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In logic, modality is the intertwined reference of the actual, the possible, and the necessary, and a modal logic is a formal language composed to speak consistently about all three. After an introduction which contextualizes the pursuit of modal logic in an era of highly-developed mathematical logic, Chapter 1 undertakes the task of introducing modality into the syntax of modern propositional logic and examines the predicaments which ensue while interpreting modal statements and attempting to determine their validity. Chapter 2 is a detailed exposition of possible-worlds interpretations of modal logic, with special attention paid to answers regarding the validity of modal formulas which interpreting statements of modal logic as referring to possible worlds can provide. Chapter 3 turns to mathematics for alternative schemes of interpretation and explores semantics for modal logic drawn from the subfield of mathematics known as topology. Finally, Chapter 4 examines a variant of modal logic, geared towards a temporal reading of modality, called tense logic. In lieu of a conclusion, Chapter 4 returns to the possible-worlds semantics and the topological semantics of previous chapters and shows how each of these may be used to interpret tense-logical statements.

An Exposition of Modal Logic

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Introduction

> “The actual is no more necessary than the possible,
>
> for the necessary is absolutely different from both.”
>
> -Kierkegaard, *Philosophical Fragments*

With these words, Kierkegaard is appealing to a distinction common to ordinary experience: being able to differentiate between the actual, the necessary, and the possible is usually considered “logical” or “reasonable” for everyday thought and practice. Although they might appear at first glance to be unrelated, a current of careful reflections and statements similar to Kierkegaard’s runs deeply through the history of Western thought, suggesting that the notions of “necessity” and “possibility” are linked together in a special way. Relating the two seems to be an inherently logical problem, because in order to reflect on either of these two notions we begin to make statements about whose relative “truth” we need some way of adjudicating. We begin to wonder about the connection of our statements to “what really is:” statements about the necessary or the possible are made relative to some actual state of affairs, even as they distinguish themselves from the actual. Just as importantly, an inner connection between necessity and possibility can be detected in the attempt to reflect on either notion individually, just at it has been time and again by authors like Kierkegaard.

For example, if we say something is necessarily so, we might mean that even under other circumstances it couldn’t be otherwise. However, then we would be speaking about the possibility of these circumstances being otherwise. Upon reflection, some aspects of the actual world appear more contingent than others, drawing us towards the discovery and formulation of invariant facts or necessary truths despite experiences of contingency and variance. Similarly, if we say something is a possibility, do we mean that it isn’t so, but it could be so? Are we saying that it isn't necessarily not so, but that it just happens to not be actual? By entertaining possibilities, we seem to be reflecting on the "actual world," yet deviating from what we could safely say is actual; like the necessary, the possible is something we can speak of in relation to actuality, or the factual, but is somehow other than what we know to be the case.

For this reason, both notions are sometimes characterized as being *counterfactual* and make their appearance in counterfactual statements (this is the terminology of Lewis [1973]). The apparent link between the notions of necessity and possibility has led to their intertwined reference as the *modal notions*. What exactly is this intertwined reference, and how does it relate to the actual world? Is there any substance to the seeming connections revealed by ordinary reflection? Above all, it seems that if there were some stable connections between the actual, the possible, and the necessary, we ought to be able to sort out our reflections and learn how to speak consistently about all three. So how are we to make statements about modalities which are at least consistent with each other and with statements about the real world? The tradition of hard-fought efforts to find precise answers to these questions encompasses the *logic of modality*, or modal logic.

In fact, modality is as old as the subject we call "logic" and was, until relatively recently, a central part of the discipline. Aristotle, commonly referenced as the founder of Western logic, included modality in a systematic exposition of the deduction or syllogism (*sullogismos*), which he lays out in the *Prior* and *Posterior Analytics.* In the former work, Aristotle creates a classification of different terms which appear in a deduction and attempts to reason through, in a systematic fashion, which types of deduction are valid and which are not. All of the deductions are "formal" because their validity is ascertainable on the basis of their arrangement into forms called figures (*schēma*) and not on the specific contents which may fill them. In brief, the figures are made up of sentences which are either premises or conclusions of some deduction. Each sentence in turn is composed of a subject, a predicate, and a constant which charaterizes the subject-predicate relationship. This important pairing, where some subject can be described as in some sense "belonging to" some predicate (e.g., in the sentence "Every man is an animal," the predicate-term "animal" is predicated of the subject-term "man"), is worth mentioning, because it circumscribes the basic design of Aristotelian logic in a way eventually seen as no longer adequate for the designs of modern logic. For their purposes, however, the figures provide an examination of how logical deduction works, and with them Aristotle exhaustively examines when and why a conclusion can be truly deduced from some premises but not from others.

Deductions made with sentences qualified modally with "it is necessary (*anankaīon*) that-P" and "it is possible (*dynatón*) that-P" (and related sentences, including their negations) are included in the *Prior Analytics* and connected works and are carefully differentiated from the assertoric or non-modal deductions (Kneale [1960], p.84). Aristotle introduces the subject of modality in his typical style:

> "Since to belong and to belong of necessity and to be possible to belong are different (for many things belong, but nevertheless not of necessity, while others neither belong of necessity nor belong at all, but it is possible for them to belong), it is clear that there will also be different deductions of each and that their terms will not be alike: rather, one deduction will be from necessary terms, one from terms which belong, and one from possible terms" [1989, p. 13].

Subsequent generations built on Aristotle's system of deduction, apart from which for centuries no "logic" could be imagined, with critiques, corrections, and additions. For this reason, modality continued to have a secure place in the considerations of ancient and medieval logicians. It was not until these older forms of logic were made defunct, replaced by a very different understanding of logic and gradually seen to be obsolete in comparison with the new, that modality slipped quietly and without struggle into a place of insignificance.

In fact, the disappearance of modality is a helpful, if not key, motif when trying to understand the historical emergence of "modern" forms of Western logic. Modern logic began with a self-conscious rupture, which sharply distinguished itself from the ancient and medieval logics that had preceded it for centuries. Breaking with all tradition, modern logic promised a radical revision of the subject from the ground up and seized upon new concepts which could actually make advances into this unknown territory. It may be surprising just how recent this break with tradition occurred: even in 1781 Immanuel Kant could still claim that "the entire field of logic had not made a single advance since Aristotle's great treatise," the *Prior Analytics* (Aristotle [1989], p. vii). Those who point to Gottlöb Frege's *Begriffsschrift* (1879) as the first indication of the arrival of modern logic, if not its actual foundation, do so because Frege (who was a sharp critic of Kant, among others) wanted to reinvision logic as a "concept-script," a language of pure thought which excluded all rhetorical inaccuracies and modeled itself on arithmetic instead, setting its sights on "the analysis of mathematical reasoning" (Kneale [1960], p. 478). Built directly into the new formal language of the "concept-script" is the rejection of the distinction between subject and predicate, hitherto the virtual hinge of all logical statements, as "dogma" which limits the advance of logical precision. In Frege's ideal language, "the whole content of a judgment is expressed by [the subject]... there will then be one and the same predicate phrase for all statements, namely [the predicate] 'is a fact'; but it will not be like ordinary predicate phrases, since its work will be simply to present the subject content in the form of a judgment" (op. cit., p. 479). Frege first introduced the symbol "$\vdash$" into formal logic, as the representation of this predicate, "is a fact," and his rearrangement of the nature of the judgment, which shifted the formal perspective on the characterization of practically every logical term: universals, particulars, and negations would now refer to the content of a judgment, not the form of judgment as such; other distinctions familiar to the old logic, such as disjunction, became part of the grammar of Frege's "concept-script."

Above all, modality could be disregarded as completely irrelevant to logic, because "it refers to grounds rather than to contents of judgments" (op. cit., p. 480). This is to say that Frege saw modality as a mistaken introduction of human concerns regarding the epistemic grounds for a judgment's content; these concerns had no place in a conception of logic in which every judgment, without exception, must be predicated as factual: "By saying that a proposition is necessary I give a hint about the grounds for my judgment. *But, since this does not affect the conceptual content of the judgment, the form of the apodictic* [modal] *judgment has no significance for us*" (italics original). Likewise "If a proposition is advanced as possible" in the sense of modal qualification, "the speaker is suspending judgment by suggesting he knows no laws from which the negation of the propostion would follow" (Frege [1879], p. 13). While Frege's vision of logic was just the beginning of a revolution for the discipline, it suggests that for the new logic, modality was to be regarded as an antiquated, conceptually imprecise holdover from a time in which "'logic' once had to do with words and with reason," studied for "philosophical insight into the mysteries of the laws of thought." The tasks laid out for logic have since required not so much deductive systems of the kind envisioned by Aristotle, but the construction of elaborate formal languages like Frege's, each of which "is like a sentence machine, a computing machine, an electronic typewriter whose output is a set of sentences" (Halmos and Givant [1998], p. 1, 3).

It is perhaps too easy to draw a straight line from Frege's global predication of "is a fact" to the much-maligned philosophy of logical positivism, whose reign reconceptualized thought and language as iterative, machinelike, and bounded, maintaining with particular vehemency that, in the words of Ludwig Wittgenstein [1922], "the world is everything that is the case" and no more (p. 25). However, the new confidence and excitement which modern logic inspired in its proponents inside and outside the mathematical community, and the radical recasting of logic's foundations in which new generations of students were being trained, should not be forgotten either. This confidence is on particular display in the work of one of the earliest modern logicians to attempt to reintroduce modality to its estranged discipline: Jan Łukasiewicz, who set out to reclaim none other than Aristotle as the actual founder of modern logic, by recasting his system of deductions into the formal logic of the twentieth century. In *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic,* Łukasiewicz [1957] lays out an interpretation of Aristotelian logic which justifies the criteria for any system of statements claiming to be logical in the modern world and derives a self-consistent version of Aristotle's assertoric (non-modal) syllogistic which meets these criteria, separating this system from an otherwise ubiquitous confusion he finds in Aristotle's thought, and also from the "philosophical prejudices" of centuries of scholastic commentary which have become "useless from the standpoint of logic" (p. 35, 38). Among the countless interpreters of Aristotle from the past, Łukasiewicz "venture[s] to say they must all have been bad mathematicians," who are more often than not guilty of "bad philsophical speculation" and a general "ignorance of logic" (op. cit., p. 8, 11). From this staunchly modern standpoint, Łukasiewicz discovered Aristotle's modal syllogistic to be riddled with "faults and inconsistencies." In the face of such glaring contradictions, Łukasiewicz recognized that his desire to "explain as well as appreciate [Aristotle's] modal syllogistic" as thoroughly as he had treated the assertoric syllogistic would require him to "establish a secure and consequent system of modal logic" beginning from the suppositions of modern logic instead (op. cit., p. 157). Compared with the extensive rapproachment with modality achieved by his contemporary C. I. Lewis, Łukasiewicz' contributions to modal logic, including the modal system he developed to grapple with Aristotle's modal syllogistic, have not subsequently been well-remembered.[1] Even still, Łukasiewicz' point

[1]For a succinct account of the modern approach to modal logic first made by C. I. Lewis, see Burgess [2013], p. 144-145. Łukasiewicz' contributions are hardly miniscule, and his modal systems still provoke interest today (cf. Font and Hájek [2002]). He also originated a different formalism for propositional logic, detailed in [1957] and known as "Polish notation," which eliminates the need for parentheses (cf. Halmos and Givant [1998], p. 17).

Furthermore, Łukasiewicz' efforts to show Aristotle's syllogistic to be rigorous, even by the then-latest standards of modern logic, helped revive genuine interest in ancient logic beyond its historical significance, and has led to many subsequent reconstructions of Aristotelian logic which try to refine the method of [1957]. Nowhere is it clearer than in Łukasiewicz' work, however, that the hard-won presuppositions which make up modern logic, as intractable as they appear in the contemporary mind, may for that very reason fundamentally interfere when trying to piece together the Aristotelian modal syllogistic into the coherent whole which Aristotle himself had in mind. Building on the advances in logic of the last half-century, Malink [2013] puts forth a model which contravenes the consensus opinion that started with Łukasiewicz by demonstrating the core set of claims in the modal syllogistic "is consistent and that, with respect to the proposed model, these claims do not contain mistakes" (p. 2). Like other more recent approaches, Malink's is much more selfof view is instructive for understanding the origin and motivation behind the basic tasks for any attempt to fashion a system of modal logic "from the standpoint of modern formal logic," as the title of [1957] suggests.

In **Chapter 1** below, we will undertake the same task of reintroducing modality into modern propositional logic and examine the predicaments which ensue. Because modality itself was relegated as contrary to logic's advance into the modern period, seeking to bring the modal notions back into formal logic at first seems like transplanting a vestigal organ into a healthy body or installing instruments into a machine that runs just fine already. From the beginning, such a task appeals to the authority of the methods of modern logic. In particular, we shall be making a clear distinction between the logical syntax, or the sequential presentation of symbols on the page, and the semantic meaning of those symbols provided by their formal interpretation. The tight clarity of the notion of *validity* which modern logic has achieved for its propositions becomes the standard for which a modern modal logic must reach, and this high standard will immediately bring us back to the problem of modality as it was posed in the first paragraphs of this introduction: when we say x is "necessarily" or "possibly" so, what exactly do we mean to say about x, and what does this mean for the validity of statements about x? Chapter 1 explores these issues from the ground up, looking to modern propositional logic itself as the source for restrictions and clarifications of the meaning of the modal notions that could bring them to the same level of formal precision. By the end of Chapter 1, many of the unique prospects which arise while working with systems of modal logic should thus be clear as well.

For a time, it seemed that the inherent ambiguity of modality when thrust onto modern logic's brightly-lit stage would prevent any further elaboration of modal logic than the kind pursued in the first chapter and opened up by logicians like Łukasiewicz, in which the modal notions were introduced through the use of special operators, and systems of modal logic were established axiomatically and then explored, to see if the statements that

conscious about developing an interpretive framework which pieces together Aristotle's modal syllogistic on its own terms, "albeit at the cost of some interpretive complexity" (ibid.).

could be derived from them according to the rules of propositional logic made any sense. As will be seen, this approach is partially illuminating but does nothing to overcome the basic superfluity of meaning the modal operators introduce into propositional logic. Beginning with Kripke [1963], however, an unexpected breakthrough occurred which provided a new scheme of interpretation that could determine the validity of modal formulas while clarifying the logical meaning of modality. Kripke's idea was that modal operators can be interpreted by relating the truth or falsity of a logical proposition in one world (presumably our world, the logically assertable world of everything "that is the case") to its truth or falsity in other *possible worlds*, the status of which required no further commentary for the purposes of constructing an entirely new formal interpretation of modal logic. Indeed, Kripke [1959] writes:

> "The basis of the informal analysis which motivated these definitions is that a proposition is necessary if and only if it is true in all 'possible worlds.' (It is not necessary for our present purposes to analyze the concept of a 'possible world' any further)" (p. 2).

Possible-worlds interpretations of modal logic are quasi-mathematical constructs which, in their developed form, consist of a *set* of "worlds," each of which are really an enumeration of propositional variables that can be judged as either true or false; a *relation* which specifies the pairwise accessibility of those worlds to one another; and a *function* which specifies every propositional variable as either true or false. However, in a possible-worlds interpretation truth-values must be specified for propositional variables, not just in one world, but in every world contained in the set of all worlds. **Chapter 2** below is a detailed exposition of possible-worlds interpretations of modal logic, with special attention paid to the answers it can provide regarding the validity of modal formulas raised in the first chapter.

The real impetus guiding the development of this paper, however, is a desire to look beyond possible-worlds semantics. Despite their breakaway success in carving out a place for necessity and possibility within the framework of modern formal logic, possible worlds do not exhaustively or definitively characterize the notion of modality. The informal insight which they cast upon the meaning of modality is not the only one that is possible, nor is it the most intuitive or relevant way to understand the modal notions.[2] Meanwhile, the formal structure of possible-worlds interpretations, while it is extremely useful as an illustration of how to provide a full semantics for modal logic, has the shortcoming of being unable to account for the vague and increasingly overtaxed notion of "worlds." **Chapter 3** explores an alternative interpretation of modal logic drawn from the subfield of mathematics known as *topology*. In more recent years, there have been many mathematical interpretations given for modality from areas ranging from Boolean algebra to probability theory. The mathematicians J.C.C. McKinsey and Alfred Tarski began to interpret modality with the help of topology as early as [1944], making these efforts at least as old as those relating to possible worlds.

This paper also explores the turn to *temporality* as a more adequate, and less eccentric, fixture for understanding and representing the logical meanings of modality. On this account of modality, when we say "it is necessary that *x*," or "it is possible that *x*," we are not just making a subtle postulation about the existence of alternate states of affairs (or worlds so-called); in a sense much more immediate to and inseparable from experience, on the temporal account these kinds of "counterfactual" statements can and do arise because of the situatedness of all our statements *in* time and so relative *to* times. This includes our statements about the actual or factual world. Arthur N. Prior was the first modern logician to dedicate himself to the logical and philosophical development of a temporal underpinning for modality. Beginning with [1955], and throughout the decade preceding Kripke's development of possible-worlds semantics, Prior made temporality the central locus for applying and understanding modal logic. In a talk given in 1954 and published later, Prior said:

[2] Pruss [2011] develops a detailed argument concerning the philosophical limitations and metaphysical problems that have become associated with possible-worlds semantics, while Kishida [2011] specifically develops topological semantics for first-order modal logic to address inadequacies in the possible-world reading.

> “When the new interpretation is employed, it becomes possible to enrich the calculus [of propositional logic] with a pair of non-truth-functional operators which cannot be attached to the timelessly-true ’propositions’ of the current interpretation. These are namely the tense operators ’It has been the case that,’ symbolized here by ’p’, and ’It will be the case that,’ symbolized here by ’F’. The functions formed by these operators are themselves propositions whose truth may vary with time” ([1958], p. 106).

In this passage, Prior indicates one strength of the temporal view of modality: its nativity. In qualifying statements as relative to times, we already intuit that these qualifications are themselves situated in time, and so can be assessed as true or false, past, present or future, “possible” or “necessary,” differently at different times. “Truth, on the face of it, is a property of propositions which is liable to alter with the time they are put forward” (Prior [1958], p. 105). Thus like the notion of “possible worlds,” the temporal reading of modality is outfitted with a loosely-delimited notion, time, which will guide the informal interpretation of modality. Why this notion is linked to modality, how this link can inform our thinking about temporal and modal notions, and why it is preferable to the similarly vague notion of “possible worlds” are all matters of speculation and debate. **Chapter 4** below will simply examine an alternative to or variant of modal logic, geared towards this temporal reading of modality, called tense logic. By altering the formal representation of the modal operators so that statements of tense logic always invoke necessity and possibility as relative to some past or future time, tense logic only alters the manner in which statements of modal logic are read and understood. The syntax of tense logic can erstwhile be dealt with autonomously of its prescribed meaning, just as before, making it formally equivalent and interpretively analogous to regular modal logic. However, this changed reading has interesting implications when we make renewed interpretations of the syntax of tense logic. Thus Chapter 4 returns to the possible-worlds semantics and the topological semantics of previous chapters, showing how each of these may be used to provide a semantics for tense logic as well.

The overall aim of this paper is thus not to break entirely new ground in the field of modal logic but to present a detailed exposition of the topic. Throughout, I supplement definitions and theorems with explications that are intended to clarify what is being done and why. For some, the level of explanatory detail may seem redundant or unnecessary, but my hope is that someone with very little background in any one of the particular subjects discussed could pierce through the formalism and gain a working understanding of the topic by reading this paper. The organization of the paper as a whole generally proceeds in a direction of increasing detail, so that the understanding requisite for subsequent topics can gradually accumulate. As this introduction has already made clear, it also moves from the much more widely-accepted understanding of modality by way of "possible worlds," to the alternative topological and/or temporal understandings of modality. Broadening our range of interpretations and understandings of modality simultaneously frees modal logic for fresh interpretation and new insights. In this way, the exposition aims to illuminate the advantages I see in forging new connections between the modal notions and areas of mathematics like topology as well as the advantages of linking modality with time. Thus the discussion is oriented in the direction where modal logic has the most potential for future development.

I am particularly indebted to the various introductory books I have consulted and with which I have become familiar; I relied upon these during my first explorations of modal logic, especially Hughes and Cresswell [1968], Hughes and Cresswell [1996], and Priest [2008]. Any one of these manuals would be the ideal place for someone who takes an interest in the subject of modality after reading this paper to continue their study.

Chapter 1

Modality and Validity

This chapter presents the basic buildling blocks of propositional logic, with which the notions of modality can be formally represented and manipulated. Because it lacks the quantification of statements and some of the more specialized symbolism of first-order mathematical logic, propositional logic is ideal for introducing the idea of a formal language and exploring the effect that the inclusion of modality has on systems of logical propositions (cf. Enderton [1972] for an extensive mathematical treatment of propositional logic). The formal definitions of the logical symbols which make up the sentences of propositional logic enable, from the beginning, a notable contrast between the *syntax* of propositional logic- the strings of symbols which are combinable according to definite rules- and its *semantics*, the commonly-understood meanings which are associated with those symbols and underpin the formal rules for building "grammatically correct" logical sentences (called well-formed formulas or wff's). When understood semantically, we think of and speak the symbols $\vee$ and $\wedge$ as represention of disjunction ("or") and conjunction ("and") respectively, but while related to the these interpretations, the formal rules which dictate how logical formulas may be built using these symbols are technically distinct from the commonsense meaning. Similarly, in section 1.1 the two modal notions, "necessity" and "possibility," will be given a place in the syntax of propositional logic by representing them formally with a square ($\Box$) and a diamond ($\Diamond$) respectively, but the formal interrelation of these two symbols, like the syntactical rules on how they may be placed correctly into a logical formula, remains distinct from the reading we have in mind concerning their use. As we shall see, while coming up with a working syntax which includes these two "modal operators" is relatively simple, the problem with building a semantics for these operators which is on par with the other symbols of propositional logic quickly becomes apparent. It is this problem that guides the material in subsequent chapters.

Statements of propositional logic help answer questions of *entailment*- this just means that, broadly speaking, propositonal logic helps demonstrate "what follows from what," as Priest [2008] writes. We are interested in knowing whether *if* such and such statements are taken for granted, *then* under what conditions we can accept other statements on a sure logical footing. By formalizing the basic components of its statements and laying out clearly the rules for connecting them, propositional logic makes it clear when and where one statement or set of statements is entailed by (e.g., when it follows from or is deducible from) another statement or set of statements. In a similar way, some statements of propositional logic can be said to be *valid* in the technical sense that they are entailed by (follow from) just any statement, even one that contradicts itself. Valid logical formulas, which are also known as *tautologies*, are always seen to be true, no matter the truth of the specific contents that they connect together. Some valid statements are quite mundane, but others can be illuminating or useful for coming up with ways of proving that a counterintuitive conclusion we might not otherwise be able to demonstrate follows from set of premises in a logically valid manner. Thus it might be thought of as a natural goal of modal logic to come up with at least one valid modal formula; here again, however, it quickly becomes apparent that defining validity and determining whether a particular statement of modal logic is valid presents unique problems which are not present in a "classical" propositional statement which omits the modal operators.

1.1 Syntax for Propositional Logic

Definition. A *formula* of propositional logic (also known as sentential or propositional calculus) is a finite string or sequence composed of symbols taken to be *primitive*, that is, as signs distinct from a particular meaning. These primitive symbols of propositional logic include the following:

(i) An unspecified number of propositional variables: $p_1, p_2, p_3, \ldots$.

(ii) The four symbols: "$\neg$", "$\vee$", "(", and ")".

The parenthesis symbols "(" and ")" serve as a kind of punctuation in logical propositions. The "$\neg$" symbol and the "$\vee$" symbol are known as negation and disjunction, respectively, and the latter is called a logical connective, since (when well-used) it "connects" two propositional variables. We take them all as syntactically primitive, however, to distinguish the symbols themselves from the meaning generally intended by them (the use of interpretation functions below will serve to reconnect each symbol and its meaning, generating a semantics for the syntax of propositional logic). Based on these commonly-understood meanings, however, we can also go ahead and derive definitions for some other commonly-used logical connective symbols. Let p and q be propositional variables. Then

[$\wedge$] $(p \wedge q) := (\neg(\neg p \vee \neg q))$

[$\supset$] $(p \supset q) := (\neg p \vee q)$

[$\equiv$] $(p \equiv q) := (\neg(\neg(\neg p \vee q) \vee \neg(\neg q \vee p)))$

where the "$\wedge$" symbol is known as conjunction, the "$\supset$" symbol as implication (or the material conditional), and the "$\equiv$" symbol as (material) equivalence. Since the definition of these last three symbols are derived from the four syntactically primitive ones, however, the syntax of propositional logical formulas is in principle reducible to finite sequences of these four original symbols, interspersed with propositional variables.

Definition. A *well-formed formula (wff)* of propositional logic is one formed according to the following recursively-defined rules:

(i) A propositional variable by itself is a wff.

(ii) If A is a wff, so is $(\neg A)$.

(iii) If A and B are wffs, so is $(A \vee B)$.

Based on the definitions of the other logical connectives, we find that if A and B are wffs, so are $(A \wedge B)$, $(A \supset B)$, and $(A \equiv B)$. These rules are called recursive since, as long as they are always used correctly, they can be applied successively to well-formed subformulas where they have already been invoked in order to make well-formed formulas of ever-increasing complexity. For convenience, we will assume hencefoward that all the formulas of propositional logic we are dealing with are well-formed and so simply refer to them as formulas. Following convention, we will refer to propositional variables using lower-case Latin letters (p, q, etc.); to formulas or subformulas using upper-case Latin letters (A, B, etc.); and to sets of formulas using capital Greek letters (Σ, Φ, etc.). And for simplicity, we will henceforward remove the outermost pair of parentheses when writing a lengthy formula as long as the contextual meaning remains clear.

Non-modal, or classical, propositional logic consists of only these symbols. In addition to these symbols, modal propositional logic introduces the modal operator as another primitive symbol.

Definition. The *modal operator* is a primitive symbol of modal propositional logic, denoted "$\Box$". From this symbol we can derive another operator, its dual, according to the following definitions. Let A be a non-modal formula. Then

[$\Diamond$] $\Diamond A := \neg\Box\neg A$

[$\Box$] $\Box A := \neg\Diamond\neg A$

Here we mean by an *operator* a symbol which, in order to be written gramatically, is written as modifying at least one other symbol or symbols, called its *argument(s)*. Both $\Box$ and $\Diamond$ are thought of as *monadic* operators because they require only one argument, a formula (represented by A in the above definition); in this way the modal operators are simular to negation, which also requires one formula as its argument in order to be written correctly. Although $\Box$ was taken to be primitive, its definition follows immediately from the definition of $\Diamond$; thus either symbol can be taken as primitive and the other derived. Because of this mutual derivability, the two symbols are called the *dual* of one another: in the formal language of propositional logic, which includes the negation sign $\neg$ as primitive, the presence of one modal operator always leads to the implied presence of the other.

Definition. A *well-formed modal formula* of modal propositional logic is one formed according to rules (i) through (iii) for a well-formed formula, with the additional rule

(iv) If A is a well-formed formula, so is $\Box A$.

From this definition it immediately follows that if A is a wff, so is $\Diamond A$. To see that this is so, let A be a formula. We can follow the chain of formation rules so far given: by (ii) and (iv), $\neg A$ and $\Box\neg A$ are both well-formed formulas; applying rule (ii) once more, the formula $\neg\Box\neg A$ is also well-formed. But by definition of $\Diamond$, this is just the formula $\Diamond A$.

1.2 Semantics for Propositional Logic

With the syntax in place, it is now possible to begin to elaborate how exactly the symbols of modal logic are supposed to be interpreted. As this section will show, semantics for a logical syntax must not only provide a careful elaboration of the intended meanings of logical formulas and their symbols, it must also address the problem of entailment: how one formula can be seen to follow from another. Trying to deduce one formula from another requires at least some sort of formal interpretive structure, and along the way to providing such an interpretation the question of validity, what makes some formulas make sense under any worthwhile interpretation, is also raised. To begin to understand the task of providing formal interpretations for systems of modal logic, it is first advisable to understand thoroughly how a semantics is provided for classical propositional logic, beginning with the definition of an interpretation function.

Definition. Given some nonempty collection of logical formulas, let $\mathfrak{Var}$ be the set of propositional variables contained in those formulas. In its very simplest conception, an *interpretation* is any function, $V : \mathfrak{Var} \to \{0,1\}$, which assigns a truth-value to each propositional variable in a formula or collection of formulas. Namely, if $V(p)$ is an interpretation of a propositional variable p, either $V(p) = 1$, and p is said to be true under the interpretation; or $V(p) = 0$, and p is said to be false under the interpretation.

A semantics for propositional logic is built by extending the interpretation function V to entire formulas or sets of formulas by recursively-defined rules which specify how the function is to evaluate subforumulas built using the various logical connectives. For any non-modal formulas A and B, these rules can be specified with exactitude:

- If $V(A) = 1$, then $V(\neg A) = 0$; otherwise, $V(\neg A) = 1$.
- If $V(A) = 1$ and $V(B) = 1$, then $V(A \wedge B) = 1$; otherwise, $V(A \wedge B) = 0$.
- If $V(A) = 1$ or $V(B) = 1$, then $V(A \vee B) = 1$; otherwise, $V(A \vee B) = 0$.
- If $V(A) = 0$ or $V(B) = 1$, then $V(A \supset B) = 1$; otherwise, $V(A \supset B) = 0$.
- If $V(A) = V(B)$, then $V(A \equiv B) = 1$; otherwise, $V(A \equiv B) = 0$.

The rules for interpreting negation, disjunction, conjunction, and implication relate the well-understood symbols of propositional logic to their intended meanings. They thereby provide a formal semantics for the syntax of propositional logic. The meaning of the notion of negation, for example, is summarized by the first rule above, enabling us to assign the truth-value of a negated proposition, given that proposition's truth-value. Given $V(p) \in \{0,1\}$ for some propositional variable p, we can specify $V(\neg p) \in \{0,1\}$ as well. The first rule listed above definitively specifies how to *interpret* the negation of p.

However, with the introduction of modal operators into a formula, the assignment of a truth-value to that formula on the basis of its propositional variables becomes problematic. In other words, given $V(p)$ for some propositional variable p, how are we to interpret $V(\Box p)$? While the respective interpretations of "it is the case that p" and "it is necessarily the case that p" are intuitively related to one another, the relationship is not unambiguous enough to formulate a simple rule like those for the connectives of classical propositional logic.

Modern modal propositional logic begins with the attempt to formulate just such a rule for interpreting a modal formula, despite the ambiguity inherent in the modal notions. To begin to retrace this attempt ourselves, we must first of all notice that it requires a treatment of the problem of entailment for formulas containing modal operators. Even in the simplest cases, the relation between the modal operator and its argument (rendered above as the question “given $V(p)$ for some p, how are we to interpret $V(\Box p)$?”) can be thought of as the relation between two logical formulas: the non-modal formula on the one hand (in this case, the lone variable p), and the original formula modified by one or more of the modal operators on the other hand (in this case, the formula $\Box p$). If our goal is to formulate a rule which relates the truth-value of one of these formulas to the truth-value of the other, the first question we might ask is, “can $\Box p$ be said to follow from p, or vice versa, and under what conditions?” To consider this problem carefully, it is worthwhile to introduce several additional definitions from classical propositional logic (cf. Miller [2015] for background and extended development of these definitions).

Definition. Let V be an interpretation function and let Σ be a set of formulas. Then V is said to *model* Σ, and we write $V \vDash \Sigma$, if and only if $V(A) = 1$ for every formula $A \in \Sigma$.

Furthermore, let Σ be a set of formulas and let B be a formula (not necessarily belonging to the set). B is called a *logical consequence* (or a *semantic consequence*) of Σ, and we write $\Sigma \vDash B$, if and only if every interpretation V of Σ which interprets all its formulas as true also interprets B as true (e.g., $\Sigma \vDash B$ if and only if $V \vDash \Sigma$ implies $V(B) = 1$ for every V). Then the formulas of Σ are called the *premises,* and B the *conclusion,* relative to one another.

If there exists some interpretation of Σ under which all of its formulas are true, but under which A is false, then A is not a logical consequence of Σ, and we write $\Sigma \nvDash A$.

Definition. A *tautology* is a formula A such that $\Sigma \vDash A$ for every set of formulas Σ (including the empty set of formulas). Then we can write $\vDash A$.

It follows that if A is a tautology then, for any interpretation function V, $V(A) = 1$, regardless of the truth-values of its propositional variables. For clarity, the following is an example of a tautology.

Example 1. Probably the simplest example of a tautology is the formula $p \vee \neg p$. Since (by definition) either the propositional variable p or its negation is true, the formula is always true regardless of the truth value of p.

Related to the definition of a tautology is the following definition.

Definition. A formula A is *satisfiable* if and only if there exists an interpretation function V such that $V(A) = 1$. Likewise, a formula A is said to be *unsatisfiable* if and only if $V(A) = 0$ for every interpretation V.

Again, for clarity, the following is an example of an unsatisfiable formula.

Example 2. Probably the simplest example of an unsatisfiable formula is the formula $p \wedge \neg p$. Since (by definition) it cannot be the case that both p and $\neg p$ are simultaneously true, the formula is always false regardless of the truth value of p.

The meaning of unsatisfiable formulas often appears contradictory, while satisfiable formulas might be interpreted as only making sense in certain situations (since they are true for some, but not necessarily all, interpretations of the propositional variable(s) they contain). On the other hand, tautologies usually appear self-evidently true when understood semantically. It follows from the definitions that a tautology A is satisfiable, but only trivially so, since $\Sigma \models A$ for every set of formulas Σ.

Constructing a truth-table is one way of determining whether formulas of classical propositional logic are the logical consequence of one another, or whether a particular formula is a tautology. The first columns of a truth-table contain all the possible permutations of truth-values for all the propositional variables in question, while successive columns build up the truth-values of subformulas according to the rules of the interpretation function, until arriving at the relevant formula(s) in their entirety.

Example 3. The following truth-table demonstrates that the formula $(p \supset q) \supset (\neg q \supset \neg p)$ is a tautology, or equivalently that $(\neg q \supset \neg p)$ is a logical consequence of $(p \supset q)$.

p	q	$\neg p$	$\neg q$	$p \supset q$	$\neg q \supset \neg p$	$(p \supset q) \supset (\neg q \supset \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since a truth-table contains every possible interpretation for all propositional variables in all (sub)formulas, once the table is constructed we can quickly evaluate for ourselves whether there are any interpretations of the variables under which the premises are true but the conclusion is false. The ease of this evaluation means a clear definition of validity for classical propositional logic is well within reach.

Definition. A non-modal formula A of classical propositional logic is said to be *valid* if and only if A is a tautology.

Although generating a truth-table for very long formulas or a large set of premises by hand could become a practical impossibility, armed with the previous definition we can imagine a truth-table large enough to determine the validity of any formula of classical propositional logic in principle. Thus for classical propositional logic the matter of validity, and its relationship to the definition of logical consequence given above, would appear to be settled.

However, the quest for a similar method of determining validity for modal formulas requires close attention to the process which occurs when we "evaluate for ourselves" whether or not a truth-table reveals a formula to be a tautology or not. Constructing such a truth-table for even a simple modal formula presents a problem, since there is no simple rule to assign a truth-value to the modal portion of the formula based on an interpretation of the propositional variables.

Example 4. The following truth-table attempts to demonstrate that the modal formula $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is a tautology (and so classically valid).

p	q	$\Box p$	$\Box q$	$p \supset q$	$\Box(p \supset q)$	$\Box p \supset \Box q$	$\Box(p \supset q) \supset (\Box p \supset \Box q)$
T	T	?	?	T	?	?	?
T	F	?	?	F	?	?	?
F	T	?	?	T	?	?	?
F	F	?	?	T	?	?	?

This example illustrates the problem with the truth-tabular approach to modal formulas. Since no unambiguous logical relationship obtains between necessity, possibility, and actuality, no explicit rule of interpretation is immediately available to introduce a modal meaning to correspond with the introduction of modal operators into the syntax of propositional logic. Therefore, whatever it will mean for a modal proposition to be valid, it cannot be valid in the sense of a non-modal logical tautology.

This conclusion has led to the distinction which considers the operators of classical logic (such as $\neg$ and $\vee$) to be *truth-functional*, since an explicit truth-tabular rule can be provided for their interpretation, but considers the modal operators to be *non-truth-functional,* since no such rule for their interpretation exists. If we return to the definition given above, a logical consequence is established when every interpretation which finds a set of premises true also finds their conclusion true. Whenever our interpretations are simple truth-functional rules, such as those given for the symbols of classical propositional logic, the semantics which reconnect these symbols with their meaning also mechanically tell us whether formulas made up of them are true or false. However, because of the relative ambiguity of the notions of modality, it is precisely such a truth-functional interpretation which is seemingly impossible for modal formulas.

Despite this impossibility, the quest to determine the validity of modal logical formulas still strives after the same clarity and rigor as the truth-functional interpretation of classical propositional logic. The goal of determining the validity of a modal formula is thus to find ways of meaningfully interpreting the modal operators such that, although the interpretations might not be expressible as simple truth-functional rules, they should nevertheless be demonstrably connected to a manipulation of the logical syntax which is considered rule-based or mechanical, analogous to the high standard of neutrality set by the recursively-defined rules for interpreting non-modal formulas.

Therefore, because the search for a way to determine the validity of modal formulas requires we make some kind of allowance for ambiguity in the modal notions themselves, a natural distinction opens up upon closer examination of the process of evaluating the validity of any logical formula. On the one hand, a semantic interpretation of the syntax, both the logical formulas and the propositional variables they contain, must be constructed. To arrive at a convincing entailment of conclusion from premises, the definition of logical consequence requires that every such interpretation which holds the premises to be true also holds the conclusion to be true. Since it derives from an attribution of meaning to the logical syntax, we might think of this as the semantic notion of validity. On the other hand, a set of recursively-defined rules must also be provided, such that the rules can be applied mechanically to the formula(s) in question as a collection of symbols, without regard to the semantics which motivates any particular interpretation. Since the mechanism of the rules themselves provide our criterion for entailment in this latter case, we might think of it as the proof-theoretic notion of validity.

Following Priest [2008], the following definition helps separate out these two simultaneous, and closely related, notions of validity, both of which will be key to determining the validity of a modal formula.

Definition. Let Σ be a set of formulas and let A be a formula. A is called a *proof-theoretic consequence* of Σ if Σ entails A according to a formal procedure which refers only to the symbols of the inference, not their meaning, and which can be applied mechanically according to a rule or set of rules. Then we write $\Sigma \vdash A$.

In the case of a truth-table containing formulas of classical propositional logic, these two notions of validity exactly coincide with one another, because the semantics provided for the formulas, given above with the help of an interpretation function, happen to be a set of recursively-defined rules. The truth table for some tautology A thus says both $\vdash A$ and $\vDash A$. Modern modal logic aims to show how a modal formula can either be the logical consequence or the proof-theoretic consequence of other modal formulas. The enterprise of modal logic has thus required the construction of schemes of interpretation, or semantics, for modal formulas, as well as the construction of mechanical procedures for determining the validity of the same formulas. Finally, in order to provide an account of validity as compelling as that of classical propositional logic, the modal logician must show that these two notions of consquence actually follow from one another.

1.3 Systems of Modal Propositional Logic

To show how modal formulas can be the logical consequence of one another, we will begin to treat certain modal formulas as self-evident axioms which then imply an entire system of theoreoms, as they are derived from those axioms according to logical rules.[1]

Definition. Let Σ be a set of formulas "closed under logical consequence," meaning that, for every formula A, if $\Sigma \vDash A$, then $A \in \Sigma$. A set of formulas $\Phi \subseteq \Sigma$ such that $\Phi \vDash \Sigma$ is then called a set of *axioms* for Σ. We call Σ the *system* axiomatized by Φ, and any $A \in \Sigma$ which is not an axiom is called a *theorem* of the system.

Making particular modal formulas into sets of axioms, and seeing what systems of theorems follow from these axioms clarifies the idea of a modal logical consequence. Additionally, the self-evidence of the axioms necessarily restricts the meaning of the modal operator, thereby reducing the formal ambiguity of the modal notions. A distinct system

[1] *nb*. The only required logical rules are *modus ponens* and the rule of uniform substitution, though others can be elaborated on this basis.

of modal logic can be generated out of the assumption of particular axioms, and taken as a whole these theorems apply a particular "kind" of modality. Until full semantic interpretations of the modal operators are achieved in later chapters, however, it will be impossible to fully envision the logical consequence of one modal formula from another here. In a similar way, the notion of proof-theoretic consequence and its relation to the demonstration of a theorem's entailment from its axioms will also require further elaboration. In this overview of several prominent modal systems, it is just possible to consider the rudimentary qualities of some common modal systems.

Names of systems, axioms, and rules here follow the catalogue laid out in Hughes and Cresswell [1996]. Alternative nomenclature for systems of modal logic abounds, however, as well as many other systems of modal logic being not considered here which have been and continue to be explored for their particular significance. Only some of the most fundamental systems of modal logic are now presented, which have clear expository value or urgent relevance for subsequent chapters. We restrict ourselves to the following seven axioms, letting A and B be arbitrary formulas.

[PC] If A is a valid formula of non-modal propositional logic, then A is an axiom

[K] $\Box(A \supset B) \supset (\Box A \supset \Box B)$

[T] $\Box A \supset A$

[4] $\Box A \supset \Box\Box A$

[B] $A \supset \Box\Diamond A$

[D] $\Box A \supset \Diamond A$

[E] $\Diamond A \supset \Box\Diamond A$

The first axiom listed, [PC], is not a formula of modal logic at all, but a more general statement conveniently referred to by Hughes and Cresswell [1996, p. 25] as an axiom schema. Since modern modal logic is built upon the foundation of non-modal propositional logic, the axiom schema [PC] holds that any tautologically-valid non-modal formula is automatically considered to be a valid statement in modal logic. Valid non-modal formulas and their logical consquences are axiomatic from the point of view of systems whose statements contain modal operators. This greatly simplifies the relationship of modal and non-modal propositional logic, at the expense of introducing at the beginning an infinite number of axioms into modal logic, since tautological combinations of the symbols of classical propositional logic alone are in principle endless. The syntactic openness of classical logic thus becomes the foundation for a similar syntactic openness in modal logic.

Altogether, we will now consider twelve systems of modal logic which are formed out of the seven axioms above. The first and most fundamental of these is known as the system K, which is formed from the axioms [PC] and [K]. K is usually considered the "minimal" system which still produces an intelligible interpretation of modality. Its only substantive axiom, [K], has had an interesting role in the history of modal logic, as it only rose to prominent place gradually and was ultimately named for Saul Kripke, the philosopher whose contributions to modern modal logic (discussed in the introduction and described in the next chapter) have been instrumental to its development.

If we keep the axioms [PC] and [K] and combine them with the axiom [T], we have the system T. This system is still extremely versatile, because its only additional axiom gives shape to the idea of necessity by asserting, for some formula A, that if it is necessarily the case that A, then it is the case that A. This assumption, which probably agrees with offhand reasoning about the nature of necessity, generates an entire system of modal logic whose theorems can be derived from the basic axioms of T. As a brief example we consider the derivation of one such theorem of T (cf. Hughes and Cresswell [1968, p. 33]).

Example 5. In this example, we show how the theorem $A \supset \Diamond A$ follows from the axioms of T. Given [T] and substituting $(\neg A)$ for A, it follows that $\Box(\neg A) \supset (\neg A)$. By the formula $(p \supset q) \supset (\neg q \supset \neg p)$, which was shown to be valid in the truth-table of Example 3, if an implication is true, then so is its contrapositive. Thus by [PC] we may extend this fact to modal formulas and write $\neg\neg A \supset \neg\Box\neg A$. Since $\neg\neg A \equiv A$, and by definition $\Diamond A := \neg\Box\neg A$, we conclude that $A \supset \Diamond A$.

An immediately interesting feature of a system of modal logic like T is the nuance to the notion of modality which is revealed when some modal formulas are held as self-evident axioms. With regards to the previous example, although possibility isn't implied by actuality in an unqualified sense, if we start by assuming that necessity implies actuality (the axiom [T]), this assumption entails that actuality implies possibility. This example is not a rigorous proof of the theorem but a kind of loose sketch which doesn't disclose or adhere to a methodical procedure for arriving at the conclusion from the premises. Such methodical procedures, which will be developed in subsequent chapters, would actually demonstrate that the theorem is a proof-theoretic consequence of the axioms of T. At the same time, while it might offer a vague restriction of the meaning of modality, the example doesn't actually provide a definitive interpretation of what necessity and possibility mean in the context of a formal modal logic. Subsequent chapters explore multiple types of interpretations, each type rendering its own, more precise, semantic meanings to the general sketch of entailment of modal formulas which this example provides. Such semantic precision will more clearly illustrate just how and why the theorems of a modal system like T are logical consequences of its axioms.

Although we cannot go into great detail concerning them now, we must mention two other "weak" systems in connection with T, which are also formed by adding a single axiom to the axioms of K. These are the systems K4, which is formed by axiomatizing [PC]+[K]+[4], and the system KB, which comes from [PC]+[K]+[B]. The systems K4 and KB will be explored alongside the system T as alternate formulations of comparable versatility.

As more axioms are added to a system of modal logic, every addition restricts the ambiguity inherent in the meaning of the modal notions, and so limits the kind of semantic interpretations we can give of modality, even while it expands the range of logical deductions we can make on the basis of the axioms (cf. the diagrams in Priest [2008], p. 37-38). The system S4 is axiomatized by [PC]+[K]+[T]+[4], and so its deductive power reaches into places the system T cannot. Making the formula [4] an axiom both raises the question of what redoubling the modal operators means and asserts an answer to the question: it claims that if a formula is necessarily true, then it is necessarily the case that it is necessarily true. The system B, which is axiomatized by [PC]+[K]+[T]+[B] and named for L.E.J. Bertus Brouwer, founder of intuitionism, is similar to S4. The axiom [B] proposes a similar and equally plausible foray into doubling the modal operators: if it is the case that A, for some formula A, then it is necessarily possible that it is the case that A. Yet this system is not reducible to S4 (for proof of this claim, see Hughes and Cresswell [1996], p. 62).

Three more systems, also explored in Chapter 2, are formed around the formula [D]. The system D is axiomatized by [PC]+[K]+[D], the system KD4 by [PC]+[K]+[D]+[4], and the system KDB by [PC]+[K]+[D]+[B]. In the system D, [T] is not a valid theorem, even though we might have begun to think of it as self-evident in all cases. In practice the system D is often used to give modality a moral or "deontological" interpretation, where the necessary becomes the morally obligatory and the possibile becomes the morally permissible. With this context in mind, the axiom [D] serves as an intelligible starting place which has met with some success (cf. Hughes and Cresswell [1996], page 43).

The final three systems to be considered involve the last axiom in the list above, [E]. One more "weak" system explored in Chapter 2, KE, is axiomatized by [PC]+[K]+[E], and another consistent system of intermediate strength is formed by [PC]+[K]+[B]+[E]. However, the last system under consideration is also another especially interesting one: the system known as S5, whose axiomatization is here given as [PC]+[K]+[T]+[E], although it has numerous others. The system S5 more restrictive than S4 in terms of the interpretation of modality it affords, but again, with this extra layer of restriction, the theorems which follow from the exploration of S5's assumptions become more illuminating.

This cursory glance at twelve systems of modal logic has looked at them solely as a set of axioms, from which we can deduce various theorems with familiar logical principles. In the following chapter, we will continue to explore these systems; however, we will broaden the view achieved so far by resuming, once again, the search for a satisfactory way of demonstrating both logical consequence and proof-theoretic consequence in different modal systems.

Chapter 2

Possible-Worlds Interpretations

This chapter details the most prevalent and influential formal semantics for modal logic so far developed. Possible-worlds semantics for statements of modal logic are based on familiar, everyday talk about possibilities, necessities, and actualities. The informal insight asks us to consider a typical counterfactual statement as a statement about other possible worlds besides the one we happen to inhabit and about which we routinely speak. For instance, a possible-worlds interpreter would view the statement "if only humans had never discovered the atom bomb," concerning something quite conceivable to ordinary reflection but something which nevertheless is not true of the reality we inhabit, as a statement about *other worlds* where these conceivable possibilities actually obtain in reality. For example, "in another world besides this one, humans never discovered the atom bomb (I wish I was there instead)..." and so on. This informal insight regarding possible worlds is appealing to the logic of modality because it intuitively presents an interpretation for statements about what is necessarily the case as well. For instance, when someone says something is necessarily the case, like "it was an absolute historical inevitability that humans would discover the atom bomb, one way or the other" on a possible-worlds interpretation they are making the assertion that "in *every* possible world, it is the case that humans discovered the atom bomb."

The logical semantics of possible-worlds itself is a formal structure which is capable of verifiably interpreting statements of modal logic on the basis of the informal insight just described. Section 2.1 builds up this formal structure and provides the rules which a given interpretation uses to interpret a particular modal formula. The remainder of the chapter aims to show that these possible-worlds interpretations provide an exhaustive account of validity for statements of modal logic. Taking the truth-functional interpretation of propositional logic seen in Chapter 1 as a guide, the goal is to demonstrate how, and under what conditions, one or more modal formula(s) may be said to "follow from" others. This means developing a semantic understanding of when one statement is a logical consequence of another as a matter of the relations between possible worlds in a given interpretive model. Additionally, in Section 2.2 a rules-based method of tableaux construction is given, so that statements may be seen to entail one another in the proof-theoretic sense as well. Section 2.3 discusses soundness and completeness in possible-worlds interpretations, demonstrating that any possible-worlds interpretation and the formal rules for applying it always follow from one another.

2.1 Worlds, Relations, and Modality

Definition. A *state of affairs* (or, more commonly, a *world*) w is a collection of propositional variables $p, q, r, \ldots$. Let $w_1, w_2, w_3, \ldots$ be distinct states of affairs (or worlds). Then a set of worlds W is some set $\{w_i : i \in \alpha\}$ where α is an index set.

The validity of any well-formed (therefore finite) modal proposition can be determined with a finite set of worlds W, but we use an index set to describe the worlds in W because it could also be the case that W contains an infinite number of worlds. Notice that the different worlds in the set W may be reduplicating the very same collection of propositional variables, so that although the worlds in the set are different from one another, no such requirement exists for the propositional variables they contain. This flexibility suggests a multitude of possibilities in specifying the set of worlds W in a given possible-worlds interpretation, which is further augmented by the following definitions.

Definition. Given a set W and some $n \in \mathbb{N}$, an n-*ary relation* on W is a subset of the Cartesian product $W^n = W \times W \times \ldots \times W$, thought of as a collection of ordered n-tuples of elements of W. More specifically, a *binary relation* on W is an n-ary relation on W where

$n = 2$. The binary relation on W is thus a subset of the Cartesian product $W^2 = W \times W$, or a collection of ordered pairs of elements of W. The generic binary relation is conventionally referred to as R and is written between each member of each ordered pair.

A set of worlds W and a binary relation R are the most basic building blocks for possible-worlds interpretations. The following example further illustrates how these two constructs are represented.

Example 6. Let $W = \{w_1, w_2, w_3\}$. Let the binary relation $R = \{(w_1, w_2), (w_2, w_3)\} \subset W \times W$. By convention, we write $w_1 R w_2$ and $w_2 R w_3$.

Taking the notion of a set of worlds composed of propositional variables, and the general definition of a binary relation, we proceed to the following definition.

Definition. A *frame* is an ordered pair $\langle W, R \rangle$, where W is a set of worlds and R is some binary relation, called an accessibility relation, which may be specified between particular pairs of worlds in W. That is, where $w_i, w_j \in W$ with i, j not necessarily distinct, we say "w_j is accessible to w_i if and only if $w_i R w_j$ is a member of the relation R. Otherwise, we say "w_j is not accessible to w_i."

A particular *class of frames* $\mathscr{X}$ can be distinguished by characterizing the relation R. Classifying frames according to particular types of accesibility relations will turn out to be key to providing a definition of validity in possible-worlds semantics.

Combining the components of a frame with an expanded interpretation function, similar to the one described in Chapter 1, completes the formal structure of possible-worlds interpretations, as the following definition states.

Definition. A *possible-worlds interpretation* (or *model*) is an ordered triple $\langle W, R, V \rangle$, where $\langle W, R \rangle$ is a frame and $V : \mathfrak{Var} \to \{0, 1\}$ is an interpretation function which assigns a truth-value to every propositional variable in every world $w_i \in W$. That is, for every propositional variable p in every world $w_i \in W$, either $V(p, w_i) = 1$ and p is said to be "true at world w_i"

under the interpretation $\langle W,R,V \rangle$; or $V(p,w_i) = 0$ and p is said to be "false at world w_i" under the interpretation $\langle W,R,V \rangle$.

Since the truth-values of all propositional variables in a given "world" are provided by V, it follows that, for any world w, the truth values of all well-formed non-modal formulas can be found by slightly modifying the recursively-defined interpretation function rules of classical propositional logic, so that the truth-values are *given at the particular world w.* Let $\langle W,R,V \rangle$ be a model. For any world $w \in W$ and any non-modal formulas A and B:

- If $V(A,w) = 1$, then $V(\neg A,w) = 0$; otherwise, $V(\neg A,w) = 1$.
- If $V(A,w) = 1$ and $V(B,w) = 1$, then $V(A \wedge B,w) = 1$; otherwise, $V(A \wedge B,w) = 0$.
- If $V(A,w) = 1$ or $V(B,w) = 1$, then $V(A \vee B,w) = 1$; otherwise, $V(A \vee B,w) = 0$.
- If $V(A,w) = 0$ or $V(B,w) = 1$, then $V(A \supset B,w) = 1$; otherwise, $V(A \supset B,w) = 0$.
- If $V(A,w) = V(B,w)$, then $V(A \equiv B,w) = 1$; otherwise, $V(A \equiv B,w) = 0$.

These definitions for the interpretation of a non-modal formula are essentially the same in possible-world semantics as they are in classical propositional logic. They all involve only a single world, and on a possible-worlds account we can think of all the statements of classical propositional logic as speaking with reference to only a single world (the "actual" world, as some accounts would have it, or the world of "everything that is the case").

However, since there can be multiple worlds in the set W, with the accesibility relation R potentially obtaining between any number of pairs of worlds, the additional possibility now opens up for offering a recursively-defined interpretation function rule for the truth value of a modal formula as well, whenever that modal formula is interpreted by a particular model $\langle W,R,V \rangle$. Reflecting the key informal insight discussed in the opening paragraphs of this chapter, these two final rules make use of the potential accessibility relation R between different states of affairs in the set W to semantically characterize necessity and possibility, and extend the interpretation function once more so that it applies to the two modal operators. They deserve close attention. Again, let $\langle W,R,V \rangle$ be a model; then for any world $w_i \in W$ and any formula A:

- If, for every $w_j \in W$ such that $w_i R w_j$, $V(A, w_j) = 1$, then $V(\Box A, w_i) = 1$; otherwise, $V(\Box A, w_i) = 0$.
- If, for some $w_j \in W$ such that $w_i R w_j$, $V(A, w_j) = 1$, then $V(\Diamond A, w_i) = 1$; otherwise, $V(\Diamond A, w_i) = 0$.

By providing an interpretation of the modal operators, these two rules summarize the fundamental advance of possible-world semantics. They maintain that, in order for the formula $\Box A$, "it is necessarily the case that A," to be true at world w, it must be true at all worlds accessible to w (not omitting w itself) by the relation R. Similarly, in order for the modal formula $\Diamond A$, "it is possibly the case that A," to be true at world w, it must be true at *at least one* world accessible to w (potentially including w itself).

2.2 Semantic Tableaux

In section 1.2, it was shown that constructing a truth-table for modal formulas in the same manner as one for the formulas of classical propositional logic was a hopeless endeavor. However, a possible-worlds interpretation (a set of worlds, the accessibility relations which obtain between those worlds, and the interpretation function defined at every world in the set) opens a new path for determining the validity of modal formulas. To begin down this path, we make use of an alternative approach to truth-tables from classical propositional logic, known as the *reductio* method.

Instead of demonstrating that every assignment of truth-values to every propositional variable and subformula of the formula in question results in a valid outcome (as with a truth-table), the *reductio* method begins by assuming that there is an assignment of truth-values which shows the formula to be false, as is done in a proof by contradiction. The method then proceeds by pursuing every possible logical consequence of the initial falsifying assumption. The end result of the *reductio* method is an assignment of truth-values to every propositional variable in the formula being tested. If there is even a single assignment of truth-values which is consistent with itself, the method has demonstrated that the original formula is not valid, since the negated formula has been demonstrated to be satisfiable. On the other hand, if the exhaustive search for a consistent assignment of truth-values is without success, the negated formula is unsatisfiable, and so we can be assured that the original formula is valid. The following example uses the *reductio* method to prove the non-modal formula from the truth-table in Example 3.

Example 7. To demonstrate that the formula $(p \supset q) \supset (\neg q \supset \neg p)$ is valid using the *reductio* method, suppose not. That is, assume that $p \supset q$, but suppose it is not the case that $\neg q \supset \neg p$. By definition of implication, the premise $p \supset q$ is true whenever it is the case that $\neg p$ is true or q is true. Since we assumed the premise to be true, at least one of these must be the case, allowing us to arrive at our first attribution of truth values: we know that either p is false or that q is true (or possibly both). Setting aside these possibilities, we now consider the conclusion, $\neg q \supset \neg p$. Since to negate the original formula we assumed that its conclusion was false, we must in turn negate the implication the conclusion contains, which (again, by definition of implication) tells us that $\neg q$ is true and so is $\neg(\neg p)$. We can then assign truth-values on this new basis: q is false and p is true. Since it is a conjunction, this new fact forces us to contradict ourselves: in order for the *reductio* assumption to work, either p is both false and true or q is both false and true (or both); no matter what, the negated formula is unsatisfiable. Therefore the original formula is valid.

The previous example walks us through the *reductio* method, but it does so in a didactic way that relies on the vagaries of ordinary language. Just as a truth-table represents the rules for interpreting non-modal formulas in an exhaustive and tabular form, thereby providing a finite and mechanical process of determining the validity of those formulas, a similar representation of the *reductio* method is sought which, through the successive application of rules, could clearly determine the validity of a modal formula under a possible-worlds interpretation. A particularly clear representation is to be found in *semantic tableaux* (cf. Priest [2008]; Hughes and Cresswell [1996] develop an analagous, though independent, test for validity using what they call semantic diagrams). A semantic tableau applies the *reductio* method to a formula of modal logic, developing a tree diagram according to a list of carefully-specified rules. The tableaux get their tree-like shape because they consist of one or more branches, where each branch is formed from a series of connected nodes. The nodes of the tableau are always marked with some information which is taken to be the case in the model which the test is building up. Usually this is some subformula or propositional variable. For instance, if a node is marked with "p" we would then assume (in the context of the model the tableau is building towards) that it is the case that the propositional variable p is true. If a node is marked "w_1Rw_2" we would similarly assume that w_2 is accessible to w_1 in the context of the model, and so on. By convention, semantic tableaux are always contructed (and read) from top to bottom. The segments connecting the nodes in the tableaux given in Figures 1 through 6 are always marked with downwards-pointing arrows to reflect this intended direction of development.

If the formula to be tested is an implication, an initial listing of the premise and negated conclusion forms the original top node of the tableau. Because everything written on a node is taken to be true, by writing the premise(s) and negated conclusion(s) we are starting out with the assumption that the formula we wish to test for validity is false. This is in accordance with the *reductio* method. Similarly, the top node of a tableau which tests other types of formulas besides implications would be marked with their negation. The method of semantic tableau is supposed to provide an automatic and exhaustive means of determining the validity of any given formula; therefore, it is important that specified rules provide *all* subsequent moves for creating branches based on the initial listing of the top node. Assuming the initial node to be true, the tableaux rules provide a way of carefully diagramming all the possible consequences which follow from that assumption, also in accordance with the reasoning of the *reductio* method. When multiple branches are generated by these rules, they are always built in the downward direction and develop independently of one another. Contrastingly, when a tableaux rule does dictate the creation of a branch, it is instructing that whenever the statement written on the topmost node appears on a given branch, the statement(s) on subsequent (lower) nodes must also appear on that same branch. Tableaux rules are activated based on the content of higher nodes, in descending order, until truth-values can be assigned *exhaustively* for all propositional variables mentioned on all branches.

The ten basic tableaux rules diagrammed in Figure 1 (page 54 below) are another representation of the ordinary recursively-defined interpretation function rules of classical propositional logic, where the rules are specified for an initial branch with some formula composed of subformulas A and B and are assumed to be true at a particular world w_i.

All the tableaux rules which involve the symbols of classical propositional logic are given in Figure 1. It is worth noting that (just like the interpretation function rules for these symbols given above) each tableaux rule involves only a single world, $w_i \in W$, of a given model $\langle W, R, V \rangle$. The successive application of the rules above is enough to complete tableaux for non-modal formulas, and any non-modal formula can be tested for validity using the tableaux rules without generating any infinite branches (branches which never terminate in an assignment of truth-values). Once again, like the truth-functional rules governing the construction of a truth table, the tableaux rules can be applied mechanically, and must be applied *exhaustively,* until they cannot be applied any more, for a given tableau to be considered finished. By repeatedly applying these rules, a tableau develops until each branch on the tableau can be characterized as either open or closed, according to the following definition.

Definition. A *closed* tableaux branch is one on which, for any formula A, the formulas A and $\neg A$ each occupy a node. A branch is *open* if it is not closed.

According to the reasoning of the *reductio* method, a formula is deemed to be valid when every branch of its tableau is closed; contrastingly, even a single open branch on a tableau verifies the formula's negation, and so demonstrates that the original formula is not valid.

Example 8. Now that the basic rules for constructing a tableau are in place, we can use them to diagram the *reductio* method we used to demonstrate the validity of the formula from Example 3, $(p \supset q) \supset (\neg q \supset \neg p)$. See Figure 2a (page 55 below) for the tableau.

Because the formula is non-modal and involves only a single world, we can omit the "**i**'s" on the tableau for simplicity. The topmost node includes the premise $p \supset q$ from the formula as well as a negation of the conclusion, $\neg(\neg q \supset \neg p)$. A quick application of the tableau rules for implication (Figure 1g) and negation of implication (Figure 1h) can test the validity of this formula. The first rule dictates the creation of a branch: one side of the branch has a node marked with $\neg p$ and the other side has a node marked with q. The next rule dictates that every branch that stems from the negation of the implication $\neg q \supset \neg p$ must have two additional nodes, one of which would be marked with p, and the other with $\neg q$. We clearly only need one of those nodes on each existing branch of the tableau to provoke a contradiction and close both branches, which are marked by an **X** to indicate that they are closed.

An open branch in an exhaustively completed tableau indicates a countermodel which falsifies the original formula, demonstrating that it is not valid. The following example shows how a tableau can also determine that a formula is not valid in the discovery of an open branch.

Example 9. In this example we construct a tableau which provides a falsifying countermodel for the formula of propositional logic $(p \supset q) \supset (\neg p \supset \neg q)$. See Figure 2b (page 55 below) for the tableau. Both branches of the tableau are created by the rule for implication generated by the premise $p \supset q$. The remaining nodes arise owing to the rules for negation of implication and double negation. Both branches of the tableau remain open, and both point to the possible truth-values for p and q which falsify the original formula. If p is false and q is true, then the entire formula $(p \supset q) \supset (\neg p \supset \neg q)$ is false; since this original formula has a falsifying interpretation, it is not valid.

In order to construct a tableau capable of testing the validity of modal formulas, four additional rules are required. These are given in Figure 3 (page 55 below) and are grounded in the framework of possible-worlds interpretations. Figures 3a and 3b are rules which apply whenever nodes appear in a tableau containing either a necessity operator or its negation. Just like the tableaux rules for the operators of classical logic, these four rules are a methodical representation of the possible-worlds semantics for the modal operators which have already been considered. If some formula A preceded by the necessity operator is true at world $w_i \in W$, *and* if there is some world $w_j \in W$ such that w_j is accessible to w_i (that is, $w_i R w_j$), then the possible-worlds interpretation of modal logic indicates that A must be true at world w_j, since otherwise $\Box A$ could not be true at w_i. The first rule represents this formally, specifying that if $\Box\mathbf{A}, \mathbf{w_i}$ is on a node which is part of a branch containing $\mathbf{w_i R w_j}$ for some $\mathbf{w_j}$ which appears written in the tableau, then a new node saying $\mathbf{A}, \mathbf{w_j}$ opens below on that branch. The entries into the tableau are here displayed in bold to distinguish them from the semantic content which they represent. The purpose of tableaux rules for the modal operators, like rules for the other operators already considered, is to provide a methodical and exhaustive procedure which could test the validity of modal formulas in the absence of recognition of the semantic interpretation of the symbols in those formulas.

Meanwhile, the second rule accounts for the relationship between the two modal operators. To offer a paraphrased semantic reading of the rule, it says that any branch which passes through a node containing the negation of a necessary formula A ("it is not necessarily the case that A") at world w_i is extended by a node which contains the possibility operator applied to the negation of the original formula ("it is possibly the case that not A") at the same w_i. Similarly, Figures 3c and 3d are rules which apply whenever nodes appear in a tableau containing either a possibility operator or its negation.

The third rule specifies (again, in a paraphrased way) that if a node in a tableaux appears containing some formula A which is qualified with the possibility operator at world w_i, then every branch which passes through that node is extended by another node. The new node states that the formula A is true at some new world w_j, where j is a number that has not yet appeared on the tree, and furthermore that this world w_j is accessible to w_i. Notice the difference from Figure 3a: that rule is only activated when the necessity operator is present and a relation to another world is present in the model up to that point. Contrastingly, Figure 3c only requires the presence of the possibility operator, and its activation creates a new world in the model which is accessible to the world written on the top node.

The final rule (in Figure 3d) again accounts for the interconvertability of the two modal operators. It should be evident that all four rules are an exhaustive formal representation of the semantic meaning of necessity and possibility enabled with possible-worlds interpretations.

Example 10. With these four additional rules in place, it is now possible to construct tableaux capable of testing the validity of simple modal formulas. We return to the modal formula encountered (unsuccessfully) in Example 4. This formula should now be recognizable as the axiom [K] required to build the systems of modal logic outlined in Chapter 1: $\Box(p \supset q) \supset (\Box p \supset \Box q)$. See Figure 4 for the tableau (page 56 below).

As with the formula from Example 8, the premise of the formula and the negated conclusion form the beginning of the tableau in Figure 4. We can set aside the premise $\Box(p \supset q)$**,**w_1 for the time being, since its tableau rule (Figure 3a) doesn't become relevant until there is another world in the model which is accessible to w_1. That leaves the tableau rule for the negation of an implication (Figure 1h), since $\neg(\Box p \supset \Box q)$**,**$w_1$ is also marked at the top of the tableau. The rule leads to the following two nodes on the tableau; we can once again set aside the first one, marked $\Box p$**,**w_1, leaving the second, marked $\neg\Box q,w_1$, to contend with. Negation of a necessity operator (Figure 3b) dictates the next node in the diagram, which as a possibility operator dictates the following node marked w_1Rw_2 and

$\neg q, w_2$. However, the creation of a world accessible to w_1 in the model triggers the rule for the necessity operator which we had hitherto been ignoring, generating the next two nodes marked p, w_2 and $p \supset q, w_2$. This last node gives contradictions to the current evaluations of p and q at w_2. Thus a closed tableau for [K] is achieved through the successive application of the rules given in Figures 1 and 3.

2.3 Worlds, Relations, and Validity

Definition. A formula A of modal logic is said to be *valid on* a frame $\langle W, R \rangle$ if and only if, for every possible-worlds interpretation $\langle W,R,V \rangle$ made from $\langle W, R \rangle$, and for every $w \in W$, it is the case that $V(A, w) = 1$. We say the formula A is *K-valid* if the choice of frame is arbitrary. K-valid formulas are thus valid on every frame.

Although this definition asserts that K-valid formulas are valid for every interpretation $\langle W,R,V \rangle$, the connection between this validity-definition and the system of modal logic known as K must still be proven. The connection must be proven in both directions: when proven one way, the system is said to be *sound;* when the connection is proven in the other direction, the system is said to be *complete*. We begin by defining soundness, in order to prove that the modal systems introduced in Chapter 1 are all sound with respect to their validity-definitions in possible-worlds semantics.

Definition. A system of modal logic is *sound* with respect to a validity-definition if every theorem A of the system is valid according to that definition.

Since a modal system's theorems can in principle be derived from its axioms according to the exhaustive supply of tableaux rules given in Figures 1 and 3, the theorems can in this sense be collectively thought of as proof-theoretic consequences of the axioms which entail them. The method of semantic tableaux has been purposefully laid out so that it can be completed in an automated fashion, with reference only to the tableaux rules and so *without* explicit reference to a possible-worlds interpretation. Recall that, in the context of possible-worlds semantics, an interpretation refers to an ordered triple $\langle W,R,V \rangle$. In contrast to the rules-based notion of proof-theoretic consequence, a theorem is seen to be a logical (or semantic) consequence of its axioms only when it is interpreted as valid on a frame or class of frames. The two notions (proof-theoretic and logical consequence) have been separated out on purpose, and by showing soundness we begin to demonstrate clearly the connection between.

To prove that a modal system is sound with respect to its validity definition is to show that, for every interpretation relevant to the definition and any formula A, if A is a proof-theoretic consequence of a set of axiom-formulas Σ, then it is a logical consequence of the same Σ (succinctly put, soundness means $\Sigma \vdash A$ implies $\Sigma \vDash A$). We begin with the soundness of K with respect to K-validity.

Theorem. *If a formula is a theorem of the system K, then it is K-valid.*

Proof. To prove that every theorem of K is K-valid is to show that they are all valid on any arbitrary frame $\langle W,R \rangle$. It suffices to show that the axioms of K are valid on any frame $\langle W,R \rangle$, because the theorems of K then follow from its axioms. Since a valid formula of classical propositional logic is valid in every world in every possible-worlds interpretation, it is clear that these formulas will be valid on every frame, verifying [PC]. To show that [K] is valid on every frame, suppose not; that is, suppose there exists a possible-worlds interpretation $\langle W,R,V \rangle$ for which $V([K], w_i) = 0$ at some world $w_i \in W$. Then the formula $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is false at w_i. For this interpretation, by definition of implication, $V(\Box(A \supset B), w_i) = 1$, $V(\Box A, w_i) = 1$, and $V(\Box B, w_i) = 0$. Recall that, in a possible-worlds interpretation, for a formula to be "necessarily true" at w_i it must be true at all worlds accessible to w_i. So, since $V(\Box B, w_i) = 0$, there exists some $w_j \in W$ such that $w_i R w_j$ and $V(B, w_j) = 0$. But then, since $V(\Box A, w_i) = 1$, by the same token $V(A, w_j) = 1$. Thus by definition of implication $V(A \supset B, w_j) = 0$. This is a contradiction, since we had already asserted that $V(\Box(A \supset B), w_i) = 1$ and we know $w_i R w_j$. Thus [K] is valid on every frame.

As the system K is axiomatized by [PC]+[K], we conclude that every theorem of K is K-valid. □

This proof is intuitively related to the semantic tableau which is given in Figure 4 (page 56 below).

Shifting our attention to the modal system T, the following example highlights why a separate validity-definition will be needed when working with separate systems of modal logic. Fortunately, these various definitions of validity are intrinsically connected to the framework of possible-worlds interpretations.

Example 11. The axiom [T] is not K-valid in the following simple possible-worlds interpretation $\langle W,R,V\rangle$ (which is thus known as a specific countermodel for [T]). Suppose A is some formula. Let $W=\{w\}$, let $R=\emptyset$, and let $V(A,w)=0$. Since world w is not accessible to itself, and there are no other worlds in the set W, it is vacuously true that $V(A,w_i)=1$ for all worlds w_i where wRw_i; thus $\Box A$ is interpreted to be *true* at w, even though A is false at w. Thus [T], the statement $\Box A\supset A$, is false under this interpretation, and is therefore not K-valid.

Although some statements in the modal system T are not K-valid, we still want to understand under what conditions they might be said to be demonstrably valid. Possible-worlds interpretations are flexible enough to discover just what these conditions might be, beginning with the following definition.

Definition. Let $\mathscr{X}$ be a class of frames. A formula A of modal logic is said to be $\mathscr{X}$*-valid* if and only if, for every frame $\langle W,R\rangle\in\mathscr{X}$ and every possible-worlds interpretation $\langle W,R,V\rangle$ made from $\langle W,R\rangle$, $V(A,w)=1$ for every $w\in W$.

With a general definition in place, we now consider the class of frames relevant to the system T. The countermodel in Example 10 is already oriented to the solution: in order for [T] to be interpreted as true in the countermodel, all that would have been necessary would have been to make the lone world w accessible to itself. We characterize the binary relation which provides for such self-accessibility *reflexive*, and the following definition establishes reflexivity as a special criterion for testing the validity of modal formulas.

Definition. Let $\mathscr{T}$ be the class of reflexive frames $\langle W, R \rangle$. In other words, for every frame in the class $\mathscr{T}$, if $w \in W$, then wRw. It follows that a formula A of modal logic is said to be $\mathscr{T}$*-valid* if and only if, for every frame $\langle W, R \rangle \in \mathscr{T}$ and every possible-worlds interpretation $\langle W, R, V \rangle$ made from $\langle W, R \rangle$, $V(A, w) = 1$ for every $w \in W$.

This definition for $\mathscr{T}$-validity gives rise to a new tableaux-rule to be used when testing for validity on reflexive frames, given in Figure 5a (page 56 below). Notice that the rule merely specifies the condition of reflexivity: when testing for $\mathscr{T}$-validity, for every world w_i which appears at some node in a tableau it becomes safe to assume that w_iRw_i.

Now we may reapproach the question of the soundness of T with respect to possible worlds semantics, by restricting ourselves to possible-worlds interpretations made with reflexive frames.

Theorem. *If a formula is a theorem of the system T, then it is $\mathscr{T}$-valid.*

Proof. It was shown in the previous proof that [PC] and [K] are K-valid, and so valid on every frame, including reflexive frames. Since the system T is axiomatized by [PC]+[K]+[T], it remains to show that [T] is valid on every reflexive frame. We proceed again by contradiction; that is, suppose there is a reflexive model $\langle W, R, V \rangle$ for which $V([T], w) = 0$ at some $w \in W$. Then the formula $\Box A \supset A$ is false at w, which means (again, by definition of implication) that $V(\Box A, w) = 1$ and $V(A, w) = 0$. By assumption, wRw, so there exists a world accessible to w for which A is false, implying $V(\Box A, w) = 0$: a contradiction. Therefore [T], and so the system T, is $\mathscr{T}$-valid. $\Box$

We have just shown that every theorem of the system T is interpreted as true in possible-worlds interpretations based on reflexive frames. This is a significant advance, as it relates the validity-testing procuedure for theorems of T based on semantic tableaux to the logical consequence already seen between the axioms of T and its theorems. The following example highlights this relationship.

Example 12. We return to a theorem of T, $A \supset \Diamond A$ which was examined in Example 5 (page 25 above). Although Example 5 gave an informal "proof" of the theorem, the tableau in Figure 6 (page 57 below) for this same statement formally demonstrates that it is a proof-theoretic consequence of the axioms of T. Because the *reductio* method aims to discover a particular counterexample which falsifies the original formula, the tableau tests the statement for a particular propositional variable p in hopes of building a countermodel which provides a non-contradictory interpretation of p. The tableau begins with representations of the premise, taken for a particular propositional variable p at world w_1, in the model, and the negation of the conclusion, $\neg\Diamond p$, also at world w_1. This latter invokes the tableaux-rule for the negation of a possibility operator (Figure 3d), leading to the following node marked $\Box\neg p, w_1$. Because the tableau is testing for $\mathscr{T}$-validity, we may use the rule from Figure 5a and assume that world w_1 is accessible to itself, e.g., $w_1 R w_1$ in the model. Thus by the tableaux-rule for necessity (Figure 3a), the final node is marked $\neg p, w_1$, closing the branch and verifying the $\mathscr{T}$-validity of the original statement.

The theorem just proven (If a formula is a theorem of the system T, then it is $\mathscr{T}$-valid) shows that this proof-theoretic relationship implies that the formula is also $\mathscr{T}$-valid. Thus from the tableau in Figure 6 it is permissible to draw the conclusion that $A \supset \Diamond A$ is valid on the class of reflexive frames $\mathscr{T}$. The previous proof generalizes this relationship and demonstrates soundness of T with respect to $\mathscr{T}$-validity.

The following definitions, theorems, and proofs establish the soundness of the other systems of modal logic introduced in Section 1.3, utilizing other classifications of the binary relation enabled by possible-worlds interpretations. Just like the exploration of $\mathscr{T}$-validity, these definitions, theorems, and proofs are interesting because they illuminate why, in the possible-worlds interpretation of modal logic, a particular type of binary relation is intrinsically connected to a particular axiom-system. We consider, in turn, all the systems which extend the system K with a single axiom: K4, KB, D, and E. Recall, however, that further systems like KBE and S5 may be constructed using multiple axioms from the original list, and so the soundness of these more complex systems are also effectively being established by way of the following definitions, theorems, and proofs.

Definition. Let $\mathscr{F}$ be the class of transitive frames $\langle W,R\rangle$. In other words, for every frame in the class $\mathscr{F}$, if $w_i,w_j,w_k \in W$ with w_iRw_j and w_jRw_k, then w_iRw_k. It follows that a formula A of modal logic is said to be *$\mathscr{F}$-valid* if and only if, for every frame $\langle W,R\rangle \in \mathscr{F}$ and every possible-worlds interpretation $\langle W,R,V\rangle$ made from $\langle W,R\rangle$, $V(A,w) = 1$ for every $w \in W$.

This definition for $\mathscr{F}$-validity gives rise to a new tableaux-rule to be used when testing for validity on transitive frames, given in Figure 5b (page 56 below).

Theorem. *If a formula is a theorem of the system K4, then it is $\mathscr{F}$-valid.*

Proof. Since the system K4 is axiomatized by [PC]+[K]+[4], it remains to show that [4] is valid on every transitive frame. Again, suppose (for a contradiction) that there is a transitive model $\langle W,R,V\rangle$ for which $V([4],w_i) = 0$ at some $w_i \in W$. Then the formula $\Box A \supset \Box\Box A$ is false at w_i, which means (again, by definition of implication) that $V(\Box A,w_i) = 1$ and $V(\Box\Box A,w_i) = 0$. If the latter formula, $\Box\Box A$, is false at w_i, then its negation $\neg\Box\Box A$ is true at w_i. By definition of the modal operators, $\neg\Box\Box A$ is equivalent to $\Diamond\Diamond\neg A$, so there exists a world $w_j \in W$ such that w_iRw_j $V(\Diamond\neg A,w_i) = 1$. By turns, this implies there exists a world $w_k \in W$ such that w_jRw_k and $V(\neg A,w_k) = 1$. Thus A is false at w_k. Since the model is transitive, w_iRw_k; this contradicts the assumption that $\Box A$ is true at w_i. Therefore [4], and the system K4, are $\mathscr{F}$-valid. $\Box$

Definition. Let $\mathscr{B}$ be the class of symmetric frames $\langle W,R\rangle$. In other words, for every frame in the class $\mathscr{B}$, if $w_i,w_j \in W$ with w_iRw_j, then w_jRw_i. It follows that a formula A of modal logic is said to be *$\mathscr{B}$-valid* if and only if, for every frame $\langle W,R\rangle \in \mathscr{B}$ and every possible-worlds interpretation $\langle W,R,V\rangle$ made from $\langle W,R\rangle$, $V(A,w) = 1$ for every $w \in W$.

This definition for $\mathscr{B}$-validity gives rise to a new tableaux-rule to be used when testing for validity on symmetric frames, given in Figure 5c (page 56 below).

Theorem. *If a formula is a theorem of the system KB, then it is $\mathscr{B}$-valid.*

Proof. Since the system KB is axiomatized by [PC]+[K]+[B], it remains to show that [B] is valid on every symmetric frame. Again, suppose (for a contradiction) that there is a symmetric model $\langle W,R,V\rangle$ for which $V([B],w) = 0$ at some $w \in W$. Then the formula $A \supset \Box\Diamond A$ is false at w, which means (again, by definition of implication) that $V(A,w) = 1$ and $V(\Box\Diamond A,w) = 0$. Thus the formula $\neg\Box\Diamond A$ and so, equivalently, the formula $\Diamond\Box\neg A$, is true at w. But then there exists a world $w_j \in W$ such that w_iRw_j and $V(\Box\neg A,w_j) = 1$. However, since the frame is symmetric, w_jRw_i, implying $V(A,w_i) = 0$, a contradiction. Therefore [B], and the system B, are $\mathscr{B}$-valid. $\Box$

Definition. Let $\mathscr{D}$ be the class of serial frames $\langle W,R\rangle$. In other words, for every frame in the class $\mathscr{D}$ and for every $w_i \in W$, there exists a $w_j \in W$ (not necessarily distinct from w_i) such that w_iRw_j. It follows that a formula A of modal logic is said to be *$\mathscr{D}$-valid* if and only if, for every frame $\langle W,R\rangle \in \mathscr{D}$ and every possible-worlds interpretation $\langle W,R,V\rangle$ made from $\langle W,R\rangle$, $V(A,w) = 1$ for every $w \in W$.

This definition for $\mathscr{D}$-validity gives rise to a new tableaux-rule to be used when testing for validity on serial frames, given in Figure 5d (page 56 below).

Theorem. *If a formula is a theorem of the system D, then it is $\mathscr{D}$-valid.*

Proof. Since the system D is axiomatized by [PC]+[K]+[D], it remains to show that [D] is valid on every serial frame. Again, suppose (for a contradiction) that there is a serial model $\langle W,R,V\rangle$ for which $V([D],w) = 0$ at some $w \in W$. Then the formula $\Box A \supset \Diamond A$ is false at w, which means (again, by definition of implication) that $V(\Box A,w) = 1$ and $V(\Diamond A,w) = 0$. Thus $\neg\Diamond A$ is true at w, and so equivalently $V(\Box\neg A,w) = 1$. Since the frame is serial, however, there exists a $w_j \in W$ such that w_iRw_j. We are forced to conclude that

$V(A,w_j) = 1$ and $V(\neg A,w_j) = 1$, a contradiction. Therefore [D], and the system D, are $\mathscr{D}$-valid. $\square$

Definition. Let $\mathscr{E}$ be the class of right Euclidean frames $\langle W,R\rangle$. In other words, for every frame in the class $\mathscr{E}$, if $w_i,w_j,w_k \in W$ with w_iRw_j and w_iRw_k, then w_jRw_k. It follows that a formula A of modal logic is said to be *$\mathscr{E}$-valid* if and only if, for every frame $\langle W,R\rangle \in \mathscr{E}$ and every possible-worlds interpretation $\langle W,R,V\rangle$ made from $\langle W,R\rangle$, $V(A,w) = 1$ for every $w \in W$.

This definition for $\mathscr{E}$-validity gives rise to a new tableaux-rule to be used when testing for validity on right Euclidean frames, given in Figure 5e (page 56 below).

Theorem. *If a formula is a theorem of the system KE, then it is $\mathscr{E}$-valid.*

Proof. Since the system KE is axiomatized by [PC]+[K]+[E], it remains to show that [E] is valid on every right Euclidean frame. Again, suppose (for a contradiction) that there is a right Euclidean model $\langle W,R,V\rangle$ for which $V([E],w_i) = 0$ at some $w_i \in W$. Then the formula $\Diamond A \supset \Box\Diamond A$ is false at w_i, which means (again, by definition of implication) that $V(\Diamond A,w_i) = 1$ and $V(\Box\Diamond A,w_i) = 0$. Since $\Diamond A$ is true at w_i, there exists a $w_j \in W$ such that w_iRw_j and $V(A,w_j) = 1$. By definition of the modal operators, $\neg\Box\Diamond A$ is equivalent to $\Diamond\Box\neg A$. So $\Diamond\Box\neg A$ is true at w_i, and there exists a w_k such that w_iRw_k and $V(\Box\neg A,w_k) = 1$. Since the frame is right Euclidean, however, w_kRw_j, and so $V(\neg A,w_j) = 1$, a contradiction. Therefore [E], and the system KE, is $\mathscr{E}$-valid. $\square$

With the question of soundness settled for the various modal systems under consideration, we now turn to the question of completeness.

Definition. A system of modal logic is *complete* with respect to a validity-definition if every formula valid by that definition is a theorem of the system.

A validity-definition presupposes a whole class of frames, and so requires an interpretation in order to see that it successfully applies to formulas which are logical consequences of the axioms associated with that definition. Meanwhile, the theoremhood or non-theoremhood of a formula from a set of axioms can be determined mechanically using the method of semantic tableaux (beginning with the axiom and the negation of the formula we wish to test for theoremhood at the top node will generate a tableau which tests this entailment; once the relation of completeness is established for the various validity-definitions, it is enough to check the formula in question by itself at the top node of a tableau which assumes the relevant class of frames). Thus to prove that a modal system is complete with respect to its validity-definition is to show that, for every interpretation relevant to the definition and any formula A, if A is a logical consequence of the set of axioms Σ, then it is a proof-theoretic consequence of the same Σ (meaning $\Sigma \models A$ implies $\Sigma \vdash A$).

Theorem. *If a formula is K-valid, then it is a theorem of the system K.*

Proof. *(by contraposition)* Assume that formula A is not a theorem of the system K. Then the semantic tableau for A has an open branch; call this open branch b. The goal is to find an interpretation $\langle W,R,V \rangle$ for which $V(A,w) = 0$ for some $w \in W$; then A can no longer be considered K-valid, since it will not be valid for every interpretation based on every frame. The required interpretation is defined as follows. Suppose $\langle W,R,V \rangle$ is an interpretation based on branch b of the tableau, and specified for A according to the following construction: W=$\{w_i : w_i$ occurs on $b\}$ and R=$\{w_iRw_j$: w_iRw_j occurs on $b\}$. Finally, where p is any propositional variable contained in A, if p,w_i is at a node on b, then $V(p,w_i) = 1$; if $\neg p,w_i$ is at a node on b, then $V(p,w_i) = 0$.

In order to prove that A is not valid according to the interpretation $\langle W,R,V \rangle$ so constructed, it is necessary to first of all demonstrate that no matter what complex form A takes, if A,w_i is on the branch b, then the interpretation $V(A,w_i) = 1$ really does follow, and if $\neg A,w_i$ is on b, then $V(A,w_i) = 0$ follows. The proof proceeds by cases which consider the possible complexity of the formula A. In cases 1 through 14, B and C are subformulas of A, and so could either be propositional variables or formulas with their own complexity. In the latter instance we can refer recursively to cases 1-14 until all subformulas of A refer back to the base case.

Case 0. Let A be a propositional variable. Suppose $\mathbf{A}, \mathbf{w_i}$ is present on branch b. Then by construction $w_i \in W$ and $V(A, w_i) = 1$, and so by definition $V(\neg A, w_i) = 0$.

Case 1. Let A be of the form $\neg B$. Suppose $\neg\mathbf{B}, \mathbf{w_i}$ is present on branch b. Then by construction $w_i \in W$ and $V(B, w_i) = 0$. Therefore $V(\neg B, w_i) = 1$.

Case 2. Let A be of the form $\neg\neg B$. Suppose $\neg\neg\mathbf{B}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for double negation (Figure 1b) so is $\mathbf{B}, \mathbf{w_i}$; thus by construction $w_i \in W$ and $V(B, w_i) = 1$.

Case 3. Let A be of the form $B \vee C$. Suppose $\mathbf{B} \vee \mathbf{C}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for disjunction (Figure 1c), either $\mathbf{B}, \mathbf{w_i}$ or $\mathbf{C}, \mathbf{w_i}$ is on b. By construction $w_i \in W$. Also by construction, $V(B, w_i) = 1$ or $V(C, w_i) = 1$ and so by definition of disjunction, $V(B \vee C, w_i) = 1$.

Case 4. Let A be of the form $\neg(B \vee C)$. Suppose $\neg(\mathbf{B} \vee \mathbf{C}), \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for disjunction-negation (Figure 1d), both $\neg\mathbf{B}, \mathbf{w_i}$ and $\neg\mathbf{C}, \mathbf{w_i}$ are on b. By construction $w_i \in W$. Also by construction, $V(B, w_i) = 0$ and $V(C, w_i) = 0$, so by definition of disjunction $V(B \vee C, w_i) = 0$.

Case 5. Let A be of the form $B \wedge C$. Suppose $\mathbf{B} \wedge \mathbf{C}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for conjunction (Figure 1e), both $\mathbf{B}, \mathbf{w_i}$ and $\mathbf{C}, \mathbf{w_i}$ are on branch b. By construction $V(B, w_i) = 1$ and $V(C, w_i) = 1$, so by definition of conjunction $V(B \wedge C, w_i) = 1$.

Case 6. Let A be of the form $\neg(B \wedge C)$. Suppose $\neg(\mathbf{B} \wedge \mathbf{C}), \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for conjunction-negation (Figure 1f), either $\neg\mathbf{B}, \mathbf{w_i}$ or $\neg\mathbf{C}, \mathbf{w_i}$ is on b. By construction $w_i \in W$. Also by construction, $V(B, w_i) = 0$ or $V(C, w_i) = 0$, so by definition of conjunction $V(B \wedge C, w_i) = 0$.

Case 7. Let A be of the form $B \supset C$. Suppose $\mathbf{B} \supset \mathbf{C}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for implication (Figure 1g), either $\neg\mathbf{B}, \mathbf{w_i}$ or $\mathbf{C}, \mathbf{w_i}$ is on b. By construction $w_i \in W$. Also by construction, $V(B, w_i) = 0$ or $V(C, w_i) = 1$, so by definition of implication $V(B \supset C, w_i) = 1$.

Case 8. Let A be of the form $\neg(B \supset C)$. Suppose $\neg(\mathbf{B} \supset \mathbf{C}), \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for implication-negation (Figure 1h), both $\mathbf{A}, \mathbf{w_i}$ and $\neg\mathbf{B}, \mathbf{w_i}$ are on b. By construction $w_i \in W$. Also by construction, $V(A, w_i) = 1$ and $V(B, w_i) = 0$, so by definition of implication $V(\neg(B \supset C), w_i) = 0$.

Case 9. Let A be of the form $B \equiv C$. Suppose $\mathbf{B} \equiv \mathbf{C}, \mathbf{w_i}$ is on branch b. By construction $w_i \in W$. By application of the tableaux rule for equivalence (Figure 1i), either $\mathbf{A}, \mathbf{w_i}$ and $\mathbf{B}, \mathbf{w_i}$ are both on the branch b, or $\neg\mathbf{A}, \mathbf{w_i}$ and $\neg\mathbf{B}, \mathbf{w_i}$ are both on b. In the first instance, by construction $V(A, w_i) = 1$ and $V(B, w_i) = 1$; in the latter instance, by construction $V(A, w_i) = 0$ and $V(B, w_i) = 0$. In either instance, by definition of equivalence $V(B \equiv C, w_i) = 1$.

Case 10. Let A be of the form $\neg(B \equiv C)$. Suppose $\neg(\mathbf{B} \equiv \mathbf{C}), \mathbf{w_i}$ is on branch b. By construction $w_i \in W$. By application of the tableaux rule for equivalence-negation (Figure 1j), either $\mathbf{A}, \mathbf{w_i}$ and $\neg\mathbf{B}, \mathbf{w_i}$ are both on the branch b, or $\neg\mathbf{A}, \mathbf{w_i}$ and $\mathbf{B}, \mathbf{w_i}$ are both on b. In the first instance, by construction $V(A, w_i) = 1$ and $V(B, w_i) = 0$; in the latter instance, by construction $V(A, w_i) = 0$ and $V(B, w_i) = 1$. In either instance, by definition of equivalence $V(B \equiv C, w_i) = 0$.

Case 11. Let A be of the form $\Box B$. Suppose $\Box\mathbf{B}, \mathbf{w_i}$ is on branch b. Then for every w_j such that $\mathbf{w_i R w_j}$ is on b, by application of the tableaux rule for necessity (Figure 3a) $\mathbf{B}, \mathbf{w_j}$ is on b. So by construction, $w_i \in W$ and for every $w_j \in W$ such that $w_i R w_j$, $V(B, w_j) = 1$. Therefore $V(\Box B, w_i) = 1$.

Case 12. Let A be of the form $\neg\Box B$. Suppose $\neg\Box\mathbf{B}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for necessity-negation (Figure 3b), $\Diamond\neg\mathbf{B}, \mathbf{w_i}$ is on b. Again, by application of the tableaux rule for possibility (Figure 3c), for some w_j it follows that $\mathbf{w_i R w_j}$ and $\neg\mathbf{B}, \mathbf{w_j}$ are also on the branch. Hence by construction $w_i, w_j \in W$, $w_i R w_j$, and $V(B, w_j) = 0$. Therefore $V(\Box B) = 0$.

Case 13. Let A be of the form $\Diamond B$. Suppose $\Diamond\mathbf{B}, \mathbf{w_i}$ is on branch b. Again, by application of the tableaux rule for possibility, for some w_j it follows that $\mathbf{w_i R w_j}$ and $\mathbf{B}, \mathbf{w_j}$ are also on the branch. Hence by construction $w_i, w_j \in W$, w_iRw_j, and $V(B, w_j) = 1$. Therefore $V(\Diamond B) = 1$.

Case 14. Let A be of the form $\neg\Diamond B$. Suppose $\neg\Diamond\mathbf{B}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for possibility-negation (Figure 3d), $\Box\neg\mathbf{B}, \mathbf{w_i}$ is on b. So for every w_j such that $\mathbf{w_iRw_j}$ is on b, by application of the tableaux rule for necessity (again) $\neg\mathbf{B}, \mathbf{w_j}$ is on b. So by construction, $w_i \in W$ and for every $w_j \in W$ such that w_iRw_j, $V(B, w_j) = 0$. Therefore $V(\Diamond A) = 0$.

Because [K] axiomatizes the system K, it holds in the constructed interpretation $\langle W, R, V\rangle$ and so true at all $w \in W$ which are on the open branch b. Since the branch b is open, $\mathbf{A}, \mathbf{w_i}$ and $\neg\mathbf{A}, \mathbf{w_i}$ are not both present on the branch. By assumption, A is not a theorem of K, so in the exhaustive application of the tableaux rules, for some $w_i \in W$ written on b, $\neg\mathbf{A}, \mathbf{w_i}$ is on b, and the above cases exhaustively show that by construction $V(A, w_i) = 0$. Thus there is an interpretation of K which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not K-valid.

By contraposition, if a formula is K-valid, then it is a theorem of the system K. □

Theorem. *If a formula is $\mathscr{T}$-valid, then it is a theorem of the system T.*

Proof. (*by contraposition*). Assume that formula A is not a theorem of the system T. Then the semantic tableau for A has an open branch; call this open branch b. Since $\mathscr{T}$-validity is restricted to reflexive frames, the proof is almost the same as for K. The only additional requirement is that the interpretation $\langle W, R, V\rangle$ (as constructed in the completeness proof for K) has a reflexive frame when it is constructed from a tableau which tests for $\mathscr{T}$-validity. By application of the tableaux-rule for $\mathscr{T}$-validity (Figure 5a), for any $i \in \mathbb{N}$, if $\mathbf{w_i}$ is written on the branch b then $\mathbf{w_iRw_i}$ is too, so by construction $w_i \in W$ and w_iRw_i, implying the frame $\langle W, R\rangle$ is reflexive. Thus there is an interpretation of T which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not $\mathscr{T}$-valid. By contraposition, if a formula is $\mathscr{T}$-valid, then it is a theorem of the system T. □

Theorem. *If a formula is $\mathscr{F}$-valid, then it is a theorem of the system K4.*

Proof. (*by contraposition*). Assume that formula A is not a theorem of the system K4. Then the semantic tableau for A has an open branch; call this open branch b. Since $\mathscr{F}$-validity is restricted to transitive frames, the only additional requirement is that the interpretation $\langle W,R,V\rangle$ (as constructed in the completeness proof for K) has a transitive frame when it is constructed from a tableau which tests for $\mathscr{F}$-validity. By application of the tableaux-rule for $\mathscr{F}$-validity (Figure 5b), for any $i,j,k \in \mathbb{N}$, if $\mathbf{w_i R w_j}$ and $\mathbf{w_j R w_k}$ are written on b, then so is $\mathbf{w_i R w_k}$, so by construction $w_i, w_j, w_k \in W$, $w_i R w_j$, $w_j R w_k$, and $w_i R w_k$ implying the frame $\langle W,R\rangle$ is indeed transitive. Thus there is an interpretation of K4 which interprets all its premises true at some w_i but interprets A as false at w_i. Therefore A is not $\mathscr{F}$-valid. By contraposition, if a formula is $\mathscr{F}$-valid, then it is a theorem of the system K4. Thus there is an interpretation of K4 which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not $\mathscr{F}$-valid. By contraposition, if a formula is $\mathscr{F}$-valid, then it is a theorem of the system K4. $\square$

Theorem. *If a formula is $\mathscr{B}$-valid, then it is a theorem of the system KB.*

Proof. (*by contraposition*). Assume that formula A is not a theorem of the system KB. Then the semantic tableau for A has an open branch; call this open branch b. Since $\mathscr{B}$-validity is restricted to symmetric frames, the only additional requirement is that the interpretation $\langle W,R,V\rangle$ (as constructed in the completeness proof for K) has a symmetric frame when it is constructed from a tableau which tests for B-validity. By application of the tableaux-rule for $\mathscr{B}$-validity (Figure 5c), for any $i,j \in \mathbb{N}$, if $\mathbf{w_i R w_j}$ is written on b, then $\mathbf{w_j R w_i}$ is also, so by construction $w_i, w_j \in W$, $w_i R w_j$, and $w_j R w_i$ implying the frame is symmetric. Thus there is an interpretation of KB which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not $\mathscr{B}$-valid. By contraposition, if a formula is $\mathscr{B}$-valid, then it is a theorem of the system KB. $\square$

Theorem. *If a formula is $\mathscr{D}$-valid, then it is a theorem of the system D.*

Proof. (*by contraposition*). Assume that formula A is not a theorem of the system D.

Then the semantic tableau for A has an open branch; call this open branch b. Since $\mathscr{D}$-validity is restricted to serial frames, the only additional requirement is that the interpretation $\langle W, R, V\rangle$ (as constructed in the completeness proof for K) has a serial frame when it is constructed from a tableau which tests for $\mathscr{D}$-validity. By finite application of the tableaux-rule for $\mathscr{D}$-validity (Figure 5d), for every $i \in \mathbb{N}$ if $\mathbf{w_i}$ is written on the branch, then possibly so is $\mathbf{w_i R w_j}$ for some $j \in \mathbb{N}$. So by construction if $w_i \in W$ then $w_i R w_j$ for some w_j implying the frame is serial. Thus there is an interpretation of D which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not $\mathscr{D}$-valid. By contraposition, if a formula is $\mathscr{D}$-valid, then it is a theorem of the system D. $\square$

Theorem. *If a formula is $\mathscr{E}$-valid, then it is a theorem of the system KE.*

Proof. (*by contraposition*). Assume that formula A is not a theorem of the system KE. Then the semantic tableau for A has an open branch; call this open branch b. Since $\mathscr{E}$-validity is restricted to right Euclidean frames, the only additional requirement is that the interpretation $\langle W, R, V\rangle$ (as constructed in the completeness proof for K) has a right Euclidean frame when it is constructed from a tableau which tests for $\mathscr{E}$-validity. By application of the tableaux-rule for $\mathscr{E}$-validity (Figure 5e) for any $i, j, k \in \mathbb{N}$, if $\mathbf{w_i R w_j}$and $\mathbf{w_i R w_k}$ are written on b, then so is $\mathbf{w_j R w_k}$, so by construction $w_i, w_j, w_k \in W$, $w_i R w_j$, $w_i R w_k$, and $w_j R w_k$ implying the frame is right Euclidean. Thus there is an interpretation of KE which interprets all its premises true at w_i but interprets A as false at w_i. Therefore A is not $\mathscr{E}$-valid. By contraposition, if a formula is $\mathscr{E}$-valid, then it is a theorem of the system KE. $\square$

Figure 1. Tableaux Rules for Non-modal Formulas

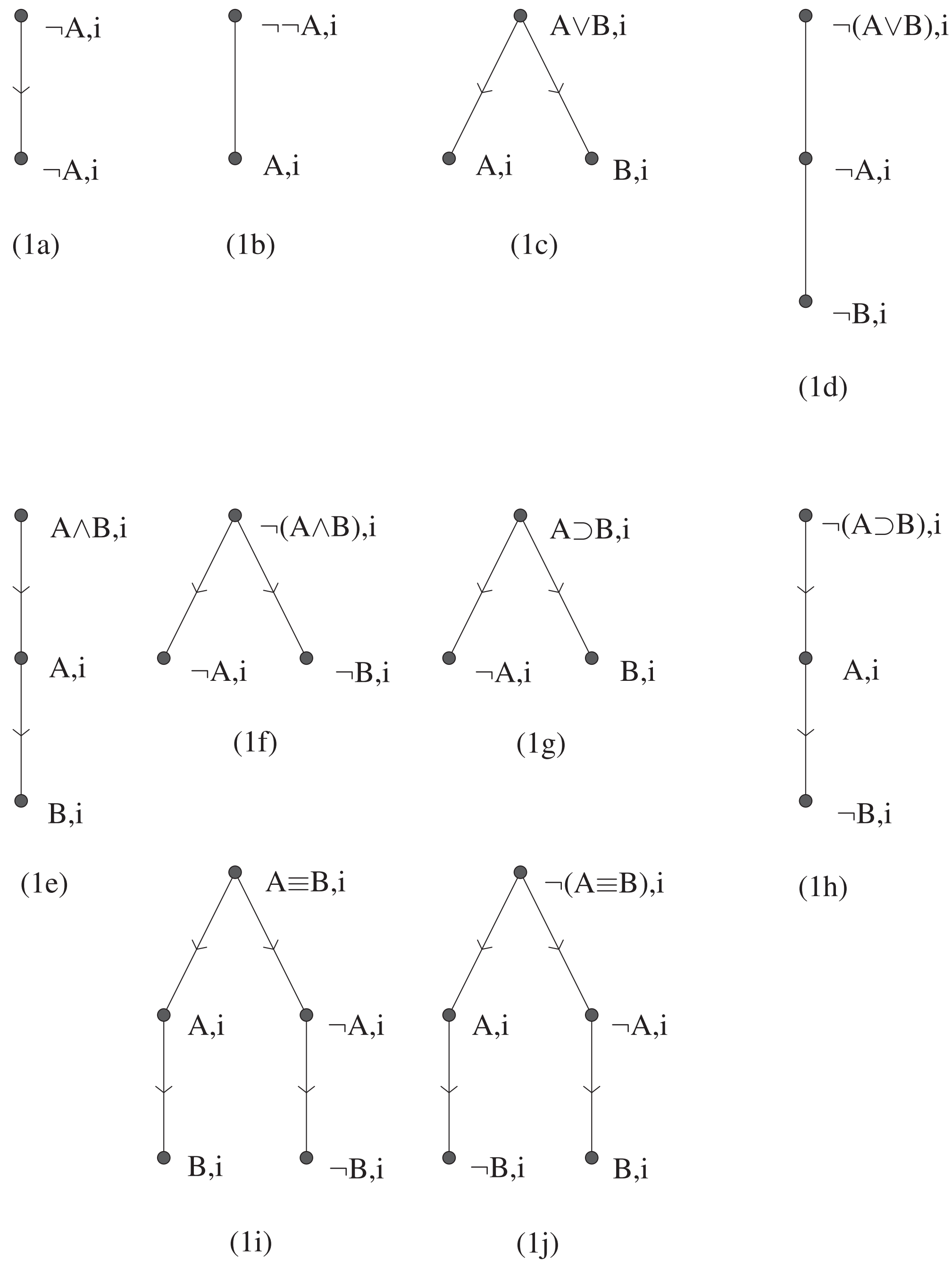


Figure 2. Tableaux for Examples 8 and 9

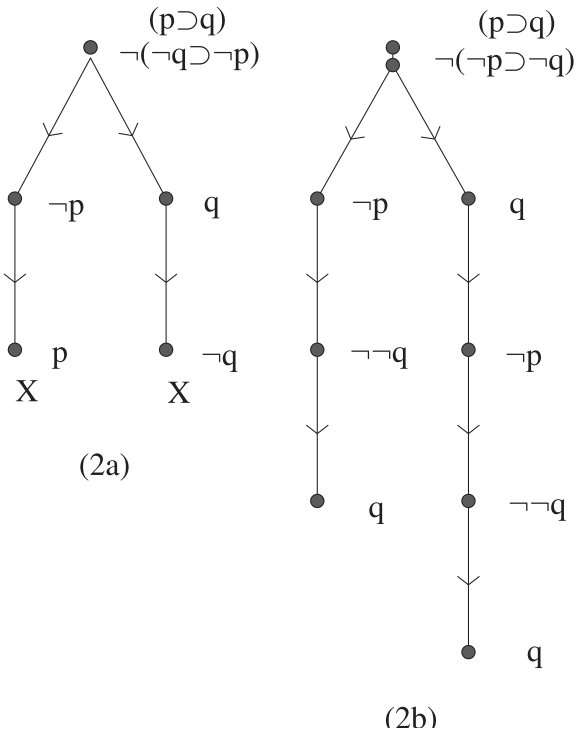


Figure 3. Tableaux Rules for Modal Formulas

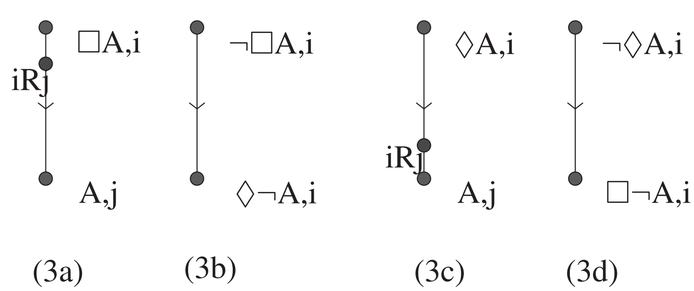


Figure 4. Tableau for Example 10

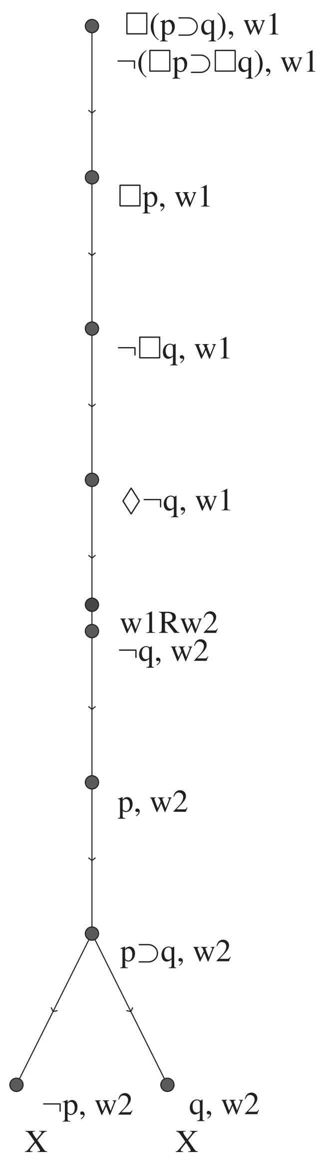


Figure 5. Tableaux Rules for Frame Classes

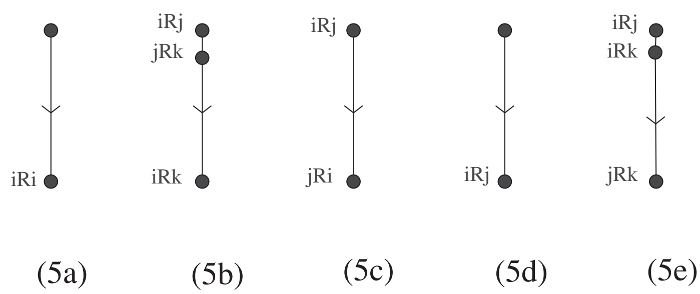
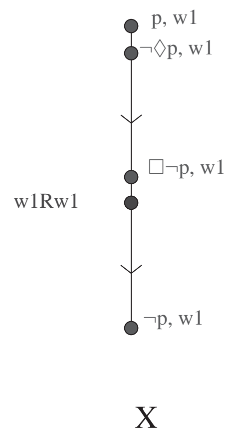


Figure 6. Tableau for Example 12



Chapter 3

Topological Interpretations

This chapter turns from the construction of possible-worlds interpretations to a branch of modern mathematics called *topology*, developing out of some of the basic topological concepts a different interpretation scheme for formulas modal logic. While topologies are abstractly defined as sets which obey certain properties, in practice these sets are visualizable as "spaces" containing sometimes-overlapping subspaces whose borders exhibit different properties. Facts about the sets in a topology reveal information which can usually be given some kind of spatial interpretation. This is for good reason, since the real number line, as well as any Euclidean space, can all be defined as topological spaces and treated as such.

Just like the idea of "possible worlds," topological interpretations rely on an informal insight to connect topology to the statements of logic; understanding this insight makes it much easier to grasp the subsequent formal elaboration. The insight of topological semantics is to identify any logical formula with a subset in the topology, which on a spatial interpretation means that the formula defines a region of the topological space. Inside this region, the formula is interpreted as "true;" outside this region, it is interpreted as "false." This conveniently suggests that a formula is topologically valid when the entire topological space interprets it as true. Section 3.1 presents the required concepts from topology required to formulate, in section 3.2, the formal structure for topological interpretations of statements of modal logic. Finally, section 3.3 explores some of some of the new perspectives which a topological interpretation brings to modality.

3.1 Interior and Closure in Topological Spaces

Definition. Let X be a set. A *topology* $\mathscr{T}$ on X is a collection of subsets of X which satisfy the following conditions:

(i). $\emptyset$ and X are elements of $\mathscr{T}$;

(ii). The intersection of finitely many open sets is an element of $\mathscr{T}$;

(iii). The union of any collection of open sets is an element of $\mathscr{T}$.

The elements of $\mathscr{T}$ are called the *open* subsets of X. We refer to the ordered pair $\langle X, \mathscr{T} \rangle$ as a *topological space*, and also call the set X a topological space wherever it is understood that a topology $\mathscr{T}$ exists on X.

This definition may appear strange to people who are unfamiliar with topology. Although it provides the concept of a topology, the definition does not concretely elaborate on what it means for the sets in the collection to be "open," except to specify that empty set $\emptyset$ and the entire topological space X must be considered open, and any finite intersection or any union of open sets also must count as an open set. The definition of a closed set in a given topological space likewise depends entirely on which sets are open, as the following definition makes clear.

Definition. A subset A of a topological space $\langle X, \mathscr{T} \rangle$ is *closed* if the complement of A, denoted $X \setminus A$ or A^C, is open in X.

Because the definition of a closed set is relative to which sets are open in a topological space, some properties about closed sets in a topological space are analagous to the criteria regarding open sets in the definition of a topology (see Adams & Franzosa [2008], p. 41). In particular, where X is a topological space, the following statements about the the collection of closed sets in X hold:

(i). $\emptyset$ and X are closed;

(ii). The intersection of any collection of closed sets is a closed set;

(iii). The union of finitely many closed sets is a closed set.

Notably, however it is possible for a subset of a topological space to be neither open or closed, or both open and closed at the same time. A basic understanding of open and closed sets in topological spaces provides the two related concepts, interior and closure, needed to provide a topological semantics for modal logic.

Definition. Let A be a subset of a topological space X. The *interior* of A, written $Int(A)$, is the union of all open sets contained in A. Similarly, the *closure* of A, written $Cl(A)$, is the intersection of all closed sets containing A. The definitions of interior and closure can be written formally as follows: where $\mathscr{T}$ is a topology defined on some topological space X, with subset $A \in \mathscr{T}$, we may write $Int(A) := \bigcup\{U \in \mathscr{T} : U \subseteq A\}$ and $Cl(A) := \bigcap\{C : X \backslash C \in \mathscr{T} \text{ and } A \subseteq C\}$.

Related to the definition of interior and closure is the characterization of points which are found in each of these special subsets. Given a particular point in a topological space, it is useful to be able to specify whether or not that point is to be found in the interior or the closure of a subset of the topology. The theorems (found in Adams and Franzosa [2008], p. 57) which describe whether a particular point belongs to the interior or closure of a given subset amount to strategic summaries of the definitions of interior and closure; we state them here without proof: let A be an open subset in some topological space $\langle X, \mathscr{T} \rangle$. For any $x \in X$, $x \in Int(A)$ if and only if there exists an open subset $U \in \mathscr{T}$ such that $x \in U$ and $U \subseteq A$. Similarly, for any $x \in X$, $x \in Cl(A)$ if and only if, for every open subset $U \in \mathscr{T}$, if $x \in U$, then there exists some $y \in U$ such that $y \in A$ (so that U intersects A).

3.2 Topological Semantics for Modal Logic

Let $\mathfrak{Var}$ be the set of all propositional variables from some collection of logical formulas. For some topological space X, let $\mathscr{P}(X)$ be the power set of X (the set of all possible subsets of X that can be formed from the elements of X). We wish to relate the propositional variables to the topology by constructing a valuation function $\upsilon : \mathfrak{Var} \to \mathscr{P}(X)$ which functions similarly to the interpretation function seen classical propositional logic in Chapter 1. For any propositional variable $p \in \mathfrak{Var}$ notice that, by definition of the valuation function, $\upsilon(p)$ is some subset of the topological space X. Topological semantics begins with the assertion that, for any point $x \in X$ in the topological space,

- $x \vDash p$ if and only if $x \in \upsilon(p)$.

In Chapter 1 logical or semantic concequence (represented by the symbol $\vDash$) was used to relate one set of logical formulas (the premise) to another (the conclusion). In topological semantics, the notion now reaches between the topological space and the statement itself, so that a point in topological space can entail a propositional variable. The logical variable p follows from the point when it is contained in a subset of the topological space (called $\upsilon(p)$), and it does not follow from x when x is not contained in the associated subset. In either case, however, the valuation function υ thus indicates "where" in the topological space the propositional variable p is true, so that for any $x \in X$, $x \vDash p$ just in case $x \in \upsilon(p)$. Since for any given propositional variable, p is either true or false, we naturally also assert that the negation of a propositional variable is true just "where" the variable itself is false (e.g., that $x \vDash \neg p$ if and only if $x \nvDash p$, and so $x \notin \upsilon(p)$).

Moving from propositional variables to formulas with their own complexity, let R and S be arbitrary well-formed formulas without specifying whether they are variables or subformulas. The expected relationship between the topological space and the truth value of formulas built from R and S continues to hold: for any $x \in X$,

- $x \vDash \neg R$ if and only if $x \nvDash R$.
- $x \vDash (R \vee S)$ if and only if $x \vDash R$ or $x \vDash S$.
- $x \vDash (R \wedge S)$ if and only if $x \vDash R$ and $x \vDash S$.

Taken together, these statements suggest that the valuation function also indicates "where" a point in the topological space must be if it is to model a conjunction or a disjunction of propositional variables or subformulas. In the case of a disjunction, it must be in the valuation subsets of at least one of the members of the disjunction; in the case of conjunction, it must belong to all of the valuation subsets.

Notice that the four bulleted rules above refer only to the semantic interpretation of the logical operators in a topological space. Following the observation in Lucero-Bryan ([2012], p. 25), we note that with these rules in place, any given well-formed formula R defines a particular subset of a topological space X. All the points found in this particular subset model the statement R and do not model $\neg R$; for this reason we think of topological semantics for R as defining "where" the statement R is true in a topological space X, and this "where" is a subset of X. As was done for the interpretation function in Chapters 1 and 2, we wish to extend the definition of the valuation function υ, which was defined over the domain of propositional variables $\mathfrak{Var}$ above, to include well-formed formulas as well. Doing so will also provide a name for the particular subsets defined by logical formulas in topological semantics. Given a valuation function of the propositional variables υ, the subset of a topological space defined by some formula R is $\upsilon(R) := \{x \in X : x \vDash R\}$. With this extention of the valuation function, we can naturally relate the semantics for logical formulas directly to various subsets in the topological space as follows.

The first relationship follows directly from the definition: for any $x \in X$ and any formula R, $x \vDash R$ if and only if $x \in \upsilon(R)$. A similar relationship holds for negations of formulas. Disjunction and conjunction follow suit. For any $x \in X$ and any formula R, since $x \vDash R$ or $x \vDash S$ implies $x \in \upsilon(R)$ or $x \in \upsilon(S)$, we conclude $x \vDash (R \vee S)$ if and only if $x \in \upsilon(R) \cup \upsilon(S)$. By a similar reasoning, $x \vDash (R \wedge S)$ if and only if $x \in \upsilon(R) \cap \upsilon(S)$.

Implication and equivalence (as well as conjunction) were seen to be derivable from disjunction and negation in Chapter 1, and so these rules for the valuation function are sufficient to provide a topological semantics for all the statements of classical propositional logic. However, the following theorem explores the subsets defined by statements containing implication and equivalence with greater detail, aiming for a clearer understanding of how they are to be interpreted topologically.

Theorem. *Let R and S be arbitrary formulas and let X be a topological space. Then the following statements hold:*

(i). For all $x \in X$, $x \vDash (R \supset S)$ if and only if $\upsilon(R) \subseteq \upsilon(S)$.

(ii). For all $x \in X$, $x \vDash (R \equiv S)$ if and only if $\upsilon(R) = \upsilon(S)$.

Proof. (i). ($\Longrightarrow$). Let $x \in X$ and assume $x \vDash (R \supset S)$. Suppose $x \in \upsilon(R)$. By definition of implication, $x \vDash (\neg R \vee S)$. So $x \vDash \neg R$ or $x \vDash S$. But $x \vDash \neg R$ would imply $x \nvDash R$, and also $x \notin \upsilon(R)$, contradicting the assumption that $x \in \upsilon(R)$. Thus $x \vDash S$, and so $x \in \upsilon(S)$. Therefore $\upsilon(R) \subseteq \upsilon(S)$. ($\Longleftarrow$). Assume $\upsilon(R) \subseteq \upsilon(S)$ and fix $x \in X$ so that $x \in \upsilon(R)$. Then $x \in \upsilon(S)$ by assumption, so $x \vDash S$. Therefore $x \vDash (\neg R \vee S)$ as well, and so by definition of implication $x \vDash (R \supset S)$.

(ii). ($\Longrightarrow$). Let $x \in X$ and assume $x \vDash (R \equiv S)$. First suppose $x \in \upsilon(R)$; by definition of equivalence, $x \vDash (R \supset S)$ and so by the first half of part (i), $x \in \upsilon(R)$. If we suppose instead that $x \in \upsilon(S)$, then by the same reasoning $x \in \upsilon(R)$. Thus $\upsilon(R) = \upsilon(S)$. ($\Longleftarrow$). Assume $\upsilon(R) = \upsilon(S)$. First, fix arbitrary $x \in X$ so that $x \in \upsilon(R)$; by the second half of part (i), $x \vDash (R \supset S)$. Since $\upsilon(R) = \upsilon(S)$, it is also the case that that $x \in \upsilon(R)$; so by the same reasoning $x \vDash (S \supset R)$. Since the choice of x was arbitrary, it follows that $x \vDash (R \equiv S)$ for any $x \in X$. $\square$

Two additional rules provide a topological interpretation of the modal operators. Like the interpretation-function rules which formalized the insight of possible-worlds semantics seen in the previous chapter, these two rules make a powerful leap of interpretation, this time by associating the modal operators with the topological concepts of interior and closure. Although it may not be immediately apparent, these valuation function rules establish the needed conditions to link modal necessity with topological interiority and to link modal possibility with topological closure.

- $x \vDash \Box R$ if and only if there exists an open subset $U \in \mathscr{T}$ such that $x \in U$ and, for every $y \in U$, $y \vDash R$.
- $x \vDash \Diamond R$ if and only if, for every open subset $U \in \mathscr{T}$, if $x \in U$ then there exists some $y \in U$ such that $y \vDash R$.

Given the subset defined by some non-modal operand R, if we wish to more explicitly identify the subsets of a topological space which are defined by a modal formula, we can extend the valuation function υ once again so that $\upsilon(\Box R)$ and $\upsilon(\Diamond R)$ are explicitly associated with interior and closure. Because the demonstration is worth considering carefully, however, we now state the extension as a theorem and prove it (as we did with implication and equivalence above).

Theorem. *Let R be an arbitrary (well-formed) formula and let $\langle X, \mathscr{T} \rangle$ be a topological space. Then the following statements hold:*

(i). For all $x \in X$, $\upsilon(\Box R) = Int(\upsilon(R))$.

(ii). For all $x \in X$, $\upsilon(\Diamond R) = Cl(\upsilon(R))$.

Proof. (i). Suppose that for some $x \in X$, $x \in \upsilon(\Box R)$. By definition $x \vDash \Box R$. So there exists an open subset $U \in \mathscr{T}$ such that $x \in U$ and, for every $y \in U$, $y \vDash R$. It follows that $x \vDash R$, hence $x \in \upsilon(R)$ and $U \subseteq \upsilon(R)$. By definition of interior, $x \in Int(\upsilon(R))$. Thus $\upsilon(\Box R) \subseteq Int(\upsilon(R))$. Now suppose that for some $x \in X$, $x \in Int(\upsilon(R))$. Let $U = Int(\upsilon(R))$. Note that $x \in U$ and that by definition of topology, as a union of open sets U is open. Also, since $U \subseteq \upsilon(R)$ it follows that for all $y \in U$, $y \vDash R$. We conclude that $x \vDash \Box R$, and so by definition $x \in \upsilon(\Box R)$. Thus $Int(\upsilon(R)) \subseteq \upsilon(\Box R)$. Therefore $\upsilon(\Box R) = Int(\upsilon(R))$.

(ii). Assume that for some $x \in X$, $x \in \upsilon(\Diamond R)$. Then by the valuation function rule for closure, for every open subset $U \in \mathscr{T}$, if $x \in U$ then there exists some $y \in U$ such that $y \vDash R$. But suppose (for a contradiction) that $x \notin Cl(\upsilon(R))$. Based on this supposition, there must be some open set $V \in \mathscr{T}$ such that $x \in V$, but for every $y \in V$, $y \notin \upsilon(R)$. But then $y \nvDash R$ for all $y \in V$, which contradicts the assumption. Thus $x \in Cl(\upsilon(R))$. The reverse argument is very similar. Now assume that for some $x \in X$, $x \in Cl(\upsilon(R))$ but suppose (for a contradiction) that $x \notin \upsilon(\Diamond R)$. Since x is in the closure of $\upsilon(R)$, by the definition of closure, for every open subset $U \in \mathscr{T}$, if $x \in U$ then there exists some $y \in U$ such that $y \in \upsilon(R)$. It follows that there exists $y \in U$ such that $y \vDash R$. But by supposition $x \notin \upsilon(\Diamond R)$, which by the valuation function rule for closure would imply that there does not exist $y \in U$ such that $y \vDash R$, a contradiction. Thus $x \in \upsilon(\Diamond(R))$. Therefore $\upsilon(\Diamond R) = Cl(\upsilon(R))$. □

This association between interior, closure, and the modal operators shows that there is ample room in topological semantics to accomodate modality. Now that modal formulas can be interpreted topologically, we can now use these semantics to help answer the same kinds of questions we asked of possible-worlds interpretations in the previous chapter. Can topological interpretations also be used demonstrate that certain modal formulas follow from others? What kinds of modal statements are valid in a topological interpretation, and what kinds are not?

3.3 Topologies and Validity

Some years before modal logic was given possible-worlds interpretations, the mathematicians J.C.C. McKinsey and Alfred Tarski had already advanced a topological interpretation of the systems of modal logic of their day (see McKinsey and Tarski [1944] and [1948]; see also Kishida [2010]). One of their most famous results is summarized in the following theorem.

Theorem. (McKinsey-Tarski). *The system $S4$ is the modal logic of all topological spaces.*

Although a proof of this theorem is not reproduced here, this section intends to discuss its significance and informally illustrate its veracity through a comparison of the axioms of the modal system $S4$ with their topological interpretations. As before, the question of establishing a firm grasp of the relationship between deductive systems of modal logic and the topological semantics which interpret them hinges on the notion of validity. The following definition lays out the meaning of validity under topological interpretations of modal logic.

Definition. A formula of modal logic R is *valid* in a topological space $\langle X, \mathscr{T} \rangle$ if and only if for every valuation function $\upsilon : \mathfrak{Var} \to \mathscr{P}(X)$ and every point $x \in X$, it is the case that $x \vDash R$. We may write $X \vDash R$, since if R is valid then $\upsilon(R) = X$ for every υ.

This definition asserts that a formula of modal logic is valid in a topological space if the formula is semantically modeled by every point in the topological space, regardless of how the various subsets of the topology are definitely associated with the formula. Informally we might say that a valid statement is interpreted as true no matter "where" in the topological space we look. Because there are thus no identifiable subsets of the topological space which would interpret the formula as false, it is said the entire topological space must model a specific formula in order for it to be considered valid.

Because of the wide range of topologies which exist, we can imagine that a given formula may be valid in some topological spaces but not valid in others. The situation is analogous to the different validity definitions which were made possible by possible-worlds interpretations, depending on the class of frame (reflexive, transitive, etc.) which was under consideration. The eventual association of these different frames with different systems of modal logic is one of the most delightfully surprising outcomes of a possible-worlds interpretation of modal logic. Because there are so many conceivable topological spaces, many of which are well-studied for their intricate and unusual properties, we should expect to find similarly interesting connections between statements (and systems) of modal logic and the various topologies which happen to find them valid.

Ultimately, though, the McKinsey-Tarski theorem accomplishes something uniquely possible in topological semantics: it identifies the modal system $S4$ as the definitive modal logic of *all topological spaces*. What this means is that for any set X, any collection of subsets of X which meets the definition of a topology outlined at the beginning of this chapter will interpret all the theorems of $S4$ as valid under topological semantics. This result is especially significant when we consider that many familiar mathematical spaces, perhaps most notably the real number line $\mathbb{R}$, qualify as topological spaces. Thus the theorems of $S4$ correspond to general topological properties which hold in $\mathbb{R}$ and related spaces, including general Euclidean space $\mathbb{R}^n$, and of all other spaces which can be defined topologically, including the rational numbers $\mathbb{Q}$. This result is the kind of strong association between a modal system and the structures of modern mathematics which originally inspired the development of modern modal logic, because it realizes the possibility of finding new bridges between modality and mathematics, and also the possibility of mutual utility between the two. Thanks to the McKinsey-Tarski theorem, topologies can be said to illuminate the contours of $S4$ from a completely different angle, and it is even possible that features of the modal system $S4$ which might appear relatively mundane or easily accessible from the standpoint of modal logic could provide an unexpected toolkit with which to think through problems regarding topological spaces.

The remainder of this chapter focuses on demonstrating the soundness of $S4$ with respect to validity in topological semantics. Because $S4$ is built up from the axioms which compose the more primitive systems K and T, it is convenient to show the soundness of these systems as well.

Theorem. *If a formula is a theorem of the system K, then it is valid for all topological spaces.*

Proof. (informal). Recall that the system K is axiomatized by [PC]+[K]. Since the theorems of K are deducible from its axioms, we will have shown the entire system is valid for all topological spaces when we show that the axioms are similarly valid.

First, consider the axiom schema [PC]: "if A is a valid formula of non-modal propositional logic, then A is an axiom." [PC] states that any formula of logic which is classically valid is taken to be axiomatic for the construction of modal systems. Now, however, it is necessary to demonstrate that all of these classically valid "axioms" of K and the other modal systems are also topologically valid for all topological spaces. Let X be a topological space and let υ be a valuation function. Let $x \in X$. Now, for that x define a function $W_x : \mathfrak{Var} \to \{0,1\}$ as follows: for any propositional variable p,

$$W_x(p) = \begin{cases} 1 & : x \in \upsilon(p) \\ 0 & : x \notin \upsilon(p) \end{cases}$$

We also recursively define W_x for negation and disjunction: for any formulas A and B,

$$W_x(\neg A) = \begin{cases} 1 & : x \notin \upsilon(A) \\ 0 & : x \in \upsilon(A) \end{cases}$$

$$W_x(A \vee B) = \begin{cases} 1 & : x \in \upsilon(A) \ or \ x \in \upsilon(B) \\ 0 & : otherwise \end{cases}$$

Notice that W_x satisfies the rules for an interpretation funtion given in Chapter 1: it interprets A as true just in case $\neg A$ is false, and it interprets $A \vee B$ as true just in case A or B is true. Since the other logical operators are derived from negation and disjunction, W_x is adequately defined to interpret any formula of classical logic. Furthermore, notice that for $x \in X$, the interpretation function W_x interprets arbitrary formula A as true only in the event that $x \in \upsilon(A)$ for the valuation function υ (e.g., just in case $x \vDash A$ in the topological interpretation). We proceed by contraposition; that is, let R be a non-modal logical formula. such that $x \nvDash R$ (then R is not topologically valid in X). But consider the interpretation $W_x(R)$ of the topologically non-valid formula R. Since $x \nvDash R$, it follows that $x \notin \upsilon(R)$. By definition of W_x, then, $W_x(R) = 0$ for this x. Then an interpretation of R exists for which R is false, implying R is not classically valid. By contraposition, if a formula is classically valid, then it is topologically valid as well. Because every classically valid formula of non-modal propositional logic is topologically valid on X, and because the choice of X was arbitrary, the axiom schema [PC] is valid for any topological space.

Next, recall the axiom [K]: $\Box(A \supset B) \supset (\Box A \supset \Box B)$ for arbitrary formulas A and B. The strategy is to show that the topological interpretation of [K] is actually a general property which holds for every topology. Let X be an arbitrary topological space, and let p and q be propositional variables for which an arbitrary valuation function υ is defined. To simplify the axiom [K], the inner implications can be rewritten according to their definition, so that (for p and q) [K] becomes $\Box(\neg p \vee q) \supset (\neg\Box p \vee \Box q)$. Since it was shown that for all $x \in X$, $x \vDash (R \supset S)$ if and only if $\upsilon(R) \subseteq \upsilon(S)$ for any formulas R and S, it follows that the statement is interpreted as $\upsilon(\Box(\neg p \vee q)) \subseteq \upsilon(\neg\Box p \vee \Box q)$ in the topology. Since it was shown that for all $x \in X$, $\upsilon(\Box R) = Int(\upsilon(R))$ for any formula R, it follows that the statement may also be written $Int(\upsilon(\neg p \vee q)) \subseteq \upsilon(\neg\Box p \vee \Box q)$. Continuing in this way, we arrive at the statement $Int(\upsilon(\neg p) \cup \upsilon(q)) \subseteq (\upsilon(\neg\Box p) \cup Int(\upsilon(q)))$. It was stated previously that for any $x \in X$, $x \vDash \neg p$ if and only if $x \nvDash p$, and so $x \notin \upsilon(p)$. But in general, if $x \in X$ and $x \notin \upsilon(R)$, for arbitrary formula R, it follows that x belongs to the complement of $\upsilon(R)$, denoted $\upsilon(R)^C$. On this basis we can rewrite the statement once again as $Int(\upsilon(p)^C \cup \upsilon(q)) \subseteq ((Int(\upsilon(p)))^C \cup Int(\upsilon(q)))$. To simplify this notation, suppose $\upsilon(p) = A$ and $\upsilon(q) = B$ where $A, B \subseteq X$. Thus we ultimately arrive at the statement

$$Int(A^C \cup B) \subseteq (Int(A)^C \cup Int(B)) \tag{3.3.1}$$

Statement (3.3.1) constitutes the topological interpretation of the original axiom [K], rewritten for maximum clarity.

We now show that statement (3.3.1) is a property of arbitrary topological space X. For some $x \in X$, assume that $x \in Int(A^C \cup B)$. Hence we can define an open subset V in the topology on X such that $x \in V$ and $V \subseteq A^C \cup B$. If $x \in Int(A)^C$ then the proof is complete; so suppose that $x \notin Int(A)^C$. Then $x \in Int(A)$. So we may define another open subset U such that $x \in U$ and $U \subseteq A$.

It follows from $V \subseteq A^C \cup B$ and $U \subseteq A$ that the intersection of the subsets, $U \cap V$, is itself a subset of the intersection of the original sets (since if $x \in U$ then $X \in A$ and if $x \in V$ then $x \in A^C \cup B$, it follows that if $x \in U$ and $x \in V$ then $x \in A$ and $x \in A^C \cup B$, so by definition $x \in A \cap (A^C \cup B)$). Thus $U \cap V \subseteq A \cap (A^C \cup B)$. This latter set can also be rewritten using distributive laws as $(A \cap A^C) \cup (A \cap B)$ or, since $A^C \cap A = \emptyset$, as $A \cap B$. Thus $U \cap V \subseteq A \cap B \subseteq B$. Recall that $x \in U$ and $x \in V$, so $x \in U \cap V$, and observe that as a finite intersection of open sets, the set $U \cap V$ is open in the topology on X. Since x belongs to an open subset of B, by definition $x \in Int(B)$. Thus $x \in (Int(A)^C \cup Int(B))$. Therefore (3.3.1), the statement $Int(A^C \cup B) \subseteq (Int(A)^C \cup Int(B))$, holds. Since the choice of X was arbitrary, statement (3.3.1) holds for any topological space, and so [K] is valid for all topological spaces.

As the system K is axiomatized by [PC]+[K], we conclude that every theorem of K is valid for all topological spaces. $\square$

Theorem. *If a formula is a theorem of the system T, then it is valid for all topological spaces.*

Proof. (informal). Let X be an arbitrary topological space, and let p be some propositional variable for which an arbitrary valuation function υ is defined. Recall that the system T is axiomatized by [PC]+[K]+[T]. Since the theorems of T are deducible from its axioms, we will have shown the entire system is valid for all topological spaces when we show that the axiom [T] is similarly valid. Recall the axiom [T]: $\Box A \supset A$ for arbitrary formula A. By reference to the valuation rules and proofs from the previous section this statement is interpreted as $Int(\upsilon(A)) \subseteq \upsilon(A)$. Since this is an elemental property of the interior, [T] is valid for all topological spaces. $\square$

Theorem. *If a formula is a theorem of the system S4, then it is valid for all topological spaces.*

Proof. (informal). Let X be an arbitrary topological space, and let p be some propositional variable for which an arbitrary valuation function υ is defined. Recall that the system

$S4$ is axiomatized by [PC]+[K]+[T]+[4]. Since the theorems of $S4$ are deducible from its axioms, we will have shown the entire system is valid for all topological spaces when we show that the axiom [4] is similarly valid. Recall the axiom [4]: $\Box A \supset \Box\Box A$ for arbitrary formula A. Again, by reference to the valuation rules and proofs from the previous section this statement is interpreted as $Int(\upsilon(A)) \subseteq Int(Int(\upsilon(A))$. This is a general property of the interior, since $Int(Int(U)) = Int(U)$ for any $U \subseteq X$. Thus [4] is valid for all topological spaces. We conclude that every theorem of S4 is valid for all topological spaces. $\Box$

The next logical step after showing the soundness of $S4$ with respect to its topological interpretation would be to show completeness, by proving that if a formula of modal logic is valid for all topological spaces, then it is a theorem of $S4$. Due to the elaborate nature of the proof the reader is referred to proofs of the McKinsey-Tarski theorem instead. The soundness proofs should already suggest the unique status of $S4$ for topological interpretations: there are topologically valid statements which are not theorems of K and T, but which are theorems of $S4$. So a completeness proof for K or T on a topological interpretation is not possible in the same way that it was on a possible-worlds interpretation.

Similarly, systems "stronger" than $S4$, like the system $S5$ axiomatized by [PC]+[K]+[T]+[E], are not sound with respect to the topological meaning of validity. The statement [E], $\Diamond A \supset \Box\Diamond A$ for arbitrary formula A, would be interpreted topologically as $Cl(A) \subseteq Int(Cl(A))$. While there may be particular cases where this statement holds, it is not a general property of interior and closure, and so the theorems of $S5$ are not in general valid for all topological spaces.

Chapter 4

Tense Logic

4.1 Syntax for Tense Logic

Tense logic is a variant of modal logic. It uses the same modal operators introduced in the syntax for modal logic but redoubles each operator so that there are two distinct pairs of operators with similar definitions. This means that there are *two* tense operators which are taken as primitive: $[F]$ and $[P]$. The presence of the negation symbol in the logical syntax implies that each of these two operators have a dual defined just the same as for modal logic. Thus the four basic modal operators of tense logic, with their definitions, are given as follows for any formula A:

[$\langle F \rangle$] $\langle F \rangle A := \neg [F] \neg A$

[$[F]$] $[F]A := \neg \langle F \rangle \neg A$

[$\langle P \rangle$] $\langle P \rangle A := \neg [P] \neg A$

[$[P]$] $[P]A := \neg \langle P \rangle \neg A$

This formalism, following the one introduced by Priest [2008], is already oriented to the particular temporal interpretation of modality which tense logic opens up, since in interpretations of tense logic to be considered shortly, "P" will stand for "past" and "F" will stand for "future" when interpreting these letters in the four operators just defined.

One additional concept is required when considering the syntax of tense logic, which relates the two pairs of operators to one another as they appear in logical formulas.

Definition. Let A be any formula of tense logic. Then the *mirror image of* A, denoted $\breve{A}$, is the formula obtained by writing all the "P"s as "F"s, and all the "F"s as "P"s, as they appear within the modal operators in A. It follows immediately that the mirror image of $\breve{A}$ is just A.

Example 13. Consider the formula $\langle P\rangle[F](p \vee q) \supset \langle F\rangle q$. The mirror image of the formula is obtained by writing a P for every F and an F for every P as they appear in the operators of the original formula. Thus the mirror image of the original formula is just $\langle F\rangle[P](p \vee q) \supset \langle P\rangle q$.

Other than the modification of the modal operators, the syntax for tense logic is the same as that for modal logic as discussed in section 1.1 above.

4.2 Possible-Worlds Semantics for Tense Logic

As with its syntax, possible-worlds semantics for tense logic are much the same as for regular modal logic, with some minor modifications. This means the interpretations still consist of the construction of models $< W, R, V >$ as outlined in section 2.1 above. Now, however, the modal operators are explicitly oriented to a temporal interpretation, so a state of affairs $w \in W$ must also be considered as temporally defined and related, perhaps as a *moment w* or a *world at time w*. Furthermore, the relation between the members of W takes on the special characteristic of temporal sequence: where $w_1, w_2 \in W$ are arbitrary worlds, whatever else characterizes the relation between them, for a possible-worlds interpretation of tense logic $w_1 R w_2$ is taken to mean "$w_1$is temporally prior to w_2."

The possible-worlds interpretation of the tense-logical operators can be summarized as follows. Let $w_i \in W$:

- If, for every $w_j \in W$ such that $w_i R w_j$, $V(A, w_j) = 1$, then $V([F]A, w_i) = 1$; otherwise, $V([F]A, w_i) = 0$.

- If, for some $w_j \in W$ such that $w_i R w_j$, $V(A, w_j) = 1$, then $V(\langle F\rangle A, w_i) = 1$; otherwise, $V(\langle F\rangle A, w_i) = 0$.

- If, for every $w_j \in W$ such that $w_j R w_i$, $V(A, w_j) = 1$, then $V([P]A, w_i) = 1$; otherwise, $V([P]A, w_i) = 0$.

- If, for some $w_j \in W$ such that $w_j R w_i$, $V(A, w_j) = 1$, then $V(\langle P \rangle A, w_i) = 1$; otherwise, $V(\langle P \rangle A, w_i) = 0$.

The interpretation for the future tense operators is identical to the interpretation of the modal operators seen in Chapter 2. In the context of a possible-worlds model, if $[F]A$ is true at $w \in W$, this can now be equivocally interpreted as "at all worlds accessible to w, A is true," or "at all future times for w, A is true." Likewise if $\langle F \rangle A$ is true at w, we could say "at some future time for w, A is true." In the possible-worlds interpretation of the past tense operators, the order of accessibility is reversed, while everything else remains the same. Thus, if in some possible-worlds interpretation, for every $w \in W$, if every $w_i \in W$ which can access w (e.g. for which $w_i R w$) $V(A, w_i) = 1$, then $[P]A$ is true at w under that interpretation, and we might say "at all times prior to w, A is true." Likewise if $\langle P \rangle A$ is true at w, we would say "at some time prior to w, A is true."

A possible-worlds interpretation of tense logic can be obtained from an existing interpretation $\langle W, R, V \rangle$ of modal logic with no additional information. The relation R in a possible-worlds interpretation becomes identically the relation for the operators $[F]$ and $\langle F \rangle$, while the same relation could be expressed for the past operators $[P]$ and $\langle P \rangle$ instead using the converse of the original relation. Expressed formally, let $w_1, w_2 \in W$. Comparing the original relation R when used as an interpretation of tense logic, where R_F is the original relation relative to the future operators, and R_P is the original relation relative to past operators, then $w_1 R w_2$ if and only if $w_1 R_F w_2$ (the two are identical), but $w_1 R w_2$ if and only if $w_2 R_P w_1$ (the two are converses). This relativity of modal statements about past and future means that for any interpretation of a statement of tense logic A, there is a corresponding interpretation of $\breve{A}$, the mirror image of A, which is obtained by taking the converse of the relation R in the original relation. Let this converse be denoted $\breve{R}$, and observe that for any

$w_1, w_2 \in W$, $w_1 R w_2$ iff $w_2 \check{R} w_1$. It follows that an interpretation of A using R is the same as an interpretation of $\breve{A}$ using $\check{R}$. The following example illustrates this connection.

Example 14. We return to the statement of tense logic, $\langle P\rangle[F](p \vee q) \supset \langle F\rangle q$, used in the previous example. To approximate this statement with English, it maintains that "if, at some time in the past, it was the case that at all future times p or q is true, then at some time in the future q is true." In that example we observed that the mirror image of this statement is $\langle F\rangle[P](p \vee q) \supset \langle P\rangle q$. This statement might be thought of as reading "if, at some time in the future, it will be the case that at all times in the past p or q is true, then at some time in the past, q was true." It is worth weighing the semantic value of these statements and considering whether they might be valid on an intuitive level. (It turns out that neither is valid, and we could construct a falsifying countermodel which shows this). For now, however, we construct an interpretation $\langle W, R, V\rangle$ for which the original statement, $\langle P\rangle[F](p \vee q) \supset \langle F\rangle q$ is true. Then we will see that the mirror image is true in an interpretation which replaces R with its converse, $\check{R}$.

Let $W = \{w_1, w_2, w_3\}$ and suppose $R = \{w_1 R w_2, w_1 R w_3, w_2 R w_3, w_3 R w_3\}$. For the construction of V it is enough to suppose that q is true at w_3. To see that this construction provides an interpretation $\langle W,R,V\rangle$ for which $\langle P\rangle[F](p \vee q) \supset \langle F\rangle q$ is true, notice that since q is always true, for every $w_i \in W$, for at least one $w_j \in W$, $w_i R w_j$ and $V(q, w_j) = 1$. Thus $\langle F\rangle q$ is true in the interpretation $\langle W,R,V\rangle$, and so the original statement is as well.

The corresponding interpretation of the mirror image, $\langle W, \check{R}, V\rangle$, is found by taking the converse of R, so that $\check{R} = \{w_2 R w_1, w_3 R w_1, w_3 R w_2, w_3 R w_3\}$. Then the mirror image $\langle F\rangle[P](p \vee q) \supset \langle P\rangle q$ is true in this interpretation by a similar reasoning, since for every world in W there is at least one world (w_3) which can access that world and for which q is true. Thus $\langle P\rangle q$ is true in the interpretation $\langle W, \check{R}, V\rangle$, and so the mirror image is true as well.

Because possible-worlds interpretations of $[F]$, $\langle F\rangle$, $[P]$, and $\langle P\rangle$ are already familiar, their tableaux rules are not difficult to provide. As might be expected, tablueaux rules for

$[F]$, $\langle F \rangle$, and their negations are identical to the analagous rules for $\Box$ and $\Diamond$. Similarly, the rules for $[P]$ and $\langle P \rangle$ are the same as these rules with the relation R reversed. All eight of these tableaux rules appear in Figure 7 (page 81 below). With these rules in place, it becomes possible to test a statement of tense logic for validity with the same procedure outlined in section 2.2.

Definition. Let Z be some system of modal logic, with $\mathscr{Z}$ being the class of frames relevant to the validity-definition of Z. Then Z^t is the system of tense logic based on Z.

An interesting feature of tense logic is that it enables one to test the validity of various tense-logical formulas under different conceptualizations of temporality based on the class $\mathscr{Z}$ of frames under consideration. In the previous chapter it was shown how these frames were connected with specific systems of modal logic. For example, it might seem reasonable to assume that moments of time carry the transitive property (e.g., if w_1, w_2 and w_3 are moments, then if w_1 is prior to w_2 and w_2 is prior to w_3, then w_1 is prior to w_3). However, the class of transitive frames was called $\mathscr{T}$ in the previous chapter, and it was demonstrated that $\mathscr{T}$-validity is linked with the modal system K4, and so it might be of especial interest to work with theorems of the system of tense logic $\mathrm{K4}^t$ or even $S4^t$ (see Section 4.3 below). Arthur Prior (especially in [1967]) engaged in the first elaborate hunt for the system of modality which was the "true" representative of temporality as we know it. The question is complicated and highly interesting; a brief and illuminating discussion takes place in Hughes and Cresswell [1996, p. 127-136].

Most fundamentally, we call K^t the system of tense logic based on the modal system K, recalling that K-validity is special since K-valid formulas are valid on every frame. Because of its foundational significance for so many other systems of modal logic, and the near-universality of K-validity, K^t provides a tense logic which makes as few assumptions about the nature of time as possible. So this system will be the tense logic we explore in this section, with the thought that such an exploration lays the main groundwork for extensions to various other systems of tense logic (it is relatively straightforward to extend possible-worlds interpretations for extensions of tense logic beyond K^t (see Priest [2008] for a brief discussion). For the remainder of this section we will be showing that K^t is sound and complete with respect to possible-worlds semantics. This involves extending the proofs given in Chapter 2 for soundness and completness of K with respect to possible-worlds semantics.

Theorem. *If a formula is a theorem of the system K^t, then it is K-valid.*

Proof. The proof consists in showing that the tense-logical versions of the axiom [K] are true for an arbitrary possible-worlds interpretation $\langle W, R, V\rangle$ of tense logic. This means showing the formula $[F](A \supset B) \supset ([F]A \supset [F]B)$ and its mirror image, $[P](A \supset B) \supset ([P]A \supset [P]B)$ are true regardless of the choice of frame. For the future-tense version of [K], the proof has already been done, since the semantics for $[F]$ are identical to those for $\Box$. To show that the past-tense version of [K] is valid on every frame, suppose not; that is, suppose there exists a possible-worlds interpretation $\langle W, R, V\rangle$ for which this formula is false at some $w_i \in W$. Then $V([P](A \supset B) \supset ([P]A \supset [P]B), w_i) = 0$. For this interpretation, by definition of implication, $V([P](A \supset B), w_i) = 1$, $V([P]A, w_i) = 1$, and $V([P]B, w_i) = 0$. Hence there must be some world $w_j \in W$ such that $w_j R w_i$ and $V(B, w_j) = 0$. But since it follows from the assumptions that $V(A, w_j) = 1$ and $V(A \supset B, w_j) = 1$, we conclude by definition of implication that $V(B, w_j) = 1$, a contradiction. $\Box$

Theorem. *If a formula is K-valid, then it is a theorem of the system K^t.*

Proof. (by contraposition). Assume formula A is not a theorem of the system of tense logic K^t. Then, as before, the semantic tableau for A has an open branch; call this open branch b. Since the tableaux rules for $[F]$ and $\langle F\rangle$ are identical to those for $\Box$ and $\Diamond$, and the rest of the tableaux rules are unchanged, the proof is almost the same as the proof for completeness of K, with the the construction of the interpretation $\langle W, R, V\rangle$, the base case, and cases 1 through 14 essentially unchanged; the only new cases are for $[P]$ and $\langle P\rangle$.

Case 15. Let A be of the form $[P]B$. Suppose $[\mathbf{P}]\mathbf{B}, \mathbf{w_i}$ is on branch b. Then for every w_j such that $\mathbf{w_j R w_i}$ is on b, by application of the tableaux rule for $[P]$ (Figure 7e) $\mathbf{B}, \mathbf{w_j}$ is on b. So by construction, $w_i \in W$ and for every $w_j \in W$ such that $w_j R w_i$, $V(B, w_j) = 1$. Therefore $V([P]B, w_i) = 1$.

Case 16. Let A be of the form $\neg[P]B$. Suppose $\neg[\mathbf{P}]\mathbf{B}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for $\neg[P]$ (Figure 7f), $\langle \mathbf{P} \rangle \neg \mathbf{B}, \mathbf{w_i}$ is on b. By application of the tableaux rule for $\langle P \rangle$ (Figure 7g), for some w_j it follows that $\mathbf{w_j R w_i}$ and $\neg \mathbf{B}, \mathbf{w_j}$ are also on the branch. Hence by construction $w_i, w_j \in W$, $w_j R w_i$, and $V(B, w_j) = 0$. Therefore $V([P]B) = 0$.

Case 17. Let A be of the form $\langle P \rangle B$. Suppose $\langle \mathbf{P} \rangle \mathbf{B}, \mathbf{w_i}$ is on branch b. By application of the tableaux rule for $\langle P \rangle$, for some w_j it follows that $\mathbf{w_j R w_i}$ and $\mathbf{B}, \mathbf{w_j}$ are also on the branch. Hence by construction $w_i, w_j \in W$, $w_j R w_i$, and $V(B, w_j) = 1$. Therefore $V(\langle P \rangle B) = 1$.

Case 18. Let A be of the form $\neg\langle P \rangle B$. Suppose $\neg\langle \mathbf{P} \rangle \mathbf{B}, \mathbf{w_i}$ is on branch b. Then by application of the tableaux rule for $\neg\langle P \rangle$ (Figure 7h), $[\mathbf{P}]\neg \mathbf{B}, \mathbf{w_i}$ is on b. So for every w_j such that $\mathbf{w_j R w_i}$ is on b, by application of the tableaux rule for $[P]$ once again, $\neg \mathbf{B}, \mathbf{w_j}$ is on b. So by construction, $w_i \in W$ and for every $w_j \in W$ such that $w_j R w_i$, $V(B, w_j) = 0$. Therefore $V(\langle P \rangle A) = 0$.

With these cases in place, it follows that there is an interpretation of K (and so K^t) which interprets all its premises as true at some $w_i \in W$ but interprets A as false at w_i. Therefore A is not K-valid (and so not valid in the tense logic K^t). □

4.3 Topological Semantics for a Future-Tense Logic

Because the formal connection of the future tense operators $[F]$ and $\langle F \rangle$, as well as their possible-worlds interpretation, are identical to those for the modal operators $\Box$ and $\Diamond$, it is intuitive that a topological interpretation of tense logic is also possible so long as the tense logic remains *future-oriented*: so long as it only involves statements about the future and is not extended to statements about the past. It is worthwhile to examine this possibility carefully, since a topological interpretation of tense logic provides a way of thinking about modality which is fully independent of the metaphor of "possible worlds," which was the goal mentioned in the introduction.

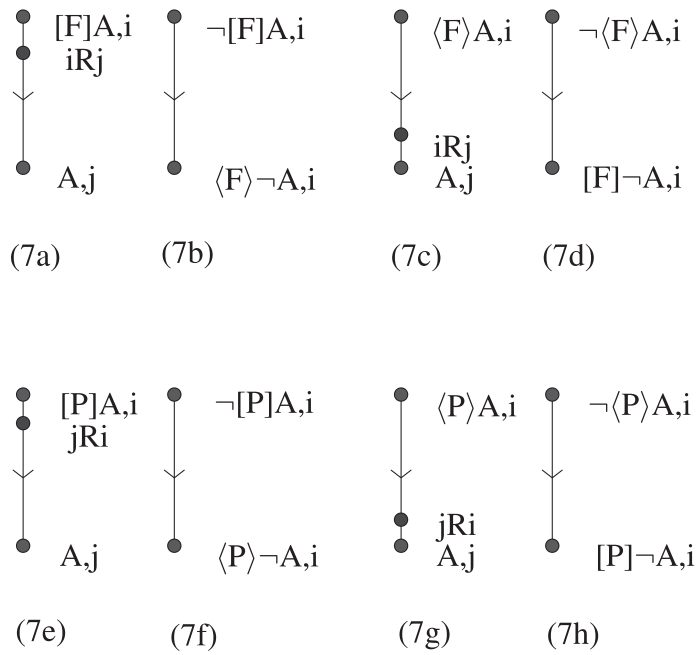
The semantics are virtually identical to those for modal logic given in Chapter 3. Let X be a topological space and $\upsilon : \mathfrak{Var} \rightarrow \mathscr{P}(X)$ be a valuation function. For any $x \in X$ and any propositional variable p or arbitrary well-formed formulas R and S, the following rules hold in a topological interpretation of future-tense logic:

- $x \vDash p$ if and only if $x \in \upsilon(p)$.
- $x \vDash \neg R$ if and only if $x \nvDash R$.
- $x \vDash (R \vee S)$ if and only if $x \vDash R$ or $x \vDash S$.
- $x \vDash (R \wedge S)$ if and only if $x \vDash R$ and $x \vDash S$.
- $x \vDash [F]R$ if and only if there exists an open subset $U \in \mathscr{T}$ such that $x \in U$ and, for every $y \in U$, $y \vDash R$.
- $x \vDash \langle F \rangle R$ if and only if, for every open subset $U \in \mathscr{T}$, if $x \in U$ then there exists some $y \in U$ such that $y \vDash R$.

Because these valuation rules are identical to those for the modal operators, we can still extend our conception of $\upsilon(R) := \{x \in X : x \vDash R\}$ as defining a particular subset "where" in a topological space X a given formula R is interpreted as true. In particular, it is still the case that the equalities $\upsilon([F]R) = Int(\upsilon(R))$ and $\upsilon(\langle F \rangle R) = Cl(\upsilon(R))$ hold for any formula R and any valuation function υ.

As the McKinsey-Tarski theorem states, because $S4$ is the modal logic of all topological spaces, when interpreting the future-tense operators topologically we make the added assumption that $S4$ is an adequate system of logic for speaking about the future, since any topologially-valid formula will also be a theorem of the system $S4$. Thus formulas which follow from the axioms of *S*4 can be read as statements oriented toward the future and interpreted in terms of interior and closure in a topological space.

Figure 7. Tableaux Rules for Tense Logic



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