-	14	-	-	-
seven	1	7	8	-	8	-	-	-
eight	1	15	2	-	6	-	-	-
nine	1	1	2	6	2	-	12	-
ten	1	1	8	6	8	-	-	-
eleven	1	1	2	14	2	-	4	-
twelve	1	5	2	2	10	-	4	-
thirteen	1	13	2	2	2	-	4	-
fourteen	1	9	2	6	6	-	-	-
fifteen	1	9	8	6	-	-	-	-

BIBLIOGRAPHY

BIBLIOGRAPHY

Alexandroff, P. S. An Introduction to the Theory of Groups. New York: Hafner Publishing Company, Inc., 1959.

Andree, Richard V. Selections from Modern Abstract Algebra. New York: Henry Holt and Company, 1958.

Burnside, W. Theory of Groups of Finite Order. [New York]: Dover Publications, Inc., 1955. 2nd ed.

Cajori, Florian. A History of Mathematics. New York: The Macmillan Company, 1924. 2nd ed.

Carmichael, Robert D. Introduction to the Theory of Groups of Finite Order. Boston: Ginn and Company, 1937.

Hall, Marshall, Jr. The Theory of Groups. New York: The Macmillan Company, 1959.

Ledermann, Walter, Introduction to the Theory of Finite Groups. New York: Interscience Publishers, Inc., 1957. 3rd ed.

Newman, James R. The World of Mathematics. 4 vols. New York: Simon and Schuster, Inc., 1956.

Struik, Dirk J. A Concise History of Mathematics. 2 vols. New York: Dover Publications, Inc., 1948.